Non-Abelian gauge theories : analogies in condensed matter systems

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Cover: the Golden Gate Bridge in San Francisco, depicted as an emergent phenomenon.

Voor mijn ouders
Voor Jacqueline

Contents

Chapter 1

Introduction

1.1 Symmetries and gauge theories

In the study of physical phenomena, conservation laws play an important role: things do not just vanish into the blue. For example, consider a bucket filled with water, and drill a hole in the bottom. The amount of water in the bucket will diminish, but one will recover the same amount when one considers the stream coming out of the hole. It is really this notion which is at the very beginning of the 19th century discipline of hydrodynamics.

The notion of conservation laws obtained a more profound meaning, when Noether realised that conservation laws are in one-to-one correspondence with symmetries. Perhaps the most simple-minded example is that of momentum conservation in Newtonian mechanics. For example, for a particle on which no forces are acting, the momentum is conserved. This is imminent from symmetry, since the absence of a force implies a constant potential, which is surely translational invariant. It is this translational symmetry which takes care of the momentum conservation.

Similarly, in Maxwell electrodynamics, conservation and symmetry go hand-in-hand as well, as related to charge conservation. The symmetry in question involves rotations of a complex scalar. When one only admits global rotations, one obtains global charge conservation. There is no reason to restrict oneself to global symmetries, however. It turns out that making symmetries local, leads to more than just a rephrasing of conservation laws. Requiring the symmetry to be local, requires in turn the introduction of a so-called gauge field, which in Maxwell theory describe photons. If one also allows this field to be dynamic, it turns out that the equations of motion for the gauge field are precisely the equations for electric and magnetic field of Maxwell theory, which say that currents and charges act as sources for electromagnetic fields.

The notion of symmetry has become more and more powerful in physics during the twentieth century, especially with the advent of quantum field theory. Indeed, the first step was to understand Maxwell dynamics as a field theory of complex scalars which are symmetric under local phase rotations, as explained above. These phase rotations are commutative, i.e., one can interchange rotations without changing the final result. These symmetries under commutative groups go under the name of Abelian gauge field theories, after the mathematician Abel.

Later on, local symmetries were generalised to non-Abelian groups, i.e., groups of non-commuting elements. For example, the rotations in three dimensions form a non-Abelian group, since rotations do not generally commute. In fact, the electroweak interaction, which governs e.g. the decay of neutrons into protons, can be described by a non-Abelian symmetry acting on two complex dimensional fields. The group of interest here is $SU(2)$, i.e., the rotations in spin space. With this notion, the very succesful Yang-Mills theory has been developed. The culmination of the study of non-Abelian gauge theories is the development in the seventies of quantum chromodynamics (QCD), which is really nothing more than the study of the group $SU(3)$. It gives a transparent framework to describe the whole zoo of particles as discovered in accelerators. QCD introduces not more than 2×6 elementary particles. The resulting unified theory, $SU(3) \times SU(2) \times U(1)$, known as the standard model, explains all phenomena encountered in accelerators. The strong predictions it makes are all confirmed, except for one. This one prediction involves the existence of the Higgs particle.

1.2 Symmetry breaking

This Higgs particle is intimately connected with another notion, which is at least as important as symmetry: namely, the spontaneous *breaking* of symmetry. It states the following: if a system is described by some theory, e.g., a theory which is symmetric under global $U(1)$, the ground state of that system might break the symmetry under consideration. This concept is actually much less esoteric than it might seem. Stronger, one ecounters symmetry breaking all the time in everyday life. For example, the chair in which you are reading this introduction, has no preference for being in some position in your room, i.e., its posititon is described by a translationally invariant theory. Still, it is where it is, and not five foot further. Obviously, the translational symmetry is broken. By the same token, consider a bunch of carbon atoms at room temperature and ambient pressure. They are similarly described by a translationally invariant theory. Yet they choose to form a quite regular lattice! The reader might object that the symmetry is not broken, since you can translate the carbon lattice by a multiple of the lattice spacing, to obtain the same result. This is correctly observed: when symmetries are (spontaneously) broken, it does not mean that all the symmetry is gone. There might be some remaining ones, like translations modulo lattice spacings. These remaining symmetries lead to physical modes: in crystals, one has phonons, and in superconductors, one has supercurrents. These modes are known as Goldstone modes. These examples already point out that symmetry is broken, when *order* occurs: crystalline order breaks translational symmetry, for example. For high energy physics, however, it is exactly this broken order parameter which still has not been found, namely, a condensate of Higgs particles. One of the reasons why condensed matter physics has an advantage relative to high energy physics, is that it is in most instances clear what the order parameter is, and that it is accessible for experiments. This is what condensed matter actually is about: the study of long wavelength, low temperature phenomena induced by states characterised by a spontaneously broken symmetry, and these occur in a plethora of possibilities.

1.3 Parallel transport

The most important problem in physics is perhaps how to unify the standard model with the other great theory of the 20th century: general relativity, describing gravity. General relativity is about an elegant geometric description of how all constituents of Nature move under the influence of gravitational forces emerging from all the other objects. All objects carry energy which is equivalent to mass, according to Einstein. Massive objects curve spacetime, and particles etc. moving in that spacetime, follow trajectories which are shortest. Rephrasing the essence of this idea, energy is a source for curvature, where the latter governs the trajectory of all physical objects. The parallel between gauge theories and gravity theories is now easily drawn: in gauge theories, one has sources which determine field strengths. In other words, the parallel is "curvature=field strength". This geometric interpretation of gauge theories and gravity has proved to be very instructive and fruitful [1]. This makes it even more tantalising that gravity and gauge theories are still not united into one single consistent framework.

Let us focus on the curvature aspect of gauge theories, implied by the geometric interpretation. In gravity, the curvature determines how matter moves. In the geometric framework, we say that curvature encodes a parallel transport structure. Now, in gauge theories, the gauge fields can be interpreted in the same way. For example, since spins are transformed by $SU(2)$ matrices, parallel transport by $SU(2)$ gauge fields can be interpreted as rotating spins.

A beautiful analogue of emergent gravity from curvature in condensed matter systems, is provided by nematic liquid crystals. Defects therein give rise to curvature, which exactly behaves like a parallel transport structure [2]. In this way, it is an excellent example of emergent gravity in condensed matter systems [3].

1.4 Motivation for this work

This thesis discusses typical examples of how structures associated with non-Abelian gauge theories can emerge in a non-obvious way in condensed matter systems. The first aspect is how parallel transport structures can be understood in the context of transport of spins by electromagnetic fields. This discussion can give a mean to understand how one can attempt to understand quark-gluon plasmas.

Secondly, the example of the Mott insulator (MI) is discussed, in which an $SU(2)$ gauge theory emerges in its full glory, having full dynamics of the gauge fields. It is shown how a condensed matter analogue of the quark-gluon plasma emerges in the MI, known as the deconfined spin liquid. The ramification of the discussion is that this mysterious deconfining state leads to predictions which, surprisingly enough, have been measured in the laboratory in the context of high-T_c superconductors.

Let us move to a more substantial level of discussion for both of the systems of interest.

1.4.1 Spin-orbit coupled systems and the hydrodynamics of QCD

In the very beginning of this introduction, hydrodynamics was already mentioned. In fact, all of everyday life understanding of transport phenomena is based on hydrodynamics. In turn, this notion is based on mass conservation, like the example of the bucket spilling water. Other examples are electric currents or magnetohydrodynamic currents in plasmas. This local mass conservation is described by a $U(1)$ theory, which is Abelian. The question arises if there is such a thing as non-Abelian hydrodynamics. If so, this concept would really help to understand colour-currents in quark-gluon plasmas, which originate from a deconfining $SU(3)$ gauge theory.

It is a surprising development that in condensed matter, non-Abelian transport phenomena in the context of spin systems has attracted much attention recently [4, 5, 6]. A main motivation is provided by spintronics [7], which is about spin currents driven by electromagnetic fields by the effect of spin-orbit coupling. Spin-orbit coupling is a prediction of the Dirac equation, when expanded to lowest order in v/c. The non-relativistic limit then shows that electromagnetic fields rotate $SU(2)$ spins. Many people [8, 9, 10] realised that this can be understood as the electromagnetic fields mimicking an $SU(2)$ parallel transport structure for the spins. The unconvenient truth is that normally, this parallel transport destroys local spin conservation, implying the absence of hydrodynamics.

Surprisingly enough, if the spin system becomes quantum coherent, i.e., becomes superfluid, hydrodynamics re-emerges from the ashes, albeit in the guise of covariant conservation. This covariant hydrodynamics hinges on the presence of an order parameter, providing rigidity. The fact that in our example the gauge fields do not have dynamics, in contrast to quark-gluon plasmas, is not important. The reason is that both in QCD as in spin-orbit coupled systems, there are no ordinary local conservation laws; only covariant conservation laws exist. It is this covariant conservation which is of importance in understanding non-Abelian hydrodynamics.

The ‘Higgs’ phase of the spin system makes it possible to discuss topological effects. These considerations are on the one hand motivated by the $SU(2)$ texture of the ’t Hooft-Polyakov monopole [11, 12], and on the other hand by the question what the spin rigid version of the Aharonov-Casher effect, relevant for spin-orbit coupled systems, would look like [9, 13]. It turns out that there exists a spin-fluid analogue of magnetic flux trapping by the Aharonov-Bohm effect. The difference with the Aharonov-Bohm effect is that in the latter case, the gauge fields are dynamical, whereas the gauge fields are fixed in the spin-orbit coupled case. We will demonstrate that it is the *absence* of screening of electromagnetic fields, which causes the charge giving rise to the electromagnetic field to be quantised in the charge quantum $\lambda = \frac{m}{\mu_0 e}$, the analogue of the magnetic flux quantum. Here, m is the mass of the particles forming the superfluid.

Then we will turn our attention to a system which is the only known candidate to become superfluid: ^{3}He. Superfluid ^{3}He is well studied [14, 15, 16], because of its rich vacuum structure, giving rise to a host of possible topological excitations. Many features of ^{3}He are also interesting since these give a condensed matter analogue of the vacuum structure of the universe. In this way, it is part of the rich subject of 'cosmology in the lab', a research programme trying to understand and simulate cosmological phenomena by studying condensed matter systems which display similar behaviour.

Our interest, however, will be focused on how spin-orbit coupling affects the topology of ^{3}He, which will provide much insight. We will also discuss if and how the richness of the order parameter will destroy the Aharonov-Casher topology making charge trapping possible.

1.4.2 The Mott insulator and the quark-gluon plasma

The spin-orbit coupled systems give a good playground to understand non-Abelian parallel transport, but are less suitable to understand gauge field dynamics, since the gauge fields are fixed by the electromagnetic fields.

A gauge theory displaying full gauge field dynamics is provided by the Mott insulator. This is a condensed matter system with precisely one electron per unit cell. The Coulomb repulsion is so strong that the electrons cannot move. Note that this is different from the band insulator, where motion of the electrons is forbidden by the Pauli principle. The fact that there is one electron per site can immediately be interpreted as a local conservation law. This constraint can be translated into gauge fields, which need to have full dynamics in order to impose the constraint exactly.

At a first glance, this would amount to a $U(1)$ theory, since particle number conservation can be interpreted as a constraint imposed by a gauge field with one phase. In fact, the structure of the Mott insulator is richer. The electron carries the quantum numbers of spin and charge, of which the charge degrees of freedom are frozen out by Coulomb blocking. Hence the relevant degrees of freedom are the excitations carrying spin, the spinons. It can be shown that in the MI the removal of a down-spinon is the same as adding an up-spinon. This symmetry is a symmetry of spinors, i.e., an $SU(2)$ gauge symmetry. These spinors are the condensed matter analogues of the quark doublets in QCD.

The question arises if the situation is similar to high-energy physics: is the deconfining state of spinons/quarks only visible at very high energy, as dictated by the asymptotic freedom in QCD? Or is there some hope that spinons have physical relevance for the long-wavelength limit? The answer is that the condensed matter analogue of the deconfined quark/gluon plasma, known the spin liquid, can exist. The proviso here is that one accepts the existence of spinonic order parameters, i.e., one beliefs that mean field vacuum expectation values of pairs of spinon operators exist.

If so, a miracle can happen: the existence of these mean fields break the $SU(2)$ gauge theory down to an effective theory, which can have lower gauge symmetry, for example, $U(1)$. The latter can still be confining [17], but there are mean field states which have the power to make the effective gauge theory deconfining, after integrating out the spinons. Put differently: there are classes of mean fields giving the condensed matter quark/gluon plasma analogue stability against gauge fluctuations.

It turns out that for the mean field theories under consideration, the familiar classification of the phases of gauge theories breaks down. For gauge theories, their behaviour is determined by the Wilson loop: if it obeys the area law, the theory is in the confining phase, whereas the perimeter law signals the deconfining phase. It is explained what classification is necessary to correctly address the various possibilities for the low-energy gauge theories. In this discussion, we summarise the ideas developed by X.-G. Wen [18]. The scheme of classifying various mean field theories will also be different from the way classical symmetry breaking is classified. This will be illustrated by considering mean fields which have different symmetries, but have the same excitation spectrum, which is impossible for classical orders. It turns out that this is due to the possibility that two different states can be gauge equivalent within the original gauge theory. The new classification scheme will involve both the gauge symmetries together and the spatial symmetries of the mean field states. The ramification is that condensed matter systems have the possibility to show deconfining states after integration over the matter fields, analogous to the quark/gluon plasma. In particular, they can show massless excitations with Dirac dispersion: the nodal fermions. These nodal fermions are then analogous to the deconfined quarks.

This deconfining state turns out to make sense experimentally as well. This becomes particularly clear in the context of the problem of high-temperature superconductivity, which emerges on removing electrons from a Mott insulator (doping). Including charge degrees of freedom, it is possible to set up a slave theory involving both spin degrees of freedom (spinons) and charge degrees of freedom (holons). Assuming a deconfining state of spinons and holons, as motivated by the possibility of a deconfining spin liquid, leads to predictions which are surprisingly well in agreement with experimental results. In the first place, the constraint structure forces the holons to have a hard core. This condition leads to phase separation for values of the doping which are consistent with the experimentally measured values for the compressibility [19]. The $SU(2)$ theory displaying phase separation denies ideas in the community that slave theories cannot incorporate inhomogeneous states.

More importantly, the constraint structure together with mean field calculations, insists that the order parameter of the high-T_c's should have an isotropic s-wave component, on top of the well-established d-wave symmetry. This s-wave admixture is measured in several experiments [20, 21, 22].

The conclusion from the second part is that an emergent deconfined state in the $SU(2)$ gauge theory of the doped Mott insulator gives results that are not in conflict with at least some experiments. This is the motivation that condensed matter systems with emergent gauge theories provide evidence that deconfining states can exist as effective low-energy theories. This fact is not evident, since the quark-gluon plasma is supposed to be found at high energies.

Part I

Spin-orbit coupled systems

Chapter 2

Introduction to spin-orbit coupled systems

It is a remarkable development that in various branches of physics there is a revival going on of the long standing problem of how non-Abelian entities are transported over macroscopic distances. An important stage is condensed matter physics. A first major development is spintronics, the pursuit to use the electron spin instead of its charge for switching purposes [4, 5, 6, 7, 23, 24], with a main focus on transport in conventional semiconductors. Spin-orbit coupling is needed to create and manipulate these spin currents, and it has become increasingly clear that transport phenomena are possible that are quite different from straightforward electrical transport. A typical example is the spin-Hall effect [4, 5, 7], defined through the macroscopic transport equation,

$$j_i^a = \sigma_{SH} \epsilon_{ial} E_l \tag{2.0.1}$$

where ϵ_{ial} is the 3-dimensional Levi-Civita tensor and E_l is the electrical field. The specialty is that since both j_i^a and E_l are even under time reversal, the transport coefficient σ_{SH} is also even under time reversal, indicating that this corresponds to a dissipationless transport phenomenon. Triggered by theoretical work, this spin-Hall effect was recently observed experimentally in various settings. An older development is the mesoscopic spin-transport analogue of the Aharonov-Bohm effect, called the Aharonov-Casher effect [13]: upon transversing a loop containing an electrically charged wire the spin conductance will show oscillations with a period set by the strength of the spin-orbit coupling and the enclosed electrical line-charge.

A rather independent development in condensed matter physics is the recent focus on the multiferroics. This refers to substances that show simultaneous ferroelectric- and ferromagnetic order at low temperatures, and these two different types of order do rather strongly depend on each other. It became clear recently that at least in an important subclass of these systems one can explain the phenomenon in a language invoking dissipationless spin transport [25, 26]: one needs a magnetic order characterized by spirals such that 'automatically' spin currents are flowing, that in turn via spin-orbit coupling induce electrical fields responsible for the ferroelectricity.

The final condensed matter example is one that was lying dormant over the last years: the superfluids realized in ^{3}He. A way to conceptualize the intricate order parameters of the A- and B-phase [14, 15] is to view these as non-Abelian ('spin-like') superfluids. The intricacies of the topological defects in these phases is of course very well known, but matters get even more interesting when considering the effects on the superflow of macroscopic electrical fields, mediated by the very small but finite spin-orbit coupling. This subject has been barely studied: there is just one paper by Mineev and Volovik [10] addressing these matters systematically.

A very different pursuit is the investigation of the quark-gluon plasma's presumably generated at the Brookhaven heavy-ion collider. This might surprise the reader: what is the relationship between the flow of spin in the presence of spin-orbit coupling in the cold condensed matter systems and this high temperature QCD affair? There is actually a very deep connection that was already realized quite some time ago. Goldhaber [27] and later Froehlich *et al.* [8], Balatskii and Altshuler [9] and others realized that in the presence of spin-orbit coupling spin is subjected to a parallel transport principle that is quite similar to the parallel transport of matter fields in Yang-Mills non-Abelian Gauge theory, underlying for instance QCD. This follows from a simple rewriting of the Pauli-equation, the Schroedinger equation taking into account the leading relativistic corrections: the spin-fields are just subjected to covariant derivatives of the Yang-Mills kind, see Eq.'s (3.1.4),(3.1.5) . However, the difference is that the 'gauge' fields appearing in these covariant derivatives are actually physical fields. These are just proportional to the electrical- and magnetic fields. Surely, this renders the problem of spin transport in condensed matter systems to be dynamically very different from the fundamental Yang-Mills theory (standard model). However, the parallel transport structure has a 'life of its own': it implies certain generalities that are even independent of the 'gauge' field being real gauge or physical.

For all the various examples we alluded to in the above, one is dealing with macroscopic numbers of particles that are collectively transporting non-Abelian quantum numbers over macroscopic distances and times. In the Abelian realms of electrical charge or mass a universal description of this transport is available in the form of hydrodynamics, be it the hydrodynamics of water, the magneto-hydrodynamics of charged plasma's, or the quantum-hydrodynamics of superfluids and superconductors. Henceforth, to get anywhere in terms of a systematic description one would like to know how to think in a hydrodynamical fashion about the macroscopic flow of non-Abelian entities, including spin.

In the condensed matter context one finds pragmatic, case to case approaches that are not necessarily wrong, but are less revealing regarding the underlying 'universal' structure: in spintronics one solves Boltzmann transport equations, limited to dilute and weakly interacting systems. In the quark-gluon plasma's one find a similar attitude, augmented by RPA-type considerations to deal with the dynamics of the gauge fields. In the multiferroics one rests on a rather complete understanding of the order parameter structure.

The question remains: what is non-Abelian hydrodynamics? To the best of our knowledge this issue is only addressed on the fundamental level by Jackiw and coworkers [28, 29] and their work forms a main inspiration for this review. The unsettling answer seems to be: *non-Abelian hydrodynamics in the conventional sense of describing the collective flow of quantum numbers in the classical liquid does not even exist!* The impossibility to define 'soft' hydrodynamical degrees of freedom is rooted in the non-Abelian parallel transport structure per se and is therefore shared by high temperature QCD and spintronics.

The root of the trouble is that non-Abelian currents do not obey a continuity equation but are instead only *covariantly conserved*: we will explain this in detail in section 3.3. It is well known that covariant conservation laws do not lead to global conservation laws, and the lack of globally conserved quantities makes it impossible to deal with matters in terms of a universal hydrodynamical description. This appears to be a most serious problem for the description of the 'non-Abelian fire balls' created in Brookhaven. In the spintronics context it is well known under the denominator of 'spin relaxation': when a spin current is created, it will plainly disappear after some characteristic spin relaxation determined mostly by the characteristic spin-orbit coupling strength of the material.

Here we will approach the subject of spin transport in the presence of spin-orbit coupling from the perspective of the non-Abelian parallel transport principle. At least to our perception, this makes it possible to address matters in a rather unifying, systematical way. It is not a-priori clear how the various spin transport phenomena identified in condensed matter relate to each other and we hope to convince the reader that they are different sides of the same non-Abelian hydrodynamical coin. Except for the inspiration we have found in the papers by Jackiw and coworkers [28, 29] we will largely ignore the subject of the fundamental non-Abelian plasma, although we do hope that the 'analogous systems' we identify in the condensed matter system might form a source of inspiration for those working on the fundamental side.

Besides bringing some order to the subject, in the course of the development we found quite a number of new and original results that are consequential for the general, unified understanding. We will start out on the pedestrian level of quantum-mechanics, discussing in detail how the probability densities of non-Abelian quantum numbers are transported by isolated quantum particles and how this relates to spin-orbit coupling (Section 3.1). We will derive equations that are governing the mesoscopics, like the Aharonov-Casher (AC) effect, while they are completely general. A main conclusion will be that already on this level the troubles with the macroscopic hydrodynamics are shimmering through: the AC effect is more fragile than the Abelian Aharonov-Bohm effect, in the sense that the experimentalists have to be much more careful in designing their machines in order to find the AC signal.

In the short section 3.3 we revisit the non-Abelian covariant conservation laws, introducing a parametrization that we perceive as very useful: different from the Abelian case, non-Abelian currents can be viewed as being composed of both a coherent, 'spin' entangled part and a factorisable incoherent part. This difference is at the core of our classification of non-Abelian fluids. The non-coherent current is responsible for the transport both in the high temperature liquid and in the multiferroic systems.

The coherent current is responsible for both the Meissner 'diamagnetic' screening currents in the 'fundamental' non-Abelian Higgs phase, but also for the non-Abelian supercurrents in true spin-superfluids like the ^{3}He A- and B phase.

The next step is to deduce the macroscopic hydrodynamics from the microscopic constituent equations and here we follow Jackiw *et al.* closely. Their 'particle based' non-Abelian hydrodynamics has just to be associated with the classical hydrodynamics of the high temperature spin-fluid and here the lack of hydrodynamical description hits full force: we hope that the high energy physicists find our simple 'spintronics' examples illuminating (Section 3.4).

As a next step we turn to the 'super' spin currents of the multiferroics (Section 3.6). As we will show, these are rooted in the 'coherent' non-Abelian currents and this renders it to be quite similar but subtly different from the 'true' supercurrents of the spin superfluid: it turns out that in contrast to the latter they can create electrical charge! This is also a most elementary context to introduce a notion that we perceive as the most important feature of non-Abelian fluid theory. In Abelian hydrodynamics it is well understood when the superfluid order sets in, its rigidity does change the hydrodynamics: it renders the hydrodynamics of the superfluid to be irrotational having the twofold effect that the circulation in the superfluid can only occur in the form of massive, quantized vorticity while at low energy the superfluid is irrotational so that it behaves like a dissipationless ideal Euler liquid. In the non-Abelian fluid the impact of the order parameter is more dramatic: its rigidity removes the multivaluedness associated with the covariant derivatives and hydrodynamics is restored!

This bring us to our last subject where we have most original results to offer: the hydrodynamics of spin-orbit coupled spin-superfluids (Section 3.7). These are the 'fixed frame' analogs of the non-Abelian Higgs phase and we perceive them as the most beautiful physical species one encounters in the non-Abelian fluid context. Unfortunately, they do not seem to be prolific in nature. The ^{3}He-superfluids belong to this category but it is an unfortunate circumstance that the spin-orbit coupling is so weak that one encounters insurmountable difficulties in the experimental study of its effects. Still we will use them as an exercise ground to demonstrate how one should deal with more complicated non-Abelian structures (Sections 5.2,5.4), and we will also address the issue of where to look for other spin-superfluids in the Conclusion and Outlook of this thesis (Chapter 9).

To raise the appetite of the reader let us start out presenting some wizardry that should be possible to realize in a laboratory when a spin-superfluid would be discovered with a sizable spin-orbit coupling: how the elusive spin-superfluid manages to trap electrical line charge (section 2.1), to be explained in Chapter 4.

2.1 The appetizer: trapping quantized electricity

Imagine a cylindrical vessel, made out of plastic while its walls are coated with a thin layer of gold. Through the center of this vessel a gold wire is threaded and care is taken that it is not in contact with the gold on the walls. Fill this container to the brim with a putative liquid that can become a spin superfluid (liquid ^{3}He would work if it did not contain a dipolar interaction that voids the physics) in its normal state and apply now a large bias to the wire keeping the walls grounded, see Fig.2.1. Since it is a capacitor, the wire will charge up relative to the walls. Take care that the line charge density on the wire is pretty close to a formidable 2.6×10^{-5} Coulomb per meter.

Having this accomplished, cool the Helium through its superfluid phase transition at 1.7 mK. Remove now the voltage and hold the end of the wire close to the vessel's wall. Given that the charge on the wire is huge, one anticipates a disastrous decharging spark but .... nothing happens!

It is now time to switch the dilution fridge. Upon monitoring the rising temperature, right at the 1.7 mK where the helium turns normal a spark jumps from the wire to the vessel, grilling the machinery into a pile of black rubble.

This is actually a joke. In section 3.7 and Chapter 4 we will present the theoretical proof that this experiment can actually be done. There is a caveat, however. The only substance that has been identified, capable of doing this trick is ^{3}He. As it turns out, in order to prevent bad things to happen *one needs a vessel with a cross sectional area that is roughly equal to the area of Alaska.* Given that there is only some 170 kg of helium on the planet, it occurs that this experiment cannot be practically accomplished.

What is going on here? This effect is analogous to magnetic flux trapping by superconducting rings. One starts out there with the ring in the

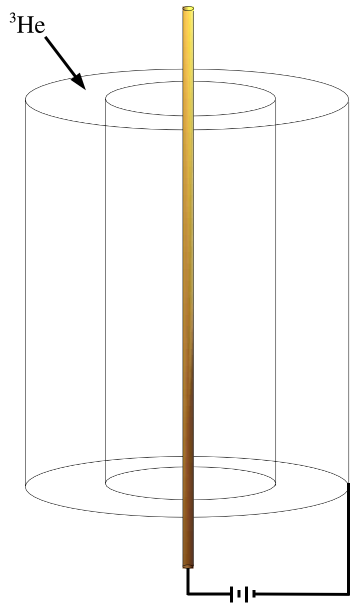

Figure 2.1: A superfluid ^{3}He container acts as a capacitor capable of trapping a quantized electrical line charge density via the electric field generated by persistent spin Hall currents. This is te analog of magnetic flux trapping in superconductors by persistent charge supercurrents.

normal state, in the presence of an external magnetic field. One cycles the ring below the transition temperature, and after switching off the external magnetic field a quantized magnetic flux is trapped by the ring. Upon cycling back to the normal state this flux is expelled. Read for the magnetic flux the electrical line charge, and for the electrical superconductor the spin-superfluid and the analogy is clear.

This reveals that in both cases a similar parallel transport is at work. It is surely not so that this can be understood by simple electro-magnetic duality: the analogy is imprecise because of the fact that the physical field enters in the spin-superfluid problem via the spin-orbit coupling in the way the vector potential enters in the superconductor. This has the ramification that the electrical monopole density takes the role of the magnetic flux, where the former takes the role of physical incarnation of the pure gauge Dirac string associated with the latter.

The readers familiar with the Aharonov-Casher (AC) effect should hear a bell ringing. This can indeed be considered as just the 'rigid' version of the AC effect, in the same way that flux trapping is the rigid counterpart of the mesoscopic Aharonov-Bohm effect. On the single particle level, the external electromagnetic fields prescribe the behavior of the particles, while in the ordered state the order parameter has the power to impose its will on the electromagnetic fields.

This electrical line-charge trapping effect summarizes neatly the deep but incomplete relations between real gauge theory and the working of spin-orbit coupling. It will be explained in great detail in section 3.7 and Chapter 4, but before we get there we first have to cross some terrain.

Chapter 3

Non-Abelian hydrodynamics

3.1 Quantum mechanics of spin-orbit coupled systems

To address the transport of spin in the presence of spin-orbit(SO) coupling we will follow a strategy well known from conventional quantum mechanical transport theory. We will first analyze the single particle quantum-mechanical probability currents and densities. The starting point is the Pauli equation, the generalization of the Schrödinger equation containing the leading relativistic corrections as derived by expanding the Dirac equation using the inverse electron rest mass as expansion parameter. We will first review the discovery by Volovik and Mineev [10], Balatskii and Altshuler [9] and Froehlich and others [8] of the non-Abelian parallel transport structure hidden in this equation, to subsequently analyze in some detail the equations governing the spin-probability currents. In fact, this is closely related to the transport of color currents in real Yang-Mills theory: the fact that in the SO problem the 'gauge fields' are physical fields is of secondary importance since the most pressing issues regarding non-Abelian transport theory hang together with parallel transport. For these purposes, the spin-orbit 'fixed-frame' incarnation has roughly the status as a representative gauge fix. In fact, the development in this section has a substantial overlap with the work of Jackiw and co-workers dedicated to the development of a description of non-Abelian fluid dynamics [28, 29]. We perceive the application to the specific context of SO coupled spin fluid dynamics as clarifying and demystifying in several regards. We will identify their 'particle based' fluid dynamics with the high temperature, classical spin fluid where the lack of true hydrodynamics is well established, also experimentally. Their 'field based' hydrodynamics can be directly associated with the coherent superflows associated with the SO coupled spin superfluids where at least in equilibrium a sense of a protected hydrodynamical sector is restored.

The development in this section might have a direct relevance to mesoscopic transport phenomena (like the Aharonov-Casher effect [9, 13]. However, we will largely ignore these applications and our first aim is to set up the system of microscopic, constituent equations that will be used in the subsequent sections to derive the various macroscopic fluid theories.

The starting point is the well known Pauli-equation describing mildly relativistic particles. This can be written in the form of a Lagrangian density in terms of spinors ψ,

$$\begin{aligned}\mathcal{L} = & \, i\hbar\psi^\dagger(\partial_0\psi) - qB^a\psi^\dagger\frac{\tau^a}{2}\psi + \frac{\hbar^2}{2m}\psi^\dagger\left(\nabla - \frac{ie}{\hbar}\vec{A}\right)^2\psi \\ & - eA_0\psi^\dagger\psi + \frac{iq}{2m}\epsilon_{ial}E_l\left\{(\partial_i\psi^\dagger)\frac{\tau^a}{2}\psi - \psi^\dagger\frac{\tau^a}{2}(\partial_i\psi)\right\} \\ & + \frac{1}{8\pi}\left(E^2 - B^2\right)\end{aligned} \tag{3.1.1}$$

where as usual

$$\vec{E} = -\nabla A_0 - \partial_0\vec{A}, \qquad \vec{B} = \nabla\times\vec{A}. \tag{3.1.2}$$

The A_μ are the usual $U(1)$ gauge fields associated with the electromagnetic fields $\vec{E}$ and $\vec{B}$. The relativistic corrections are present in the terms containing the quantity q, proportional to the Bohr magneton, and the time-like first term $\propto B$ is the usual Zeeman term while the space-like terms $\propto E$ corresponds with spin-orbital coupling.

