Dancing with the stars

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Appendix A

Derivation of tomographic equivalence

We now explicitly prove the assertion that Eq. 2.24 is equal to Eq. 2.26, that is,

$$\int P(\boldsymbol{\tau} \mid \boldsymbol{n_e}) P(\boldsymbol{n_e} \mid \boldsymbol{\tau}_{\text{obs}})\, \mathrm{d}\boldsymbol{n_e} = \int P(\boldsymbol{n_e}, \boldsymbol{\tau} \mid \boldsymbol{\tau}_{\text{obs}})\, \mathrm{d}\boldsymbol{n_e}. \tag{A.1}$$

We note that we sometimes use the notation $\mathcal{N}[a \mid m_a, C_a]$ which is equivalent to $a \sim \mathcal{N}[m_a, C_a]$.

We define the matrix representation of the DRI operator in Eq. 2.6, $\boldsymbol{\Delta}_* \boldsymbol{n_e} = \{\Delta_{\mathbf{x}}^{\hat{\boldsymbol{k}}} n_e \mid (\mathbf{x}, \hat{\boldsymbol{k}}) \in \mathcal{S}_*\}$, and likewise let $\boldsymbol{\Delta}$ be the matrix representation over the index set $\mathcal{S}_{\text{obs}}$. Similarly, the matrix representation of the FED kernel – the Gram matrix – is $\boldsymbol{K} = \{K(\mathbf{x}, \mathbf{x}') \mid \mathbf{x}, \mathbf{x}' \in \mathcal{X}\}$. Using these matrix representation we have the following joint distribution,

$$P(\boldsymbol{n_e}, \boldsymbol{\tau}, \boldsymbol{\tau}_{\text{obs}}) = \mathcal{N}\begin{bmatrix} \bar{n}_e & \boldsymbol{K} & \boldsymbol{K}\boldsymbol{\Delta}_*^T & \boldsymbol{K}\boldsymbol{\Delta}^T \\ 0\,, & \boldsymbol{\Delta}_*\boldsymbol{K} & \boldsymbol{\Delta}_*\boldsymbol{K}\boldsymbol{\Delta}_*^T & \boldsymbol{\Delta}_*\boldsymbol{K}\boldsymbol{\Delta}^T \\ 0 & \boldsymbol{\Delta}\boldsymbol{K} & \boldsymbol{\Delta}\boldsymbol{K}\boldsymbol{\Delta}_*^T & \boldsymbol{\Delta}\boldsymbol{K}\boldsymbol{\Delta}^T + \sigma^2 \boldsymbol{I} \end{bmatrix}. \tag{A.2}$$

Let us first work out the left-hand side (LHS) of Eq. A.1. Because $\boldsymbol{\tau} = \boldsymbol{\Delta}_* \boldsymbol{n_e}$, and using standard Gaussian identities we have,

$$P(\boldsymbol{\tau} \mid \boldsymbol{n_e}) = \mathcal{N}[\underbrace{\boldsymbol{\Delta}_*\boldsymbol{K}\boldsymbol{K}^{-1}(\boldsymbol{n_e} - \bar{n}_e)}_{\boldsymbol{\Delta}_*(\boldsymbol{n_e} - \bar{n}_e)}, \underbrace{\boldsymbol{\Delta}_*\boldsymbol{K}\boldsymbol{\Delta}_* - \boldsymbol{\Delta}_*\boldsymbol{K}\boldsymbol{K}^{-1}\boldsymbol{K}\boldsymbol{\Delta}_*}_{\boldsymbol{0}}]. \tag{A.3}$$

Similarly, the second distribution on the LHS is,

$$\begin{aligned} P(\boldsymbol{n_e} \mid \boldsymbol{\tau}_{\text{obs}}) = \mathcal{N}[&\bar{n}_e + \boldsymbol{K}\boldsymbol{\Delta}^T(\boldsymbol{\Delta}\boldsymbol{K}\boldsymbol{\Delta}^T + \sigma^2\boldsymbol{I})^{-1}\boldsymbol{\tau}_{\text{obs}}, \\ &\boldsymbol{K} - \boldsymbol{K}\boldsymbol{\Delta}^T(\boldsymbol{\Delta}\boldsymbol{K}\boldsymbol{\Delta}^T + \sigma^2\boldsymbol{I})^{-1}\boldsymbol{\Delta}\boldsymbol{K}]. \end{aligned} \tag{A.4}$$

We now can use standard Gaussian identities [e.g. Weiss and Freeman, 2001] to evaluate the integral on the LHS,

$$\int P(\boldsymbol{\tau} \mid \boldsymbol{n_e}) P(\boldsymbol{n_e} \mid \boldsymbol{\tau}_{\text{obs}})\, \mathrm{d}\boldsymbol{n_e} = \mathcal{N}[\boldsymbol{\Delta}_* \boldsymbol{K} \boldsymbol{\Delta}^T (\boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}^T + \sigma^2 \boldsymbol{I})^{-1} \boldsymbol{\tau}_{\text{obs}}, \boldsymbol{\Delta}_* \boldsymbol{K} \boldsymbol{\Delta}_*^T - \boldsymbol{\Delta}_* \boldsymbol{K} \boldsymbol{\Delta}^T (\boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}^T + \sigma^2 \boldsymbol{I})^{-1} \boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}_*^T] \tag{A.5}$$

In order to work out the right-hand side (RHS), we simply condition Eq. A.2 on $\boldsymbol{\tau}_{\text{obs}}$ and then marginalise $\boldsymbol{n_e}$ by selecting the corresponding sub-block of the Gaussian,

$$P(\boldsymbol{n_e}, \boldsymbol{\tau} \mid \boldsymbol{\tau}_{\text{obs}}) = \mathcal{N}\left[\begin{pmatrix} \bar{n}_e \\ 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{K}\boldsymbol{\Delta}^T \\ \boldsymbol{\Delta}_*^T \boldsymbol{K} \boldsymbol{\Delta}^T \end{pmatrix} \left(\boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}^T + \sigma^2 \boldsymbol{I}\right)^{-1} \boldsymbol{\tau}_{\text{obs}}, \begin{pmatrix} \bar{K} & \boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}_*^T \\ \boldsymbol{\Delta}_* \boldsymbol{K} \boldsymbol{\Delta}^T & \boldsymbol{\Delta}_* \boldsymbol{K} \boldsymbol{\Delta}_*^T \end{pmatrix} - \begin{pmatrix} \boldsymbol{K}\boldsymbol{\Delta}^T \\ \boldsymbol{\Delta}_* \boldsymbol{K} \boldsymbol{\Delta}^T \end{pmatrix} \left(\boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}^T + \sigma^2 \boldsymbol{I}\right)^{-1} \begin{pmatrix} \boldsymbol{\Delta} \boldsymbol{K} & \boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}_*^T \end{pmatrix}\right] \tag{A.6}$$

Marginalising over $\boldsymbol{n_e}$ is equivalent to neglecting the sub-block corresponding to $\boldsymbol{n_e}$. Therefore, the RHS is,

$$\int P(\boldsymbol{n_e}, \boldsymbol{\tau} \mid \boldsymbol{\tau}_{\text{obs}})\, \mathrm{d}\boldsymbol{n_e} = \mathcal{N}\left[\boldsymbol{\Delta}_* \boldsymbol{K} \boldsymbol{\Delta}^T \left(\boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}^T + \sigma^2 \boldsymbol{I}\right)^{-1} \boldsymbol{\tau}_{\text{obs}}, \boldsymbol{\Delta}_* \boldsymbol{K} \boldsymbol{\Delta}_*^T - \boldsymbol{\Delta}_* \boldsymbol{K} \boldsymbol{\Delta}^T \left(\boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}^T + \sigma^2 \boldsymbol{I}\right)^{-1} \boldsymbol{\Delta} \boldsymbol{K} \boldsymbol{\Delta}_*^T\right]. \tag{A.7}$$

■

Appendix B

Derivation of the $\Delta\mathrm{TEC}$ variance function and its limits

We derive the ΔTEC variance function $\sigma^2_{\Delta\mathrm{TEC}}(d)$ for zenith observations ($\boldsymbol{k} = \boldsymbol{k}' = \hat{\boldsymbol{z}}$) by considering a baseline between an antenna-of-interest at $\mathbf{x}_i = \mathbf{x}_j$ and a reference antenna at $\mathbf{x}_0 = \mathbf{0}$. To use the Pythagorean theorem later, we assume that this baseline lies in the plane of the local horizon, i.e. perpendicular to the zenith. Without loss of generality, we can orient the coordinate axes such that this baseline lies along the $\hat{\mathbf{x}}$ direction, so that $\mathbf{x}_i - \mathbf{x}_0 = d\hat{\mathbf{x}}$. Here $d \triangleq ||\mathbf{x}_i||$ is the distance between the two antennae. We then take the general covariance function $K_{\Delta\mathrm{TEC}}\left(\left[\mathbf{x}_i, \mathbf{x}_0, \hat{\mathbf{k}}\right], \left[\mathbf{x}_j, \mathbf{x}_0, \hat{\mathbf{k}}'\right]\right)$, and find that in this particular case

$$\sigma^2_{\Delta\mathrm{TEC}}(d) \triangleq K_{\Delta\mathrm{TEC}}\left([\mathbf{x}_i, \mathbf{x}_0, \hat{\mathbf{z}}], [\mathbf{x}_i, \mathbf{x}_0, \hat{\mathbf{z}}]\right) \tag{B.1}$$

$$= \sum_{p_1=0}^{1} \sum_{p_2=0}^{1} (-1)^{p_1+p_2}$$

$$\int_0^b \int_0^b K_{n_e}\left(||\mathbf{x}_{(1-p_1)i} - \mathbf{x}_{(1-p_2)i} + \hat{\mathbf{z}}(s_1 - s_2)||\right) \mathrm{d}s_1 \mathrm{d}s_2, \tag{B.2}$$

where K_{n_e} is an arbitrary stationary and isotropic kernel (such as the Exponentiated Quadratic and Matérn $\frac{3}{2}$ kernels considered earlier) for the FED. The two terms where p_1 and p_2 are equal give the same contribution, as do the two terms for which p_1 and p_2 are unequal. By subsequently applying the Pythagorean theorem in this last case (i.e. $p_1 = 0$ and $p_2 = 1$, and vice versa), we find

$$\sigma^2_{\Delta\mathrm{TEC}}(d) = 2 \int_0^b \int_0^b K_{n_e}\left(|s_1 - s_2|\right) - K_{n_e}\left(\sqrt{d^2 + (s_1 - s_2)^2}\right) \mathrm{d}s_1 \mathrm{d}s_2. \tag{B.3}$$

We manipulate this result to obtain a more insightful expression. First, we note the (implicit) presence of three parameters with dimension length: ionospheric thickness b, reference antenna distance d, and FED kernel half-peak distance h. We perform transformations to dimensionless coordinates $u_1 = \frac{s_1}{h}$ and $u_2 = \frac{s_2}{h}$ to reveal that the *shape* - though not the absolute scale - of the function $\sigma^2_{\Delta\mathrm{TEC}}(d)$ is governed only by the length-scale ratios $\frac{b}{h}$ and $\frac{d}{h}$, and the particular functional form of K_{n_e}.