The recognition that this has much to do with a non-Abelian parallel transport structure, due to Mineev and Volovik [10], Goldhaber [27] and Froehlich et al. [8] is in fact very simple. Just redefine the magnetic- and electric field strengths as follows,

$$A_0^a = B^a \qquad\qquad A_i^a = \epsilon_{ial}E_l\,. \tag{3.1.3}$$

Define covariant derivatives as usual,

$$D_i = \partial_i - i\frac{q}{\hbar}A_i^a\frac{\tau^a}{2} - i\frac{e}{\hbar}A_i \tag{3.1.4}$$

$$D_0 = \partial_0 + i\frac{q}{\hbar}A_0^a\frac{\tau^a}{2} + i\frac{e}{\hbar}A_0\,, \tag{3.1.5}$$

and it follows that the Pauli Lagrangian becomes,

$$\begin{aligned}\mathcal{L} &= i\hbar\psi^\dagger D_0\psi + \psi^\dagger\frac{\hbar^2}{2m}\vec{D}^2\psi \\ &+ \frac{1}{2m}\psi^\dagger\left(2eq\frac{\tau^a}{2}\vec{A}\cdot\vec{A}^a + \frac{q^2}{4}\vec{A}^a\cdot\vec{A}^a\right)\psi \\ &+ \frac{1}{8\pi}\left(E^2 - B^2\right).\end{aligned} \tag{3.1.6}$$

Henceforth, the derivatives are replaced by the covariant derivatives of a $U(1)\times SU(2)$ gauge theory, where the $SU(2)$ part takes care of the transport of spin. Surely, the second and especially the third term violate the $SU(2)$ gauge invariance for the obvious reason that the non-Abelian 'gauge fields' A_μ^a are just proportional to the electromagnetic $\vec{E}$ and $\vec{B}$ fields. Notice that the second term just amounts to a small correction to the electromagnetic part (third term). The standard picture of how spins are precessing due to the spin-orbit coupling to external electrical- and magnetic fields, pending the way they are moving through space can actually be taken as a literal cartoon of the parallel transport of non-Abelian charge in some fixed gauge potential!

To be more precise, the SO problem actually corresponds to fixing a particular gauge in the full $SU(2)$ gauge theory. The electromagnetic fields have to obey the Maxwell equation

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \tag{3.1.7}$$

and this in turn implies

$$\partial^\mu A_\mu^a = 0\,. \tag{3.1.8}$$

Therefore, the SO problem is 'representative' for the $SU(2)$ gauge theory in the Lorentz gauge and we do not have the choice of going to another gauge as the non-Abelian fields are expressed in terms of real electric and magnetic fields. This is a first new result.

By varying the Lagrangian with respect to $\psi^\dagger$ we obtain the Pauli equation in its standard Hamiltonian form,

$$i\hbar D_0\psi = -\frac{\hbar^2}{2m}D_i^2\psi - \frac{1}{2m}\left(2eq\frac{\tau^a}{2}\vec{A}\cdot\vec{A}^a + \frac{q^2}{4}\vec{A}^a\cdot\vec{A}^a\right)\psi \qquad (3.1.9)$$

where we leave the electromagnetic part implicit, anticipating that we will be interested to study the behavior of the quantum mechanical particles in fixed background electromagnetic field configurations. The wave function ψ can be written in the form,

$$\psi = \sqrt{\rho}\, e^{(i\theta + i\varphi^a\tau^a/2)}\chi \qquad (3.1.10)$$

with the probability density ρ, while θ is the usual Abelian phase associated with the electromagnetic gauge fields. As imposed by the covariant derivatives, the $SU(2)$ phase structure can be parametrised by the three non-Abelian phases φ^a, with the Pauli matrices τ^a acting on a reference spinor χ. The careful reader might object that the number of degrees of freedom does not match: the spinor on the left-hand side has four real degrees of freedom, whereas the right hand side has five (viz., ρ, θ and the three τ^a). This redundancy can be removed, however. For example, if the reference spinor is chosen $\chi = (1, 0)$, the third Pauli matrix $e^{i\varphi_3\tau^3/2}$ can be absorbed in the Abelian phase $e^{i\theta}$. This Abelian $U(1)$ phase is gauge, as phases of the wave function make no physical difference. Hence it can be chosen zero, by which the redundant fifth degree of freedom vanishes.

Hence, with regard to the wavefunction there is no difference whatever between the Pauli-problem and genuine Yang-Mills quantum mechanics: this is all ruled by parallel transport.

Let us now investigate in further detail how the Pauli equation transports spin-probability. This is in close contact with work in high-energy physics and we develop the theory along similar lines as Jackiw *et al.*[29]. We introduce however a condensed matter inspired parametrization that we perceive as instrumental towards laying bare the elegant meaning of the physics behind the equations. We derive several new results. Let us dwell a little longer on the level of quantum mechanics.

A key ingredient of our parametrization is the introduction of a non-Abelian phase velocity, an object occupying the adjoint together with the vector potentials.The equations in the remainder will involve time and space derivatives of θ, ρ and of the spin rotation operators $e^{i\varphi^a\tau^a/2}$. Let us introduce the operator S^a as the non-Abelian charge at time t and at position $\vec{r}$, as defined by the appropriate $SU(2)$ rotation

$$S^a \equiv e^{-i\varphi^a\tau^a/2} \, \frac{\tau^a}{2} \, e^{i\varphi^a\tau^a/2} \, . \tag{3.1.11}$$

The temporal and spatial dependence arises through the non-Abelian phases $\varphi^a(t, \vec{r})$. The non-Abelian charges are, of course, $SU(2)$ spin 1/2 operators:

$$S^a S^b = \frac{\delta^{ab}}{4} + \frac{i}{2}\epsilon^{abc} S^c. \tag{3.1.12}$$

It is illuminating to parametrize the derivatives of the spin rotation operators employing non-Abelian velocities $\vec{u}^a$ defined by,

$$\begin{aligned} \frac{im}{\hbar}\vec{u}^a S^a &\equiv e^{-i\varphi^a\tau^a/2}(\nabla e^{i\varphi^a\tau^a/2}) \qquad \text{or} \\ \vec{u}^a &= -2i\frac{\hbar}{m}\mathrm{Tr}\left\{ e^{-i\varphi^a\tau^a/2}(\nabla e^{i\varphi^a\tau^a/2})S^a \right\} , \end{aligned} \tag{3.1.13}$$

which are just the analogs of the usual Abelian phase velocity

$$\vec{u} \equiv \frac{\hbar}{m}\nabla\theta = -i\frac{\hbar}{m}e^{-i\theta}\nabla e^{i\theta} \tag{3.1.14}$$

and as the latter this phase velocity is the scale parameter for the propagation of spin probability in non-Abelian quantum mechanics, or either for the hydrodynamical flow of spin-superfluid.

In addition we need the zeroth component of the velocity

$$\begin{aligned} iu_0^a S^a &\equiv e^{-i\varphi^a\tau^a/2}(\partial_0 e^{i\varphi^a\tau^a/2}) \qquad \text{or} \\ u_0^a &= -2i\mathrm{Tr}\left\{ e^{-i\varphi^a\tau^a/2}(\partial_0 e^{i\varphi^a\tau^a/2})S^a \right\} \end{aligned} \tag{3.1.15}$$

being the time rate of change of the non-Abelian phase, while is the exact analog of the time derivative of the Abelian phase representing matter-density fluctuation,

$$u_0 \equiv \partial_0\theta = -i\frac{\hbar}{m}e^{-i\theta}\partial_0 e^{i\theta} \, . \tag{3.1.16}$$

It is straightforward to show that the definitions of the spin operators S^a, Eq.(3.1.11) and the non-Abelian velocities u^a_μ, Eq.(3.1.13), (3.1.15), imply in combination,

$$\partial_0 S^a = -\epsilon^{abc}u_0^b S^c \quad \nabla S^a = -\frac{m}{\hbar}\epsilon^{abc}\vec{u}^b S^c. \tag{3.1.17}$$

It is easily checked that the definition of the phase velocity Eq. (3.1.13) imply the following identity,

$$\nabla \times \vec{u}^a + \frac{m}{2\hbar}\epsilon_{abc}\vec{u}^b \times \vec{u}^c = 0 \ , \tag{3.1.18}$$

which has as Abelian analog,

$$\nabla \times \vec{u} = 0 \ . \tag{3.1.19}$$

as the latter controls vorticity, the former is in charge of the topology in the non-Abelian 'probability fluid'. It, however, acquires a truly quantum-hydrodynamical status in the rigid superfluid where it becomes an equation of algebraic topology. This equation is well known, both in gauge theory and in the theory of the ^{3}He superfluids where it is known as the Mermin-Ho equation[30].

3.2 Spin transport in the mesoscopic regime

Having defined the right variable, we can now go ahead with the quantum mechanics, finding transparent equations for the non-Abelian probability transport. Given that this is about straight quantum mechanics, what follows does bear relevance to coherent spin transport phenomena in the mesoscopic regime. We will actually derive some interesting results that reveal subtle caveats regarding mesoscopic spin transport. The punchline is that the Aharonov-Casher effect and related phenomena are intrinsically fragile, requiring much more fine tuning in the experimental machinery than in the Abelian (Aharonov-Bohm) case.

Recall the spinor definition Eq. (3.1.10); together with the definitions of the phase velocity, and it follows that the vanishing of the imaginary part of the Pauli equation implies,

$$\partial_0 \rho + \vec{\nabla} \cdot \left[\rho \left(\vec{u} - \frac{e}{m}\vec{A} + \vec{u}^a S^a - \frac{q}{m}\vec{A}^a S^a \right) \right] = 0 \tag{3.2.1}$$

and this is nothing else than the non-Abelian continuity equation, imposing that probability is covariantly conserved. For non-Abelian parallel transport this is a weaker condition than for the simple Abelian case where the continuity equation implies a global conservation of mass, being in turn the condition for hydrodynamical degrees of freedom in the fluid context. Although locally conserved, the non-Abelian charge is not globally conserved and this is the deep reason for the difficulties with associating a universal hydrodynamics to the non-Abelian fluids. The fluid dynamics will borrow this motive directly from quantum mechanics where its meaning is straightforwardly isolated.

Taking the trace over the non-Abelian labels in Eq. (3.2.1) results in the usual continuity equation for Abelian probability, in the spintronics context associated with the conservation of electrical charge,

$$\partial_0 \rho + \nabla \cdot \left[\rho \left(\vec{u} - \frac{e}{m} \vec{A} \right) \right] = 0 \,, \tag{3.2.2}$$

where one recognizes the standard (Abelian) probability current,

$$\vec{J} = \rho \left(\vec{u} - \frac{e}{m} \vec{A} \right) = \frac{\hbar}{m} \rho \left(\nabla \theta - \frac{e}{\hbar} \vec{A} \right) . \tag{3.2.3}$$

From Abelian continuity and the full non-Abelian law Eq. (3.2.1) it is directly seen that the non-Abelian velocities and vector potentials have to satisfy the following equations,

$$\nabla \cdot \left[\rho \left(\vec{u}^a - \frac{q}{m} \vec{A}^a \right) \right] = \frac{q}{\hbar} \rho \epsilon^{abc} \vec{u}^b \cdot \vec{A}^c \tag{3.2.4}$$

and we recognize a divergence – the quantity inside the bracket is a conserved, current-like quantity. Notice that in this non-relativistic theory this equation contains only space like derivatives: it is a static constraint equation stating that the non-Abelian probability density should not change in time. The above is generally valid but it is instructive to now interpret this result in the Pauli-equation context. Using Eq.(3.1.3) for the non Abelian vector potentials, Eq (3.2.4) becomes,

$$\partial_i \left[\rho \left(u_i^a - \frac{q}{m} \epsilon_{ail} E_l \right) \right] = -\frac{q}{\hbar} \rho \left(u_a^b E_b - u_b^b E_a \right) . \tag{3.2.5}$$

As a prelude to what is coming, we find that this actually amounts to a statement about spin Hall probability currents. When the quantity on the r.h.s. would be zero, $j_i^a = \rho u_i^a = \frac{\rho q}{m} \epsilon_{ail} E_l + \nabla \times \vec{\lambda}$, the spin Hall equation modulo an arbitrary curl and thus the spin Hall relation exhibits a "gauge invariance".

Let us complete this description of non-Abelian quantum mechanics by inspecting the real part of the Pauli equation in charge of the time evolution of the phase,

$$\partial_0\theta - eA_0 + u_0^a S^a - qA_0^a S^a = -\frac{1}{\hbar}\left(\frac{m}{2}\left[\vec{u} - \frac{e}{m}\vec{A} + \vec{u}^a S^a - \frac{q}{m}\vec{A}^a S^a\right]^2 \right.$$
$$\left. + \frac{1}{2m}\left[2eqS^a\vec{A}\cdot\vec{A}^a + \frac{q^2}{4}\vec{A}^a\cdot\vec{A}^a\right]\right) + \frac{\hbar}{4m}\left[\frac{\nabla^2\rho}{\rho} - \frac{(\nabla\rho)^2}{2\rho^2}\right]. \tag{3.2.6}$$

Tracing out the non-Abelian sector we obtain the usual equation for the time rate of change of the Abelian phase, augmented by two $SU(2)$ singlet terms on the r.h.s.,

$$\begin{aligned}\partial_0\theta - eA_0 &= \frac{\hbar}{4m}\left[\frac{\nabla^2\rho}{\rho} - \frac{(\nabla\rho)^2}{2\rho^2}\right] \\ &- \frac{1}{\hbar}\left(\frac{m}{2}\left[\left(\vec{u} - \frac{e}{m}\vec{A}\right)^2 + \frac{1}{4}\vec{u}^a\cdot\vec{u}^a - \frac{q}{2m}\vec{u}^a\cdot\vec{A}^a\right]\right).\end{aligned} \tag{3.2.7}$$

Multiplying this equation by S^b and tracing the non-Abelian labels we find,

$$u_0^a - qA_0^a = -\frac{m}{\hbar}\left(\vec{u} - \frac{e}{m}\vec{A}\right)\cdot\left(\vec{u}^a - \frac{q}{m}\vec{A}^a\right). \tag{3.2.8}$$

It is again instructive to consider the spin-orbit coupling interpretation,

$$u_0^a = qB_a - \frac{m}{\hbar}\left(u_i - \frac{e}{m}\vec{A}_i\right)\cdot\left(u_i^a - \frac{q}{m}\epsilon_{ial}E_l\right) \tag{3.2.9}$$

ignoring the spin orbit coupling this just amounts to Zeeman coupling. The second term on the right hand side is just expressing that spin orbit coupling can generate uniform magnetization, but this requires both matter current (first term) and a *violation* of the spin-Hall equation! As we have just seen such violations if present *necessarily* take the form of a curl.

To appreciate further what these equations mean, let us consider an experiment of the Aharonov-Casher[13] kind. The experiment consists of an electrical wire oriented, say, along the z-axis that is charged, and is therefore producing an electrical field E_r in the radial direction in the xy plane. This wire is surrounded by a loop containing mobile spin-carrying but electrically neutral particles (like neutrons or atoms). Consider now the spins of the particles to be polarized along the z-direction and it is straightforward to demonstrate that the particles accumulate a holonomy $\sim E_r$. It is easily seen that this corresponds with a special case in the above formalism. By specializing to spins lying along the z-axis, only one component $\vec{u}^z, u_0^z$ of the non-Abelian phase velocity $\vec{u}^a, u_0^a$ has to be considered, and this reduces the problem to a $U(1)$ parallel transport structure; this reduction is rather implicit in the standard treatment. Parametrise the current loop in terms of a radial (r) and azimuthal (ϕ) direction. Insisting that the electrical field is entirely along r, while the spins are oriented along z and the current flows in the ϕ direction so that only $u_\phi^z \neq 0$, Eq. (3.2.5) reduces to $\partial_\phi \left(\rho(u_\phi^z - (q/m)E_r) \right) = 0$. $J_\phi^z = \rho u_\phi^z$ corresponds with a spin probability current, and it follows that $J_\phi^z = (q\rho/m)E_r + f(r,z)$ with f an arbitrary function of the vertical and radial coordinates: this is just the quantum-mechanical incarnation of the spin-Hall transport equation Eq.(2.0.1)! For a very long wire in which all vertical coordinates are equivalent, the cylindrical symmetry imposes z independence, and since we are at fixed radius, f is a constant. In the case where the constant can dropped we have $u_\phi^z = \partial_\phi \theta^z = (q/m)E_r$ the phase accumulated by the particle by moving around the loop equals $\Delta\theta^z = \oint d\phi u_\phi^z = L(q/m)E_r$: this is just the Aharonov-Casher phase. There is the possibility that the Aharonov-Casher effect might not occur if physical conditions make the constant f nonzero.

Inspecting the 'magnetization' equation (3.2.9), assuming there is no magnetic field while the particle carries no electrical charge, $u_0^a = -(m/\hbar)\vec{u}\cdot (\vec{u}^a - (q/m)\epsilon_{ial}E_l) = 0$, given the conditions of the ideal Aharonov-Casher experiment. Henceforth, the spin currents in the AC experiment do not give rise to magnetization.

The standard AC effect appears to be an outcome of a rather special, in fact fine tuned experimental geometry, hiding the intricacies of the full non-Abelian situation expressed by our equations Eq.(3.2.5,3.2.9). As an example, let us consider the simple situation that, as before, the spins are polarized along the z-direction while the current flows along ϕ such that only u_ϕ^z is non zero. However, we assume now a stray electrical field along the z-direction, and it follows from Eq.(3.2.5),

$$\partial_\phi \left(\rho(u_\phi^z - \frac{q}{m} E_r) \right) = -\frac{q}{\hbar} u_\phi^z E_z. \tag{3.2.10}$$

We thus see that if the field is not exactly radial, the nonradial parts will provide corrections to the spin Hall relation and more importantly will invalidate the Aharonov-Casher effect! This stray electrical field in the z-direction has an even simpler implication for the magnetization. Although no magnetization is induced in the z-direction, it follows from Eq. (3.2.9) that this field will induce a magnetization in the radial direction since $u_0^r = -u_\phi (q/m) \varepsilon_{\phi r z} E_z$. This is finite since the matter phase current $u_\phi \neq 0$.

From these simple examples it is clear that the non-Abelian nature of the mesoscopic spin transport underlying the AC effect renders it to be a much less robust affair than its Abelian Aharonov-Bohm counterpart. In the standard treatment these subtleties are worked under the rug and it would be quite worthwhile to revisit this physics in detail, both experimentally and theoretically, to find out if there are further surprises. This is however not the aim of this chapter. The general message is that even in this rather well behaved mesoscopic regime one already finds the first signs of the fragility of non-Abelian transport. On the one hand, this will turn out to become lethal in the classical regime, while on the other hand we will demonstrate that the coherent transport structures highlighted in this section will acquire hydrodynamical robustness when combined with the rigidity of non-Abelian superfluid order.

3.3 Spin currents are only covariantly conserved

It might seem odd that the quantum equations of the previous section did not have any resemblance to a continuity equation associated with the conservation of spin density. To make further progress in our pursuit to describe macroscopic spin hydrodynamics an equation of this kind is required, and it is actually straightforward to derive using a different strategy (see also Jackiw *et al.*[28, 29]).

Let us define a spin density operator,

$$\Sigma^a = \rho S^a \tag{3.3.1}$$

and a spin current operator,

$$\begin{aligned} \vec{j}^a &= -\frac{i\hbar}{2m}\left[\psi^\dagger \frac{\tau^a}{2}\nabla\psi - (\nabla\psi)^\dagger \frac{\tau^a}{2}\psi\right] \\ &\equiv \vec{j}^a_{NC} \quad + \vec{j}^a_C \,. \end{aligned} \tag{3.3.2}$$

We observe that the spin current operator can be written as a sum of two contributions. The first piece can be written as

$$\vec{j}^a_{NC} = \rho \vec{u} S^a \,. \tag{3.3.3}$$

It factorizes in the phase velocity associated with the Abelian mass current $\vec{u}$ times the non-Abelian charge/spin density Σ^a carried around by the mass current. This 'non-coherent' (relative to spin) current is according to the simple classical intuition of what a spin current is: particles flow with a velocity $\vec{u}$ and every particle carries around a spin. The less intuitive, 'coherent' contribution to the spin current needs entanglement of the spins,

$$\vec{j}^a_C = \frac{\rho}{2}\vec{u}^b\{S^a, S^b\} = \frac{\rho}{4}\vec{u}^a \tag{3.3.4}$$

and this is just the current associated with the non-Abelian phase velocity $\vec{u}^a$ highlighted in the previous section.

The above expressions for the non-Abelian currents are of relevance to the 'neutral' spin fluids, but we have to deal with the gauged matter currents, say in the presence of SO-coupling. Obviously we have to substitute covariant derivatives for the normal derivatives,

$$\vec{J}^a = -\frac{i\hbar}{2m}\left[\psi^\dagger\frac{\tau^a}{2}\vec{D}\psi - (\vec{D}\psi)^\dagger\frac{\tau^b}{2}\psi\right] \tag{3.3.5}$$

$$= \vec{J}S^a + \frac{\rho}{4}\left(\vec{u}^a - \frac{q}{m}\vec{A}^a\right)$$

$$\equiv \vec{J}^a_{NC} \quad + \quad \vec{J}^a_C\ , \tag{3.3.6}$$

where the gauged version of the non-coherent and coherent currents are respectively,

$$J^a_{NC} = \vec{J}S^a \tag{3.3.7}$$

$$J^a_C = \frac{\rho}{4}\left(\vec{u}^a - \frac{q}{m}\vec{A}^a\right) \tag{3.3.8}$$

with the Abelian (mass) current $\vec{J}$ given by Eq. (3.2.3).

It is a textbook exercise to demonstrate that the following 'continuity' equation holds for a Hamiltonian characterized by covariant derivatives (like the Pauli Hamiltonian),

$$D_0\Sigma^a + \vec{D}\cdot\vec{J}^a = 0 \tag{3.3.9}$$

with the usual non-Abelian covariant derivatives of vector-fields,

$$D_\mu B^a = \partial_\mu B^a + \frac{q}{\hbar}\epsilon^{abc}A^b_\mu B^c\,. \tag{3.3.10}$$

Eq. (3.3.9) has the structure of a continuity equation, except that the derivatives are replaced by covariant derivatives. It is well known[31] that in the non-Abelian case such covariant 'conservation' laws fall short of being real conservation laws of the kind encountered in the Abelian theory. Although they impose a local continuity, they fail with regard to global conservation because they do not correspond with total derivatives. This is easily seen by rewriting Eq. (3.3.9) as

$$\partial_0 \Sigma^a + \nabla \cdot \vec{J}^a = -\frac{q}{\hbar} \epsilon^{abc} A_0^b \Sigma^c - \frac{q}{\hbar} \epsilon^{abc} \vec{A}^b \cdot \vec{J}^c \,. \tag{3.3.11}$$

The above is standard lore. However, using the result Eq. (3.2.4) from the previous section, we can obtain a bit more insight in the special nature of the phase coherent spin current, Eq. (3.3.8). The equation (3.2.4) can be written in covariant form as

$$\vec{D} \cdot \vec{J}_C^a = 0 \,, \tag{3.3.12}$$

involving only the space components and therefore

$$D_0 \Sigma^a + \vec{D} \cdot \vec{J}_{NC}^a = 0 \,. \tag{3.3.13}$$

Since Σ^a is spin density, it follows rather surprisingly that the *coherent part of the spin current cannot give rise to spin accumulation*! Spin accumulation is entirely due to the non-coherent part of the current. Anticipating what is coming, the currents in the spin superfluid are entirely of the coherent type and this 'non-accumulation theorem' stresses the rather elusive character of these spin supercurrents: they are so 'unmagnetic' in character that they are even not capable of causing magnetization when they come to a standstill due to the presence of a barrier!

As a caveat, from the definitions of the coherent- and non-coherent spin currents the following equations can be derived

$$\rho \left(\nabla \times \vec{J}_{NC}^a \right) = 4 \frac{m}{\hbar} \epsilon^{abc} \vec{J}_C^b \times \vec{J}_{NC}^c + \frac{q}{\hbar} \rho \epsilon^{abc} \vec{A}^b \times \vec{J}_{NC}^c \tag{3.3.14}$$

$$\begin{aligned} \rho \left(\nabla \cdot \vec{J}_{NC}^a \right) &= -\frac{1}{2} \frac{\partial \rho^2}{\partial t} S^a - 4 \frac{m}{\hbar} \epsilon^{abc} \vec{J}_C^b \cdot \vec{J}_{NC}^c \\ &\quad - \frac{q}{\hbar} \rho \epsilon^{abc} \vec{A}^b \cdot \vec{J}_{NC}^c \,. \end{aligned} \tag{3.3.15}$$

From these equations it follows that the coherent currents actually do influence the way that the incoherent currents do accumulate magnetization, but only indirectly. Similarly, using the divergence of the Abelian covariant spin current together with the covariant conservation law, we obtain the time rate of precession of the local spin density

$$\partial_0 \Sigma^a = \frac{\partial \rho}{\partial t} S^a + 4\frac{m}{\hbar \rho}\epsilon^{abc} \vec{J}^b_C \cdot \vec{J}^c_{NC} - \frac{q}{\hbar}\epsilon^{abc} A^b_0 \Sigma^c \tag{3.3.16}$$

demonstrating that this is influenced by the presence of coherent- and incoherent currents flowing in orthogonal non-Abelian directions.

This equation forms the starting point of the discussion of the (lack of) hydrodynamics of the classical non-Abelian/spin fluid.

3.4 Particle based non-Abelian hydrodynamics, or the classical spin fluid

We have now arrived at a point that we can start to address the core-business of this chapter: what can be said about the collective flow properties of large assemblies of interacting particles carrying spin or either non-Abelian charge? In other words, what is the meaning of spin- or non-Abelian hydrodynamics? The answer is: if there is no order-parameter protecting the non-Abelian phase coherence on macroscopic scales *spin flow is non-hydrodynamical*, i.e. macroscopic flow of spins does not even exist.

The absence of order parameter rigidity means that we are considering classical spin fluids as they are realized at higher temperatures, i.e. away from the mesoscopic regime of the previous section and the superfluids addressed in Section 3.7. The lack of hydrodynamics is well understood in the spintronics community: after generating a spin current it just disappears after a time called the spin-relaxation time. This time depends on the effective spin-orbit coupling strength in the material but it will not exceed in even the most favorable cases the nanosecond regime, or the micron length scale. Surely, this is a major (if not fundamental) obstacle for the use of spin currents for electronic switching purposes. Although spin currents are intrinsically less dissipative than electrical currents it takes a lot of energy to replenish these currents, rendering spintronic circuitry as rather useless as competitors for Intel chips.

Although this problem seems not to be widely known in corporate head quarters, or either government funding agencies, it is well understood in the scientific community. This seems to be a different story in the community devoted to the understanding of the quark-gluon plasma's produced at the heavy ion collider at Brookhaven. In these collisions a 'non-Abelian fire ball' is generated, governed by high temperature quark-gluon dynamics: the temperatures reached in these fireballs exceed the confinement scale. To understand what is happening one of course needs a hydrodynamical description where especially the fate of color (non-Abelian) currents is important. It seems that the theoretical mainstream in this pursuit is preoccupied by constructing Boltzmann type transport equations. Remarkably, it does not seem to be widely understood that one first needs a hydrodynamical description, before one can attempt to calculate the numbers governing the hydrodynamics from microscopic principle by employing kinetic equations (quite questionable by itself given the strongly interacting nature of the quark-gluon plasma). The description of the color currents in the quark-gluon plasma is suffering from a fatal flaw: *because of the lack of a hydrodynamical conservation law there is no hydrodynamical description of color transport.*

The above statements are not at all original in this regard: this case is forcefully made in the work by Jackiw and coworkers [28, 29] dealing with non-Abelian 'hydrodynamics'. It might be less obvious, however, that precisely the same physical principles are at work in the spin-currents of spintronics: spintronics can be viewed in this regard as 'analogous system' for the study of the dynamics of quark-gluon plasma's. The reason for the analogy to be precise is that the reasons for the failure of hydrodynamics reside in the parallel transport structure of the matter fields, and the fact that the 'gauge fields' of spintronics are in 'fixed frame' is irrelevant for this particular issue.

The discussion by Jackiw *et al.* of classical ('particle based') non-Abelian 'hydrodynamics' starts with the covariant conservation law we re-derived in the previous section, Eq. (3.3.13). This is still a microscopic equation describing the quantum physics of a single particle and a coarse graining procedure has to be specified in order to arrive at a macroscopic continuity equation. Resting on the knowledge about the Abelian case this coarse graining procedure is unambiguous when we are interested in the (effective) high temperature limit. The novelty as compared to the Abelian case is the existence of the coherent current $\vec{J}^a_C$ expressing the transport of the *entanglement* associated with the non-Abelian character of the charge; Abelian theory is special in this regard because there is no room for this kind of entanglement. By definition, in the classical limit quantum entanglement cannot be transported over macroscopic distances and this implies that the expectation value $\langle \vec{J}^a_C \rangle$ cannot enter the macroscopic fluid equations. Although not stated explicitly by Jackiw *et al.*, this particular physical assumption (or definition) is the crucial piece for what follows – the coherent current will acquire (quantum) hydrodynamic status when protected by the order parameter in the spin-superfluid.

What remains is the non-coherent part, governed by the pseudo–continuity equation (3.3.13). Let us first consider the case that the non-Abelian fields are absent (e.g., no spin-orbit coupling) and the hydrodynamical status of the equation is immediately obvious through the Ehrenfest theorem. The quantity $\Sigma^a \rightarrow \langle \rho S^a \rangle$ becomes just the macroscopic magnetization (or non-Abelian charge density) that can be written as $n\vec{Q}$, i.e. the macroscopic particle density $n = \langle \rho \rangle$ times their average spin $\vec{Q} = \langle \vec{S} \rangle$. Similarly, the Abelian phase current $\rho \vec{u}$ turns into the hydrodynamical current $n\vec{v}$ where $\vec{v}$ is the velocity associated with the macroscopic 'element of fluid'. In terms of these macroscopic quantities, the l.h.s. of Eq. (3.2.9) just expresses the hydrodynamical conservation of uniform magnetization in the absence of spin-orbit coupling. In the presence of spin orbit coupling (or gluons) the r.h.s. is no longer zero and, henceforth, uniform magnetization/color charge is no longer conserved.

Upon inserting these expectation values in Eqs. (3.2.2), (3.3.13) one obtains the equations governing classical non-Abelian fluid flow,

$$\partial_t n + \nabla \cdot (n\vec{v}) = 0 \tag{3.4.1}$$

$$\partial_t Q^a + \vec{v} \cdot \nabla Q^a = -\varepsilon_{abc} \left(c A^0_b + \vec{v} \cdot \vec{A}^b \right) Q^c. \tag{3.4.2}$$

Eq. (3.4.1) expresses the usual continuity equation associated with (Abelian) mass density. Eq. (3.4.2) is the novelty, reflecting the non-Abelian parallel transport structure, rendering the substantial time derivative of the magnetiziation/color charge to become dependent on the color charge itself in the presence of the non-Abelian gauge fields. To obtain a full set of hydrodynamical equations, one needs in addition a 'force' (Navier-Stokes) equation expressing how the Abelian current $n\vec{v}$ accelerates in the presence of external forces, viscosity, etcetera. For our present purposes, this is of secondary interest and we refer to Jackiw *et al.*[28, 29]for its form in the case of an ideal (Euler) Yang-Mills fluid.