Furthermore, for stationary covariance functions, we have $K_{n_e} = \sigma^2_{n_e} C_{n_e}$, where C_{n_e} is the corresponding dimensionless correlation function.
These considerations enable us to express the ΔTEC structure function as a dimensionless, shape-determining double integral appended by dimensionful prefactors; i.e.

$$\sigma^2_{\Delta\mathrm{TEC}}(d) = 2\sigma^2_{n_e} h^2 \int_0^{\frac{b}{h}} \int_0^{\frac{b}{h}} C_{n_e}(h\,|u_1 - u_2|) - C_{n_e}\left(h\sqrt{\left(\frac{d}{h}\right)^2 + (u_1-u_2)^2}\right) \mathrm{d}u_1 \mathrm{d}u_2. \tag{B.4}$$

We first note that the variance of ΔTEC is simply proportional to the variance of n_e. Secondly, we note that $h\,|u_1 - u_2| < h\sqrt{\left(\frac{d}{h}\right)^2 + (u_1-u_2)^2}$ for any non-zero d, so that $C_{n_e}(h\,|u_1 - u_2|) > C_{n_e}\left(h\sqrt{\left(\frac{d}{h}\right)^2 + (u_1-u_2)^2}\right)$ for all monotonically decreasing correlation functions C_{n_e} (or, equivalently, covariance functions K_{n_e}). With the integrand always positive, we see that the integral must be a strictly increasing function of $\frac{b}{h}$ (which occurs in the integration limits). Therefore, we conclude that for stationary, isotropic, and monotonically decreasing (SIMD) FED kernels with HPD h, the ΔTEC variance increases monotonically with the thickness of the ionosphere b. Simply put: thicker SIMD ionospheres cause larger ΔTEC variations.

Let us now consider three limits of the ΔTEC zenith variance function, that all do not require K_{FED} to decrease monotonically. In the short-baseline limit, i.e. $\frac{d}{h} \to 0$, we have $C_{n_e}\left(h\sqrt{\left(\frac{d}{h}\right)^2 + (u_1-u_2)^2}\right) \to C_{n_e}(h\,|u_1 - u_2|)$. We therefore find that $\sigma^2_{\Delta\mathrm{TEC}} \to 0$ irrespective of other parameters, recovering that the variance of ΔTEC vanishes near the reference antenna. In the long-baseline limit, i.e. $\frac{d}{h} \gg \frac{b}{h} > 1$, we see that $\sqrt{\left(\frac{d}{h}\right)^2 + (u_1-u_2)^2} \approx \frac{d}{h}$, since $(u_1-u_2)^2 < \left(\frac{b}{h}\right)^2 \ll \left(\frac{d}{h}\right)^2$. Assuming $C_{n_e}(d) \approx 0$ when $\frac{d}{h} \gg 1$, the integrand reduces to $C_{n_e}(h\,|u_1 - u_2|) - C_{n_e}\left(h \cdot \frac{d}{h}\right) \approx C_{n_e}(h\,|u_1 - u_2|)$. We find that in this case,

$$\sigma^2_{\Delta\mathrm{TEC}} \approx 2\sigma^2_{n_e} h^2 \int_0^{\frac{b}{h}} \int_0^{\frac{b}{h}} C_{n_e}(h\,|u_1 - u_2|)\, \mathrm{d}u_1 \mathrm{d}u_2. \tag{B.5}$$

This is the plateau value of the ΔTEC variance that our model predicts for the long-baseline limit.

Another way to arrive at the plateau value expression of Equation B.5 is by considering the statistical properties of TEC first. In a computation analogous to the one for ΔTEC in Section 2.3, one can derive the general TEC covariance function K_{TEC}. The variance of $\tau_i^{\hat{\boldsymbol{z}}}$ (the TEC of antenna i while observing towards the zenith $\hat{\boldsymbol{z}}$) is straightforwardly shown to be

$$\mathbb{V}\left(\tau_i^{\hat{\boldsymbol{z}}}\right) = \sigma^2_{n_e} h^2 \int_0^{\frac{b}{h}} \int_0^{\frac{b}{h}} C_{n_e}(h\,|u_1 - u_2|)\, \mathrm{d}u_1 \mathrm{d}u_2. \tag{B.6}$$

We highlight the absence of a dependence on i at the RHS. As a ΔTEC is simply a TEC differenced with a TEC for a reference antenna observing in the same direction, we have

$$\sigma^2_{\Delta\mathrm{TEC}} = \mathbb{V}\left(\tau_i^{\hat{\boldsymbol{z}}} - \tau_0^{\hat{\boldsymbol{z}}}\right) = \mathbb{V}\left(\tau_i^{\hat{\boldsymbol{z}}}\right) + \mathbb{V}\left(\tau_0^{\hat{\boldsymbol{z}}}\right), \tag{B.7}$$

where the second equality only holds when the TECs are independent. This is exactly the scenario considered in the long-baseline limit. Plugging in Equation B.6 recovers the plateau level. We can find a general upper bound to the variance of ΔTEC in terms of physical parameters. To this end, we note that the integrand in Equation B.4 is maximised when, over the full range of integration, the value of the first term is 1 whilst the second term is equal to the infimum of the correlation function. Calling $\inf_{\mathbb{R}} \{C_{n_e}(r) : r \in \mathbb{R}_{>0}\} \triangleq I$, we find the inequality,

$$\sigma^2_{\Delta\mathrm{TEC}} \leq 2\sigma^2_{n_e} h^2 \int_0^{\frac{b}{h}} \int_0^{\frac{b}{h}} 1 - I \, \mathrm{d}u_1 \mathrm{d}u_2 = 2(1-I)\sigma^2_{n_e} b^2. \tag{B.8}$$

For strictly positive FED kernels that decay to zero at large distances (such as the EQ and Matérn kernels considered in this work), we find $\sigma^2_{\Delta\mathrm{TEC}} \leq 2\sigma^2_{n_e} b^2$. Kernels resulting in anti-correlated FEDs produce the constraint $\sigma^2_{\Delta\mathrm{TEC}} \leq 4\sigma^2_{n_e} b^2$ or tighter. By measuring $\sigma_{\Delta\mathrm{TEC}}(d)$, one can bound the product $\sigma_{n_e} b$ from below. The strongest bound is obtained for large d.

Appendix C

Factoring commutative DI dependence from the RIME

We consider the effect of directionally referencing spatially referenced commutative Jones scalars. In particular, we study phases of the Jones scalars, and assume amplitudes of one, but the same idea extends to amplitude by considering log-amplitudes that can be treated like a pure-imaginary phase.

Let $g(\mathbf{x},\mathbf{k}) = e^{i\phi(\mathbf{x},\mathbf{k})}$ be a Jones scalar, and consider the necessarily non-unique decomposition of phase into DD and DI components,

$$\phi(\mathbf{x},\mathbf{k}) = \phi^{\mathrm{DD}}(\mathbf{x},\mathbf{k}) + \phi^{\mathrm{DI}}(\mathbf{x}). \tag{C.1}$$

This functional form only specifies that the DI term is not dependent on $\mathbf{k}$. The differential phase to which the RIME is sensitive is found by spatially referencing the phase,

$$\Delta_0\phi(\mathbf{x},\mathbf{k}) = \phi(\mathbf{x},\mathbf{k}) - \phi(\mathbf{x}_0,\mathbf{k}) \tag{C.2}$$

$$= \Delta_0\phi^{\mathrm{DD}}(\mathbf{x},\mathbf{k}) + \Delta_0\phi^{\mathrm{DI}}(\mathbf{x}). \tag{C.3}$$

Directionally referencing the differential phase to direction $\mathbf{k}_0$, we have

$$\Delta_0^2\phi(\mathbf{x},\mathbf{k}) = \Delta_0\phi(\mathbf{x},\mathbf{k}) - \Delta_0\phi(\mathbf{x},\mathbf{k}_0) \tag{C.4}$$

$$= \Delta_0^2\phi^{\mathrm{DD}}(\mathbf{x},\mathbf{k}). \tag{C.5}$$

We see that the DI phase has disappeared, and we are left with the doubly differential phase for the DD term. We then assume that there was a remnant DI component in $\phi^{\mathrm{DD}}(\mathbf{x},\mathbf{k})$. Then by induction we have that $\Delta_0^2\phi^{\mathrm{DD}}(\mathbf{x},\mathbf{k})$ must be free of DI terms, and $\Delta_0^2\phi(\mathbf{x},\mathbf{k})$ must be free of all DI components. It follows that directionally referencing phase guarantees that all DI components are removed.

Furthermore, assuming we were to directionally reference a doubly differential phase. We assume that the doubly differential phase is referenced to direction $\mathbf{k}_0'$, with $\Delta'^2_0\phi(\mathbf{x},\mathbf{k}) \triangleq$

$\Delta_0\phi(\mathbf{x},\mathbf{k})-\Delta_0\phi(\mathbf{x},\mathbf{k}_0')$. Then we see

$$\Delta'^2_0\phi(\mathbf{x},\mathbf{k})-\Delta'^2_0\phi(\mathbf{x},\mathbf{k}_0) = \Delta_0\phi(\mathbf{x},\mathbf{k})-\Delta_0\phi(\mathbf{x},\mathbf{k}_0')-(\Delta_0\phi(\mathbf{x},\mathbf{k}_0)-\Delta_0\phi(\mathbf{x},\mathbf{k}_0')) \tag{C.6}$$

$$= \Delta_0\phi(\mathbf{x},\mathbf{k})-\Delta_0\phi(\mathbf{x},\mathbf{k}_0) \tag{C.7}$$

$$= \Delta_0^2\phi(\mathbf{x},\mathbf{k}). \tag{C.8}$$

That is, directionally referencing a doubly differential phase produces a new doubly differential phase, referenced to the new direction, $\mathbf{k}_0$. This trick thus allows us to set a well-defined reference direction to phases that have undergone DI calibration before DD calibration.

Appendix D

Recursive Bayesian estimation

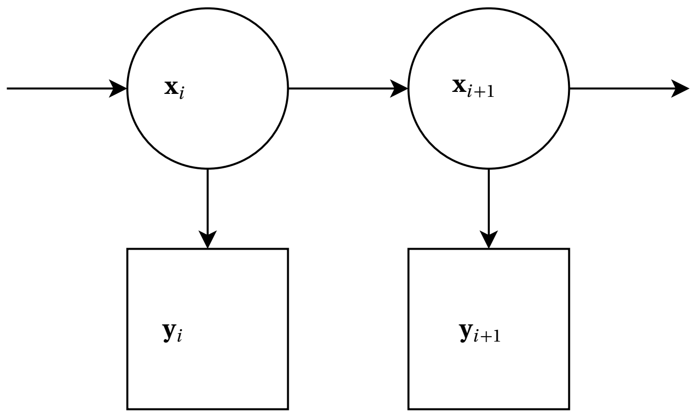

Figure D.1: Casual graph depicting a hidden Markov model.

Recursive Bayesian estimation is a method of performing inference on a hidden Markov model [HMM; Rabiner and Juang, 1986]. Let $\mathbf{y}$ be an observable and $\mathbf{x}$ be a hidden variable. The HMM assumptions on $\mathbf{x}$ and $\mathbf{y}$ are, firstly, that the hidden random variable is only conditionally dependent on its previous state, and secondly, the observable is conditionally independent of all other random variables except for the current hidden state. These assumptions are depicted in **Figure** D.1, where i is the sequence index. This paradigm is often given a notion of causality or time, but this is not necessary in any way. The sequence index is an abstract notion that simply explains how the set of hidden variables are traversed. For example, we assume that the observations are frames of a movie, and the hidden variable is the plot contained in each movie frame. The HMM assumption is that the movie plot is linear, and the picture encodes what is going on in the movie at a given point in time. With recursive Bayesian estimation the movie can be watched with frames randomly ordered and the complete plot is still rendered completely.

There are two distinct types of information propagation in a hidden Markov model. Information can flow in the direction of the arrows, or against them. This leads to the notion of the forward and backwards equations that describe how belief in hidden variables is propagated forward, and for revising, we believe in previously visited hidden variables.