Jackiw et al coined the name 'Fluid-Wong Equations' for this set of equations governing classical non-Abelian fluid flow. These would describe a hydrodynamics that would be qualitatively similar to the usual Abelian magneto-hydrodynamics associated with electromagnetic plasma's were it not for Eq. (3.4.2): this expression shows that the color charge becomes itself dependent on the flow. This unpleasant fact renders the non-Abelian flow to become non-hydrodynamical.

We perceive it as quite instructive to consider what this means in the spintronics interpretation of the above. Translating the gauge fields into the physical electromagnetic fields of the Pauli equation, Eq. (3.4.2) becomes,

$$\partial_t Q^a + \vec{v} \cdot \nabla Q^a = \left(\left[\vec{B} + \vec{v} \times \vec{E} \right] \times \vec{Q} \right)_a \tag{3.4.3}$$

where $\vec{Q}(\vec{r})$ has now the interpretation of the uniform magnetization associated with the fluid element at position $\vec{r}$. The first term on the r.h.s. is just expressing that the magnetization will have a precession rate in the comoving frame, proportional to the external magnetic field $\vec{B}$. However, in the presence of spin-orbit coupling (second term) this rate will also become dependent on the velocity of the fluid element itself when an electrical field $\vec{E}$ is present with a component at a right angle both to the direction of the velocity $\vec{v}$ and the magnetization itself. This velocity dependence wrecks the hydrodynamics.

The standard treatments in terms of Boltzmann equations lay much emphasis on quenched disorder, destroying momentum conservation. To an extent this is obscuring the real issues, and let us instead focus on the truly hydrodynamical flows associated with the Galilean continuum. For a given hydrodynamical flow pattern, electromagnetic field configuration and initial configuration of the magnetization, Eq. (3.4.3) determines the evolution of the magnetization. Let us consider two elementary examples. In both cases we consider a Rashba-like[32] electromagnetic field configuration: consider flow patterns in the xy directions and a uniform electrical field along the z direction while $\vec{B} = 0$.

a. Laminar flow

Consider a smooth, non-turbulent laminar flow pattern in a 'spin-fluid tube' realized under the condition that the Reynold's number associated with the mass flow is small. Imagine that the fluid elements entering the tube on the far left have their magnetization $\vec{Q}$ oriented in the same direction (Fig.3.1). Assume first that the velocity $\vec{v}$ is uniform inside the tube and it follows directly from Eq. (3.4.3) that the $\vec{Q}$'s will precess with a uniform rate when the fluid elements move trough the tube. Assuming that the fluid elements arriving at the entry of the tube have the same orientation at all times, the result is that an observer in the lab frame will measure a static 'spin spiral' in the tube, see Fig.3.2. This simple example sheds light on an interesting and somewhat confusing issue. It is easy to prove that an ordered magnet characterized by a finite spiral pitch can also be viewed as a state carrying a spin current with a magnitude proportional to the pitch. One might want to view this particular kind of spin flow as just a realization of the above, except that the flow pattern is completely fixed by the presence of the conventional magnetic (spin density) order. We will analyze this in further detail in the next section. Finally, we leave it to the reader to find out the peculiarity that the spiral pattern actually will not change when the flow in the tube acquires a typical laminar, non-uniform velocity distribution, with the velocities vanishing at the walls.

b. Turbulent flow

Let us now consider the case that the fluid is moving much faster, such that downstream of an obstruction in the flow turbulence arises in the matter current. In Figure 3.3 we have indicated a typical stream line showing that the flow is now characterized by a finite vorticity in the region behind the obstruction. Let us now repeat the exercise, assuming that fluid elements arrive at the obstruction with aligned magnetization vectors. Following a fluid element when it traverses the region with finite circulation it is immediately obvious that even for a fixed precession rate *the non-Abelian charge/magnetization becomes multivalued when it has travelled around the vortex!* Henceforth, at long times the magnetization will average away and the spin current actually disappears at the 'sink' associated with the rotational Abelian flow. This elementary example highlights the essence of the problem dealing with non-Abelian 'hydrodynamics': the covariant conservation principle underlying everything is good enough to ensure a *local* conservation of non-Abelian charge so that one can reliably predict how the spin current evolves over infinitesimal times and distances. However, it fails to impose a *global* conservation. This is neatly illustrated in this simple hydrodynamical example: at the moment the mass flow becomes topologically non-trivial it is no longer possible to construct globally consistent non-Abelian flow patterns with the consequence that the spin currents just disappear.

Although obscured by irrelevant details, the above motive has been recognized in the literature on spin flow in semiconductors where it is known as D'yakonov-Perel spin relaxation. We hope that the analogy with spin-transport in solids is helpful for the community that is trying to find out what is actually going on in the quark-gluon fireballs. Because one has to deal eventually with the absence of hydrodynamics we are pessimistic with regard to the possibility that an *elegant* description will be found, in a way mirroring the state of spintronics. We will instead continue now with our exposition of the remarkable fact that the rigidity associated with order parameters is not only simplifying the hydrodynamics (as in the Abelian case) but even making it possible for hydrodynamics to exist!

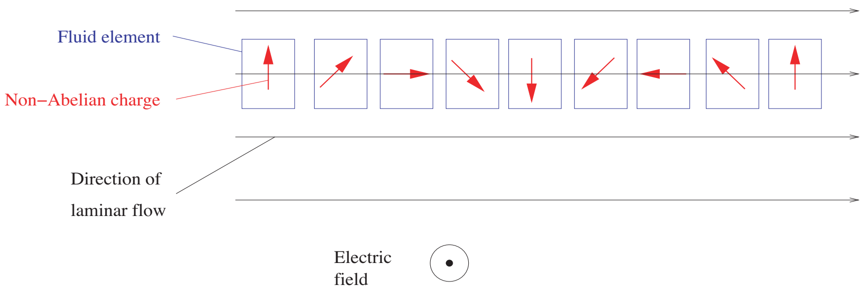

Figure 3.1: Laminar flow of a classical spin fluid in an electric field. The fluid elements (blue) carry non-Abelian charge, the red arrows indicating the spin direction. The flow lines are directed to the right and the electric field is pointing outwards of the paper. Due to Eq. (3.4.3), the spin precesses as indicated.

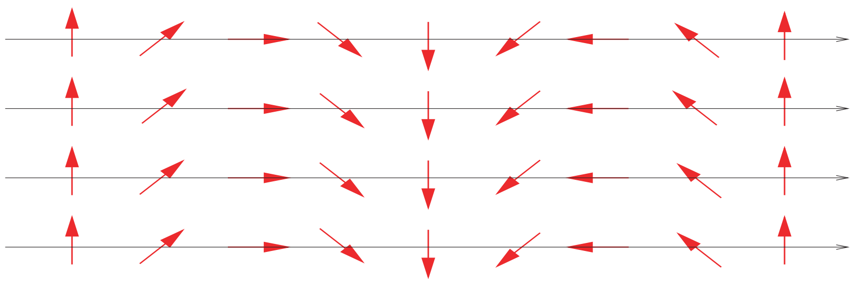

Figure 3.2: The laminar flow of a parallel transported spin current, Figure 3.1, can also be viewed as a static spin spiral magnet.

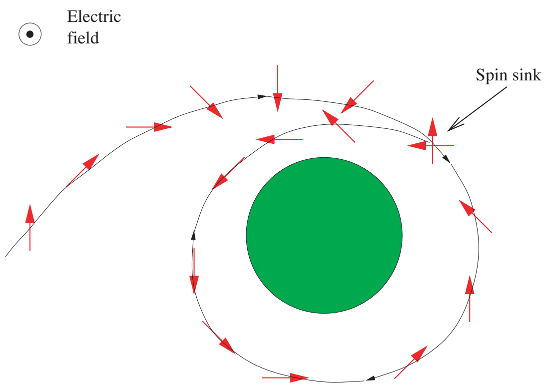

Figure 3.3: Turbulent spin flow around an obstruction in an electric field. It is seen that only the "mass" is conserved. The change in spin direction after one precession around the obstruction causes a spin sink. Hence it is precisely the parallel transport, or the covariant conservation, which destroys hydrodynamic conservation for non-Abelian charge.

3.5 Electrodynamics of spin-orbit coupled systems

Before we address the extremely interesting and novel effects in multiferroics and spin superfluids, we pause to obtain the electrodynamics of spin orbit coupled systems. From the Pauli Maxwell Lagrangian (3.1.1) we see that the spin current couples directly to the electric field and will thus act as a source for electric fields. In order to see how this comes about let us obtain the electrodynamics of a spin-orbit coupled system. We presuppose the usual definition of electromagnetic fields in terms of gauge potentials, which implies the Maxwell equations

$$\nabla \cdot \vec{B} = 0 \,, \qquad \nabla \times \vec{E} + \partial_0 \vec{B} = 0 \,. \tag{3.5.1}$$

If we vary the Lagrangian with respect to the scalar electromagnetic potential, we obtain

$$\partial_i E_i = 4\pi q \epsilon_{ial} \left(\chi^\dagger \partial_i J^a_l \chi \right) \tag{3.5.2}$$

where we suppose that the charge sources are cancelled by the background ionic lattice of the material or that we have a neutral system. This term is extremely interesting because it says that the "curl" of spin currents are sources for electric fields. In fact, the electric field equation is nothing but the usual Maxwell equation for the electric displacement $\nabla \cdot \vec{D} = 0$ where $\vec{D} = \vec{E} + 4\pi\vec{P}$ with

$$P_i = -\epsilon_{ial}\chi^\dagger J_l^a \chi \,. \tag{3.5.3}$$

The spin current acts as a polarization for the material. The physical origin of this polarization is relativistic. In the local frame the moving spins in the current produce a magnetic field as they are magnetic moments. When you Lorentz transform into the lab frame, part of this field becomes electric. On the other hand, it can be shown that $\nabla \cdot \vec{P} = 0$ unless the spin current has singularities. Thus, in the absence of singularities spin currents cannot create electric fields.

Varying the Lagrangian (3.1.1) with respect to the vector potential we obtain

$$\begin{aligned}\left(\nabla \times \vec{B}\right)_i &= 4\pi \vec{J}_{em} - 4\pi\left(\nabla \times q\vec{\Sigma}\right)_i + \partial_0 E_i \\ &\quad - 4\pi q \epsilon_{lai}\partial_0\left(\chi^\dagger j_l^a \chi\right) \\ &= 4\pi \vec{J}_{em} - 4\pi\left(\nabla \times q\vec{\Sigma}\right)_i + \partial_0 D_i \,.\end{aligned} \tag{3.5.4}$$

The first term on the right contains the usual electromagnetic current

$$\vec{J}_{em} = 4\pi e\rho\left(u_i + u_i^a \chi^\dagger S^a \chi\right) \tag{3.5.5}$$

which includes the motion of particles due to the advance of the Abelian and the non-Abelian phases. The term containing the non-Abelian velocity(the coherent spin current) in this electromagnetic current will only contribute when there is magnetic order $\langle S^a \rangle \neq 0$. The second term is conventional since it is the curl of the magnetization which generates magnetic fields. The third is the Maxwell displacement current in accordance with our identification of the polarization from the spin current.

3.6 Spin hydrodynamics rising from the ashes I: the spiral magnets.

Recently the research in multiferroics has revived. This refers to materials that are at the same time ferroelectric and ferromagnetic, while both order parameters are coupled. The physics underlying this phenomenon goes back to the days of Lifshitz and Landau[33]. Just from considerations regarding the allowed invariants in the free energy it is straightforward to find out that when a crystals lacks an inversion center (i.e., there is a net internal electric field) spin-spin interactions should exist giving rise to a spiral modulation of the spins (helicoidal magnets). The modern twist of this argument is[26]: the spin spiral can be caused by magnetic frustration as well, and it now acts as a cause (instead of effect) for an induced ferroelectric polarization. Regarding the microscopic origin of these effects, two mechanisms have been identified. The first one is called 'exchange striction' and is based on the idea that spin-phonon interactions of the kind familiar from spin-Peierls physics give rise to a deformation of the crystal structure when the spin-spiral order is present, and these can break inversion symmetry[34]. The second mechanism is of direct relevance to the present subject matter. As we already explained in the previous section, a spiral in the spin-density can be viewed at the same time as a spin current. In the presence of the magnetic order parameter this spin current acquires rigidity (like a supercurrent) and therefore it can impose its will on the 'gauge' fields. In the spin-orbital coupling case, the 'gauge' field of relevance is the physical electrical field, and henceforth the 'automatic' spin currents associated with the spiral magnet induce an electrical field via the spin-orbit coupling, rendering the substance to become a ferroelectric[25].

This substance matter is rather well understood [26] and the primary aim of this section is to explain how these 'spiral magnet' spin currents fit into the greater picture of spin-hydrodynamics in general. Viewed from this general perspective they are quite interesting: they belong to a category of non-Abelian hydrodynamical phenomena having no analogy in the Abelian universe. On the one hand these currents are spontaneous and truly non-dissipative and in this regard they are like Abelian supercurrents. They should not be confused with the Froehlich 'super' currents associated with (Abelian) charge density waves: these require a time dependence of the density order parameter (i.e., the density wave is sliding) while the spiral magnet currents flow also when the non-Abelian density (the spiral) is static. At the same time they originate entirely in the factorisable non-coherent current sector $\vec{J}^a_{NC}$ because this current is communicating with the spin density; we already learned that the coherent non-Abelian phase current $\vec{J}^a_C$ is completely detached from the spin density and this current 'rules the waves' in the spin-superfluids, or either the non-Abelian Higgs phase. Since these two varieties of 'rigid' currents cannot be distinguished by merely their dissipationless nature, a more fanciful physical measure has to be invoked in order to distinguish them. This observable turns out to be surprising in the spin-orbit coupling context: the spin-spiral 'super' spin current can induce monopole sources of electrical charge. This is embodied by the observation of Mostovoy that a vortex in the (2D) spin system has a monopole electrical charge at its core, with a total charge determined by the spin-orbit coupling. On the other hand, we will present a theorem in the next chapter demonstrating that no configuration of coherent spin supercurrents exist in the spin superfluid that can act as a source of electrical fields. The phase coherent spin fluid can *quantize* the electrical line charge but not *cause* electrical charge.

Last but no least, the spiral magnet currents offer a minimal context to illustrate the most fundamental feature of non-Abelian hydrodynamics: the rigidity of the order parameter is capable of restoring hydrodynamical degrees of freedom that are absent in the 'normal' fluid at high temperature. This is so simple that we can explain it in one sentence. One directly recognizes the XY spin vortex in the turbulent flow of Fig. 3.3, but in the presence of spin density order the 'spiral' spin pattern associated with the vortex has to be single valued, and this in turns renders the spin current to be single valued: spin currents do not get lost in the ordered magnet!

To become more explicit, consider an ordered XY-magnet with a local order parameter that is the expectation value of the local spin operator

$$\langle S_x + iS_y \rangle = Se^{i\theta} \,. \tag{3.6.1}$$

In general a spin state of an XY-magnet is given by

$$\prod_{\text{lattice sites}} g(\vec{x})|\uparrow\rangle \tag{3.6.2}$$

where we have chosen $S = 1/2$ spins for explicitness, but similar results hold for larger spin. The ket $|\uparrow\rangle$ describes a spinor in the $+z$ direction and $g(\vec{x})$ is an $SU(2)$ rotation matrix in the xy-plane:

$$g(\vec{x}) = e^{i\theta(\vec{x})\tau_z/2} \tag{3.6.3}$$

where τ_z is the Pauli matrix in the z-direction. The ground state in the ordered side of the phase diagram for the XY-magnet is given when $\theta(\vec{x})$ and hence $g(\vec{x})$ are constant independent of $\vec{x}$. Besides the ground state, XY-magnets have excited metastable states corresponding to spin vortices. These are easily constructed by choosing

$$\theta(\vec{x}) = n\phi \ , \quad n \, \text{integer} \ , \quad \phi = \arctan\left(\frac{y}{x}\right) \ . \tag{3.6.4}$$

Now we can compute the spin current in this state. The coherent spin current is given by

$$\vec{J}^a_C = \frac{\hbar\rho}{2m}\vec{u}^a = -i\frac{\hbar\rho}{2m}\left[g^{-1}\frac{\tau^a}{2}\nabla g - (\nabla g^{-1})\frac{\tau^a}{2}g\right] \ . \tag{3.6.5}$$

For our case

$$\begin{aligned} g^{-1}\frac{\tau_x}{2}g &= \frac{1}{2}\left[\tau_x\cos\theta + \tau_y\sin\theta\right] \\ g^{-1}\frac{\tau_y}{2}g &= \frac{1}{2}\left[-\tau_x\sin\theta + \tau_y\cos\theta\right] \end{aligned} \tag{3.6.6}$$

we have the appropriate $O(2)$ or $U(1)$ rotation. We also have for the vortex $\theta = n\varphi$

$$\begin{aligned} J^a_c &= \frac{n\hbar\rho}{8m}\nabla\varphi\left[e^{-in\varphi\tau^z/2}\left\{\tau^a,\tau^z\right\}e^{in\varphi\tau^z/2}\right] \\ &= \frac{n\hbar\rho}{4m}(\nabla\varphi)\delta^{az} \ . \end{aligned} \tag{3.6.7}$$

We derived in the previous section that spin currents contribute to Maxwell's equations and in particular Gauss' law is

$$\partial_i E_i = 4\pi q \epsilon_{ial}\langle \partial_i J^a_l \rangle \tag{3.6.8}$$

where q measures the coupling between spin currents and electric fields via spin orbit coupling. Hence using that for $\phi = \arctan(y/x)$

$$\nabla \times \nabla\phi = 2\pi\delta^{(2)}(\vec{r}) \tag{3.6.9}$$

we find for the spin current of the vortex

$$\partial_i E_i = 2\pi^2 nq\frac{\hbar\rho}{m}\delta^{(2)}(\vec{r}) \ . \tag{3.6.10}$$

Therefore spin vortices in XY-magnets produce electric fields!

3.7 Spin hydrodynamics rising from the ashes II: the spin superfluids

Even without knowing a proper physical example of a spin-orbit coupled spin-superfluid one can construct its order parameter theory using the general principles discovered by Ginzburg and Landau. One imagines a condensate formed from electrically neutral bosons carrying a $SU(2)$ spin triplet quantum numbers. This condensate is characterized by a spinorial order parameter,

$$\Psi = |\Psi| \, e^{(i\theta + i\varphi^a \tau^a/2)} \chi \tag{3.7.1}$$

where $|\Psi|$ is the order parameter amplitude, nonzero in the superfluid state, while θ is the usual $U(1)$ phase associated with particle number, while the three non-Abelian phases φ^a, with the Pauli matrices τ^a acting on a reference spinor χ keep track of the $SU(2)$ phase structure. According to the Ginzburg-Landau recipe, the free energy of the system should be composed of scalars constructed from Ψ, while the *gradient structure should be of the same covariant form as for the microscopic problem* – 'parallel transport is marginal under renormalization. Henceforth, we can directly write down the Ginzburg-Landau free energy density for the spin superfluid in the presence of spin orbit coupling,

$$\begin{aligned} \mathcal{F} = {} & i\hbar\psi^\dagger D_0 \psi + \psi^\dagger \frac{\hbar^2}{2m} \vec{D}^2 \psi + m^2 |\Psi|^2 \\ & + w|\Psi|^4 + \frac{1}{2m} \psi^\dagger \frac{q^2}{4} \vec{A}^a \cdot \vec{A}^a \psi \\ & + \frac{1}{8\pi} \left(E^2 - B^2 \right) . \end{aligned} \tag{3.7.2}$$

We now specialize to the deeply non-relativistic case where the time derivatives can be ignored, while we consider electrically neutral particles ($e = 0$) so that the electromagnetic gauge fields drop out from the covariant derivatives.

Well below the superfluid transition the amplitude $|\Psi|$ is finite and frozen and one can construct a London-type action. After some algebra we obtain that

$$\mathcal{L}_{\text{spin-vel}} = -\frac{m}{8} \rho \left(\vec{u}^a - \frac{m}{2} \rho \vec{u}^2 - \frac{q}{m} \vec{A}^a \right)^2 + \frac{q^2}{8m} \vec{A}^a \cdot \vec{A}^a \, . \tag{3.7.3}$$

Using the spin identities written in section 3.3, this can be rewritten as

$$\mathcal{L}_{\text{spin-vel}} = -2\vec{J}^a_C \cdot \vec{J}^a_C - 2\vec{J}^a_{NC} \cdot \vec{J}^a_{NC} - \frac{q}{m} \left(\vec{A}^a \right)^2 + \frac{q^2}{8m} \vec{A}^a \cdot \vec{A}^a \, . \tag{3.7.4}$$

We see that the Ginzburg-Landau action is a sum of the spin coherent and non-coherent square currents. The spin noncoherent part has to do with mass or $U(1)$ currents, but since the particles carry spin they provide a spin current but only if $\langle S^a \rangle \neq 0$, that is if there is magnetic order. The coherent part is a bona fide spin current originating in coherent advance of the non-Abelian phase associated with the spin direction.

In order to make contact with the Helium literature we will write our spin operators and the coherent spin currents in terms of $SO(3)$ rotation matrices via

$$R^a_{\ b}(\vec{\varphi})\ \frac{\tau^b}{2} = e^{-i\varphi^a\tau^a/2}\ \frac{\tau^a}{2}\ e^{i\varphi^a\tau^a/2}. \tag{3.7.5}$$

Here, $R^a_{\ b}(\vec{\varphi})$ is an $SO(3)$ rotation matrix around the vector $\vec{\varphi}$ by an angle $|\vec{\varphi}|$. Using this, we obtain that the spin operator is a local $SO(3)$ rotation of the Pauli matrices

$$S^a = R^a_{\ b}(\vec{\varphi})\ \frac{\tau^b}{2}\ . \tag{3.7.6}$$

In terms of the rotation operators, the spin velocities related to advance of the non-Abelian phase are

$$\vec{u}^a = \frac{\hbar}{m}\epsilon_{abc}[\nabla R^b_{\ d}(\vec{\varphi})]R^d_{\ c}(\vec{\varphi})\,. \tag{3.7.7}$$

It is also easily seen that

$$u^a_0 = \epsilon_{abc}[\partial_0 R^b_{\ d}(\vec{\varphi})]R^d_{\ c}(\vec{\varphi})\,. \tag{3.7.8}$$

If we look at the expressions for $\vec{u}^a$ and u^a_0 in terms of the spin rotation matrix for the spin-orbit coupled spin superfluid (3.7.7, 3.7.8), we see them to be the exact analog of the spin velocity and spin angular velocity of ^{3}He-B (5.2.1) reproduced in the section 5.2. We define g through

$$\begin{aligned} R_{\alpha i}(\vec{\varphi})\ \frac{\tau^i}{2} &= e^{-i\varphi^a\tau^a/2}\ \frac{\tau_\alpha}{2}\ e^{i\varphi^a\tau^a/2} \\ &= g^{-1}\frac{\tau_\alpha}{2}g = S_\alpha\ , \end{aligned} \tag{3.7.9}$$

that is

$$g = e^{i\varphi^a\tau^a/2}\ , \tag{3.7.10}$$

which is an $SU(2)$ group element. We now have the spin velocities and angular velocities expressed as

$$\begin{aligned} \omega_{\alpha i} &= -i\mathrm{Tr}\left\{S_\alpha g^{-1}\partial_i g\right\} = -i\mathrm{Tr}\left\{g^{-1}\frac{\tau_\alpha}{2}\partial_i g\right\}, \\ \omega_\alpha &= -i\mathrm{Tr}\left\{S_\alpha g^{-1}\partial_0 g\right\} = -i\mathrm{Tr}\left\{g^{-1}\frac{\tau_\alpha}{2}\partial_0 g\right\}. \end{aligned} \tag{3.7.11}$$

The first is proportional to the coherent spin current and the second to the effective magnetization. If we define the spin superfluid density via

$$\rho = \frac{1}{\gamma^2}\chi_B c^2 \ , \tag{3.7.12}$$

we have the following Lagrangian that describes the low energy spin physics, written in a way that is quite analogous to that of ^{3}He-B,

$$\begin{aligned} L(\vec{\varphi},\vec{E},\vec{B}) \;&=\; \frac{1}{2\gamma^2}\chi_B\left(\vec{\omega}^2 + 2\gamma\vec{\omega}\cdot\vec{B}\right) - \frac{1}{2\gamma^2}\chi_B c^2\left(\omega_{\alpha i}^2 - \frac{4\mu}{\hbar c}\omega_{\alpha i}\epsilon_{\alpha i k}E_k\right) \\ &\quad + \frac{1}{8\pi}\left(E^2 - B^2\right) \ . \end{aligned} \tag{3.7.13}$$

From the lagrangian (3.7.13) we obtain the spin equations of motion for the spin superfluid by varying with respect to the non-Abelian phase

$$\partial_0\left[\frac{\partial L}{\partial(\partial_0 g)}\right] + \partial_i\left[\frac{\partial L}{\partial(\partial_i g)}\right] - \frac{\partial L}{\partial g} = 0 \ . \tag{3.7.14}$$

We evaluate

$$\begin{aligned} \frac{\partial L}{\partial g} &= \frac{\partial g^{-1}}{\partial g}\frac{\partial \omega_\alpha}{\partial g^{-1}}\frac{\partial L}{\partial \omega_\alpha} + \frac{\partial g^{-1}}{\partial g}\frac{\partial \omega_{\alpha i}}{\partial g^{-1}}\frac{\partial L}{\partial \omega_{\alpha i}} \\ &= -ig^{-2}\frac{\tau_\alpha}{2}(\partial_0 g)\frac{1}{\gamma^2}\chi_B\left(\omega_\alpha + 2\gamma B_\alpha\right) \\ &\quad + ig^{-2}\frac{\tau_\alpha}{2}(\partial_i g)\frac{1}{\gamma^2}\chi_B c^2\left(\omega_{\alpha i} - \frac{2\mu}{\hbar c}\epsilon_{\alpha i k}E_k\right) \end{aligned} \tag{3.7.15}$$

$$\begin{aligned} \frac{\partial L}{\partial(\partial_0 g)} &= \frac{\partial \omega_\alpha}{\partial(\partial_0 g)}\frac{\partial L}{\partial \omega_\alpha} \\ &= ig^{-1}\frac{\tau_\alpha}{2}\frac{1}{\gamma^2}\chi_B\left(\omega_\alpha + \gamma B_\alpha\right) \end{aligned} \tag{3.7.16}$$

$$\begin{aligned} \frac{\partial L}{\partial(\partial_i g)} &= \frac{\partial \omega_{\alpha i}}{\partial(\partial_i g)}\frac{\partial L}{\partial \omega_{\alpha i}} \\ &= -ig^{-1}\frac{\tau_\alpha}{2}\frac{1}{\gamma^2}\chi_B c^2\left(\omega_{\alpha i} - \frac{2\mu}{\hbar c}\epsilon_{\alpha i k}E_k\right) \end{aligned} \tag{3.7.17}$$

which yields the rather formidable equation of motion

$$\begin{aligned} 0 = \partial_0&\left[ig^{-1}\frac{\tau_\alpha}{2}\left(\omega_\alpha + \gamma B_\alpha\right)\right] + \partial_i\left[-ig^{-1}\frac{\tau_\alpha}{2}c^2\left(\omega_{\alpha i} - \frac{2\mu}{\hbar c}\epsilon_{\alpha i k}E_k\right)\right] \\ &+ ig^{-2}\frac{\tau_\alpha}{2}(\partial_0 g)\left(\omega_\alpha + \gamma B_\alpha\right) - ig^{-2}\frac{\tau_\alpha}{2}(\partial_i g)c^2\left(\omega_{\alpha i} - \frac{2\mu}{\hbar c}\epsilon_{\alpha i k}E_k\right) \ . \end{aligned} \tag{3.7.18}$$

After some straightforward algebra this equation reduces to the fairly simple equation

$$\partial_0 \left(\omega_\alpha + \gamma B_\alpha\right) - c^2 \partial_i \left(\omega_{\alpha i} - \frac{2\mu}{\hbar c}\epsilon_{\alpha i k} E_k\right) = 0. \tag{3.7.19}$$

The solution of this equation of motion gives the spin velocities and angular velocities as a function of space and time.

Similarly, by varying the Lagrangian (3.7.13) with respect to the electromagnetic potentials, we obtain the Maxwell equations for the electromagnetic fields "created" by the spin velocities and angular velocities,

$$\partial_k E_k = 4\pi \partial_k \left(\frac{2c\mu}{\hbar\gamma^2}\chi_B \epsilon_{\alpha i k}\omega_{\alpha i}\right), \tag{3.7.20}$$

$$\begin{aligned} \left(\nabla \times \vec{B}\right)_\alpha = & -4\pi \left(\nabla \times \frac{1}{\gamma}\chi_B \omega_\alpha\right) \\ & + \partial_0 \left(E_\alpha - 4\pi \frac{2c\mu}{\hbar\gamma^2}\chi_B \epsilon_{\beta i \alpha}\omega_{\beta i}\right). \end{aligned} \tag{3.7.21}$$

We like to draw the reader's attention to the fact that Mineev and Volovik derived these results already in the seventies [10] in the context of ^{3}He-B. We show here that these hold in the general case of an $SU(2)$ spin superfluid, and will demonstrate in section 5.4 that similar equations can be derived for the case of superfluid ^{3}He-A as well.

Chapter 4

Charge trapping by spin superfluids

We now go back to the trick of charge trapping in superfluids we used previously to wet your appetite. How does this magic trick work? At the heart of our idea lies the spin vortex solution. Let us first briefly sketch the argument, and then prove it. The straight wire causes an electric field of

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}, \tag{4.0.1}$$

where $\hat{r}$ is a radial unit vector in the xy plane perpendicular to the cylinder axis z. The azimuthal angle is φ. We now need to determine the electric field in the superfluid region. Because of the symmetry of the problem, this electric field will be radial. Lets call it E_i. This electric field will drive a spin current, which will be a source of electric field itself if it has a singularity, which because of radial symmetry will lie on the wire. The symmetry of the problem suggests that the spins will be polarized along the axis of the cylinder. By solving the equations of motion in the presence of an electric field and no magnetic field, we obtain that when the spin current and spin angular velocity satisfy the Spin Hall relation for spin direction $\alpha = z$

$$\omega_\alpha = 0, \quad \omega_{z\varphi} = \frac{2\mu}{\hbar c^2}E_r, \tag{4.0.2}$$

with the magnetic moment of the He-atoms

$$\mu = g\frac{m_e}{m_{He}}\mu_B, \tag{4.0.3}$$

whereas the other spin superfluid velocities vanish. Since the electric fields do not depend on the z-coordinate and only have a radial component, the equations of motion (3.7.19) are satisfied. In our case, written in cylindrical coordinates,

$$\vec{\omega}_z = \frac{2\mu}{\hbar c^2}\epsilon_{zik}E_k \sim \hat{\varphi}\ . \tag{4.0.4}$$

We see that the electric field leads to a *spin vortex*, i.e., z-polarised spins flowing around the wire. This is nothing different from vortices in Bose superfluids induced by rotation. This might cause some concern as we have an $SU(2)$ superfluid rather than a $U(1)$. Why does the spin vortex have stability? First the geometry and topology of the experimental set up does not allow the formation of $SU(2)$ monopoles and restricts the only topological objects that exist to obey the experimental cylindrical symmetry. One possible argument relies on a result from mathematics. Gauge theories coupled to matter are known to mathematicians as bundle theories. One way to classify them is by using Chern classes [1, 35]. The Chern classes do not depend on the gauge chosen, or the configuration of the matter fields, but are a property of the bundle. The ramification is that if the topology of the gauge field is cylindrical, the matter field has cylindrical topology as well.