The joint distribution of the hidden random variables and observables in a chain of length T can be written out as a product of conditional distributions and a marginal using the product rule of probability distributions [Kolmogorov, 1960]. Because of the HMM assumptions the joint distribution is

$$p(\mathbf{x}_{0:T}, \mathbf{y}_{0:T}) = p(\mathbf{x}_0) \prod_{i=1}^{T} p(\mathbf{x}_i \mid \mathbf{x}_{i-1}) p(\mathbf{y}_i \mid \mathbf{x}_i). \tag{D.1}$$

We first consider propagating information forward. This is done in two steps, typically called the predict and update steps. For the predict step, we consider how belief in the absence of new observables is propagated. For this, we apply the Chapman-Kolmogorov identity[1] for Markovian processes,

$$p(\mathbf{x}_{i+1} \mid \mathbf{y}_{0:i}) = \mathbb{E}_{\mathbf{x}_i \mid \mathbf{y}_{0:i}} \left[p(\mathbf{x}_{i+1} \mid \mathbf{x}_i) \right], \tag{D.2}$$

which gives us the probability distribution of the hidden variables at time $i+1$ in terms of the so-called state transition distribution $p(\mathbf{x}_{i+1} \mid \mathbf{x}_i)$ and posterior distribution $p(\mathbf{x}_i \mid \mathbf{y}_{0:i})$ at index i. The current prior belief can be understood as the expectation of the state transition distribution over the measure of the current posterior belief.

The update step is simply an application of the Bayes theorem with the prior defined by the predict step,

$$p(\mathbf{x}_i \mid \mathbf{y}_{0:i}) = \frac{p(\mathbf{y}_i \mid \mathbf{x}_i) \mathbb{E}_{\mathbf{x}_{i-1} \mid \mathbf{y}_{0:i-1}} \left[p(\mathbf{x}_i \mid \mathbf{x}_{i-1}) \right]}{p(\mathbf{y}_i \mid \mathbf{y}_{0:i-1})}, \tag{D.3}$$

where the denominator is the Bayesian evidence of the newly arrived data given all previous data, and it is independent of the hidden variables. Equation D.3 gives a recurrence relation for propagating our belief forward, therefore this is called the forward equation.

We now assume that we are at index T, and wish to use all acquired information to revise our belief in the previously visited hidden variables at indices $i < T$. The trick is to realise that $p(\mathbf{x}_i \mid \mathbf{x}_{i+1} \mid \mathbf{y}_{0:T}) = p(\mathbf{x}_i \mid \mathbf{x}_{i+1} \mid \mathbf{y}_{0:i})$ as a result of the Markov properties. In this case, again using the product rule, we find that the joint conditional distribution of a pair of sequential hidden states given the whole sequence of data is

$$p(\mathbf{x}_i, \mathbf{x}_{i+1} \mid \mathbf{y}_{0:T}) = \frac{p(\mathbf{x}_{i+1} \mid \mathbf{x}_i) p(\mathbf{x}_i \mid \mathbf{y}_{0:i}) p(\mathbf{x}_{i+1} \mid \mathbf{y}_{0:T})}{p(\mathbf{x}_{i+1} \mid \mathbf{y}_{0:i})}. \tag{D.4}$$

Marginalising the second hidden parameter, we arrive at the recurrence relation

$$p(\mathbf{x}_i \mid \mathbf{y}_{0:T}) = p(\mathbf{x}_i \mid \mathbf{y}_{0:i}) \int \frac{p(\mathbf{x}_{i+1} \mid \mathbf{x}_i) p(\mathbf{x}_{i+1} \mid \mathbf{y}_{0:T})}{p(\mathbf{x}_{i+1} \mid \mathbf{y}_{0:i})} \, d\mathbf{x}_{i+1}. \tag{D.5}$$

This can be solved by starting at T and solving this equation iteratively backwards, therefore Equation D.5 is called the backward equation. Most importantly, we note that the backward equation does not require conditioning on data, as was done in the update step.

[1] If b is conditionally independent of a, then $p(a \mid c) = \int p(a \mid b) p(b|c) \mathrm{d}b = \mathbb{E}_{b|c}[p(a \mid b)]$ is the Chapman-Kolmogorov identity.

When the transition and likelihood are assumed to be Gaussian, for example, in a linear dynamical system, the forward and backward equations are equivalent to the well-known Kalman filter equations and Rauch smoother equations [Rauch, 1963], respectively.

Appendix E

Jones scalar variational expectation

Let $\mathbf{g} \in \mathbb{C}^{N_{\text{freq}}}$ be an observed complex Jones scalar vector, with amplitudes $g \in \mathbb{R}^{N_{\text{freq}}}$ and phases $\phi \in \mathbb{R}^{N_{\text{freq}}}$. We assume that the Jones scalars have complex Gaussian noise, described by the observational covariance matrix $\mathbf{\Sigma}$. Thus, we have that the observational likelihood of the Jones scalars (*cf* Eq. 3.9) is

$$p(\mathbf{g} \mid \phi, g, \mathbf{\Sigma}) = \mathcal{N}_{\mathbb{C}}[\mathbf{g} \mid g e^{i\phi}, \mathbf{\Sigma}], \tag{E.1}$$

where $\mathcal{N}_{\mathbb{C}}$ is the complex Gaussian distribution, which is defined as the Gaussian distribution of the stacked real and imaginary components. We define the residuals $\delta R = \text{Re}[\mathbf{g}] - g\cos\phi$ and $\delta I = \text{Im}[\mathbf{g}] - g\sin\phi$, and the stacked residuals $\delta\mathbf{g} = (\delta R, \delta I)^T$. Then the log-likelihood becomes

$$\log p(\mathbf{g} \mid \phi, g, \mathbf{\Sigma}) = -N_{\text{freq}} \log 2\pi - \frac{1}{2}\log|\mathbf{\Sigma}| - \frac{1}{2}\delta\mathbf{g}^T \mathbf{\Sigma}^{-1} \delta\mathbf{g}, \tag{E.2}$$

$$= -N_{\text{freq}} \log 2\pi - \frac{1}{2}\log|\mathbf{\Sigma}| - \frac{1}{2}\text{Tr}[\mathbf{\Sigma}^{-1}\delta\mathbf{g}\delta\mathbf{g}^T] \tag{E.3}$$

$$= -N_{\text{freq}} \log 2\pi - \frac{1}{2}\log|\mathbf{\Sigma}| - \frac{1}{2}\text{vec}[\mathbf{\Sigma}^{-1}]^T \text{vec}[\delta\mathbf{g}\delta\mathbf{g}^T] \tag{E.4}$$

where we used the fact that the trace of a scalar is a scalar, and that $\text{Tr}[A^T B] = \text{Tr}[BA^T] = \text{vec}[A]^T \text{vec}[B]$.