The stability of the vortex can also be obtained from the fact that only a vortex centered on the wire, with spin parallel to it, satisfies the equations of motion as shown above, and that such a solution is an energy minimum. From the Lagrangian in the previous section the momentum conjugate to the non-Abelian or spin phase is

$$\mathcal{H} = \frac{\chi_B c^2}{2\gamma^2}\left(\omega_{\alpha i}^2 - \frac{4\mu}{\hbar c^2}\omega_{\alpha i}\epsilon_{\alpha ik}E_k\right) + \frac{1}{8\pi}E^2\ . \tag{4.0.5}$$

When the vortex solution and thus the Spin Hall relation is valid we have energy density

$$\mathcal{H}_{SH} = \left(\frac{1}{8\pi} - \frac{2\chi_B c^2}{\gamma^2}\frac{\mu^2}{\hbar^2 c^4}\right)E^2\ . \tag{4.0.6}$$

If there is no vortex we have energy density

$$\mathcal{H}_{\text{no-vortex}} = \frac{1}{8\pi}E^2 \tag{4.0.7}$$

which is bigger than the energy density $\mathcal{H}_{SH}$ corresponding to a vortex present and thus the solution with the vortex is favored. If we have a vortex solution and perturb around by $\delta\omega_{\alpha i}$ the energy changes by

$$\delta\mathcal{H} = \frac{\chi_B c^2}{2\gamma^2} \left(\delta\omega_{\alpha i}\right)^2 \tag{4.0.8}$$

which is a positive quantity and we see that the vortex solution is stable against perturbations as they increase the energy of the system. We can rephrase the above reasoning in a more sophisticated way: the cylindrical topology of the fixed-frame gauge fields imposes the same vortex-type topology on the matter field, because of the parallel transport structure originating from spin-orbit coupling!

The vortex topology can be classified by winding numbers. Indeed, from the definition of the spin supercurrent in chapter 3.7 we have

$$\vec{\omega}_z = -\nabla\theta. \tag{4.0.9}$$

Therefore the spin current must satisfy the quantization condition

$$\oint \vec{\omega}_z \cdot d\vec{l} = 2\pi N \tag{4.0.10}$$

when we integrate around the cylinder where N is an integer. This quantisation is not quite shocking, since any order parameter theory has this condition. However, bearing in mind the magnetic flux trapping in superconductors, it is interesting to integrate the spin current after substituting the spin-Hall equation. By Gauss' law, one obtains that the very same phase velocity integral becomes

$$\oint \vec{\omega}_z \cdot d\vec{l} = \frac{e}{m_{He}} \mu_0 \lambda 2\pi. \tag{4.0.11}$$

In other words, the charge density is quantised in units of

$$\lambda = N\lambda_0 = N\frac{m_{He}}{\mu_0 e} = 2.6 \times 10^{-5} C/m! \tag{4.0.12}$$

This experiment is the rigid realisation of the Aharonov-Casher phase [13], for which our application is inspired by Balatskii and Altshuler [9]. The rigidity is provided by the superfluid density, forcing the integral winding number. Our idea is actually the spin superfluid analogue of

Aharonov-Bohm flux trapping [36] with superconducting rings. The quantization of magnetic flux is provided by the screening of electromagnetic fields, causing zero total superconducting current. The latter, being defined covariantly, consists of a $U(1)$ superfluid velocity and a gauge field. Calculation of the line integral

$$0 = \oint J_i^{sc} dx_i = \oint \partial_i \phi - \oint A_i dx_i = 2\pi n - \Phi_{sc}, \qquad (4.0.13)$$

leads to the flux quantisation condition. In the above argument, the gauge fields A_i have dynamics, leading to screening of the A_i in the superconducting ring.

In our case, the gauge fields are fixed by the electromagnetic fields, such that there cannot be screening effects. Still, the spin-Hall equations, which solve the equations of motion (3.7.19), lead to a vanishing superconducting current. The gauge fields, being unscreened, play now a quite different role: these are necessary to force the topology of the superfluid order parameter to be $U(1)$. The result is the same: quantisation of electric flux, determined by the charge on the wire.

Charge trapping in spin superfluids and in spiral magnets both originate from the coherent part of the spin current. In this sense, there is not too much difference between the two effects. On the other hand, there is a subtle, but important distinction. For spiral magnets there is no need for electric fields to impose the supercurrent, since they are wired in by the magnetic spiral order. In contrast in the spin superfluids, an electric field is necessary to create a coherent spin current since there is no magnetisation.

The question which surely is nagging the reader's mind, is whether one can actually *perform* our experiment. The answer is threefold. To begin with, nobody knows of the existence of a material exhibiting an $SU(2)$-order parameter structure. Fortunately, the existence of two spin superfluids is well-established: ^{3}He-A and ^{3}He-B.

We will show that ^{3}He-B has an order parameter structure similar to that of the pure spin superfluid. The effect of dipolar locking will destroy the spin vortex caused by the electric field, however, see Section (5.3). Then we will show that ^{3}He-A has, for subtle reasons, the wrong topology to perform our experiment. We will also demonstrate that the small spin-orbit coupling constant forces us to use an amount of ^{3}He with which one can cover Alaska, turning our experiment into a joke. In the outlook of this work, we will discuss how the organic superconductors [37, 38] might meet the desired conditions.

To make these discussions more substantial, let us first consider the secrets of ^{3}He more closely.

Chapter 5

Superfluid ^{3}He

5.1 Order parameter structure of ^{3}He

As is well-known, ^{3}He is a fermionic atom carrying spin $\frac{1}{2}$. In field theory, we describe it with an operator $c_{p\alpha}$, where p is momentum and α is spin. In the normal phase, it is a Fermi liquid, but for low temperatures and/or high pressures, ^{3}He displays a BCS-like instability towards pairing. Indeed, the condensate wave function Ψ displays an order parameter which transforms under both spin and orbital angular momentum:

$$\langle\Psi| \sum_{\mathbf{p}} \mathbf{p} c_{\mathbf{p}\alpha} c_{-\mathbf{p}\beta} |\Psi\rangle = A_{\mu i} (i\sigma^\mu \sigma^2)_{\alpha\beta}, \tag{5.1.1}$$

so the order parameter describes a p-wave state. The $A_{\mu i}$ carry a spatial index i and an internal spin index μ. The numbers $A_{\mu i}$ transform as a vector under the spin rotation group $SO(3)^S$ acting on the index μ and the orbital rotation group $SO(3)^L$ acting on the index i. We can reconstruct the wave function Ψ from the $A_{\mu i}$ as follows. First we rewrite them as a vector decomposition with amplitudes a_{kl} in the following way:

$$A_{\mu i} = \sum_{k,l} a_{kl} \lambda^k_\mu \lambda^l_i. \tag{5.1.2}$$

The $\lambda^{k,l}$ are vectors. Then the wave function in momentum space $\Psi(\mathbf{p}) = \langle \mathbf{p}|\Psi\rangle$ is the decomposition

$$\Psi(\mathbf{p}) = \sum_{k,l} a_{kl} Y_{\mathrm{L}=1,k}(\mathbf{p})\ \chi_{\mathrm{S}=1,l} \quad , \tag{5.1.3}$$

where $Y_{\mathrm{L}=1,k}$ is a triplet spherical harmonic and $\chi_{\mathrm{S}=1,l}$ is a triplet spinor. This means that the order parameter has $3\times3\times2$ real degrees of freedom. Indeed, following Volovik [15] and Leggett [14], there exist two mean-field states.

The first one is an isotropic state with vanishing total angular momentum $J = L + S = 0$. In order to have zero projection of the total spin $m_J = m_l + m_s = 0$, we have for the coefficients in the decomposition (5.1.2)

$$a_{+-} = a_{-+} = a_{00} = \Delta_B. \tag{5.1.4}$$

This state is called the B-phase of ^{3}He, or the BW-state, after Balian and Werthamer [39]. This means that the order parameter looks like

$$A_{\alpha i} = \Delta_B \delta_{\alpha i}. \tag{5.1.5}$$

There is still a degeneracy, however. Indeed, both the spin and orbit index transform under $SO(3)$, which leads to an order parameter manifold

$$R_{\alpha i} = R^L_{ij} R^S_{\alpha\beta}\delta_{\beta j}, \qquad \text{or} \qquad R = R^S(R^L)^{-1}. \tag{5.1.6}$$

So the matrix $R \in SO(3)$ labels all degenerate vacua and describes a *relative* rotation of spin and orbital degrees of freedom. Including also the $U(1)$ phase of the matter field, the order parameter manifold of ^{3}He-B is

$$G_B = SO(3)_{rel} \times U(1)_{matter}. \tag{5.1.7}$$

This will be the starting point of our considerations for ^{3}He-B, in which we will often drop the $U(1)$ matter field.

The second one is the A-phase, which has just one non-vanishing amplitude in (5.1.2), viz.,

$$a_{0+} = \sqrt{2}\Delta_A, \tag{5.1.8}$$

which corresponds to a state with $m_s = 0$ and $m_l = 1$. The quantisation axes are chosen along the $\hat{z}$-axis, but this is just arbitrary. This is known as the ^{3}He-A phase, or the Anderson-Brinkman-Morel (ABM) state [40]. The order parameter is

$$A_{\alpha i} = \Delta_A \hat{z}_\alpha(\hat{x}_i + i\hat{y}_i). \tag{5.1.9}$$

Rotations of the quantisation axis of ^{3}He-A lead to the same vacuum, which tells us how to describe the degeneracy manifold. The vector describing spin, called the $\hat{d}$-vector in the literature [14], can be any rotation of the $\hat{z}$-axis:

$$\hat{d}_\alpha = R^S_{\alpha\beta}\hat{z}_\beta. \tag{5.1.10}$$

Since only the direction counts in which the $\hat{d}$-vector points, its order parameter manifold is the 2-sphere S^2. The orbital part of the order parameter is called the $\hat{l}$ vector, which is in the "gauge"(5.1.9) simply $\hat{z}$. Again, the orientation is arbitrary, so that any rotation R^L and gauge transformation $e^{i\phi}$ leads to a correct vacuum state,

$$\hat{e}^{(1)}_i + i\hat{e}^{(2)}_i = e^{i\phi}R^L_{ij}(\hat{x}_j + i\hat{y}_j), \tag{5.1.11}$$

where $\hat{l} = e^{(1)} \times e^{(2)}$ is invariant under $e^{i\phi}$. This phase communicates with the phase of the matter field, so that the order parameter has a relative $U(1)_{rel} = U(1)_{matter-orbital}$. For the determination of the order parameter manifold for He-A, we need to observe that the order parameter does not change if we perform the combined transformation $\hat{d} \to -\hat{d}$ and $(\hat{e}^{(1)}_i + i\hat{e}^{(2)}_i) \to -(\hat{e}^{(1)}_i + i\hat{e}^{(2)}_i)$. This means that we have to divide out an extra $\mathbb{Z}_2$ degree of freedom. In summary, the order parameter manifold for He-A is

$$G_A = (S^2_s \times SO(3)_l)/\mathbb{Z}_2, \tag{5.1.12}$$

where s refers to the spin and l to the orbit. The intricateness of the order parameter already indicates that there is a lot of room for various kinds of topological excitations and other interesting physics. For extensive discussions, we recommend the books of Grigory Volovik [15, 16]. What counts for us, however, is how the topology is influenced by switching on fixed frame gauge fields.

5.2 ^{3}He-B

As discussed above, the order parameter of ^{3}He is described by an $SO(3)$ matrix R. The question is now if R admits spin vortex solutions. In principle, it does, because $SU(2)$ rotations are like $SO(3)$ rotations, since they are both representations of angular momentum, as we learned in freshman quantum mechanics courses. This means that, in principle, all considerations for the $SU(2)$ case apply to ^{3}He-B as well. In particular, the spin superfluid velocity (3.7.11) has a similar expression, but now with $g = R \in SO(3)$. It reads

$$\omega_{\alpha i} = \frac{1}{2}\epsilon_{\alpha\beta\gamma} R_{\beta j} \partial_i R_{\gamma j}. \tag{5.2.1}$$

Inspired by the $SU(2)$ case, which was effectively Abelianised, we try a vortex solution around the z-axis (assuming the electric field is radial)

$$R = \exp(i\theta J_3) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.2.2}$$

where J is the generator of total angular momentum, and $\theta = \arctan(\frac{x_2}{x_1})$. With the help of the $SO(3)$ analogue of Eq. (3.7.11), the superfluid velocities (5.2.1)are readily calculated to be

$$\begin{aligned} \omega_1^3 &= -(\partial_1 R_{1k})R_{2k} = \frac{x_2}{r^2} = \frac{2\mu}{\hbar c^2}E_2 \\ \omega_2^3 &= -(\partial_2 R_{1k})R_{2k} = -\frac{x_1}{r^2} = -\frac{2\mu}{\hbar c^2}E_1 \\ \omega_3^1 &= -(\partial_3 R_{2k})R_{3k} = 0 \\ \omega_1^2 &= -(\partial_3 R_{1k})R_{3k} = 0, \end{aligned} \tag{5.2.3}$$

where $r^2 = x_1^2 + x_2^2$. Since the groups $SO(3)$ and $SU(2)$ give the same equations of motion (3.7.19), we see that the Ansatz (5.2.2) satisfies these as well, giving a spin-Hall current for the z-polarised spin. In other words, ^{3}He-B is a possible candidate for our quantised spin vortex.

This result can also be understood by topological means, in the following way. The equation of motion for the $SU(2)$ case tells us, that the vacuum manifold for the spin becomes $U(1)$ instead of $SO(3) \simeq SU(2)$. Only if we were allowed to change the orientation of the wire, described by a point on S^2, we would obtain the full $SO(3)$. This is the translation of the mathematical fact that $SO(3)/S^2 \simeq U(1)$, merely saying that a rotation is fixed by an axis of rotation and the angle of rotation about that particular axis. The implication is that we need to calculate the fundamental group of G_B/S^2 instead of G_B itself:

$$\pi_1(SO(3)/S^2) = \pi_1(U(1)) = \mathbb{Z}\,, \tag{5.2.4}$$

leading to the existence of vortices in a cylindrical set up, i.e., the inclusion of radial electric fields induces vortices.

There is however one effect which destroys our spin vortex solution. This effect, known as dipolar locking, will be discussed in the next section.

5.3 Dipolar locking

In the 1970s, Leggett described in his seminal article about ^{3}He many important properties of this interesting system [14]. One of them is how the spin part of the condensate wave function $\Psi(\vec{x})$ interacts with its orbital motion by a $\vec{S} \cdot \vec{L}$ interaction. According to Leggett, the contribution of the Cooper pairs to the dipolar energy is

$$\begin{aligned} E_{dip} &= -g_{dip} \int d\vec{x} \frac{1}{x^3} \left(|\Psi(\vec{x})|^2 - 3|\vec{x} \cdot \Psi(\vec{x})|^2 \right) \\ &= g_{dip} \int \frac{d\Omega}{4\pi} 3|\hat{n} \cdot (A_{\alpha i} n_\alpha)|^2 - \text{constant}, \end{aligned} \tag{5.3.1}$$

remembering that the spin order parameters carry a spatial index, cf. (5.1.9), (5.1.5). We used the notation $\hat{n} = \frac{\vec{x}}{|x|}$. On inserting the order parameters (5.1.9) and (5.1.5), we obtain for both phases the dipole locking Lagrangians

$$\begin{aligned} L_{dip,B} &= -g_{dip} \left((\text{Tr} R)^2 + \text{Tr}(R)^2 \right), \\ L_{dip,A} &= -g_{dip} (\hat{l} \cdot \hat{d})^2. \end{aligned} \tag{5.3.2}$$

For the ^{3}He-A part, we do not need to solve the equations of motion in order to infer that the orbital and spin vector wish to be aligned. For the B-phase, we give a derivation of the Leggett angle. A general matrix $R \in SO(3)$ can be described by three Euler angles. For the trace, only one of them is important, let's say it is called θ. Then

$$L_{dip,B} = -g_{dip} \left\{ (1 + 2\cos\theta)^2 + 2(\cos^2\theta - \sin^2\theta) \right\}, \tag{5.3.3}$$

which leads to the static equation of motion

$$0 = \frac{dL_{dip,B}}{d\theta} = 4\cos\theta + 1, \tag{5.3.4}$$

with the Leggett angle as solution,

$$\theta_L = \arccos(-\frac{1}{4}) \simeq 104^o. \tag{5.3.5}$$

The Leggett angle tells us that one degree of freedom is removed from the order parameter of ^{3}He-B so that

$$SO(3)_{rel} \to G_{B,dip} = S^2, \tag{5.3.6}$$

but $\pi_1(S^2) = 0$, as any closed path on the sphere can be continuously shrunk to a point.

Now we can also understand that dipolar locking destroys vortices, even in a cylindrical set up, i.e. with a radial electric field, since

$$\pi_1(G_{B,dip}/S^2) = \pi_1(e) = 0\,. \tag{5.3.7}$$

The "division" by the manifold S^2 translates the fact that different vortices in the ^{3}He-B manifold are only equivalent to each other up to different orientations of the cylindrical wire, being described by S^2. Another way to understand the destruction of vortices beyond the dipolar length, is that the $U(1)$ vortex angle θ is fixed to the Leggett angle, as depicted in Figure 5.1.

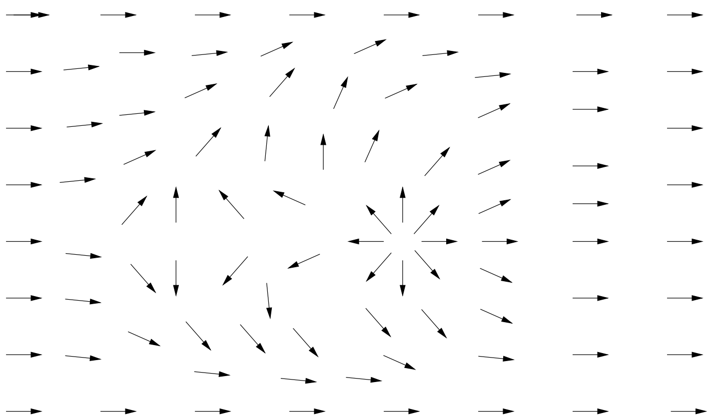

Figure 5.1: The destruction of the spin vortex by dipolar locking. The $U(1)$ degree of freedom is indicated by an arrow. In the center where the electric field is located, the angle follows a vortex configuration of unit winding number, corresponding to one charge quantum. Since the electric field, decaying as $\frac{1}{r}$, is not able to compete with the dipolar locking at long distances, the $U(1)$ angle becomes fixed at the Leggett angle, indicated by a horizontal arrow.

The fact that the vortices are destroyed, even though the spin-orbit coupling energy is higher than the dipolar locking energy [10], is due to the fact that small energy scales do play a role at large distances. This is similar to spontaneous symmetry breaking in, for example, an XY-antiferromagnet. A small external field is enough to stabilise domain walls at long wavelenghts.

5.4 ^{3}He-A

In the discussion of the pure spin superfluids and of ^{3}He-B, we used the fact that the order parameter has a matrix structure, namely $SU(2)$ and $SO(3)$, respectively. For the $SU(2)$ case we had to transform from the fundamental spinor representation to the adjoint matrix representation. Since both representations are $SU(2)$, the physics did not change fundamentally. The resulting equations of motion were equations for group elements g, with the ramification that spin vortex states lower the energy with respect to the trivial solution, cf. Eq.(4.0.6). As a result, the vacuum manifolds in both cases become $U(1)$ instead of $SU(2)$ (pure spin superfluid) or $SO(3)$ (^{3}He-B without dipolar locking). The topological protection of the spin vortex solution followed from the fact that $U(1)$ is characterised by the winding numbers, $\pi_1(U(1)) = \mathbb{Z}$.

For the case of ^{3}He-A, matters are different, since the spin order parameter for ^{3}He-A is a vector in S^2 instead of a matrix in $SO(3)$. Although $SO(3)$ acts on S^2, these manifolds are not the same. What we will prove is that as a result, spin vortices do *not* lower the energy in the presence of an electric field, as opposed to the ^{3}He-B and pure spin superfluids. The consequence is that the vacuum manifold remains S^2, and since $\pi_1(S^2) = 0$, spin vortices are not protected. The presence of dipolar locking will not change matters.

Let us prove our assertions by deriving the equations of motion from the Lagrangian for ^{3}He-A.

The free energy functional [15] for ^{3}He-A is quite analogous to that of liquid crystal[41], as the A phase is both a superfluid and liquid crystal is some sense. Besides the bulk superfluid energy, there are also gradient energies present in the free energy, of which the admissible terms are dictated by symmetry:

$$\begin{aligned} F_{grad} &= \gamma_1(\partial_i A_{\alpha j})(\partial_i A_{\alpha j})^* + \gamma_2(\partial_i A_{\alpha i})(\partial_j A_{\alpha j})^* + \gamma_3(\partial_i A_{\alpha j})(\partial_j A_{\alpha i})^* \\ A_{\alpha i} &= \Delta_A \hat{d}_\alpha e^{i\phi_{rel}}(\hat{e}_i^{(1)} + i\hat{e}_i^{(2)}). \end{aligned} \tag{5.4.1}$$

This then leads to

$$\begin{aligned} F_{grad}^{London} &= \frac{1}{2}K_{ijmn}\partial_i\hat{e}_m\partial_j\hat{e}_n + C_{ij}(v_s)_i\epsilon_{jkl}\partial_k\hat{e}_l \\ &+\frac{1}{2}\rho_{ij}(\partial_i\hat{d}_\alpha)(\partial_j\hat{d}_\alpha) + g_{dip}(\hat{d}_\alpha\hat{e}_\alpha)^2. \end{aligned} \tag{5.4.2}$$

The coefficients K_{ijmn} and C_{ij} are the liquid crystal like parameters[41].The superfluid velocity v_s is the Abelian superfluid velocity coming from the relative $U(1)$ phase.

We are going to prove that ^{3}He-A does not have topologically stable spin vortices, and that dipolar locking does not stabilise these. Generically, the spin stiffness tensor ρ_{ij} is given by [15]

$$\rho_{ij} = \rho^{||}\hat{l}_i\hat{l}_j + \rho^{\perp}\left(\delta_{ij} - \hat{l}_i\hat{l}_j\right), \tag{5.4.3}$$

but it becomes fully diagonal when the neglect anisotropies in the spin wave velocities, i.e., $\rho^{||} = \rho^{\perp}$. We are also take that the $K_{ij,mn}$ and and C_{ij} are fully diagonal, since this will not change the nature of the universal low energy physics. Including now spin-orbit coupling and kinetic terms the ^{3}He-A Lagrangian is

$$\begin{aligned} L^A(\psi_{\alpha j}, \vec{E}, \vec{B}) &= -\frac{\hbar^2}{2mc^2}\left\{|\partial_0\hat{e}_j|^2 + (\partial_0 d_\alpha)^2 + \frac{2\mu m n_s}{\hbar^3 c}\epsilon_{\alpha\beta\gamma}\hat{d}_\beta\partial_0\hat{d}_\gamma B_\alpha\right\} \\ &+\frac{\hbar^2}{2m}\left\{|\partial_i\hat{e}_j|^2 + (\partial_i d_\alpha)^2 - \frac{2\mu m n_s}{\hbar c^2}\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha i k}\hat{d}_\beta\partial_i\hat{d}_\gamma E_k\right\} \\ &+\frac{1}{8\pi}\left(E^2 - B^2\right) - \frac{1}{2}g_{dip}\left(\hat{d}\cdot\hat{l}\right)^2. \end{aligned} \tag{5.4.4}$$

The strategy for solving the equations of motion is as follows: first we demonstrate that a spin vortex is possible without dipolar locking, but that is does not gain energy with respect to the constant solution. Then we show that the spin vortex is not stabilised on switching on the dipolar locking.

Without dipolar locking a spin-only action is obtained, leading to an equation of motion which resembles (3.7.19),

$$\partial_i\left[\partial_i d_j - \frac{2\mu m n_s}{\hbar c^2}\epsilon_{\alpha i k}(\epsilon_{\alpha\beta j})d_\beta E_k\right] = 0. \tag{5.4.5}$$

Let us choose a reference vector D_ν, such that $d_j = R_{j\nu}D_\nu$. Again, R is an $SO(3)$ matrix, describing the superfluid phase of the S^2 variable d. In this way, the equation of motion for the group element R reads

$$\partial_i\left[\partial_i R_{j\nu} - \frac{2\mu m n_s}{\hbar c^2}\epsilon_{\alpha i k}(\epsilon_{\alpha\beta j})R_{\beta\nu}E_k\right] = 0. \tag{5.4.6}$$

Using cylindrical coordinates, the demonstration that the spin vortex Ansatz for R is a solution to this equation of motion is analogous to the proof that a spin vortex exists in ^{3}He-B, cf. Eq.(5.2.3). On the other hand, this equation also admits constant R, i.e., the equation (5.4.5) admits a constant D_μ as well. Substituting both solutions back into the energy functional Eq.(5.4.4), there turn out to be no energy differences between the spin vortex and the constant solution. In mathematical terms, the vacuum manifold in the presence of a cylindrical electric field remains S^2. In plain physics language: the electric field does not prevent phase slips to occur.

The presence of dipolar locking makes matters even worse, since the equations of motion become The equations of motion for e and d involving dipolar locking,

$$\begin{aligned}
\frac{\hbar^2}{2m}\partial_i^2 \hat{e}_j^{(1)} &= -g_{dip}(\epsilon_{abc}\hat{e}^{(1)}\hat{e}_c^{(2)}\hat{d}_a)\epsilon_{kjm}\hat{e}_m^{(2)}\hat{d}_\alpha \\
\frac{\hbar^2}{2m}\partial_i^2 \hat{e}_j^{(2)} &= -g_{dip}(\epsilon_{abc}\hat{e}^{(1)}\hat{e}_c^{(2)}\hat{d}_a)\epsilon_{kmj}\hat{e}_m^{(1)}\hat{d}_\alpha \\
\partial_i\left[\partial_i d_j - \frac{2\mu m n_s}{\hbar c^2}\epsilon_{\alpha ik}(\epsilon_{\alpha\beta j})d_\beta E_k\right] &= -2g_{dip}(\epsilon_{abc}\hat{e}^{(1)}\hat{e}_c^{(2)}\hat{d}_a)\epsilon_{jlm}\hat{e}_l^{(1)}\hat{e}_m^{(2)}.
\end{aligned} \tag{5.4.7}$$

It is clear that in general, a vortex configuration for $\hat{d}$ is not a solution, since the left hand side of the equation for $\hat{d}$ is annihilated, whereas the right hand side is not. Instead, the orbital and spin vectors will perform some complicated dance, set in motion by the electric field.

The conclusion as to whether our charge trapping experiment is possible, is that it will not work for ^{3}He-A.

5.5 Baked Alaska

In the search for an experimental realisation of the proposed charge trapping experiment, it turned out that ^{3}He-B admits spin vortex solutions only at short wavelengths. But if there were a way to circumvent dipolar locking in some ideal world, nothing would stop us from performing the actual experiment.

Or... does it? It turns out that the numbers which Nature gave us conspire to obstruct matters. It is really hidden in the fact that electric fields are so strong, and spin-orbit coupling so weak. Let us first confess that in the previous considerations we did not regard a very important part of our charge trapping device, namely, the wire itself. The charge stored on it is hugely repelling indeed, giving rise to an enormous charging energy.

First, we calculate the Coulomb energy stored in the wire. Let $\rho(x)$ be the charge density distribution, which we approximate by a step function of the radius. Then,

$$\begin{aligned} W_{\text{Coulomb}} &= \frac{1}{8\pi\epsilon_0}\int \frac{\rho(x)\rho(x)}{\|\mathbf{x}-\mathbf{x}'\|}d\mathbf{x}d\mathbf{x}' \\ &= \frac{1}{8\pi\epsilon_0}\frac{Q_{\text{tot}}^2}{\pi a^2 L}I. \end{aligned} \tag{5.5.1}$$

We integrated over the center-of-mass coordinate, and (with the definitions $\mathbf{u}=\mathbf{x}-\mathbf{x}'$ and $q=L/a$) we introduced

$$\begin{aligned} I &\equiv \int_0^L du_z \int_0^a 2\pi du_\perp u_\perp \frac{1}{\|\mathbf{u}\|} \\ &= 2\pi\left\{-\frac{1}{2}L^2 + a\int_0^L du_z\sqrt{1+\left(\frac{u_z}{a}\right)^2}\right\} \\ &= 2\pi\left\{-\frac{1}{2}L^2+\frac{a^2}{2}\left(q\sqrt{1+q^2}+\ln(q+\sqrt{1+q^2})\right)\right\} \\ &\simeq 2\pi\frac{a^2}{2}\ln(2q) \quad \text{for } L>>a. \end{aligned} \tag{5.5.2}$$

We used the standard integral $\int d\tau\sqrt{1+\tau^2} = \frac{1}{2}\tau\sqrt{1+\tau^2}+\frac{1}{2}\ln(\tau+\sqrt{1+\tau^2})$. Hence

$$W_{\text{Coulomb}} = \frac{1}{8\pi\epsilon_0}\lambda^2 L\ln\left(\frac{2L}{a}\right). \tag{5.5.3}$$

For the parameters under estimation, $W_{\text{Coulomb}}/L \simeq 1\text{J/m}$, which is really enormous, since the coupling constant of electric fields is so huge.

The question is now if the superfluid is strong enough to keep the charge trapped. Indeed, if it doesn't, the system can lower its energy by simply discharging the wire, causing a big spark, and destroying the superfluid. This is analogous to magnetic flux trapping in superconducting rings with the Aharonov-Bohm effect [36]. The flux trapped in a ring is a metastable state, but the superconducting condensate is strong enough to keep it there.

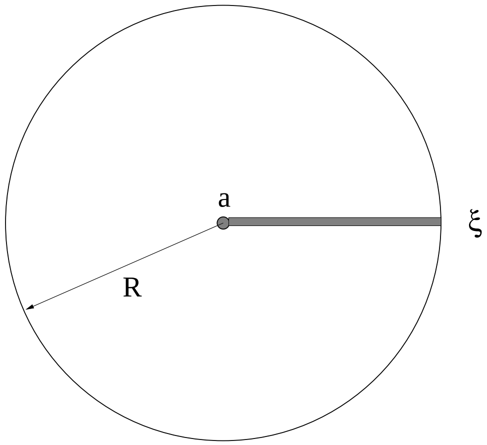

Figure 5.2: View from the top of our container. The container radius is R, and the wire has radius a. Now, the Coulomb energy of the wire has to make a tiny region of superfluid normal again, in order to make phase slips happen, removing the topological constraint. The region in which this should happen, needs to be of the width of the coherence length ξ, but it has to extend over the whole radius of the container.