Now we assume that the phases are linearly modelled according to

$$\phi(\nu) = \sum_i^M f_i(\nu) a_i, \tag{E.5}$$

$$\triangleq f^T(\nu) a \tag{E.6}$$

where $f_i(\nu)$ is the i-th basis function of a set of M linearly independent functions depending on frequency ν, and a_i is the i-th parameter in the linear model. In the second line we have written the sum as a dot product. We furthermore assume that the parameter vector, a, is a (potentially correlated) Gaussian random variable,

$$a \sim \mathcal{N}[\mu, \Gamma]. \tag{E.7}$$

This induces a distribution on phase via Eq. E.5.

The variational expectation of the Jones scalars with respect to a is the expectation of the log-likelihood,

$$\begin{aligned}\mathbb{E}_a\left[\log p(\mathbf{g} \mid \phi, g, \mathbf{\Sigma})\right] = &-N_{\text{freq}} \log 2\pi - \frac{1}{2}\log|\mathbf{\Sigma}| \\ &- \frac{1}{2}\text{vec}[\mathbf{\Sigma}^{-1}]^T \text{vec}\left[\mathbb{E}_a[\delta\mathbf{g}\delta\mathbf{g}^T]\right],\end{aligned} \tag{E.8}$$

where we used the fact that expectation and vectorisation commute. Now we need to evaluate the expectation, $\mathbb{E}_a\left[\delta\mathbf{g}\delta\mathbf{g}^T\right]$. In order to do this, we realise that there are three explicit expectations that must be done corresponding to sub-blocks of $\delta\mathbf{g}\delta\mathbf{g}^T$ consisting of the real-real, real-imaginary, and imaginary-imaginary outer products of residual vectors,

$$\mathbb{E}_a\left[\delta R \delta R^T\right] = \int \delta R \delta R^T \,\mathrm{d}p(a) \tag{E.9}$$

$$\triangleq I_{rr} \tag{E.10}$$

$$\mathbb{E}_a\left[\delta R \delta I^T\right] = \int \delta R \delta I^T \,\mathrm{d}p(a) \tag{E.11}$$

$$\triangleq I_{ri} \tag{E.12}$$

$$\mathbb{E}_a\left[\delta I \delta I^T\right] = \int \delta I \delta I^T \,\mathrm{d}p(a) \tag{E.13}$$

$$\triangleq I_{ii}, \tag{E.14}$$

where $p(a) = \mathcal{N}[\mu, \Gamma]$. All three of the above integrals involve Gaussian expectations of trigonometric functions. For example, the (i, j)-th element of the integral I_{rr} is,

$$(I_{rr})_{(i,j)} = \int (\text{Re}[\mathbf{g}]_i - g_i \cos\phi(\nu_i))(\text{Re}[\mathbf{g}]_j - g_j \cos\phi(\nu_j))\,\mathrm{d}p(a). \tag{E.15}$$

To evaluate this integral, we use angle addition formulae to reduce all powers of trigonometric functions. We then use the characteristic function of the Gaussian distribution, which gives the relation, $\mathbb{E}_{X\sim\mathcal{N}[\mu,\Gamma]}[e^{if^TX}] = e^{if^T\mu - \frac{1}{2}f^T\Gamma f}$, on the exponential form of cosine and sine. Applying the above, we evaluate all integral matrix elements,

$$\begin{aligned}(I_{rr})_{(i,j)} = &\text{Re}[\mathbf{g}]_i \text{Re}[\mathbf{g}]_j - g_i \text{Re}[\mathbf{g}]_j h_1(f(\nu_i)) - \text{Re}[\mathbf{g}]_i g_j h_1(f(\nu_j)) \\ &+ g_i g_j h_3(f(\nu_i), f(\nu_j))\end{aligned} \tag{E.16}$$

$$\begin{aligned}(I_{ri})_{(i,j)} = &\text{Re}[\mathbf{g}]_i \text{Im}[\mathbf{g}]_j - g_i \text{Im}[\mathbf{g}]_j h_1(f(\nu_i)) - \text{Re}[\mathbf{g}]_i g_j h_2(f(\nu_j)) \\ &+ g_i g_j h_5(f(\nu_i), f(\nu_j))\end{aligned} \tag{E.17}$$

$$\begin{aligned}(I_{ii})_{(i,j)} = &\text{Im}[\mathbf{g}]_i \text{Im}[\mathbf{g}]_j - g_i \text{Im}[\mathbf{g}]_j h_2(f(\nu_i)) - \text{Im}[\mathbf{g}]_i g_j h_2(f(\nu_j)) \\ &+ g_i g_j h_4(f(\nu_i), f(\nu_j))\end{aligned} \tag{E.18}$$

where,

$$h_1(x) = \mathbb{E}_a[\cos x^T a] = e^{-\frac{1}{2}x^T\Gamma x}\cos x^T \mu \tag{E.19}$$

$$h_2(x) = \mathbb{E}_a[\sin x^T a] = e^{-\frac{1}{2}x^T\Gamma x}\sin x^T \mu \tag{E.20}$$

$$h_3(x, y) = \mathbb{E}_a[\cos x^T a \cos y^T a] = \frac{1}{2}(h_1(x+y) + h_1(x-y)) \tag{E.21}$$

$$h_4(x, y) = \mathbb{E}_a[\sin x^T a \sin y^T a] = \frac{1}{2}(h_1(x-y) - h_1(x+y)) \tag{E.22}$$

$$h_5(x, y) = \mathbb{E}_a[\cos x^T a \sin y^T a] = \frac{1}{2}(h_2(x+y) + h_2(x-y)). \tag{E.23}$$

This completes the evaluation of Eq. E.8 for an arbitrary linear phase model. For example, in a phase model consisting only of DDTEC, we have $M = 1$, $f_1(\nu) = \frac{\kappa}{\nu}$ and $a_1 = \Delta_0^2\tau$. It is simple to calculate the variational expectation of other linear phase models including terms such as clock-like terms, constant-in-frequency terms, and higher order ionospheric terms.