However, spin-orbit coupling is too weak to do so with our Aharonov-Casher analogue. In fact, the only thing the system needs to do, is to destroy the spin superfluid, not in the whole container, but just a small strip of the order of the coherence length ξ, which is of the order of $0.01\mu m$ [42].

We now need to estimate the energy density of the fluid. To do this, we perform Landau theory for the superfluid order parameter ψ,

$$\delta F = \int \left\{ a|\psi|^2 + \frac{1}{2}b|\psi|^4 \right\} d\mathbf{x}. \tag{5.5.4}$$

This expression is zero when there is no superfluid. There is no kinetic term, since ψ is parallel transported by the electric field: indeed, if it satisfies the equations of motion, the kinetic term vanishes, cf. Eq. (3.7.19). Hence, we are only left with the potential energy terms. From Landau theory, we know the saddle point value for ψ in terms of $a = \alpha(T - T_c)$ and b, viz.,

$$|\psi|^2 = \frac{-a}{b} \Rightarrow \delta F = -V\frac{\alpha^2}{2b}(T - T_c)^2, \tag{5.5.5}$$

where $V = \pi R^2 L$ is the volume of the container. Note that R is the unknown variable in our problem. From Landau and Lifschitz we obtain the BCS-parameters

$$a(T) = \frac{6\pi^2}{7\zeta(3)}\frac{k_B T_c}{\mu}(k_B T_c)\left(1 - \frac{T}{T_c}\right), \qquad b = \alpha\frac{k_B T_c}{\rho}, \tag{5.5.6}$$

where ρ is the superfluid density. For low temperatures $T << T_c$ we have $\mu \simeq \varepsilon_F$, so

$$\delta F \simeq 3.52(nk_B T_c V)\frac{k_B T_c}{\varepsilon_F}. \tag{5.5.7}$$

We use experimental values [43] $\varepsilon_F/k_B = 0.312K$ and $T_c = 3mK$. From the Fermi gas relation $\rho = p_F^3/3\pi^2\hbar^2$ we then obtain $\rho \approx 15$ mol/liter. This leaves us with an estimate

$$\frac{\delta F}{V} \sim 34 \text{ J/m}^3.$$

The question we need to ask is: how big does the container radius R need to be, in order to remain in the metastable, charge trapped state? The estimate per unit length L is

$$\frac{W_{Coulomb}}{L} = \frac{\delta F}{V} R\xi. \tag{5.5.8}$$

Due to the enormously small ξ and the enormously big $W_{Coulomb}$, this leads to a truly disappointing radius of

$$R \simeq 1000km, \tag{5.5.9}$$

enough to cover Alaska, and much more than the total amount of ^{3}He on Earth (180 liters). There might be enough ^{3}He on the Moon, but still it is a "only in your wildest dreams" experiment. Is there no way out? In the concluding chapter 9 , we give a direction which might provide some hope.

Part II

The doped Mott insulator

Chapter 6

Projective symmetry

In the first part of this thesis the parallel transport aspect of non-Abelian gauge theories was considered. It was shown how spin-orbit coupling appears as an $SU(2)$ Yang-Mills theory in disguise: the electromagnetic fields coupling to spin act as a non-Abelian parallel transport structure. Central to this construction is the fixed-frame nature of the parallel transport fields as they are completely determined by the electromagnetic fields applied to the spin system. This implies that the analogy with ordinary Yang-Mills gauge theory is not complete since the fields providing the parallel transport do not obey the dynamics of $SU(2)$ gauge fields, and hence do not impose conservation laws for spin-orbit coupled systems. It was shown that only in the phase coherent phase, analogous to the Higgs phase in Yang-Mills theory, conservation laws emerge, giving hydrodynamic status to the spin-Hall equation, familiar from spintronics.

In the second part of this work a system is studied in which non-Abelian gauge theory appears in its full glory: the non-Abelian gauge fields are now introduced as fields imposing local constraints, thereby requiring the gauge fields to become fully dynamic. This dynamics give rise to the phenomenon of confinement, familiar from quantum chromodynamics (QCD) in high-energy physics. Quantum chromodynamics is the $SU(3)$ gauge theory describing quarks and gluons, mediating the gauge force. This gauge theory is in a confining phase, meaning that up to very high energies, only hadrons like protons and mesons can exist. Only if there were a deconfining phase would the quarks and gluons acquire an existence of their own.

In this part of the thesis, we are going to explain how an $SU(2)$ gauge theory emerges in a condensed matter system known as the Mott insulator (MI) [18, 44, 45]. It is shown how a deconfining state can emerge in the context of an $SU(2)$ gauge theory, following the work of theorists Lee, Wen and Nagaosa [18, 44]. The deconfined "quarks" are called spinons and holons, forming the meat-and-potato electrons after the confinement transition. The stability of the deconfined state against gauge fluctuations is provided by the *assumption* that a mean field state of condensed spinons exists. This Higgs-like phase is able to give mass to gauge fluctuations, preventing confinement, at least in some cases [44].

This chapter starts with the explanation of how the $SU(2)$ gauge theory emerges from the Mott insulator, the parent state of high-T_c compounds. Next, we give an example of a special class of mean field states, or Higgs phases, in which the miracle of deconfinement happens. The stability of this state will be explained.

In the course of the story, the concept of quantum order is discussed, as introduced by X.-G. Wen [18]. This novel idea is the gauge theory counterpart of the classical concept of symmetry breaking. The motivation for the introduction of quantum order is that, on the one hand, mean fields form a measure of order. On the other hand, it turns out that mean field states can be gauge equivalent, but a gauge symmetry can never be broken [46]. Hence the Landau-Ginzburg-Wilson paradigm for classical order parameter theories, as classified by broken symmetries, does not work. In order to still be able to classify order in quantum systems obeying a gauge symmetry, X.-G. Wen introduces the projective symmetry group (PSG). This classification exploits the fact that mean-field theories with different symmetries can still be equivalent up to a gauge transformation, thus leading to the same physical state. In the context of $SU(2)$ gauge theory it is shown that some quantum orders, as classified by the PSG, give stability to a deconfined spin liquid.

Let us now set the stage for the $SU(2)$ gauge theory for high-T_c superconductors, which emerges from Mott insulator states.

6.1 $SU(2)$-gauge theory of the half-filled Mott insulator

Since the discovery by Bednorz and Müller of compounds being able to support superconductivity up to much high temperatures than previously encountered, a number of compounds showing the same exciting phenomenon have been discovered. Some well known examples are $La_{2-x}Sr_xCuO_4$, $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$, $YBa_2Cu_3O_{6+x}$ and $Bi_2Sr_2CaCuO_{8+\delta}$. Soon after the discovery, it was realised what the relevant physics of these compounds is [47]. The high-T_c materials consist of weakly coupled copper-oxide planes, the copper-oxide bonds forming the bonds in a square lattice. The relevant physics is that of strongly repelling electrons in the d-shells of the copper atoms.

For zero doping, i.e., $x = 0$ in the above materials, the electrons are unable to move due to this repulsion, and form the state known as the Mott insulator.

On the other hand, on introducing dopant atoms, like strontium or oxygen, the electron wave functions become overlapping, enabling tunneling to nearby sites. This might be connected to the fact that for dopings of 8%-25% superconductivity emerges, although the parent compound is an insulator instead of a metal, as for conventional superconductors like Al. In the early days of this research area it was realised that the question of high-T_c is how superconductivity emerges on doping a Mott insulator.

One thing is clear: the explanation will probably be quite different from the explanation of conventional superconductivity. The latter involves a weakly interacting electron system, i.e., a conducting Fermi liquid, which becomes unstable towards superconductivity by electron-phonon coupling. On the other hand, in the high-T_c materials one is dealing with strongly interacting electrons, with the Mott insulating ground state.

The textbook starting approach in understanding this problem, is writing down a Hamiltonian capturing both the Coulomb repulsion U and tunneling t between sites, the Hubbard Hamiltonian [48]. In the limit of half filling, there is one electron on each site. Because of the Coulomb repulsion, these are not able to move, leading to the Mott insulator state. Virtual hopping is still possible. By the Pauli exclusion principle, these virtual processes are only possible when the spins of adjacent electrons are anti-aligned. So the effective Hamiltonian is the Heisenberg Hamiltonian,

$$H_{heis} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \tag{6.1.1}$$

where the operators $\mathbf{S}_i$ describe the spins of the electrons. The effective Heisenberg coupling constant $J = \frac{4t^2}{U}$ is positive. A key observation is that this Hamiltonian is obviously invariant under global $SU(2)$, i.e., it describes spin singlet states.

The spin operators $\mathbf{S}$, being composed of two fermionic electrons, are bosonic in nature. Let us now assume that these spins are not the fundamental particles, just as we can assume that baryons and mesons are not the fundamental particles in nature, as is done in QCD. In high-energy physics, this approach is chosen to more concisely describe the zoo of particles discovered in accelerators. In condensed matter, the objective is different: from the elementary particles forming the spinons, one hopes to find other states than the Mott insulator, providing a possible handle to understand how superconductivity emerges on doping the Mott state.

Let us now exploit this idea of the spinon degrees of freedom as the relevant ones by introducing the fermionic, chargeless spin-$\frac{1}{2}$ operators $f_{i\alpha}$, with spin index α. The spin operator $\mathbf{S}$ on site i is then represented in terms of the Pauli matrices τ^l as

$$\mathbf{S}_i^l = \frac{1}{2} f_{i\alpha}^\dagger \tau_{\alpha\beta}^l f_{i\beta}. \tag{6.1.2}$$

At a first glance, this spin operator, and every Hamiltonian expressed in terms of $\mathbf{S}_i$, is invariant under the $U(1)$ gauge transformation

$$f_i \to f_i e^{i\theta_i}. \tag{6.1.3}$$

This is the usual local symmetry connected with local charge conservation. To see this, we have to realise that this $U(1)$ symmetry means that in decomposing $\mathbf{S}_i$ according (6.1.2) degrees of freedom are introduced which are not physical: no matter what phase θ_i is chosen, the physical spin operator remains the same. This redundancy can be projected out by including the constraint

$$f_{i\alpha}^\dagger f_{i\alpha} = 1, \tag{6.1.4}$$

which can be implemented by including a minimal coupling to a $U(1)$ gauge field A_i in the Hamiltonian by

$$H_{constr} = \sum_i A_i (f_{i\alpha}^\dagger f_{i\alpha} - 1). \tag{6.1.5}$$

Hence the $U(1)$ gauge symmetry corresponds with a local conservation law.

The gauge symmetry of the Mott insulator is actually larger than just $U(1)$: it is $SU(2)$, corresponding to a particle-hole symmetry. This is demonstrated by introducing the spinor

$$\psi_i = \begin{pmatrix} f_{i\uparrow} \\ f^\dagger_{i\downarrow} \end{pmatrix}. \tag{6.1.6}$$

The spin operators can then be rewritten as $SU(2)$ invariant combinations of the ψ:

$$\mathbf{S}_i^+ = \frac{1}{2}(\psi^\dagger_{1i}\psi^\dagger_{2i} - \psi^\dagger_{2i}\psi^\dagger_{1i}) \tag{6.1.7}$$

$$\mathbf{S}_i^z = \frac{1}{2}(\psi^\dagger_i\psi_i - 1). \tag{6.1.8}$$

The spin operators $\mathbf{S}_i$ are invariant under local $SU(2)$ transformations,

$$\psi_i \to g_i\psi_i, \quad \tau^l \to g_i\tau^l g_i^\dagger, \quad g_i \in SU(2), \tag{6.1.9}$$

and so is the Hamiltonian (6.1.1) when expressed in ψ_i. The transformation g_i interchange the spinons $f_{i\uparrow}$ and $f^\dagger_{i\downarrow}$, which is nothing more than interchanging particles and holes. The $SU(2)$ symmetry then expresses the particle-hole symmetry of the system.

Again, the Hilbert space is enlarged, since the fermion number per site can be different from one. To make sure that the correct Hilbert space with one spinon per site is obtained, constraint equations similar to the $U(1)$ case have to be introduced. Since $SU(2)$ has three degrees of freedom, viz. the three Pauli matrices, three constraint equations should be imposed:

$$f^\dagger_{i\alpha}f_{i\alpha} = 1, \quad f_{i\alpha}f_{i\beta}\epsilon_{\alpha\beta} = 0. \tag{6.1.10}$$

These constraints will lead to an $SU(2)$ gauge theory, as these are implemented by three gauge fields a^l_{0i}. This is done by adding the term

$$H_{constr} = \sum_i a^3_{0i}(f^\dagger_{i\alpha}f_{i\alpha} - 1) + [(a^1_{0i} + ia^2_{0i})f_{i\alpha}f_{i\beta}\epsilon_{\alpha\beta} + h.c.] \tag{6.1.11}$$

to the original Hamiltonian. These constraints can be reformulated in terms of the $SU(2)$ spinors ψ. Since the physical Hilbert space has precisely one spinon per site, the operator $S^z_i = \frac{1}{2}f^\dagger_{i\alpha}\sigma^3_{\alpha\beta}f_{i\beta}$ has eigenvalues

$\pm\frac{1}{2}$ on that Hilbert space. Translating this into the reformulation (6.1.7), this means

$$\psi^\dagger\psi\,|\text{phys}\rangle = 0 \quad \text{or} \quad \psi^\dagger\psi\,|\text{phys}\rangle = 2\,|\text{phys}\rangle\,. \tag{6.1.12}$$

In other words, one f-fermion per site corresponds to an even number of ψ-fermions per site. Now we demonstrate that the Hilbert space of one fermion per site is a singlet under the $SU(2)$ transformations $\psi_i^\dagger\tau^l\psi_i$. This can be seen by observing that $\psi_i^\dagger\tau^+\psi_i = f_{i\uparrow}^\dagger f_{i\downarrow}^\dagger$, $\psi_i^\dagger\tau^-\psi_i = f_{i\downarrow}f_{i\uparrow}$ and $\psi_i^\dagger\tau^3\psi_i = f_{i\uparrow}^\dagger f_{i\uparrow} + f_{i\downarrow}^\dagger f_{i\downarrow} - 1$ act as zero on the states $f_{i\alpha}^\dagger\,|\text{vac}\rangle$. The constraints (6.1.10) then translate into the requirement that the physical states are $SU(2)$ gauge singlets,

$$\psi_i^\dagger\tau\psi_i\,|\text{phys}\rangle = 0. \tag{6.1.13}$$

These constraints are, again, implemented by the gauge fields a_{0i}^l.

Imposing the constraints (6.1.10), (6.1.13) exactly, means that the gauge coupling is infinite. This does not mean that the gauge fields do not have dynamics. Instead, by integrating out the fermion fields, the gauge coupling can be renormalised to an effectively finite value.

When the gauge fields become dynamical, we should worry about confinement issues, since Polyakov showed that a compact $U(1)$-gauge theory on a lattice is always confining in $2+1$ dimensions[12]. Hence it is to be expected that an $SU(2)$-gauge theory, having more degrees of freedom, confines as well. So how can any reality be attached to the construction of bosonic spins consisting of fermionic spinons, if only the confining state has reality?

In this chapter, it will be shown that the deconfined state can gain physical relevance after assuming that mean field theory has physical meaning, i.e., one may assume the existence of the fermionic vacuum expectation values

$$\chi_{ij} = \langle f_{i\uparrow}^\dagger f_{j\uparrow} + f_{i\downarrow}^\dagger f_{j\downarrow}\rangle \tag{6.1.14}$$

$$\Delta_{ij} = \langle f_{i\uparrow} f_{j\downarrow} - f_{i\downarrow} f_{j\uparrow}\rangle. \tag{6.1.15}$$

The first one is a hopping amplitude, inspired by the staggered flux spin liquid states proposed by Affleck and Marston [49, 50]. The order parameter Δ is going to play the role of a superconducting amplitude with d-wave symmetry. If these expectation values exist, the hope is that a Higgs-like mechanism will give mass to gauge fields. One might expect that the gauge fields, being massive, will then not be able to fluctuate, thus preventing confinement.

6.2 $SU(2)$ mean field theory: the deconfined spin liquid

We will devote this section to the question of whether a deconfined spin liquid is possible by arguing that a mean field state exists which gives mass to gauge fluctuations.

Let us now assume that the expectation values (6.1.14) and (6.1.15) exist, and let us presuppose deconfinement, by neglecting fluctuations in the gauge fields a^l_{0i}. This is equivalent to replacing of the exact constraints (6.1.13) by the mean field constraints

$$\langle \psi_i^\dagger \tau^l \psi_i \rangle = 0. \tag{6.2.1}$$

Let us first obtain a manifestly $SU(2)$-gauge invariant mean field theory, by grouping the mean fields as follows:

$$U_{ij} = \begin{pmatrix} -\chi^*_{ij} & \Delta_{ij} \\ \Delta^*_{ij} & \chi_{ij} \end{pmatrix}. \tag{6.2.2}$$

We arrive at a central result of this section, namely the mean-field Hamiltonian for the half-filled state,

$$H_{mean} = -\frac{3}{8} J \sum_{\langle ij \rangle} \left[\frac{1}{2} \mathrm{Tr}(U^\dagger_{ij} U_{ij}) + (\psi_i^\dagger U_{ij} \psi_j + h.c.) \right] + \sum_i a^l_{0i} \left[\psi_i^\dagger \tau^l \psi_i \right]. \tag{6.2.3}$$

Provided that the mean-field Ansatz U_{ij} acquires the form

$$U_{ij} = \tau^\mu \eta_{ij\mu}, \quad \eta_{ij0} \text{ imaginary}, \quad \eta_{ijl} \text{ real}, \tag{6.2.4}$$

the Hamiltonian (6.2.3) is manifestly $SU(2)$ invariant under the transformation

$$\psi_i \to g_i \psi_i, \quad U_{ij} \to \tilde{U}_{ij} = g_i U_{ij} g^\dagger_j, \quad a^l_{i0} \tau^l \to \tilde{a}^l_{0i} \tau^l = g_i a^l_{i0} \tau^l g^\dagger_j. \tag{6.2.5}$$

The assumption of deconfinement, as speculative as it is, leads to predictions which make physical sense. To make this point, let us consider an Ansatz describing the d-wave superconductor at zero doping, (dSC)

$$\begin{aligned} U_{\mathbf{i},\mathbf{i}+\hat{x}} &= -\chi \tau^3 + \Delta \tau^1, \\ U_{\mathbf{i},\mathbf{i}+\hat{y}} &= -\chi \tau^3 - \Delta \tau^1, \\ a^l_0 &= 0. \end{aligned} \tag{6.2.6}$$

Although the MI is not superconducting, the Ansatz Eq. (6.2.6) is called "dSC"since the order parameter Δ couples $f_{i\uparrow}$ and $f_{i\downarrow}$. The a_0^l can be chosen to be zero [18]. The dispersion for the fermions is readily calculated, and reads

$$E_f = \sqrt{\varepsilon_f^2 + \eta_f^2}, \tag{6.2.7}$$

where

$$\varepsilon_f = -\frac{3J}{4}(\cos k_x + \cos k_y)\chi, \;\; \eta_f = -\frac{3J}{4}(\cos k_x - \cos k_y)\Delta. \tag{6.2.8}$$

This dispersion shows Dirac quasiparticles at $(\frac{1}{2}\pi, \frac{1}{2}\pi)$, i.e., massless excitations with linear dispersion carrying a spin quantum number. As opposed to electrons, they do not carry a charge quantum number, remiscent of the fact that the charge degree of freedom is frozen out in the MI. In the context of high-T_c superconductors, similar kinds of quasiparticles have been measured and confirmed [51], and are known as nodal fermions, since these occur along the line describing the gap node. These nodal excitations are the condensed matter analogues of the quarks in QCD. In a similar way to the deconfined quarks and gluons, the nodal fermions are given real existence by the magic of the long-wavelength limit, in which mean field theory becomes exact. Beyond the confinement transition the spinons are glued back into the original Heisenberg spins by gauge fluctuations, in analogy to the confinement of quarks and gluons into the hadrons in QCD.

Out of the spinons one can make a spin liquid as well. Let us now consider the following mean-field Ansatz, which is known in the literature as the *staggered flux phase* [49, 50, 52] (SFP):

$$\begin{aligned} U_{\mathbf{i},\mathbf{i}+\hat{x}} &= -\chi\tau^3 - i(-)^I\Delta, \\ U_{\mathbf{i},\mathbf{i}+\hat{y}} &= -\chi\tau^3 + i(-)^I\Delta, \\ a_0^{1,2,3} &= 0. \end{aligned} \tag{6.2.9}$$

The quantity I is defined as $I = i_x + i_y$. The SFP is a spin liquid, describing spinons hopping around the plaquettes of the square copper-oxide lattice. It breaks translation symmetry, since the hopping fluxes

$$\Phi_{hop} = \frac{\pi}{4}\sum_{\text{plaquette}} \text{Arg}(U_{ij}^{11}) = \pm\frac{\Delta}{\chi}\pi \tag{6.2.10}$$

show a bipartite staggered pattern. By a Fourier transformation to momentum space, the dispersion is readily calculated. It turns out to be identical to the dSC-dispersion

$$E_f = \sqrt{\varepsilon_f^2 + \eta_f^2}. \tag{6.2.11}$$

This is surprising, since the SFP breaks translation symmetry, whereas the dSC does not. In the framework of classical Landau-Ginzburg-Wilson theory, this is impossible: in general, different symmetry broken states give rise to different excitations. What is going on here? In fact, the two states above are two sides of the same coin. This is seen after applying the following site-dependent transformation,

$$g_i = \exp\left(-i\frac{\pi}{4}(-)^I \tau^1\right) \tag{6.2.12}$$

by which the SFP is mapped to the dSC. In fact, all states connected to the dSC by a gauge transformation

$$g_i = \exp\left(-i\theta_i \tau^1\right) \tag{6.2.13}$$

are equivalent. This can be pictured nicely by the concept of the isospin sphere. An $SU(2)$-gauge group element $g_{\mathbf{i}}$ can be written as follows:

$$g_{\mathbf{i}} = \begin{pmatrix} z_{\mathbf{i}1} & -z_{\mathbf{i}2}^* \\ z_{\mathbf{i}2} & z_{\mathbf{i}1}^* \end{pmatrix} \tag{6.2.14}$$

where the complex numbers $z\mathbf{i}$ are parametrised by three angles, viz.,

$$z_{\mathbf{i}1} = e^{i\alpha_{\mathbf{i}}} e^{-i\frac{\phi_{\mathbf{i}}}{2}} \cos\frac{\theta_{\mathbf{i}}}{2}, z_{\mathbf{i}2} = e^{i\alpha_{\mathbf{i}}} e^{i\frac{\phi_{\mathbf{i}}}{2}} \sin\frac{\theta_{\mathbf{i}}}{2}. \tag{6.2.15}$$

The z's are grouped in the vector $z_{\mathbf{i}} = (z_{\mathbf{i}1}, z_{\mathbf{i}2})$.

The isospin vector $\mathbf{I_i}$ turns out to be a useful definition:

$$\mathbf{I_i} = z_{\mathbf{i}}^\dagger \tau z_{\mathbf{i}} = (\cos\phi_{\mathbf{i}} \sin\theta_{\mathbf{i}}, \sin\phi_{\mathbf{i}} \sin\theta_{\mathbf{i}}, \cos\theta_{\mathbf{i}}) \tag{6.2.16}$$

The angle θ can then be interpreted as the latitude on the isospin sphere, whereas the angle ϕ is the longitude, cf. Figure 6.1. The north and south pole of the sphere correspond to a staggered flux phase, with $A-B$ and $B-A$ staggering respectively, while the equator corresponds to the d-wave superconductor. For half filling, the rotations on the isospin

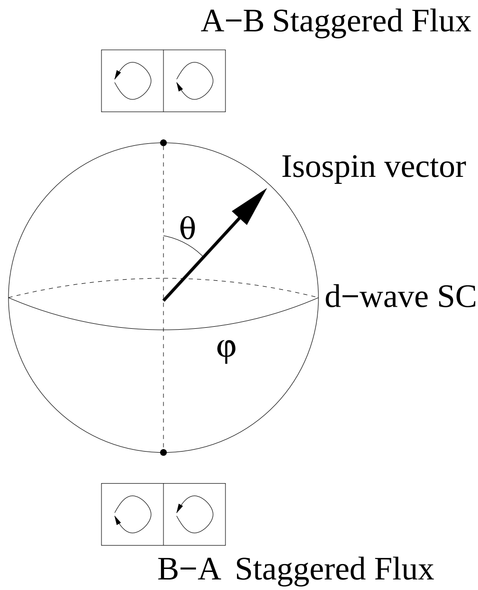

Figure 6.1: The isospin sphere. States on the north and south pole are staggered flux phases, whereas states on the equator are dSC. In between, the isospin vector describes a mix.

sphere correspond to pure gauge transformations, meaning that spinon flux phases and d-wave superconductors are gauge-equivalent.

We have thus demonstrated the fact that the SFP is not translationally invariant and must be viewed as a gauge artefact. This leads to the conclusion that different mean field states are able to describe the same physics. The reason is that in applying mean field theory, we neglect the fluctuating nature of the gauge fields. In turn, this implies that the projection onto the physical Hilbert space of one f-fermion per site, is not exact. Instead, one should project out the doubly occupied and empty sites of the mean field wave function $\Psi_{mean}^{(U_{ij},a_{0i}^l)}$ described by the Ansatz (U_{ij}, a_{0i}^l). This projection onto the singly occupied sites is achieved by the operator

$$\mathcal{P} \equiv \langle \text{vac}| \prod_i f_{i\alpha}. \tag{6.2.17}$$

This projection is equivalent to the projection onto the Hilbert space with an even number of ψ-fermions per site. Mean field states connected by the gauge transformation (6.2.5) lead to the same physical wave function:

$$\Psi_{phys}^{(U_{ij},a_{0i}^l)} = \mathcal{P}\Psi_{mean}^{(U_{ij},a_{0i}^l)} = \mathcal{P}\Psi_{mean}^{(U'_{ij},a'^l_{0i})}. \tag{6.2.18}$$

The gauge equivalent mean field states leading to the same physical wave function $\Psi_{phys}^{(U_{ij},a_{0i}^l)}$ constitute a new gauge group, which is not necessarily the same as the original gauge group, being $SU(2)$ for the case of the Mott insulator. This effective gauge group is called *projective symmetry group PSG*, since different states in the same new gauge group lead to the same physical state after projection. This new gauge volume will be an extension of the usual symmetry breaking classification schemes, including the notion of the Higgs phase in gauge theory. This construction of the projective symmetry group actually works for every gauge group, not just the group $SU(2)$ of relevance for the Mott insulator, although we restrict ourselves to the latter when discussing the Mott insulator.

On the other hand, it is also different from determining the phases in a pure gauge theory, as determined by the Wilson loop

$$W[\Gamma] = \langle \exp iq \oint_\Gamma a_\mu^l \tau^l dx_\mu \rangle, \tag{6.2.19}$$

where the loop Γ consists of paths of length T along the time direction and those of length R along the spatial direction. It is related to the gauge potential $V(R)$ between two static gauge charges $\pm q$ with opposite sign put at the distance R as

$$W[\Gamma] = \exp[-V(R)T]. \tag{6.2.20}$$

There are two types of behaviour of $W[\Gamma]$, i.e., the area law $W[\Gamma] \sim \exp(\alpha RT)$, or a perimeter law $W[\Gamma] \sim \exp(\beta(R+T))$, where α and β are some constants. In the first case, the potential $V(R)$ increases linearly with R, hence the gauge charges can never be free. This is the confining phase. On the other hand, in the perimeter phase, the potential grows with a lower power, thus allowing free charges. This phase is the deconfining state of the gauge theory.

For the mean field theories of the Mott insulator, a variable similar to the Wilson loop can be defined, with the important difference that it is constituted by the matter fields U_{ij} instead of the gauge fields. Let us define the $SU(2)$ flux operator $P(C_i)$ through a closed loop C_i with base point i by

$$P(C_i) = (iU_{ij})(iU_{jk})...(iU_{ki}) = \prod_{C_i} iU_{ij}. \tag{6.2.21}$$

Since $P(C_i)$ is an element of $SU(2)$, we may write it in terms of three $SU(2)$ fluxes Φ_i^l:

$$P(C_i) = e^{i\Phi_i^l \tau^l}. \qquad (6.2.22)$$

This flux operator will be important for determining which gauge fields can obtain mass.

Let us consider what these $SU(2)$ fluxes are for the dSC and SFP mean fields. Since the dSC Ansatz has the form $\chi_{ij}\tau^3 + \Delta_{ij}\tau^1$, Eq.(6.2.21) becomes for this case

$$P(C_i, \text{dSC}) = c_0(C_i)\tau^0 + c_2(C_i)\tau^2, \qquad (6.2.23)$$

where c_0 and c_2 are some constants, due to the fact that the vacuum expectation values in (6.2.6) are constant, implying that in contrast with the Wilson loop, these fluxes do not decay according to some perimeter or area law, but are constant instead. The question arises how a sensible classification can be made. To answer this, let us consider the $SU(2)$ flux variable $\Phi(C_i) = \Phi_l(C_i)\tau^l$. For the dSC case, the flux operator can be rewritten

$$P(C_i, \text{dSC}) = \exp(i\Phi(C_i)) = \exp(i\Phi_2(C_i)\tau^2), \qquad (6.2.24)$$

expressing the fact that the dSC matter fields lead to a flux Φ_2 in the τ^2 direction. On the other hand, the SFP, having the form $\chi_{ij}\tau^3 + \Delta_{ij}\tau^1$ has a flux pointing in the τ^3 direction:

$$P(C_i, \text{SFP}) = c_0'\tau^0 + c_3'\tau^3 \propto \exp i\Phi_3(C_i)\tau^3. \qquad (6.2.25)$$

Although the dSC leads to the same quasiparticle spectrum as the SFP after a gauge transformation, the fluxes are not the same. But since the flux operators are not gauge invariant quantities, by gauge transformations $g_i \in SU(2)$, these can be rotated

$$\Phi(C_i) \to g_i \Phi(C_i) g_i^\dagger. \qquad (6.2.26)$$

For example, the dSC flux can be rotated in the τ^3 direction by the transformation $g_i = \exp(i\frac{\pi}{4})\tau^1$. The class of mean fields for which the fluxes of loops originating from the *same* base point can be rotated to point in the same direction, is called the class of collinear Ansatzes. Fluxes for loops with different base points cannot be compared, since they can be rotated independently by $SU(2)$ gauge transformations. Hence it only makes sense to compare fluxes from loops with the same base point.

Let us now show how one can use the flux operator to determine which gauge fluctuations obtain mass. Having established that the SFP leads to a τ^3 flux, we can work backwards, and choose a gauge in which the mean field state acquires the form

$$\bar{U}_{ij} = ie^{i\phi_{ij}\tau^3}, \tag{6.2.27}$$

since this leads precisely to the expression (6.2.25). One immediately sees that (6.2.27) is not invariant under the whole $SU(2)$. Instead, the Ansatz is invariant under a subgroup $U(1)$, generated by $e^{i\theta_i\tau^3}$. This subgroup leaving the Ansatz invariant is called the invariant gauge group (IGG), and will be important in classifying PSGs.

Now we will demonstrate that the low lying gauge fluctuations for the collinear Ansatz are also described by a $U(1)$ gauge field, exploiting the notion of the flux operator (6.2.21).

The mean field energy (6.2.3) depends on $\bar{U}_{ij}$, which can gauge fluctuate: gauge equivalent states lead to the same physics. Put differently, the state $\bar{U}_{ij}$ gives the same physical state as the gauge transformed $\bar{U}_{ij}e^{ia^l_{ij}\tau^l}$. The requirement for the gauge fluctuations is that the mean field energy (6.2.3) is invariant under gauge transformations $e^{i\theta_i\tau^3}$:

$$H_{mean}(\bar{U}_{ij}e^{ia^l_{ij}\tau^l}) = H_{mean}(\bar{U}_{ij}e^{i\theta_i\tau^3}e^{ia^l_{ij}\tau^l}e^{-i\theta_j\tau^3}), \tag{6.2.28}$$

which for $a^{1,2}_{ij}$ reduces to

$$H_{mean}(\bar{U}_{ij}e^{ia^3_{ij}\tau^3}) = H_{mean}(\bar{U}_{ij}e^{i(a^3_{ij}+\theta_i-\theta_j)\tau^3}). \tag{6.2.29}$$

This expression means that a mass term a^3_{ij} is incompatible with Eq. (6.2.29), so the gauge field a^3_{ij} is massless. On the other hand, the other two fields $a^{1,2}_{ij}$ do obtain masses. This will be demonstrated by exploiting the flux operator (6.2.21). Let $P_A(i)$ be the $SU(2)$ flux through a loop A with base point i. Assuming that all gauge invariant terms which can appear do appear, H_{mean} will contain the term

$$H_{mean} = \mathrm{Tr}\left[P_A(i)iU_{i,i+\hat{x}}P_A(i+\hat{x})iU_{i+\hat{x},i}\right] + \dots \tag{6.2.30}$$

Let us now write $iU_{i,i+\hat{x}}$ as $\chi e^{i\phi_{ij}\tau^3}e^{ia^l_{\hat{x}}}\tau^l$, and note that $U_{i,i+\hat{x}} = U^\dagger_{i+\hat{x},i}$, cf. Eq.(6.2.2). Expanding to quadratic order in $a^l_{\hat{x}}$, a term

$$H_{mean} = -\frac{1}{2}\chi^2\mathrm{Tr}\left([P_A(i), a^l_{\hat{x}}\tau^l]\right) + \dots \tag{6.2.31}$$

is obtained. Now the flux operator shows its merit: since for the collinear Ansatz $P_A \propto \exp i\Phi_A^3(i)\tau^3$, mass terms for a_{ij}^1 and a_{ij}^2 are generated, such that at low energies a $U(1)$ gauge theory remains.

The above discussion can be summarised in a result which is general [18, 44]. The collinear Ansatz is invariant under $U(1)$ gauge transformations, which coincides with the gauge group describing the massless gauge fluctuations.

Although the $SU(2)$ gauge fluctuations are now restrained to a compact $U(1)$-theory, the confinement/deconfinement issue is still not settled, since Polyakov showed in 1977 [17] that a compact $U(1)$ theory in 2+1 dimensions is always in the confining phase. The argument is that the screened interactions between the gauge charges give rise to a finite creation probability for the instantons in the theory, destroying deconfinement. This can still be the case after integrating out the fermion fields, thereby obtaining an effective action for the gauge fields. For example, when the fermion fields are fully gapped, the effective interaction between the gauge charges after integrating out the fermion fields will be confining [44].

For the case of the dSC/SFP, however, the spinon matter fields show gapless points in the dispersion. It turns out that these Dirac quasiparticles give rise to logarithmic interactions between the topological excitations, rather than screened interactions[53, 54]. Then the question arises of whether there exists a Kosterlitz-Thouless-like phase transition in 3D. The first results predicted the absence of such a transition [54], implying that the monopoles, being liberated, always confine the spinons and holons together into electrons. This is similar to the $SU(3)$ theory of QCD, where the confining interactions between the gluons only allows us to detect hadrons and mesons, not the constituent quarks.

Numerical calculations by Sudbø and collaborators [55] show that such a transition exists, admitting liberation of the spinon fields, giving stability to the $U(1)$ spin liquid [56, 57, 58], as pursued by Zou, Baskaran and Anderson [59]. The ramification of those results is that the assumption of a deconfining phase in the $SU(2)$ gauge theory coupled to matter is justified, at least for collinear Ansatzes showing nodal quasiparticles. In turn, since confinement is caused by fluctuations in a_{0i}^l, the existence of a deconfined phase means that those fluctuations can be neglected, and the a_{0i}^l may be treated as mean fields.

In summary, we have explained how the miracle of deconfinement can happen: as soon as a mean field exists, the gauge symmetry is effectively lowered from $SU(2)$ to $U(1)$. The latter gauge theory can still be confining, but when the mean field state admits nodal quasiparticles, these are protected against gauge fluctuations in the effective theory, remaining after integration over the fermion fields.

In the next section, we will explain Xiao-Gang Wen's ideas about the deeper principle of quantum order underlying this interesting phenomenon, and how quantum order can be classified by the projective symmetry group mentioned before.

6.3 Classification of projective symmetry groups

In the previous section, it was shown how two seemingly different mean field states lead to the same physics. Both the SFP and the dSC lead to nodal quasiparticles, the condensed matter analogue of deconfined quarks. These quasiparticles were argued to gain physical existence, i.e., they not only exist on mean field level, but are preserved on projecting onto the physical Hilbert space of singly occupied sites of the parent Mott insulator. This preservation is interpreted as protection against gauge fluctuations. Projection onto the physical Hilbert space is accomplished by making the constraints (6.2.1) exact, i.e., by including all the gauge fluctuations. So if the nodal quasiparticles are protected against gauge fluctuations, these will be preserved after projection onto the physical state.

As was shown in the previous section, this protection is due to deconfinement. The deconfinement emerges by the combination of two reasons: the mean field states dSC and SFP lead to a $U(1)$ gauge theory, and the nodal excitation spectrum of the matter field leads to a deconfining gauge field-mediated interaction.

The combination of this invariance of mean field states and protection against gauge fluctuations leads to the idea that there is a ordering principle at work. For classical systems, order is classified by breaking of symmetry groups, leading to predictions of collective excitations (Goldstone bosons). As observed earlier, since dSC and SFP have different symmetries, classical order cannot be the protecting principle, the more so since gauge symmetries cannot be broken spontaneously [46]. This new kind of order at work, called quantum order, should be related to the gauge invariance of mean field states, which might have different symmetries. In the previous section we already showed that the translation symmetry breaking of the SFP is a gauge artefact. It was shown that the symmetry breaking can be absorbed by a gauge transformation (6.2.14) into the dSC Ansatz. As mentioned before, this observation will lead to the classification of quantum order by the projective symmetry group.

Let us elaborate more on this invariance by considering the following example. The SFP, which breaks translational symmetry breaking,

$$U_{i,i+\hat{x}} = -\chi\tau^3 - i(-)^I\Delta, U_{i,i+\hat{y}} = -\chi\tau^3 + i(-)^I\Delta \tag{6.3.1}$$

is changed under a parity transformation P_{xy} into

$$P_{xy}(U_{i,i+\hat{x}}) = -\chi\tau^3 - i(-)^I\Delta, P_{xy}(U_{i,i+\hat{y}}) = -\chi\tau^3 + i(-)^I\Delta. \tag{6.3.2}$$

This parity transformation can be annihilated by performing a gauge transformation $g_i = i(-)^I\tau^1$, since

$$g_i(U_{i,i+\hat{x}})g^\dagger_{i+\hat{x}} = [i(-)^I\tau^1](-\chi\tau^3 - i(-)^I\Delta)[-i(-)^{I+1}\tau^1] = -\chi\tau^3 + i(-)^I\Delta. \tag{6.3.3}$$

Hence the parity transformation P_{xy} is equivalent to $SU(2)$ gauge transformation $g_i = i(-)^I\tau^1$. The gauge equivalence of the SFP to the dSC can now be understood in a different way. Working out the gauge transformation connecting those,

$$g_i = \exp(-i\frac{\pi}{4}(-)^I\tau^1) = \frac{1}{2}\sqrt{2}(1 - i(-)^I\tau^1), \tag{6.3.4}$$

we see that this transformation corresponds to the application of a combination of the identity transformation and the parity operation P_{xy}. Apparently, it is this combination which gauges away the translation symmetry breaking of the SFP. Stated the other way around, the dSC is invariant under the combination of breaking translation symmetry towards the SFP, combined with the gauge transformation (6.2.14). This illustrates the idea that Ansatzes with different symmetries which are gauge equivalent, should be considered to have the same quantum order.

The above example gives the inspiration for the classification of quantum order. States with different symmetries are equivalent if they reside in the same gauge volume. In line with the discussion of the previous section, this gauge volume should coincide with the effective gauge theory as dictated by the flux operator loop determined by the mean field Ansatz. For the dSC/SFP case, we have already seen that the low-lying gauge fluctuations are $U(1)$, and the gauge symmetry connecting those states, is also a $U(1)$ transformation, since it only involves τ^1.

This can be formalised in the following way. Let us consider some mean field state described by some group element $U_{ij} \in G$, where G is the symmetry group of the microscopic Hamiltonian describing the system. For the Mott insulator, $G = SU(2)$, but the construction of the PSG works for any group. The projective symmetry group (PSG) is a combination of a gauge symmetry G_s in the gauge group G and an element s in spatial symmetry group SG. The projective symmetry group is referred to that way, since projection of mean field states residing in the same gauge volume onto the physical Hilbert space lead to the same physical state, according to (6.2.17).

This can be expressed mathematically by introducing the definitions

$$\begin{aligned} G_s(U_{ij}) &\equiv G_s(i)U_{ij}G_s^\dagger(j) \\ s(U_{ij}) &\equiv U_{s(i),s(j)}. \end{aligned} \tag{6.3.5}$$

The combination $G_s s$ is an element of the PSG if and only if the requirement

$$G_s s(U_{ij}) = U_{ij} \tag{6.3.6}$$

is met. Hence the elements in the PSG of a mean field state are characterised by pairs (G_s, s) with $G_s \in G$ and $s \in SG$, such that (6.3.6) holds. For any s, Eq. (6.3.6) can be understood as an equation for $G_s(i)$ and $G_s(j)$. In other words, if the SG of an Ansatz is given, one can solve for the group elements G_s, and the PSG is obtained. Unfortunately, for a fixed symmetry transformation s, there can be many gauge transformations G_s fulfilling (6.3.6), i.e. the mapping from G_s to s is not one-to-one.

In the solution of this problem, the IGG as determined by the flux operators (6.2.21) plays a special role, forming the key for the construction of the PSG. For the collinear Ansatzes like dSC/SFP, the IGG leaving the flux operator (6.2.21) invariant, turned out to be $U(1)$, which was also demonstrated to be the gauge theory describing gauge fluctuations. On the other hand, in some gauge, the collinear Ansatz can be written as a gauge group element invariant under the IGG, cf. (6.2.27). According to (6.3.6), the spatial symmetry connected with the elements of IGG, is simply the unit transformation. Then a theorem from mathematical group theory can be applied [60], saying that for each symmetry transformation s, the different possible choices of G_s satisfying (6.3.6) are related by a gauge transformation living in IGG. Put differently, if $G_s s \in PSG$, and

$G'_s s \in PSG$, then this is the case if and only if $G'_s = hG_s, h \in IGG$. This leads to the unique classification of the PSG, since each element in the PSG can now be written in terms of an element in IGG and an element from SG,

$$SG = PSG/IGG. \tag{6.3.7}$$

In this way, the quantum order of a mean field state is characterised by the effective gauge theory described by IGG, together with the symmetry transformations $s \in SG$, where the latter are translated into elements G_s of the gauge group by the requirement (6.3.6).

This construction is independent of the choice of the gauge group G. In particular, it works for the case at hand, the Mott insulator with $G = SU(2)$.

Let us now point out the three distinct classes of the $SU(2)$ mean field theory.

6.3.1 Collinear flux

In his classification of mean fields with gauge group $SU(2)$, Xiao-Gang Wen considers symmetric spin liquids, these being spin liquids obeying the lattice symmetries of translation, rotation, parity and time reversal. The dSC/SFP state obeys all of these, since these can all be absorbed in gauge transformations as follows:

$$\begin{aligned} G_{T_x}(i) &= G_{T_y}(i) = i(-)^I \tau^1, G_{P_{xy}}(i) = i(-)^I \tau^1 \\ G_{P_x}(i) &= G_{P_y}(i) = \tau^0, \qquad G_{IGG}(i) = e^{i\theta_i \tau^3}. \end{aligned} \tag{6.3.8}$$

The example of $G_{P_{xy}}$ was discussed in the previous section, and the other transformations are checked similarly. For completeness, the $U(1)$ elements of IGG are recorded as well.

Since the SFP/dSC state is classified as a collinear state by the flux operator (6.2.21), it could be shown that the low-lying gauge degrees of freedom are $U(1)$, explaining why the dSC/SFP is called a $U(1)$ spin liquid.

Although there are many $U(1)$ spin liquids possible, only the states related by rotations on the isospin sphere obey (6.3.8). Hence the IGG $U(1)$ together with (6.3.8) gives the unique classification of the quantum order of the SFP. On the other hand, Wen showed that there are at least 4 gauge inequivalent $U(1)$ spin liquids showing nodal fermions, whereas there are several quantum orders showing either gapless or a fully gapped dispersion. Finally, there is also a $U(1)$ quantum order showing gapless, but quadratic dispersion at a finite number of points in reciprocal space. Hence, because of the possibility of a gapped spectrum for some $U(1)$ spin liquids, not all these quantum orders are stable against gauge fluctuations, as discussed before.

6.3.2 Trivial flux

Next to the $U(1)$ spin liquids with collinear flux, there are $SU(2)$ mean field states showing the full $SU(2)$ as IGG. One such a state is a sibling of the SFP, characterised by its plaquette fluxes as in Eq.(6.2.10). An $SU(2)$ liquid is obtained by taking $\Delta = \chi$, as will be shown. The plaquette fluxes show a staggered pattern of fluxes $\pm\pi$. This justifies the name π-flux liquid (πFL) for the state

$$\begin{aligned} U_{i,i+\hat{x}} &= -\chi(\tau^3 + i(-)^I), \\ U_{i,i+\hat{y}} &= -\chi(\tau^3 - i(-)^I). \end{aligned} \tag{6.3.9}$$

It has the dispersion relation

$$E_k = \frac{1}{2}\sqrt{2}\chi\sqrt{\cos^2 k_x + \cos^2 k_y}, \tag{6.3.10}$$

also showing nodal Dirac quasiparticle dispersion.

Although the only difference with the SFP seems to be the equality of Δ and χ, the quantum order is different. After applying gauge transformations, the πFL is proportional to τ^0, the unit element in $SU(2)$,

$$\begin{aligned} U_{i,i+\hat{x}} &= i\chi, \\ U_{i,i+\hat{y}} &= i(-)^{i_x}\chi, \end{aligned} \tag{6.3.11}$$

Hence the flux operator is proportional to τ^0 as well, classifying it as a trivial flux state. Clearly, this quantum order has $IGG = SU(2)$. It can be shown that the symmetry group of this state is again the full lattice symmetry group, since these can all be absorbed by the gauge transformations

$$\begin{aligned} G_{T_x}(i) &= (-)^{i_x} G_{T_y}(i) = \tau^0,\ G_{P_{xy}}(i) = (-)^{i_x i_y}\tau^0 \\ (-)^{i_x}\quad G_{P_x}(i) &= (-)^{i_y} G_{P_y}(i) = \tau^0,\ \ G_{IGG}(i) = e^{i\theta_i^l \tau^l}, \end{aligned} \tag{6.3.12}$$

satisfying (6.3.6).

The different physics is obviously related to the fact that for $IGG = SU(2)$ all gauge fluctuations are massless, leading to confinement. In turn, this means that after projection onto the physical spin wave function, the nodal fermions will not survive. Hence, one cannot expect $SU(2)$ spin liquids to be physical.

6.3.3 Non-collinear flux

Finally, there is a class of spin liquids which is always protected against gauge fluctuations. This is the class having as IGG the discrete subgroup $\mathbb{Z}_2$ of $SU(2)$. This class of spin liquids are described by the mean field states carrying non-collinear flux.

An example is provided by a mean field state including longer links,

$$\begin{aligned} U_{i,i+\hat{x}} &= U_{i,i+\hat{y}} = -\chi\tau^3 \\ U_{i,i+\pm\hat{x}+\hat{y}} &= \Delta\tau^1 \pm \lambda\tau^2 \\ a_0^{2,3} &= 0,\ a_0^1 \neq 0. \end{aligned} \tag{6.3.13}$$

Let us demonstrate that this Ansatz shows a non-collinear flux through loops with the same base point. Take two triangular loops $(i, i+\hat{y}, i-\hat{x})$ and $(i, i+\hat{y}, i+\hat{x})$ through the same base point i. The corresponding flux operators are

$$\begin{aligned} U_{i,i+\hat{y}}U_{i+\hat{y},i-\hat{x}}U_{i-\hat{x},i} &= -\chi^2(\Delta\tau^1 + \lambda\tau^2) \\ U_{i,i+\hat{x}}U_{i+\hat{x},i+\hat{y}}U_{i+\hat{y},i} &= -\chi^2(\Delta\tau^1 - \lambda\tau^2). \end{aligned} \tag{6.3.14}$$

As opposed to the $U(1)$ spin liquid, the fluxes cannot be rotated towards the same direction, since they cannot be written as the exponent of a single Pauli matrix.

The state (6.3.13) can be shown to have a gap [18], hence this state is called the $\mathbb{Z}_2$-gapped state. This gap in the fermion spectrum is not important anymore to prevent confinement, as the gauge symmetry is discrete, making all gauge fluctuations massive.

6.4 Concluding remarks

The examples shown worked as follows. One starts out with some mean field state, figures out what the IGG of that particular state is, and then determines the elements G_s for every s in the symmetry group SG. To obtain *all* PSG's, it is not necessary to list all possibilities for the mean field states U_{ij}. This would become very cumbersome when one also includes longer links than just nearest neighbour links. Wen has shown that it is possible to obtain all PSG's just by studying the flux operator (6.2.21), under the assumption of some IGG. The flux operator is able to classify all gauge inequivalent choices of G_s for every $s \in SG$. This leads to the result that there are at most 196 $\mathbb{Z}_2$ spin liquids for $G = SU(2)$, whereas there is a countably infinite number of $U(1)$ spin liquids.

Chapter 7

The doped Mott insulator: lessons from the empty limit

In the previous chapter, we have extensively studied the Mott insulator at half filling. Since charge excitations have a large gap, due to Coulomb repulsion, we reformulated the Mott insulator in terms of a constrained Heisenberg Hamiltonian. We parametrised the theory in terms of fermionic degrees of freedom, transforming under local $SU(2)$, associated with the redundant fermions and their local particle-hole symmetry.

The reformulation in terms of these spinful fermions, known as spinons, put us in the position to describe spin liquid states. We discussed the idea of Xiao-Gang Wen, that all the mean-field spin liquid states can be classified within one framework, the projective symmetry group PSG. It basically stated that the possible mean field states group in gauge volumes associated with the gauge group. Furthermore, this PSG is a powerful mathematical framework to describe quantum order, i.e., the order associated with the various gauge equivalent groups of mean field states. The projective symmetry group explained why spin flux states (FL) and d-wave superconducting states (dSC) are indistinguishable states of matter for zero doping. They both support nodal fermionic excitations, an expression of the fact that they are gauge fixes describing the same quantum order.

We ended our discussion of the PSG with the announcement that for non-zero doping, the symmetry between dSC and FL is broken, singling out the dSC. This presupposes that the effects of doping can be incorporated in the $SU(2)$ gauge theory as well, a fact first realised by Lee and Wen [45]. This chapter will review carefully the Lee and Wen approach of describing doping in the framework of $SU(2)$ gauge theory. This will take some space, because we have to explain some novel results related to improving the original formulation due to Wen, Lee and coworkers [45].

In Chapter 8 we will continue discussing the breaking of $SU(2)$ symmetry, and what the properties of the resulting mean field theory are. It turns out that only with the lessons from this Chapter, it is possible to formulate the theory consistently for non-zero doping.

We start out this discussion with introducing a first-order approximation of a Heisenberg model with doping: the $t-J$-model. It is just a theorists' playground, but it does capture the essence of the physics of the doped Mott insulator.

Subsequently we show how one describes electronic degrees of freedom for the doped case, rather than the spin degrees of freedom in the Mott insulator. Indeed, charge degrees of freedom, holons, are also to be incorporated in the $SU(2)$ framework, by carefully taking into account that the physical Hilbert space has to consist of singly occupied or empty sites only, because the strong Coulomb interactions project out the doubly occupied sites. One lesson following from these projection constraints, imposed by $SU(2)$ gauge fields, is that when one wishes to do mean field theory, one needs all the constraints, and not just one. In this regard the original formulation seems to be flawed [45]. In Chapter 8, this will lead to the suprising conclusion that the superconducting order parameter does not have a pure d-wave gap structure, but a $d+s$-wave form. In this way, $SU(2)$ theory is the only explanation on the market which gives an immediate and transparent explanation of experiments detecting an s-wave admixture [22, 20].

We continue studying the extreme case of the empty limit: only holes, and no electrons. The full $SU(2)$ gauge theory is certainly able to do this, since it can describe the absence of a charge exactly. However, the mean-field theory becomes unphysical in this limit. Indeed, when a lot of vacancies are present, $SU(2)$ gauge fluctuations become eventually completely dominant, confining the spinons and holons into electrons. We show, however, that as an energy density functional, the mean field theory is a fair description, since it gives the correct energy in the empty limit.

We learn from the empty limit an important lesson regarding the way the holons have to be described within mean field theory: one has to assume that the bosons have a hard core. In Chapter 8, we will show that this leads to phase separation below certain dopings. This will denounce thoughts in the community, that slave theories and phase separation are mutually exclusive.

7.1 Slave boson formulation of the doped Mott insulator

As is widely accepted, the problem of high-temperature superconductivity, is the problem of doping a Mott insulator. The parent compounds of high-T_c are insulating, due to a large Coulomb repulsion. By removing electrons however , the charges get mobile, and it is believed that this physics is at the origin of the superconductivity. This forms the motivation to include the hopping of projected electrons $\tilde{c}_{i\sigma}$ in the original Heisenberg Hamiltonian:

$$H_{t-J} = \sum_{<ij>} J \left(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right) - \sum_{ij} t_{ij} \left(\tilde{c}^{\dagger}_{i\sigma} \tilde{c}_{j\sigma} + h.c. \right), \qquad (7.1.1)$$

where the hoppings t_{ij} are the wave function overlaps of electrons at sites i and j. Without loss of generality, we take $t_{ij} = t$ for nearest neighbour sites, and $t_{ij} = 0$ otherwise. The spin operators are still given in terms of the fermion operators,

$$\mathbf{S}_i^+ = \frac{1}{2}(\psi^{\dagger}_{1i}\psi^{\dagger}_{2i} - \psi^{\dagger}_{2i}\psi^{\dagger}_{1i}) \qquad (7.1.2)$$

$$\mathbf{S}_i^z = \frac{1}{2}(\psi^{\dagger}_i \psi_i - 1). \qquad (7.1.3)$$

Importantly, the Hilbert space of the $t - J$ Hamiltonian is formed from three states only: a spin up electron, $c^{\dagger}_{\uparrow} |\text{vac}\rangle$, a spin down electron $c^{\dagger}_{\downarrow} |\text{vac}\rangle$ and a vacancy $|\text{vac}\rangle$. Consequently, the electron operators in (7.1.1) are projected electron operators $\tilde{c}_{i\alpha} = c_{i\alpha}(1 - n_{i\bar{\alpha}})$, where $\bar{\alpha}$ denotes a spin opposite from α. The tilde is dropped from now on and the projection is kept implicit. Bear in mind that one should take care that all physics takes place in this projected Hilbert space!

Let us now describe these electrons in the $SU(2)$ gauge theory. As mentioned at previous occasions, in everyday life, one does not encounter

$SU(2)$ doublets: the deconfined spinons and holons. This means that the confined phase describing electrons, corresponds to a phase consisting of $SU(2)$ gauge singlets. To construct these singlets, introduce the $SU(2)$ doublet describing holons:

$$h_i = \begin{pmatrix} b_{1i} \\ b_{2i} \end{pmatrix}. \tag{7.1.4}$$

Then the appropriate $SU(2)$ singlet describing the projected electron is

$$\begin{aligned} c_{\uparrow i} &= \frac{1}{\sqrt{2}} h_i^\dagger \psi_i = \frac{1}{\sqrt{2}} (b_{1i}^\dagger f_{\uparrow i} + b_{2i}^\dagger f_{\downarrow i}^\dagger), & (7.1.5) \\ c_{\downarrow i} &= \frac{1}{\sqrt{2}} h_i^\dagger \overline{\psi}_i = \frac{1}{\sqrt{2}} (b_{1i}^\dagger f_{\downarrow i} - b_{2i}^\dagger f_{\uparrow i}^\dagger). & (7.1.6) \end{aligned}$$

Let us first demonstrate that this reparametrisation does indeed describe projected electrons. From the half-filled case, we know that a spin up electron should be described by a single up spinon, $f_{\uparrow i}^\dagger |\text{vac}\rangle$. Acting with a down electron creation operator should annihilate this sate,

$$(h_i^\dagger \psi_i)^\dagger f_{\downarrow i}^\dagger |\text{vac}\rangle = (b_{1i} f_{\uparrow i}^\dagger + b_{2i} f_{\downarrow i}) f_{\uparrow i}^\dagger |\text{vac}\rangle = 0.$$

This works since the bosons annihilate the vacuum. Eq. (7.1.5) as it stands, however, is not an operator equality, since the Hilbert space of the $SU(2)$ theory is larger. To arrive at the correct physics, we have to impose constraints to make the mapping to the states of the original $t-J$-Hamiltonian exact. In the spirit of the previous chapter, we again require that the physical states $|\text{phys}\rangle$ of the $t-J$ model are $SU(2)$ singlets,

$$\left(\psi_i^\dagger \tau^l \psi_i + h_i^\dagger \tau^l h_i \right) |\text{phys}\rangle = 0. \tag{7.1.7}$$

Let us now identify the states satisfying (7.1.7). We can borrow the spin up and spin down fermion states already identified in the half-filled case, $f_{\uparrow i}^\dagger |\text{vac}\rangle$ and $f_{\downarrow i}^\dagger |\text{vac}\rangle$. Since these do not involve holons, we only need to care about the ψ-part of the constraint, hence

$$\begin{aligned} \left(\psi_i^\dagger \tau^1 \psi_i + h_i^\dagger \tau^1 h_i \right) f_{\uparrow i}^\dagger |\text{vac}\rangle &= \left(\psi_i^\dagger \tau^1 \psi_i \right) f_{\uparrow i}^\dagger |\text{vac}\rangle \\ &= (f_{\uparrow i}^\dagger f_{\downarrow i}^\dagger + f_{\downarrow i} f_{\uparrow i}) f_{\uparrow i}^\dagger |\text{vac}\rangle = 0, \\ \left(\psi_i^\dagger \tau^3 \psi_i + h_i^\dagger \tau^3 h_i \right) f_{\uparrow i}^\dagger |\text{vac}\rangle &= \left(\psi_i^\dagger \tau^3 \psi_i \right) f_{\uparrow i}^\dagger |\text{vac}\rangle \\ &= (f_{\uparrow i}^\dagger f_{\downarrow i}^\dagger + f_{\downarrow i} f_{\uparrow i}) f_{\uparrow i}^\dagger |\text{vac}\rangle = 0, \end{aligned}$$

and similarly for τ^2, and for $f^\dagger_{\downarrow i}\,|\text{vac}\rangle$. Hence, the $f^\dagger_{\sigma i}$ are indeed the desired gauge singlets. But how should we now incorporate the holes? As a first attempt, let us try $|\text{vac}\rangle$. This fails, since an empty site is turned into a pair singlet site:

$$\left(\psi_i^\dagger \tau^1 \psi_i + h_i^\dagger \tau^1 h_i\right)|\text{vac}\rangle = f^\dagger_{\uparrow i} f^\dagger_{\downarrow i}\,|\text{vac}\rangle \neq 0, \tag{7.1.8}$$

Conversely, a doubly occupied site is turned into an empty site,

$$\left(\psi_i^\dagger \tau^1 \psi_i + h_i^\dagger \tau^1 h_i\right) f^\dagger_{\downarrow i} f^\dagger_{\uparrow i}\,|\text{vac}\rangle = -\,|\text{vac}\rangle\,. \tag{7.1.9}$$

This guides us toward the correct answer of Wen and Lee [45]: let us accompany an empty site with a b_1 boson, and a doubly occupied site with a b_2 boson, so that a hole in the $t-J$ model is represented by

$$|0\rangle_i = \frac{1}{\sqrt{2}}\left(b^\dagger_{1i} + b^\dagger_{2i} f^\dagger_{\downarrow i} f^\dagger_{\uparrow i}\right)|\text{vac}\rangle_i\,. \tag{7.1.10}$$

Indeed, it is easy to check that (7.1.10) *does* satisfy all the constraints (7.1.7). Observe that we needed all the three constraints to arrive at this expression. This last fact is overlooked in the original Wen and Lee formulation, and will turn out to be very important. In the expression for the hole, $SU(2)$ gauge theory already reveals some of its powers: it captures the fact associated with the particle-hole symmetry intrinsic to spin that the singletness of pure vacuum should be treated on precisely the same footing as the spin singletness of either the empty or doubly occupied spinon configuration.

7.2 The empty limit: the importance of being hard core

In the previous section, we have described how one can introduce charge degrees of freedom in the doped Mott insulator. By introducing a holon $SU(2)$ doublet, it was possible to recover the physical electron. It was also shown how one can make a mapping from the slave boson operator states to the Hilbert space of the $t-J$ model, by including the exact constraints (7.1.7). Solving for exact constraints is however extremely difficult. To do this, one needs to introduce gauge fields, that have to be treated exactly, in principle. In order to make progress, we are going to put forward a mean field theory, and treat the constraints on a mean field level. For the mean-field theory, we have to construct a trial wave function, incorporating bosons and spinons. The best inspiration for such a wave function, follows from inspecting the empty limit, as counterintuitive as it may seem. In a sense, it is unphysical to treat the state with doping $x = 1$ in the gauge theory way, since gauge fluctuations already restore confinement into electrons at a much lower doping level. It turns out, however, that the gauge theory does not only give the correct energy in the empty limit, but it also gives a guide line of how to construct the low-doping mean field wave function.