A simplification follows if the observational errors are assumed uncorrelated, which leads to a reduction of the number of operations required to compute the variational expectation. In this case, only the diagonals of I_{rr} and I_{ii} are needed, and I_{ri} is not needed at all.

Appendix F

Derivation of the M-step

Here we derive the maximisation step of Algorithm 1. This is an extension of Shumway and Stoffer [1982] to time varying observational covariance and state transition covariance. In order to do this we must linearise the forward equation. Let the set of HMM states be $X = \{x_i \mid i = 1..n\}$ and the set of observations be $Y = \{y_i \mid i = 1..n\}$. Following similar notation as in Shumway and Stoffer [1982] we'll term x_i^s be the state at step i conditioned on data up to an including step s. Therefore if $i > s$ then x_i^s is the forecast of the state. Consequently the conditional mean and covariance is μ_i^s and Γ_i^s. We'll also need the joint-covariance $\Gamma_{i,i-1}^s$ which is the covariance of $\{x_i, x_{i-1}\}$ conditioned on data up to step s. The recursive equations for all these components are given in [e.g. Shumway and Stoffer, 1982] and follow from the forward and backward equations in Appendix D.

Since at each time step the state is a Gaussian variable, using standard identities it follows that the posterior distribution of $x_i - x_{i-1}$ conditioned up to step n is given by,

$$x_i - x_{i-1} \sim \mathcal{N}\left(x_i - x_{i-1} \mid \mu_i^n - \mu_{i-1}^n, \Gamma_i^n + \Gamma_{i-1}^n - \Gamma_{i,i-1}^n - \Gamma_{i-1,i}^n\right). \tag{F.1}$$

The joint log-likelihood of the data and state is the sum of the prior component and the likelihood component,

$$\begin{aligned}\log L(X,Y) \triangleq & -\frac{1}{2}\log|\Gamma_0| - \frac{1}{2}(x_0-\mu_0)^T\Gamma_0^{-1}(x_0-\mu_0) \\ & -\frac{1}{2}\sum_{i=1}^{n}\log|\Omega_i| - \frac{1}{2}\sum_{i=1}^{n}(x_i-x_{i-1})^T\Omega_i^{-1}(x_i-x_{i-1}) \\ & -\frac{1}{2}\sum_{i=1}^{n}\log|\Sigma_i| - \frac{1}{2}\sum_{i=1}^{n}(y_i-f_i(x_i))^T\Sigma_i^{-1}(y_i-f_i(x_i)) + \text{const.}\end{aligned} \tag{F.2}$$

where f_i is the forward function from the HMM state to observable and time i. In general, the function f_i can change over time, e.g. deterministic input control parameters to f_i can be time dependent. Let us define the linear map $D_i(y) = \nabla_x f_i|_{\bar{x}_i} y$ for some $\bar{x}_i$. The forward function then satisfies $f_i(x_i) = f_i(\bar{x}_i) + D_i(x_i - \bar{x}_i) + o(|x_i - \bar{x}_i|^2)$. From this Eq. F.2 becomes,

$$\begin{aligned}\log L(X,Y) = &-\frac{1}{2}\log|\Gamma_0| - \frac{1}{2}(x_0-\mu_0)^T\Gamma_0^{-1}(x_0-\mu_0)\\ &-\frac{1}{2}\sum_{i=1}^{n}\log|\Omega_i| - \frac{1}{2}\sum_{i=1}^{n}(x_i-x_{i-1})^T\Omega_i^{-1}(x_i-x_{i-1})\\ &-\frac{1}{2}\sum_{i=1}^{n}\log|\Sigma_i|\\ &-\frac{1}{2}\sum_{i=1}^{n}(y_i-f_i(\bar{x}_i)-D_i(x_i-\bar{x}_i))^T\Sigma_i^{-1}(y_i-f_i(\bar{x}_i)-D_i(x_i-\bar{x}_i))\\ &+\text{const.}\end{aligned} \tag{F.3}$$

which is exactly equal to Eq. F.2 when $x_i = \bar{x}_i$ and f_i quadratic in the neighbourhood of $\bar{x}_i$. Defining $\delta y_i = y_i - f_i(\bar{x}_i) + D_i\bar{x}_i$ then Eq. F.3 becomes,

$$\begin{aligned}\log L(X,Y) = &-\frac{1}{2}\log|\Gamma_0| - \frac{1}{2}(x_0-\mu_0)^T\Gamma_0^{-1}(x_0-\mu_0)\\ &-\frac{1}{2}\sum_{i=1}^{n}\log|\Omega_i| - \frac{1}{2}\sum_{i=1}^{n}(x_i-x_{i-1})^T\Omega_i^{-1}(x_i-x_{i-1})\\ &-\frac{1}{2}\sum_{i=1}^{n}\log|\Sigma_i|\\ &-\frac{1}{2}\sum_{i=1}^{n}(\delta y_i - D_i x_i)^T\Sigma_i^{-1}(\delta y_i - D_i x_i)\\ &+\text{const.}\end{aligned} \tag{F.4}$$

Taking the expectation of Eq. F.4 with respect to the state conditioned on all data we find [cf. Shumway and Stoffer, 1982],

$$\begin{aligned}\mathbb{E}_X[\log L(X,Y)] = &-\frac{1}{2}\log|\Gamma_0| - \frac{1}{2}\sum_{i=1}^{n}\log|\Omega_i| - \frac{1}{2}\sum_{i=1}^{n}\log|\Sigma_i|\\ &-\frac{1}{2}(\mu_0^n-\mu_0)^T\Gamma_0^{-1}(\mu_0^n-\mu_0) - \frac{1}{2}\mathrm{Tr}[\Gamma_0^{-1}\Gamma_0^n]\\ &-\frac{1}{2}\sum_{i=1}^{n}(\mu_i^n-\mu_{i-1}^n)^T\Omega_i^{-1}(\mu_i^n-\mu_{i-1}^n)\\ &-\frac{1}{2}\sum_{i=1}^{n}\mathrm{Tr}[\Omega_i^{-1}(\Gamma_i^n+\Gamma_{i-1}^n-\Gamma_{i,i-1}^n-\Gamma_{i-1,i}^n)]\\ &-\frac{1}{2}\sum_{i=1}^{n}(\delta y_i - D_i\mu_i^n)^T\Sigma_i^{-1}(\delta y_i - D_i\mu_i^n)\\ &-\frac{1}{2}\sum_{i=1}^{n}\mathrm{Tr}[\Sigma_i^{-1}D_i\Gamma_i^n D_i^T]\\ &+\text{const.}\end{aligned} \tag{F.5}$$