The exact wave function describing the empty limit is simple,

$$|0\rangle = \prod_i \left(b^\dagger_{i1} + b^\dagger_{i2} f^\dagger_{\downarrow i} f^\dagger_{\uparrow i} \right) |\mathrm{vac}\rangle \,. \tag{7.2.1}$$

Deconfinement or spin-charge separation implies that the system loses its knowledge about the three particle correlation $b^\dagger_{i2} f^\dagger_{\downarrow i} f^\dagger_{\uparrow i}$ and the best one can do is to look for a holon-spinon product wave function. The best choice is obviously

$$|0\rangle_{MF} = \prod_i \left[\frac{1}{2} \left(b^\dagger_{i1} + b^\dagger_{i2} \right) \left(1 + f^\dagger_{\downarrow i} f^\dagger_{\uparrow i} \right) \right] |\mathrm{vac}\rangle \,. \tag{7.2.2}$$

This wave function still has to satisfy the mean-field version of the constraint (7.1.7),

$$\left\langle \psi^\dagger_i \tau^l \psi_i + h^\dagger_i \tau^l h_i \right\rangle = 0, \tag{7.2.3}$$

where the brackets in this case stand for the expectation value relative to the state $|0\rangle_{MF}$. This brings us to the main point: the constraints are only satisfied when the bosons have the hard core. Indeed, the mean field wave function (7.2.2) obeys

$$\langle 0| \psi^\dagger_i \tau^3 \psi_i |0\rangle_{MF} + \langle 0| h^\dagger_i \tau^3 h_i |0\rangle_{MF} = 0 + 0 = 0. \tag{7.2.4}$$

If we were to take soft-core bosons, the state $(b^\dagger_{1i}{}^2 + (b^\dagger_{1i}) |\mathrm{vac}\rangle$ would be possible. It does not satisfy the constraints, however, since then

$$\langle 0| \psi^\dagger_i \tau^3 \psi_i |0\rangle_{MF} + \langle 0| h^\dagger_i \tau^3 h_i |0\rangle_{MF} = 0 + 1 \neq 0. \tag{7.2.5}$$

So, if we are to take the Hilbert space constraints seriously, we need to accept that the holons have an infinite hard core. This is consistent with the fact that the holons carry electric charge. Since the Coulomb repulsion is taken infinite to arrive at the $t - J$ model in the first place, this means that there can be at most one b-boson per site.

The reason to stress this hard-core nature of the bosons, is that in the original formulation [45], the bosons were taken to be non-interacting. The empty limit exercise, being transparent in this regard, shows that this is inconsistent for appreciable dopings. This is also illustrated by the fact that the hard-core bosons give the correct energy in the empty limit. Indeed, let us first show that with the original projected electrons, the exact energy of the $t - J$ Hamiltonian (7.1.1) is zero in this limit. The empty state is described in the projected electron formulation simply by $|\text{vac}\rangle$, implying vanishing n_i on this state. Since the spin operator $\mathbf{S}_i^l$ is given by $c_{i\alpha}^\dagger \frac{1}{2}\tau_{\alpha\beta}^l c_{i\beta}$, the energy of the Heisenberg part vanishes. Since there are no electrons around, the hopping part vanishes as well, and we conclude that the total energy vanishes.

Let us now demonstrate that the Ansatz (7.2.2) yields the same result. Firstly, the Heisenberg term vanishes. The only component of spin that could contribute is the $l = 3$ component, since the others vanish when acting on $\left(1 + f_{\downarrow i}^\dagger f_{\uparrow i}^\dagger\right)|\text{vac}\rangle$. However, since $\mathbf{S}_i^3$ is the number of up spins minus the number of down spins, it vanishes as well. Furthermore, since the number operator n_i in the slave boson representation reads $c_{i\alpha}^\dagger c_{i\alpha} = \frac{1}{2}\psi_i^\dagger h_i h_i^\dagger \psi_i$, it also gives zero contribution, because of the hard core condition. Similarly, the hoppings vanish for the same reason.

This would not be the case if the bosons were assumed to have no hard core, since for weakly interacting bosons there is no restriction on the hoppings. This is inconsistent with the $t - J$ model, as hopping is only allowed between occupied and empty sites. In conclusion, the hard core condition is a sufficient condition for the empty state to have the correct energy in the empty limit.

So far, we have derived an exact expression for the hole creation operator in the $SU(2)$ gauge theory, taking seriously all three constraints. We considered the empty limit next, since it is easy to construct the mean field theory in this case. Treating the constraints correctly, we arrived at the conclusion that the bosons should be treated as having a hard core. In the next section, we will show how our mean field Ansatz (7.2.2) generalises to mean field wave functions describing intermediate dopings.

7.3 $SU(2)$ mean field theory

In the previous section, we showed that a correct description of the empty limit requires that the bosons should be treated as hard-core. The empty limit considerations leading us to that conclusion, turns out to be very useful to find out the structure of the mean field wave function at intermediate dopings. In fact, hard-core bosons are just like XY spins, and the straightforward generalisation of (7.2.2) becomes obvious,

$$|\Psi_0\rangle_{holons} = \prod_i \left(\alpha_i + \beta_i(u_i b_{i1}^\dagger + v_i b_{i2}^\dagger)\right)|\text{vac}\rangle\,, \tag{7.3.1}$$

where the complex numbers α and β obey the normalisation condition $|\alpha|^2 + |\beta|^2 = 1$.

To actually calculate matters, we have to derive the slave boson version of the Hamiltonian (7.1.1), with the decomposition (7.1.5). To decouple terms quartic in the spinons, we use the spin liquid Ansatzes from the previous Chapter 6, namely

$$\chi_{ij} = \langle f_{i\uparrow}^\dagger f_{j\uparrow} + f_{i\downarrow}^\dagger f_{j\downarrow}\rangle \tag{7.3.2}$$

$$\Delta_{ij} = \langle f_{i\uparrow} f_{j\downarrow} - f_{i\downarrow} f_{j\uparrow}\rangle. \tag{7.3.3}$$

In order to impose the three constraints (7.2.3), we need to incorporate the Lagrange multipliers a_{0i}^l into the mean field Hamiltonian, just as in the half filled case in Chapter 6. The difference is that these fields in the doped case also couple to the bosons describing doping. Bearing these remarks in mind, it is a straightforward exercise to derive the mean field Hamiltonian

$$\begin{aligned} H_{mf} &= -\mu\sum_i h_i^\dagger h_i - \sum_i a_{0i}^l\left(\frac{1}{2}\psi_{\alpha i}^\dagger \tau^l \psi_{\alpha i} + h_i^\dagger \tau^l h_i\right) \\ &+ \sum_{<ij>} \frac{3J}{8}\left(|\chi_{ij}|^2 + |\Delta_{ij}|^2 + \psi_i^\dagger U_{ij}\psi_j + h.c.\right) \\ &+ \sum_{<ij>} t\left(h_i^\dagger U_{ij} h_j + h.c.\right), \end{aligned} \tag{7.3.4}$$

where U_{ij} was already defined in (6.2.2), while the holons are still exact.

Of course, the Hamiltonian (6.2.3) is manifestly $SU(2)$ invariant under the transformation

$$\psi_i \to g_i\psi_i,\quad h_i \to g_i h_i,\quad U_{ij} \to \tilde{U}_{ij} = g_i U_{ij} g_j^\dagger,\quad a_{i0}^l\tau^l \to \tilde{a}_{0i}^l \tau^l = g_i a_{i0}^l \tau^l g_j^\dagger. \tag{7.3.5}$$

To already harvest some results from our considerations, let us consider the saddle point Lagrange multiplier equations

$$\frac{\partial}{\partial a_0^l} \langle H_{mf} \rangle = 0, \quad l = 1, 2, 3. \tag{7.3.6}$$

These are precisely the mean field constraint equations (7.2.3):

$$\left\langle f_{\uparrow i}^\dagger f_{\downarrow i}^\dagger + f_{\downarrow i} f_{\uparrow i} \right\rangle = \left\langle b_{1i}^\dagger b_{2i} + b_{2i}^\dagger b_{1i} \right\rangle \tag{7.3.7}$$

$$-i \left\langle f_{\uparrow i}^\dagger f_{\downarrow i}^\dagger - f_{\downarrow i} f_{\uparrow i} \right\rangle = -i \left\langle b_{1i}^\dagger b_{2i} - b_{2i}^\dagger b_{1i} \right\rangle \tag{7.3.8}$$

$$\left\langle f_{\alpha i}^\dagger f_{\alpha i} - 1 \right\rangle = \left\langle b_{2i}^\dagger b_{2i} - b_{1i}^\dagger b_{1i} \right\rangle . \tag{7.3.9}$$

These constraint equations already convey an important message. The third equation tells us something about the deviation from half-filling, which was an important point already made by Lee, Wen and Nagaosa [44]. As soon as the average fermion occupation number deviates from unity, i.e., deviates from half filling, there is a difference between b_1 bosons and b_2 bosons. In plain physics language: as soon as Fermi pockets form, the difference between empty sites and spinon pair singlets becomes physical.

The first two equations acquire a novel interpretation. For non-zero dopings, the holon expectation values are non-zero as well. However, looking at the left-hand side of the equations, one needs to conclude that a superfluid order parameter appears with an s-wave structure. In other words, taking seriously all constraint equations, and having convinced oneself that doping must be described by a superposition of both empty and doubly occupied sites, one has to face an extra order parameter with a superconducting s-wave symmetry. Rephrased in physical language: within the framework of $SU(2)$ theory, doping induces s-wave pairing.

One could argue that there are some left-over degrees of freedom, so that one could gauge away the s-wave component. This is not the case, however. Let us exploit the isospin representation, introduced in Chapter 6. Using the isospin angles φ and θ, cf. Fig 6.1, we parametrise the holon wave function (7.3.1) by $u_i = \cos(\frac{\theta_i}{2})$ and $v_i = \sin(\frac{\theta_i}{2})$. Further, choose $\beta_i \to \beta_i e^{i\varphi_i}$ such that β_i is real. This parametrisation is instructive, since $\theta_i = \frac{\pi}{2}$ makes the expectation values for b_1 with vacancies indistinguishable from b_2 with a spinon pair, reproducing the particle-hole symmetric empty state (7.2.2). Moreover, this corroborates the point that the equator on the $SU(2)$-isospin sphere (i.e.,$\theta_i = \frac{\pi}{2}$) corresponds to the particle-hole symmetric d-wave superconductor. Calculating the expectation values explicitly, the equations (7.2.3) become

$$\left\langle f^\dagger_{\uparrow i} f^\dagger_{\downarrow i} + f_{\downarrow i} f_{\uparrow i} \right\rangle = |\beta_{0i}|^2 \sin(\theta_i)\cos(\varphi_i) \tag{7.3.10}$$

$$\left\langle f^\dagger_{\uparrow i} f^\dagger_{\downarrow i} - f_{\downarrow i} f_{\uparrow i} \right\rangle = |\beta_{0i}|^2 \sin(\theta_i)\sin(\varphi_i) \tag{7.3.11}$$

$$\left\langle f^\dagger_{\alpha i} f_{\alpha i} - 1 \right\rangle = |\beta_{0i}|^2 \cos(\theta_i). \tag{7.3.12}$$

On the one hand, this illustrates once again that equator states are particle-hole symmetric, cf. Eq.(7.3.12). Secondly, and more importantly, we see that there is no way to gauge away the s-wave component. In other words, as soon as there is a superconducting order parameter ($\sin(\theta) \neq 0$), there is an s-wave component linearly increasing with doping x:

$$\Delta_s = \frac{1}{2} x \sin(\theta). \tag{7.3.13}$$

The only freedom is to choose its phase to be real by choosing $\varphi_i = 0$, implying zero a^2_{0i} [18]. This is a first result of our empty-limit exercise, which at first sight looks trivial. Conversely, the first equation tells us that we cannot neglect the Lagrange multiplier a^1_{01}, which accounts for the s-wave admixture. This has been ignored in the original formulations of the mean field theory [44, 45, 61]. This flaw leads to a disaster, as we will show in the next section.

7.4 The empty limit in mean-field theory

One could wonder if it is really wrong to leave out the first constraint. Probably one remains very closely to the "correct" mean field state when ignoring it? This is not the case: it leads to nonsensical results. The bright side is the ease by which the constraint a^1_{0i} is incorporated. Let us fix the holon density $\left\langle h^\dagger_i h_i \right\rangle = 1$, and choose the Hubbard-Stratonovich fields to be homogeneous, $\Delta_{ij} = \Delta, \chi_{ij} = \chi$. Since the empty state makes no distinction between empty and doubly occupied sites, we have $\theta_i = \theta = \frac{1}{2}\pi$. Inserting these assumptions into (7.3.4), we obtain an energy density functional E_{mf} for the empty limit.

Let us first ignore a_{01}^1. Then the mean field equations are

$$\frac{\partial E_{mf}}{\partial \chi} = 2\chi = \sum_k \frac{\chi(\cos k_x + \cos k_y)^2}{E_k}, \tag{7.4.1}$$

$$\frac{\partial E_{mf}}{\partial \Delta} = 2\Delta = \sum_k \frac{\Delta(\cos k_x - \cos k_y - a_0^1)(\cos k_x - \cos k_y)}{E_k} \tag{7.4.2}$$

$$a_0^1 = 0. \tag{7.4.3}$$

where the dispersion E_k is given by

$$E_k = \sqrt{(\chi_k - a_0^3)^2 + (\Delta_k - a_0^1)^2},$$
$$\chi_k = -\frac{3J}{4}(\cos k_x + \cos k_y)\chi,$$
$$\Delta_k = -\frac{3J}{4}(\cos k_x - \cos k_y)\Delta. \tag{7.4.4}$$

In the empty limit, the holons cannot move, so there are no mean field equations and dispersions governing those. The above mean field equations can be solved numerically to yield the unphysical result $\chi = \Delta = \frac{\sqrt{2}}{4}$, identical to the result for half filling. But this is clearly nonsense: the empty limit is neither a spin liquid nor a superconductor. Also, since Δ and χ are non-zero, the total energy will be nonzero, in flagrant contrast with the correct result being zero, as pointed out earlier.

Taking a_0^1 into account, however, the above mean field equations are extended with the saddle point equation for a_0^1,

$$1 = \sum_k \frac{(\Delta(\cos k_x - \cos k_y) - a_0^1)}{E_k}, \tag{7.4.5}$$

where the number 1 is the boson density. Solving the new system of equations numerically, we obtain the correct result $\chi = \Delta = 0$ and $a_0^1 = \frac{1}{2}$. Therefore, the Lagrange multiplier is of central importance, and the mean fields vanish, as they should. Substituting this solution in the Hamiltonian (7.3.4), we recover the correct energy for the empty limit. In other words, things go dramatically wrong if a_0^1 is ignored. In the last section we will show that our mean field theory is performing well as an energy density functional at intermediate dopings. To calculate dynamic properties, one has to be careful, because of the confinement issues we already mentioned.

7.5 Dynamical properties of the empty limit

It is interesting to consider the dynamic properties following from the slave boson theory in the empty limit. It turns out that although this theory is a good energy density functional, it is less trustworthy with regard to dynamical properties revealed through the propagators. This is of course due to the mean field treatment ignoring fluctuations of the gauge fields. Indeed, the Lagrange multipliers a^l_{0i} should be given dynamics, causing fluctuations confinement of the spinons and holons to electrons. Since we ignored the fluctuations, we can not expect the mean field theory to describe the physical electron of the empty limit.

The starting point for the study of electron dynamics is the single electron propagator

$$\begin{aligned} G(x,y;t-t') &= \langle T(c_{x\uparrow}(t)c^\dagger_{y\uparrow}(t'))\rangle \qquad (7.5.1) \\ &= \langle c_{x\uparrow}e^{-i\hat{H}(t-t')/\hbar}c^\dagger_{y\uparrow}\rangle\Theta(t-t')e^{iE_0(t-t')/\hbar} \\ &\quad -\langle\langle c^\dagger_{y\uparrow}e^{-i\hat{H}(t'-t)/\hbar}c_{x\uparrow}\rangle\Theta(t'-t)e^{-iE_0(t-t')}. \end{aligned}$$

Here the $c_{i\sigma}$ again describe projected electron operators. Since in the empty limit there are by definition no electrons in the vacuum,

$$G(x,y;t-t') = \sum_{kk'} e^{ik'x-iky}\langle c_{k'\uparrow}e^{-i\hat{H}(t-t')/\hbar}c^\dagger_{k\uparrow}\rangle\Theta(t-t')e^{iE_0(t-t')/\hbar}. \qquad (7.5.2)$$

Let us first show that for the exact expression (7.2.1) describing the empty limit of the $t-J$ model, one obtains a free particle dispersion.

We need to know the time evolution operator $e^{-i\hat{H}(t-t')/\hbar}$. The Hamiltonian operator has no Heisenberg part for projected electrons, and it also vanishes on the state $c^\dagger_{x\uparrow}|0\rangle$, where $|0\rangle$ is the wave function (7.2.1). So we only need to calculate the effect of $H_t = -t\sum_{ij}\psi^\dagger_i h_i h^\dagger_j \psi_j$ on

$$c^\dagger_{x\uparrow}|\text{empty}\rangle = \frac{1}{V}\sum_m e^{imx}(b_{1m}f^\dagger_{\uparrow m} + b_{2m}f_{\downarrow m})|0\rangle. \qquad (7.5.3)$$

In Fourier space, the result is

$$\langle c_{k'\uparrow}H_t c^\dagger_{k\uparrow}\rangle = -2t\delta_{kk'}(\cos k_x + \cos k_y) \equiv \varepsilon_k\delta_{kk'}. \qquad (7.5.4)$$

Including the chemical potential and using a contour integral expression for $\Theta(t-t')$, we conclude that the propagator is

$$G(k,\omega) = \frac{1}{\hbar\omega - (\varepsilon_k - \mu) + i\eta}, \qquad (7.5.5)$$

the correct result for the propagator of a free particle. This means that the wave function (7.2.1) encodes the right physics.

But what to expect from the mean field theory? Without a_1^0, one obtains the expected errors, namely a d-wave dispersion. Repairing this with a nonzero a_0^1 leads, however, to both bad news and good news. The bad news is that one does not obtain the desired free particle dispersion, but the good news is that the dispersionless spectrum makes possible a vanishing of the ground state energy.

Let us consider the expectation values relative to the mean field wave function (7.2.2),

$$\langle 0|(b_{1m}^\dagger f_{\uparrow m} + b_{2m}^\dagger f_{\downarrow m}^\dagger) e^{-iH(t-t)'/\hbar}(b_{1n}^\dagger f_{\uparrow n}^\dagger + b_{2n}^\dagger f_{\downarrow n})|0\rangle_{MF} \tag{7.5.6}$$

Defining $\Delta t = (t-t)'/\hbar$, the correlator (7.5.6) splits in a boson and a fermion part,

$$(7.5.6) = \langle b_{1m}^\dagger e^{-iH_{bos}\Delta t} b_{1n}\rangle\langle f_{\uparrow m} e^{-iH_{fer}\Delta t} f_{\uparrow n}^\dagger\rangle \tag{7.5.7}$$

where $\langle . \rangle$ are expectation values with respect to (7.2.2). Now we make a Bogoliubov transform to diagonalise the mean field Hamiltonian (7.3.4),

$$\begin{aligned} f_{q\uparrow} &= u_q \gamma_{q0} + v_q \gamma_{q1}^\dagger & (7.5.8) \\ f_{q\downarrow}^\dagger &= -v_q \gamma_{q0} + u_q \gamma_{q1}^\dagger, & (7.5.9) \end{aligned}$$

where the Bogoliubons $\gamma_{q\sigma}$ are supposed to annihilate the mean field ground state, $\gamma_{q\sigma}|0\rangle_{MF} = 0$, while they have an energy E_q. It follows directly that

$$\langle f_{\uparrow a} e^{-i\Delta t \sum_s E_s(\gamma_{s0}^\dagger \gamma_{s0} + \gamma_{s1}\gamma_{s1}^\dagger)} f_{\uparrow q'}^\dagger\rangle = e^{i\Delta \sum_s E_s} \delta_{qq'} e^{-i\Delta t E_q} u_q^2. \tag{7.5.10}$$

For the bosons a free particle dispersion is obtained, just as in the exact case, except that the hoppings have been renormalised to become $t\chi$: $E_p^b = -2t\chi(\cos p_x + \cos p_y) - \mu$. Using the mean field solution without a_0^1, we have $E_q^f = \sqrt{\chi_q^2 + \Delta_q^2}$. Including the correlator $< f^\dagger f >$ as well, the result is

$$G(k, k'; \omega) = \delta_{kk'} \sum_{qp} \frac{(u_q - v_q)^2}{\hbar\omega - (E_q^f - E_p^b + i\eta)}, \tag{7.5.11}$$

with $(u_q - v_q)^2 = 1 - \frac{\Delta_q}{E_q^f}$. This expression is similar to the result of Lee, Wen, Nagaosa and Ng. This is unphysical, since the free electron on the square lattice obeys the dispersion associated with the d-wave superconductor.

Including a_0^1 does improve matters. Following the steps above, but now with $\chi = \Delta = 0$, and $a_0^1 = \frac{1}{2}$, the dispersions are modified to become $E_q^f = -a_0^1$ and $E_p^b = -\mu - a_0^1$. The coherence factors are $v_q = -u_q = -\frac{1}{\sqrt{2}}$. The spinon dispersion becomes

$$G(k, k'; \omega) = \delta_{kk'} \frac{1}{\hbar\omega - (\mu + 1) + i\eta}. \tag{7.5.12}$$

Since the bosons form a Bose Mott insulator, there are no low lying states associated with those and, henceforth, the spinon propagator corresponds directly with the electron propagator. Unfortunately, it does not yield the correct free electron dispersion result. The reason for this is that in the mean field theory the gauge fluctuations, causing confinement, are completely ignored by construction. Although the mean field energy is insensitive to this, the electron propagator surely is not. Indeed, propagators probe systems directly if meat-and-potato electrons are present. We, however, ignored these fluctuations, thus excluding the existence of those electrons. Hence there are good reasons that the theory should fail to describe the excitations of the empty limit.

The conclusion of this chapter is that there is some bad news, and quite some good news. The bad news is that by treating the gauge fields as non-dynamical mean field Lagrange multipliers, we ignored confinement issues. Consequently, when one probes our slave boson gauge theory with real electrons one does not obtain the correct dynamics - not for high dopings, at least.

The good news is that it is possible to include doping into the $SU(2)$ gauge theory. In paying much attention to the projection constraints, one obtains the correct description of a hole in terms of the hard-core bosons. This description leads automatically to the best mean field description: it gives the correct energy, even in the confining empty limit. This illustrates our earlier claim that we may regard the mean field theory as a good "density functional". As an important consequence of our mean field description together with the projection constraints, one inevitably obtains the conclusion that the holons should be treated as hard core, reminding of the infinite Coulomb repulsion needed to arrive at the $t-J$ model. The second important point was that in the mean field theory, one needs to consider all the three constraints, and not just one of them. This leads to a second striking conclusion: a d-wave superconducting state *needs* to have an s-wave admixture in its order parameter structure!

As academic as our empty limit discussion might seem, the fruit which we are going to harvest in the next chapter, cannot be tasted if we do not convince the reader of the inevitability of these matters, the motivation for this chapter. Indeed, let us now show what the beautiful ramifications of the lessons from the empty limit are.

Chapter 8

The phase separated $d+s$-wave superconductor

In the previous chapter, we have discussed how one should describe doping within the context of $SU(2)$ gauge theory. By studying this carefully, we convinced ourselves of two important lessons. The first one is that a hole in the $t-J$ model is described in the $SU(2)$ theory by a superposition of a vacancy and a spin pair singlet state. This implied that doping induces s-wave pairing. The second message is that the holons describing doping should be hard core bosons, instead of gaseous, non-interacting particles employed by Wen and Lee. This hard-core is necessary condition in order to account for the fact that there can be at most one charge per site, as has been clear from very the beginning.

We arrived at these conclusions by studying the seemingly irrelevant limit of no electrons. The first lesson learnt from that trivial exercise was that it taught us how to construct a mean field theory describing intermediate doping. In this chapter, we show that the pay-off is already considerable for the low doping regime of interest for the high-T_c superconductors, being between 5% and 30 %. As it turns out, the s-wave nature induced by the holons will affect the order parameter structure in the superconducting doping regime.

This interferes in an interesting way with the empirical developments in high-T_c superconductivity. The $SU(2)$ mean field theory predicts a

$d + s$ order parameter. There is strong experimental evidence from c-axis tunneling [20], [21] and Raman scattering [22, 62] that in Bi2212 there is an s-wave component in the gap, which is in largeness comparable [21] to our prediction, and grows with doping [22], also in accord with our prediction. Further, π-phase shift experiments for YBCO point out that the s-wave component therein cannot be fully explained by the orthorombicity of the crystal [63, 64]. As far as we are aware, $SU(2)$ gauge theory, in our formulation, is the only theory explaining these results in at least an elegant way.

It appears that the above experimental findings are largely ignored because all existing mechanism theories predict either a d-wave or an s-wave, and the $SU(2)$ gauge theory is stand-only with regard to its insistence on a $d + s$-symmetry. In the narrow context of slave-like theories, Ogata and coworkers excluded $d + s$ in the related context of Gutzwiller projected wave function Ansatzes [65, 66].

The second claim is somewhat of a sociological surprise: it is generally assumed that slave theories cannot explain phase separation phenomena. We will demonstrate that our mean field theory accounts for phase separation. It turns out that the hard-core nature of the bosons is fully responsible for this. The predicted compressibility and critical doping are in accord with chemical potential shift measurements [19]. These phase separation tendencies of the $SU(2)$ gauge theory puts the door ajar for more intricate phenomena, like the stripe order [67, 68, 69] that has been observed in experiments [70, 71, 72, 73, 74].

Our results are summarised in the mean-field phase diagram, reflecting the phase separated $d + s$-wave superconductor. Another ramification we make quantitative. The s-wave admixture in the superconducting gap is shown to shift the gap nodes along the Fermi surface. We predict how this node shift behaves as a function of doping, an effect which might be just within the resolution of present day angle resolved photo-emission experiments.

8.1 $SU(2)$ energy density functional

In the previous chapters, we already set the stage for the mean field theory description of the doped Mott insulator. Let us first summarise the main points so far, we give an executive summary, to help the reader in understanding the way the results are obtained. Then we derive a mean field energy expression for homogeneous mean field states, being the working horse for the remainder of this chapter.

8.1.1 Summary of the previous results

Let us first quickly remind the reader of the basic ingredients, as discussed in the previous chapters. Our starting point is the slave boson Hamiltonian (7.3.4), derived from the $t-J$ model,

$$\begin{aligned} H_{mf} &= -\mu \sum_i h_i^\dagger h_i - \sum_i a_{0i}^l \left(\frac{1}{2} \psi_{\alpha i}^\dagger \tau^l \psi_{\alpha i} + h_i^\dagger \tau^l h_i \right) \\ &+ \sum_{<ij>} \frac{3J}{8} \left(|\chi_{ij}|^2 + |\Delta_{ij}|^2 + \psi_i^\dagger U_{ij} \psi_j + h.c. \right) \\ &+ \sum_{<ij>} t \left(h_i^\dagger U_{ij} h_j + h.c. \right), \end{aligned} \tag{8.1.1}$$

where the holons h described the charge sector of the theory, whereas ψ desribes the spinons, cf. (7.1.4) and (6.1.6), respectively. We also remind the reader that the matrix U_{ij} in (6.2.2), defined as

$$U_{ij} = \begin{pmatrix} -\chi_{ij}^* & \Delta_{ij} \\ \Delta_{ij}^* & \chi_{ij} \end{pmatrix}, \tag{8.1.2}$$

incorporates the hopping amplitudes χ_{ij} and pairing amplitudes Δ_{ij} of the spinons, as defined in Eqs. (6.1.14) and (6.1.15). Therefore, a mean field Ansatz is described by U_{ij}, the projective symmetry idea. The Lagrange multipliers a_{0i}^l enforce the single-occupancy constraints, required to give a faithful representation of the $t-J$-model Hilbert space. Away from the mean-field solution, the a_{0i}^l become dynamical, turning them into fluctuating gauge fields, which kindles confinement issues. The previous chapter 6 gave a summary of the ideas of Lee and Wen [44] that for zero doping, an effective $U(1)$ gauge theory remains, which is deconfining for the SF-spin liquid. This justification is continued, since doping explicitly breaks $SU(2)$ symmetry, as we will show. In the previous chapter, we argued that this approach leads to a correct energy density functional description, but that it should not be trusted as soon as one asks dynamical questions, concerning correlation functions. But in this chapter, we only rely on vacuum properties of the Hamiltonian (7.3.4), which turns out to give remarkable agreements with experiment.

Inspired by the spin liquid ideas of many people [47, 75, 76], an idea having some experimental support [51] , we introduced in Chapter 6 three mean field states, namely the staggered flux phase [49, 50], the d-density wave state [77, 78] and the d-wave superconductor. We recall their descriptions here.

The d-wave superconductor is in the projective symmetry group represented by

$$\begin{aligned} &\text{dSC} \\ U_{\mathbf{i},\mathbf{i}+\hat{x}} &= -\chi\tau^3 + \Delta\tau^1, \quad U_{\mathbf{i},\mathbf{i}+\hat{y}} = -\chi\tau^3 - \Delta\tau^1, \\ a_0^3 &= 0, \quad a_0^{1,2} \neq 0, \; < b_1 >=< b_2 >\neq 0. \end{aligned} \tag{8.1.3}$$

The a_{0i}^l are generally non-zero, because of the doping. The dispersion for the fermions is (at least for homogeneous $a_{0\mathbf{i}}$) readily calculated to be

$$\begin{aligned} E_k &= \sqrt{(\chi_k - a_0^3)^2 + (\Delta_k - a_0^1)^2}, \\ \chi_k &= -\frac{3J}{4}(\cos k_x + \cos k_y)\chi, \\ \Delta_k &= -\frac{3J}{4}(\cos k_x - \cos k_y)\Delta. \end{aligned} \tag{8.1.4}$$

One should notice that the spinons have gapless Dirac dispersions at the points $(k_x, k_y) = (\pm\frac{1}{2}\pi, \pm\frac{1}{2}\pi)$, i.e., at those points we have nodal fermions consistent with experiments [79, 80].

The following two phases also support these Dirac quasiparticles. The first one is the staggered flux phase,

$$\begin{aligned} &\text{SFP} \\ U_{\mathbf{i},\mathbf{i}+\hat{x}} &= -\chi\tau^3 - i(-)^I\Delta, \quad U_{\mathbf{i},\mathbf{i}+\hat{y}} = -\chi\tau^3 + i(-)^I\Delta, \\ a_0^{1,2,3} &= 0, \; < b_1 >=< b_2 >= 0. \end{aligned} \tag{8.1.5}$$

The third phase has Fermi pockets, with nodes radially shifted from

$(\frac{1}{2}\pi, \frac{1}{2}\pi)$:

dSC with pockets

$$\begin{aligned} U_{\mathbf{i},\mathbf{i}+\hat{x}} &= -\chi\tau^3 - i(-)^I\Delta, \;\; U_{\mathbf{i},\mathbf{i}+\hat{y}} = -\chi\tau^3 + i(-)^I\Delta, \\ a_0^{1,2} &= 0, \; a_0^3 \neq 0, \; < b_1 > \neq 0, \; < b_2 > = 0. \end{aligned} \quad (8.1.6)$$

We remind the reader that the *dSC* state is referred to that way, since in the Hamiltonian 7.3.4 that particular U_{ij} couples $f_{\uparrow i}$ with $f_{\downarrow i}$.