We next take the derivative with respect to μ_0, Γ_0, Ω_i, and Σ_i and set to zero and solve. Some derivative identities are required. Assuming A is symmetric and positive definite, then using Einstein summation notation we have,

$$\nabla_{A_{ij}} A^{-1}_{lq} = -A^{-1}_{li} A^{-1}_{jq} \tag{F.6}$$

$$\nabla_{A_{ij}} \log|A| = A^{-1}_{ij} \tag{F.7}$$

$$\nabla_{A_{ij}} x_l A^{-1}_{lq} y_q = -(A^{-1}_{il} x_l)(A^{-1}_{jq} y_q) = -(A^{-1}x)(A^{-1}y)^T \tag{F.8}$$

$$\nabla_{A_{ij}} \mathrm{Tr}[A^{-1}B] = -A^{-1}_{il} B^T_{lq} A^{-1}_{qj} = -A^{-1}B^T A^{-1}. \tag{F.9}$$

Applying these identities as needed we have,

$$\nabla_{\mu_0} \mathbb{E}_X[\log L(X,Y)] = \Gamma_0^{-1}(\mu_0^n - \mu_0) = 0 \tag{F.10}$$

$$\implies \mu_0 = \mu_0^n \tag{F.11}$$

$$\begin{aligned}\nabla_{\Gamma_0} \mathbb{E}_X[\log L(X,Y)] &= -\frac{1}{2}\nabla_{\Gamma_0} \log|\Gamma_0| - \frac{1}{2}\nabla_{\Gamma_0}(\mu_0^n - \mu_0)^T \Gamma_0^{-1}(\mu_0^n - \mu_0) \\ &\quad - \frac{1}{2}\nabla_{\Gamma_0} \mathrm{Tr}[\Gamma_0^{-1}\Gamma_0^n] \end{aligned} \tag{F.12}$$

$$= -\frac{1}{2}\Gamma_0^{-1} + \frac{1}{2}\Gamma_0^{-1}\Gamma_0^n\Gamma_0^{-1} = 0 \tag{F.13}$$

$$\implies \Gamma_0 = \Gamma_0^n \tag{F.14}$$

$$\begin{aligned}\nabla_{\Omega_i} \mathbb{E}_X[\log L(X,Y)] &= -\frac{1}{2}\sum_{i=1}^{n} \nabla_{\Omega_i} \log|\Omega_i| \\ &\quad - \frac{1}{2}\sum_{i=1}^{n} \nabla_{\Omega_i}(\mu_i^n - \mu_{i-1}^n)^T \Omega_i^{-1}(\mu_i^n - \mu_{i-1}^n) \\ &\quad - \frac{1}{2}\sum_{i=1}^{n} \nabla_{\Omega_i} \mathrm{Tr}[\Omega_i^{-1}(\Gamma_i^n + \Gamma_{i-1}^n - \Gamma_{i,i-1}^n - \Gamma_{i-1,i}^n)] \end{aligned} \tag{F.15}$$

$$\begin{aligned} &= -\frac{1}{2}\Omega_i^{-1} + \frac{1}{2}\Omega_i^{-1}(\mu_i^n - \mu_{i-1}^n)(\mu_i^n - \mu_{i-1}^n)^T \Omega_i^{-1} \\ &\quad + \frac{1}{2}\Omega_i^{-1}(\Gamma_i^n + \Gamma_{i-1}^n - \Gamma_{i,i-1}^n - \Gamma_{i-1,i}^n)\Omega_i^{-1} = 0 \end{aligned} \tag{F.16}$$

$$\begin{aligned} \implies \Omega_i &= (\mu_i^n - \mu_{i-1}^n)(\mu_i^n - \mu_{i-1}^n)^T \\ &\quad + (\Gamma_i^n + \Gamma_{i-1}^n - \Gamma_{i,i-1}^n - \Gamma_{i-1,i}^n) \end{aligned} \tag{F.17}$$

$$\begin{aligned}\nabla_{\Sigma_i} \mathbb{E}_X[\log L(X,Y)] &= -\frac{1}{2}\sum_{i=1}^{n} \nabla_{\Sigma_i} \log|\Sigma_i| \\ &\quad - \frac{1}{2}\sum_{i=1}^{n} \nabla_{\Sigma_i}(\delta y_i - D_i\mu_i^n)^T \Sigma_i^{-1}(\delta y_i - D_i\mu_i^n) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n}\nabla_{\Sigma_i}\mathrm{Tr}[\Sigma_i^{-1}D_i\Gamma_i^n D_i^T] && \text{(F.18)}\\
&=-\frac{1}{2}\Sigma_i^{-1}+\frac{1}{2}\Sigma_i^{-1}(\delta y_i-D_i\mu_i^n)(\delta y_i-D_i\mu_i^n)^T\Sigma_i^{-1}\\
&\quad+\frac{1}{2}\Sigma_i^{-1}D_i^T\Gamma_i^n D_i\Sigma_i^{-1} && \text{(F.19)}\\
&=0 && \text{(F.20)}\\
\Longrightarrow\ \Sigma_i &=(\delta y_i-D_i\mu_i^n)(\delta y_i-D_i\mu_i^n)^T+D_i^T\Gamma_i^n D_i && \text{(F.21)}
\end{aligned}
$$

This completes the derivation of the update equations of the M-step.