The three mean field phases above describe nodal fermions, but at different points in k-space. For zero doping, all the three Ansatzes become the same, supporting Dirac quasiparticles at $(\frac{1}{2}\pi, \frac{1}{2}\pi)$, since the Lagrange multipliers and boson densities vanish. This is an expression of the fact that for zero doping, these states describe the same quantum order, as realised by Wen [18]. In fact, for zero doping, these three Ansatzes are different representatives of the same projective symmetry group, and because of this, these can be smoothly morphed into each other by $SU(2)$ transformations. Indeed, in Chapter 6 we showed that the transformation responsible for this, reads

$$g_i = \exp\left(-i\frac{\pi}{4}(-)^I\tau^1\right). \quad (8.1.7)$$

The way the $SU(2)$ mean field theory is set up, is as follows. The spinons and holons are considered to be separate systems. As long as one rotates the spinons and holons together, $SU(2)$ gauge symmetry gives the same mean field properties. If one fixes a gauge for the spinons, and then starts to rotate the holons independently, the mean field results for the energy will be different. The strategy we choose is to fix the spinon gauge at the dSC mean field state, whereas the holons will be rotated by the group element g_i. We showed in Eq.(6.2.14) that we can decompose g_i into "Euler angles" α_i, θ_i and ϕ_i, and encoded these in the useful concept of isospin, as defined in (6.2.16). We pictured this concept in the isospin sphere, cf. Fig. 6.1. Only the angle θ_i turned out to be physical, whereas the other ones are gauge. For equator states, we have $\theta = \frac{1}{2}\pi$, corresponding to $< b_1 > = < b_2 >$, whereas for $\theta = 0$, $< b_1 > \neq 0$ and $< b_2 > = 0$. The latter means that the symmetry between empty sites and doubly occupied sites is broken.

8.1.2 Mean field energy

Having recapitulated the main ingredients, let us derive the energy density functional. Inspired by the empty limit, our starting point is the mean field wave function,

$$|\Psi_0\rangle_{MF} = \prod_i \left(\alpha_i + \beta_i e^{i\varphi_i}(u_i b^\dagger_{i1} + v_i b^\dagger_{i2})\right) |\text{vac}\rangle \, |\text{F}\rangle \, . \tag{8.1.8}$$

The ket $|F\rangle$ describes the mean field spinon state, and the boson density $|\beta|^2$ is the density of physical holes.

The important point of $SU(2)$ gauge theory is that the particle-hole symmetry is broken upon doping. Indeed, a hole is described by a superposition of vacancies and spin pair singlets $f^\dagger_{i\downarrow} f^\dagger_{i\uparrow}$, accompanied by their own boson. In the particle-hole symmetric state, the b_1 and b_2 boson are equal, meaning that this should correspond with $\theta = \frac{1}{2}\pi$. This is the motivation for the parametrisation $u_i = \cos(\frac{1}{2}\theta_i)$ and $v_i = \sin(\frac{1}{2}\theta_i)$.

The phase φ is the same as the phase φ_i of g_i, which is gauge. From now on, we gauge fix $\varphi_i = 0$ everywhere. The transformation (8.1.7) mapping the SFP into the dSC corresponds with $\varphi_i = \frac{1}{2}\pi + I\pi$, as the reader can verify.

For theoreticians, it is natural to first study spatially homogeneous mean field states. However, in the course of time it has become clear that strongly interacting electron systems tend to form inhomogeneous states, like stripes. Being aware of this complication, let us nevertheless study homogeneous states. Still, this exercise turns out to be instructive, in this regard. The reason is that we treat the Hamiltonian (7.3.4) in the grand canonical ensemble, instead of the canonical ensemble. The original formulation of the $SU(2)$ mean field theory [44, 45, 61] rested on the canonical ensemble as well.

To account for condensation of the holons at finite doping, the temperature was taken to be finite, to find out that the particle number constraint leads to Bose-Einstein condensation of the holons, by treating μ simply as a Lagrange multiplier. We prefer a different approach, since considering first finite temperature is a detour given in by the unphysical assumption that the holons form a non-interacting gas. On the other hand, hard-core bosons are not only more physical, but also easy to treat at zero temperature. In Chapter 7 we made the point that the bosons are interacting, making them superfluid at zero temperature. The last motive is the possibility of phase separation, i.e., the possibility of coexistence of phases with different densities at the same chemical potential in the same volume, giving rise to the need of performing the Maxwell construction. To anticipate this, it is necessary to take the chemical potential for the holons as control parameter, instead as the Lagrange multiplier enforcing a fixed density.

Let us substitute the mean field wave function (8.1.8) in the slave Hamiltonian (7.3.4), we obtain an expression for the mean field energy per site $\frac{1}{N}\langle H_{mf}\rangle = e_{MF}$. ($N$ is the number of lattice sites.)

$$\begin{aligned} e_{MF} &= -\frac{1}{N}\sum_k E_k + \frac{3}{4N}J(|\chi|^2+|\Delta|^2) \\ &\quad - 2t\chi|\alpha|^2|\beta|^2 - (\mu + a_0^1\sin\theta + a_0^3\cos\theta)|\beta|^2. \end{aligned} \tag{8.1.9}$$

The homogeneity of the Ansatz is expressed in the fact that the lattice site subscript i is dropped. It is important to observe that the isospin latitude angle appears in the mean-field energy, expressing the fact that for non-zero doping, $SU(2)$ gauge symmetry is broken. Hence θ is not gauge, but has acquired physical meaning. The kinetic part of the energy is gauge invariant, leading to the fact that only the hopping amplitude shows up in the holon hoppings. From this density functional, we derive the saddle point equations for the dynamical variables $\chi, \Delta, a_0^1, a_0^3$ and the hole density $|\beta|^2 = \rho(\chi) = 1 - |\alpha|^2$. To simplify matters a bit, we take the isospin angle θ as an external parameter, controlling the density of b_2 relative to b_1.

The saddle point equations in the grand canonical ensemble are easily derived, with the homogeneous forms of (7.3.10 and (7.3.12),

$$\begin{aligned} 2\chi &= \frac{1}{\chi}\sum_k \frac{\chi_k(\chi_k - a_0^3)}{E_k} + 2\left(\frac{4t}{3J}\frac{\partial}{\partial\chi}\rho(\chi)(1-\rho(\chi)\ )\right) \\ 2\Delta &= \frac{1}{\Delta}\sum_k \frac{\Delta_k(\Delta_k - a_0^1)}{E_k} \\ \rho(\chi)\sin(\theta) &= \sum_k \frac{(\Delta_k - a_0^1)}{E_k} \\ \rho(\chi)\cos(\theta) &= \sum_k \frac{(\chi_k - a_0^3)}{E_k} \\ 0 &= \rho(\chi)\left(\rho(\chi) - \frac{1}{2}\left(1 + \frac{\mu + a_0^1\sin\theta + a_0^3\cos\theta}{2t\chi}\right)\right). \end{aligned} \tag{8.1.10}$$

As already discussed after (7.3.10), these equations give rise to an important law which provides a linear relationship between the s-wave spinon pairing $\Delta_s \equiv \left\langle f^\dagger_{i\downarrow} f^\dagger_{i\uparrow} \right\rangle$ and the doping $x = \rho(\chi)$,

$$\Delta_s = \frac{1}{2} x \sin(\theta), \tag{8.1.11}$$

as a direct ramification of the full constraint structure.

In order to find the solutions to the saddle point equations (8.1.10), the energy (8.1.10) is minimised numerically using the simulated annealing method [81, 82].

We point out that the fifth equation admits both zero and non-zero solutions for the density ρ. As a function of μ, the mean field energy will tell which one is more favourable. In Section 8.2, we will show that the system chooses between these two by a first order phase transition, and not a second order one! This means that the mean field theory 8.1.10 implies a phase separation regime, for the usual Maxwell construction reasons, when transforming to the canonical ensemble.

8.2 The $SU(2)$ mean field phase diagram

We have now arrived at the point where we can collect the results. Our first step was to prove that one needs all $SU(2)$ constraints to project onto the $t - J$ model Hilbert space, while this constraint structure is also required for the mean field description of the $SU(2)$ gauge theory. This will bring us to the first result: that the superconducting order parameter *needs* to have an s-wave component! Then we spent effort in proving that the holons need to be hard-core in order to respect the full set of $SU(2)$ constraints. We now show that the superfluid hard-core holon condensate displays phase separation behaviour, as expected for hard-core interacting systems. We thereby achieve an intrinsic connection between slave theories on the one hand, and the observation of inhomogeneous states on the other hand, a connection that is traditionally considered as absent.

Let us first discuss the numerical verification of a claim inferred in the literature [45], namely that at finite hole density the superconducting state ($\theta = \pi/2$) is preferred over the flux phase ($\theta = 0, \pi$) (cf. the inset in Fig. 8.3). This is a natural ramification of the breaking of $SU(2)$-symmetry for non-zero doping. The $\theta = \frac{\pi}{2}$ states, characterised by $a^3_0 = 0$, are energetically more favourable, being consistent with the instability of the flux state towards d-wave superconductivity, as already understood in the early nineties [83].

The reader might already have noticed a kink in e_{MF} as a function of doping in the inset figure 8.3. In other words: there is a first-order phase transition. Indeed, our main result is that generically this mean-field theory predicts phase separation at small chemical potential. The system stays initially at half-filling and pending the ratio of J/t at some finite μ a level crossing takes place to a state with a finite doping level, cf. Fig.8.3.

We stress here that this phase separation behaviour is eventually coming from the hard-core nature of the holons: also the "wrong"mean field states (SFP and pure dSC with pockets) exhibit first order behaviour. We conclude that, due to the Hilbert space restrictions on the $SU(2)$ description of the holons, the theory insists on inhomogeneous states for low doping. As a function of increasing J/t the width of this phase separation regime is increasing (see Fig.8.4) and we find that for $J/t \simeq 4$ the phase separation is complete. This is consistent with exact diagonalization studies on the t-J model indicating a complete phase separation for $J/t \geq 3.5$ [68],[84]. This is quite remarkable and it reveals that the gauge mean field theory has to be a remarkably accurate quantitative theory of the density functional kind: it is a good description of the empty limit and the Mott insulator, and gives a fair prediction of phase separation tendencies. We stress that as a rule less severe demands on physical reality are put on density functional theory , instead of full dynamical theories.

The significance of this finding is that for this most sophisticated version of spin-charge separation theories, phase separation is natural feature, as it is in the empirical reality. It is well understood that these macroscopic phase separated states are an artefact of the oversimplified $t-J$ model. By taking the long-range Coulomb interaction into account this will turn immediately into the microscopic inhomogeneity [67],[69],[85], of the kind that are seen in STM-experiments [86]. To see how well this slave theory handles the 'big numbers' in this regard, we show in Fig. 8.3 the electronic incompressibility $1/\kappa = \frac{\partial^2 E_{\text{MF}}}{\partial x^2} = \frac{\partial \mu}{\partial x}$ according to the $SU(2)$ theory, to find that it compares remarkably well with the experimental results due to Fujimori and coworkers[19]. Firstly, it is seen that independent of the ratio J/t, the compressibility is right on spot of the experiments: the slope of μ vs. holon density is the same as the measured slope. The doping at which phase separation occurs, namely 13 %, is cor-

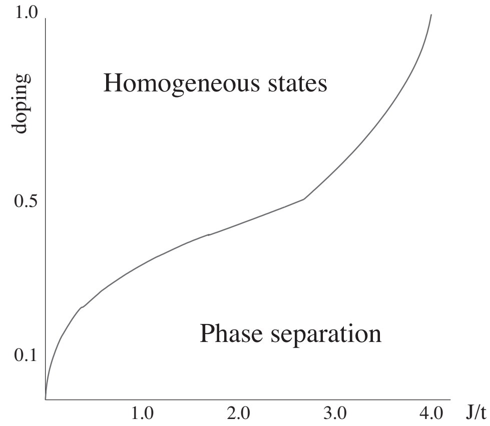

Figure 8.1: Phase diagram as a function of doping x and the ratio J/t, according to the Maxwell construction. For dopings below a critical doping line $x_c(J/t)$, homogeneous states are metastable against phase separation, stripes etc. For $J \simeq 4t$, total phase separation takes place and the system becomes a mixture of Mott insulating and empty regions.

rect only for the value $J/t = 0.1$, which is too small. Indeed, from ARPES measurements a ratio of $J/t = 0.3$ is more realistic, but for those samples phase separation takes place at dopings of about 17 %, and not 21 %, as found in the $SU(2)$ mean field theory for $J/t = 0.3$.

Let us now focus on the nature of the superconducting order parameter found elevated doping levels. As expected, the s-wave component becomes increasingly important, cf. Eq. (8.1.11). To further emphasize this , we compare in the inset of Fig.8.3 the energy of a state where we have fixed the Lagrange multiplier $a_0^1 = 0$ such that the s-wave component vanishes, with the best $d + s$ mean-field state, finding that the former is indeed a false vacuum. To mimick the average behavior of the superconducting order parameter also in the micro-phase separated states at low dopings, we calculate matters now in the false (uniform) vacuum of the canonical ensemble, fixing the average density, simplifying the mean field equations (8.1.10). Indeed, $\rho(\chi)$ becomes now a fixed ρ. Furthermore, since the state $\theta = \frac{1}{2}\pi$ is favoured, we take the mean field Ansatz U_{ij} to be the SC one, as stated before, and consequently we have $a_0^3 = 0$. This enables us to map out the phase diagram as a function of doping and the ratio J/t, leading to three phases. The first, for low doping, is the phase-separated, underdoped $d + s$-wave superconductor. It is a mixture of charged, superconducting islands in an insulating sea without charges, where the full $SU(2)$ symmetry is restored. This reminds the reader of the STM-pictures from S.C. Davis' group [86]. For intermediate dopings, the homogeneous, overdoped $d+s$-wave superconductor is found, whereas for high dopings, the d-wave gap vanishes, leaving behind a pure s-wave superconductor. Hence, although d-wave superconductivity leads to an s-wave admixture, the reverse is not true.

We find that the regime where phase separation is important, the s-wave component is not negligible. This is consistent with Raman measurements [22], where the superconducting gap was found to have both d- and s-wave components. Although screening effects in Raman scattering make it difficult to compare our results directly to theirs, their results indicate that the ratio $r = \Delta_s/\Delta_d$ grows with doping, as it does in our approach. Looking to Figure 8.2, we find in the phase separated overdoped regime s-wave admixtures of about $r = 10 - 20\%$, consistent with c-axis tunneling experiments [20].

We predict that at a doping level that appears to be higher than can be achieved in cuprate crystals a phase transition occurs to a pure s-wave superconductor. As we already alluded to, the gauge fluctuations should become more severe as well, for increasing doping and at some doping level a transition should occur to a confining "electron like"system.

As we learnt from the empty limit, it still make more sense than the result obtained by disregarding the first constraint equation (7.3.10), since that would lead to the unphysical result $\chi = \Delta = \frac{1}{\sqrt{2}}$, giving the wrong vacuum energy, as we discussed in Section 7.4. In other words, our mean field theory is a remarkably good density functional theory, but unevitably the theory fails completely in dynamic regards. In the confined phase, at sufficiently high dopings, we need an approach which is qualitatively different from slave theories, let alone that one can get away with the mean field version. On the other hand, in the superconducting doping regime, we find some promising experimental support for our results with regard to the $d + s$ structure of the order parameter, meaning that confinement physics might not be overwhelmingly important in the low-doping part of the phase diagram.

Having said this, the experimental support for an s-wave admixture makes it possible to come up with a falsifiable prediction for photo-emission experiments. The s-wave component induces a shift in the nodes, as can be inferred from the spinon dispersion E_k: the hopping vanishes along the line $k_y = \pi - k_x$, so that the locus of the node can be readily calculated to be $k_y - \frac{1}{2}\pi = \arccos(a_0^1/(3J/2)\Delta)$, which shows a doping-dependent behaviour as well, cf. the black line in Figure 8.2. The node shifts might be able to explain the U-shaped gap as measured in Bi2212 [87], since the twinning of samples mixes regions of node shifts $+\delta$ with $-\delta$, smearing out the V-shape of the gap to a U-shape.

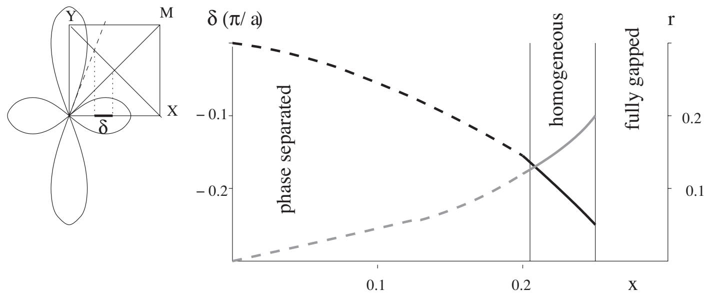

Figure 8.2: In this figure, we show how the s-wave admixture $r = \Delta_s/\Delta_d$ grows as function of doping (light gray line) for $J = 0.3t$. The leftmost figure shows the symmetry of the gap function in reciprocal space, indicating the shift of the gap node along the Fermi surface, coinciding with the Brillouin zone boundary Y-X. This shift is denoted by δ, and grows with doping. The black line plots δ as function of doping in units of π/a, where a is the lattice spacing. Up to a doping of 20% the results of the false vacuum homogeneous solution in the canonical ensemble are used, exploiting the Maxwell construction. To remind the reader of the fact that the homogeneous states are false vacuum states, the lines are dashed. Finally, the lines end where the d-wave gap vanishes. Here r is infinite and the Fermi surface is fully gapped.

In summary, we have argued that in order to achieve consistency in the $SU(2)$ slave boson theory, one has to implement hard-core bosons instead of non-interacting Bogoliubov bosons to describe the charge sector. As a result, one obtains phase separation at lower dopings consistent with the experimental observations. Also the compressibility matches very well. Furthermore, by inspecting the empty limit, we showed that an s-wave component in the superconducting order parameter is implied when d-wave superconductivity occurs, at least for dopings where homogeneous states exist, a ramification of the constraint structure. This finds its origin eventually in the particle-hole symmetry central to the gauge structure of the $SU(2)$ theory: to describe physical spin singlets, "no fermions" are indistinguishable from an "s-wave spinon pair". This is reflected in the constraint equations, required to reduce the $SU(2)$ Hilbert space to the Hilbert space of the $t-J$model. The constraint equations tell us that as soon as d-wave superconductivity emerges, one necessarily has an s-wave component. This s-wave admixture is in accord with Raman [22] and c-axis tunneling experiments [20, 21]. We also predict a node shift in the gap function, that might be measurable by photoemission.

8.3 Conclusion: strong correlations and inhomogeneous systems

In this chapter, we derived some important results, just using the mean field version of the $SU(2)$ gauge theory. Two important lessons from Chapter (7) lead to two important results. The first is that by just respecting the appropriate constraints, a superconducting gap with an s-wave component emerges. This result is already engrained in the particle-hole symmetric formulation captured in $SU(2)$ theory. Although the theory does not deal with confinement issues, it explains experiments indicating the d+s wave symmetry remarkably well, within an elegant theoretical framework.

Secondly, the physics of the problem enforces the holons to have infinite hard core, in order to model the Coulomb repulsion correctly within the physics of the $t-J$ model. Hence, we should regard the holons as a superfluid, instead of as a Bose gas. We have shown that this approach leads to phase separation behaviour. In this way we demonstrate that when the theory is formulated correctly, one finds that it unifies different schools of thought: inhomogeneities and slave boson constructions are actually going hand in hand.

This inhomogeneity seems to be ubiquitous for strongly correlated, complex systems, see, e.g., Balatsky [88]. One way this tendency towards inhomogeneity/phase separation is manifest in experiments is by the extreme sensitivity of the strongly correlated electron systems in oxides to disorder. A clear example of this is provided by the manganites, showing a first order transition towards impurity-induced “puddles” [89, 90].

Inspired by the projective symmetry group, we will propose an inhomogeneous mean field state in the outlook of this thesis. Therein we will discuss how experiments inspire the idea that inhomogeneous $SU(2)$ mean field states connecting spin liquid states with superconducting states, incorporate both stripes and protection of nodal fermions. If this idea works, this would mean another unification in the theory of high-T_c, since the communis opinio is that the "weak"nodal fermions should be destroyed by the "strong"charge ordering effects. That would be nothing less than the culmination of Wen's idea of quantum order.

8.4 Appendix: colour figures for section 8.2

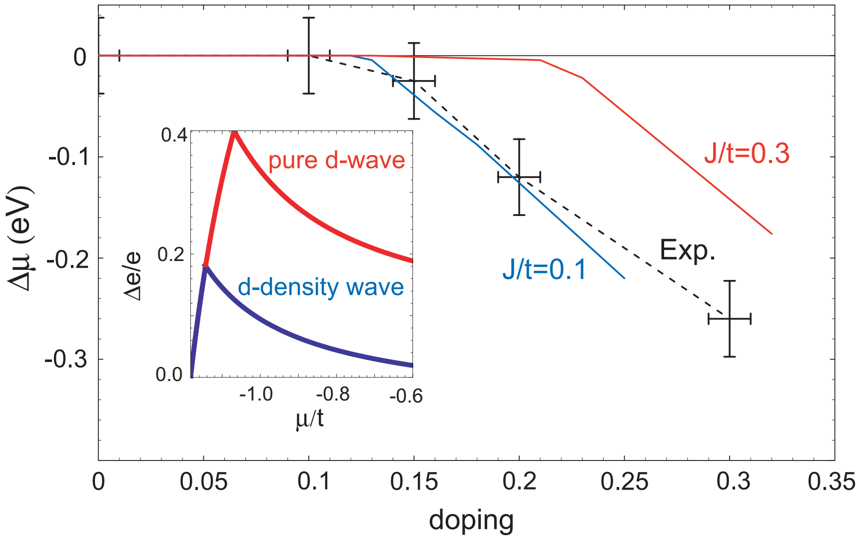

Figure 8.3: Chemical potential shift as a funtion of doping, showing the phase separation behaviour of the reformulated mean field theory. Indeed, the chemical potential starts to shift for appreciable dopings only. The blue line are the numerical results for $J/t = 0.1$, and agrees very well with the experimental results from Fujimori [19] (dotted line). The red line depicts the results for $J/t = 0.3$. The critical doping changes, but not the compressibility. The inset shows that ignoring a_0^1, i.e., ignoring the s-wave component, gives a false vacuum. Indeed, there is a positive relative energy difference between the pure dSC and our mean field theory, growing with doping (red line). The inset also shows that for non-zero doping, the DDW/SFP state (blue line) is higher in energy than the superconducting states.

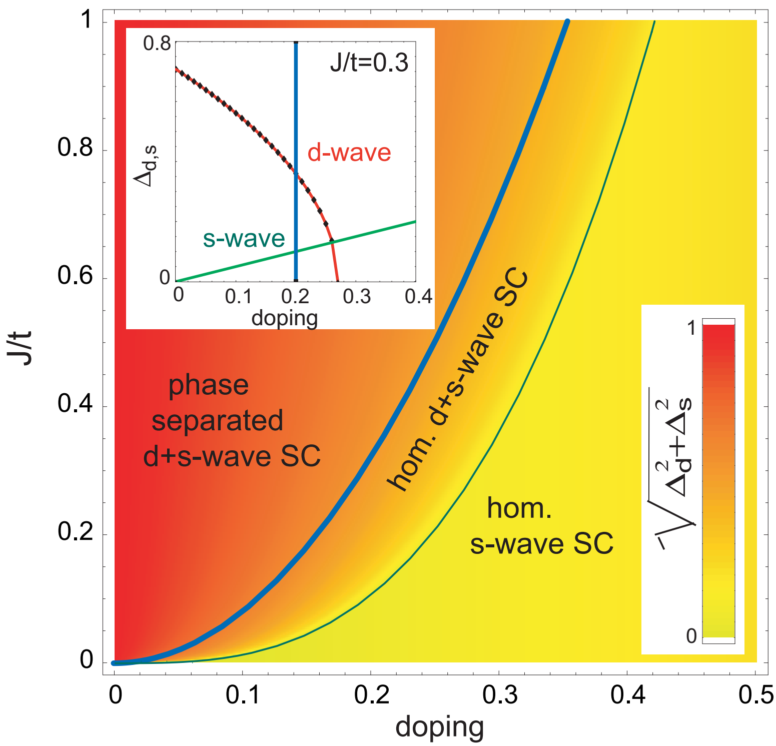

Figure 8.4: Zero temperature phase diagram of the $t - J$ model according to $SU(2)$-mean field theory. It incorporates three phases, viz. the phase separated $d + s$-wave superconductor for low dopings, the homogeneous $d + s$-wave superconductor for intermediate dopings, and a homogeneous s-wave superconductor at high dopings. The bold line shows indicates the border of the phase separation region. The phase separation tendency grows for increasing J/t, to become complete at $J/t = 4$. The colors indicate the total superconducting gap. For zero doping, there is only a d-wave component, whereas the s-wave admixture grows linearly with doping, so that the total gap is non-zero even beyond the critical $x - J/t$ line (rightmost line), where Δ_d vanishes. The inset shows the d-wave and the growth of the s-wave component separately for $J/t = 0.3$. The blue line indicates the doping level where phase separation terminates, computed by imposing uniformity (canonical ensemble) for $x < 0.2$.

Chapter 9

Conclusions and outlook

Let us conclude this thesis with a summary of the main results, and by giving an outlook on the perspectives and ideas this work opens up. The main theme of this thesis has been the investigation of analogues of non-Abelian gauge theory structures in condensed matter systems. In high-energy physics these are quite familiar, as symmetries become larger in approaching higher energy scales. Non-Abelian gauge theories are less natural in condensed matter systems, since these describe emergent *low*-energy states. Yet there are examples of such kind of structures in condensed matter systems.

One aspect of finding such an analogue is parallel transport: spin-orbit coupling can be interpreted as an $SU(2)$ parallel transport structure [8, 9]. This gives another context of colour-currents in quark-gluon plasmas: for spin-orbit coupled systems, the non-Abelian charges in this case are the spins, being transported by electromagnetic fields appearing in the guise of non-Abelian gauge fields. Although the analogy is not complete since the "gauge" fields do not obey dynamics since these are fixed by the electromagnetic fields, it is still an excellent context to address whether and how non-Abelian hydrodynamics can exist. If so, it would provide an excellent mean to figure out how the non-Abelian explosions at CERN work.

A condensed-matter system in which a full gauge theory is truly emergent, is the Mott insulator. The non-Abelian gauge structure arises from imposing both the single-occupancy constraint and the particle-hole symmetry locally in a slave theory [45]. This structure is captured in the gauge group $SU(2)$. To keep the constraints exact, it is necessary to keep the full gauge field dynamics, leading to a full non-Abelian gauge theory coupled to matter.

Both aspects turned out to lead to physical predictions, and both analogues give more insight as to what the contours of answers to questions in high-energy physics might be. Let us focus on those aspects separately, highlighting the main results, and giving an outlook for both.

9.1 Parallel spin transport and non-Abelian hydrodynamics

The notion of parallel transport of non-Abelian quantities leads to the question of whether a hydrodynamic description of, e.g., the quark-gluon plasma exists. The answer turned out to be no and yes. The negative part of the answer is based on the fact that the gauge fields change the colour/spin. This translates into the statement that spin/colour currents are only covariantly conserved, i.e., there is only a local conservation law. But since local conservation not necessarily imply global conservation, an effective long-wavelength scale hydrodynamic description is impossible. On the other hand, we have shown in section 3.7 that when the non-Abelian matter condenses into a phase coherent state, the resulting phase stiffness restores hydrodynamics! This is an example of emergent non-Abelian hydrodynamics in condensed-matter systems. These equations incorporate the spin-Hall equation, familiar from spintronics, with the difference that this equation has now acquired hydrodynamic status, absent in ordinary spintronics.

The first example of hydrodynamics rising from the ashes is the ordered XY-magnet, an example inspired by the work on spiral magnets by Mostovoy [26]. The hydrodynamic currents can be wired in by spiral order, creating a singularity. This singularity corresponds with an electric field. The second example is given by spin superfluids. In the case of the pure $SU(2)$ spin superfluid, an electric field sets a hydrodynamic spin current in motion. Inspired by $SU(2)$ topological textures like the 't Hooft-Polyakov monopole, the case of a cylindrical electric field was considered. We demonstrated that the cylindrical topology of the electric field is inherited by the spin current, giving rise to the spin vortex.

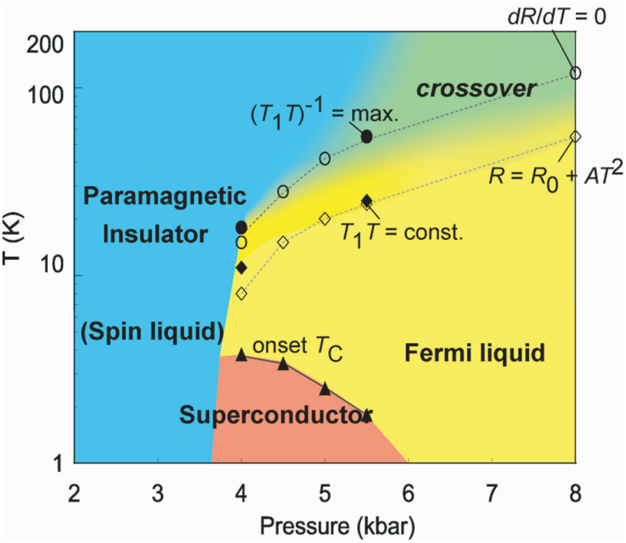

Figure 9.1: The phase diagram of the highly frustrated κ-(ET)$_2$Cu$_2$(CN)$_3$, as proposed by Kanoda [38]. The spin liquid state shows linear specific heat, which might signal the presence of a spinon Fermi surface. This would amount to making a spinon Fermi liquid out of an insulator. Then the interesting possibility is that this spinon metal might be unstable against an $S = 1$ spin superfluid.

This is the rigid version of the Aharonov-Casher effect [13], as devised by Balatsky and Altshuler [9]. The very same topology, provided by the rigidity of the spin superfluid, gives a beautiful ramification: the winding number of the vortex corresponds with quantisation of the charge that causes the electric fields. Put differently: charge is trapped by the spin superfluid in quanta of $\lambda_0 = \frac{m}{\mu_0 e}$, which is 2.6×10^{-5}C/m for ^{3}He . This charge trapping effect is reminiscent of, but not quite the same as Aharonov-Bohm flux trapping with superconducting rings. In the latter, the Maxwell gauge field is dynamical, becoming screened in the bulk. In the spin superfluids, in contrast, the electric field is not a dynamical gauge field. But since the total current vanishes, and since the field is necessary to create a spin vortex current in the bulk, a quantisation condition is still obtained.

Then the question was whether our proposed experiment in Figure 2.1 can be performed. We considered the two possible candidates, ^{3}He-A and ^{3}He-B. ^{3}He-B resembles the pure spin superfluid most, its order parameter manifold being described by the group $SO(3)$. The effect of dipolar locking destroys the spin vortex, however. Matters are even worse for ^{3}He-A, since the spin order parameter is described by a vector, not a matrix. Although a cylindrical electric field will set a spin vortex into motion, the vortex will decay, since it is energetically not more favourable than the situation without vortex.

Even in the absence of dipolar locking, the numbers Nature provided us with, conspire against the charge trapping experiment. The spin-orbit coupling constant is inversely proportional to mass, making it small for the ^{3}He-atoms. Hence, if the amount of superfluid is too small, it is more advantageous for the wire to discharge by an enormous spark, than to remain trapped by the spin vortex. Actually it turned out that we need an amount of ^{3}He enough to cover Alaska, rendering our experiment merely into a joke.

9.1.1 Outlook: organic superconductors

The problems signaled, point the direction in which we can look for a solution. We have concluded that the heavier the constituent particles are, the smaller the spin-orbit coupling gets. Hence, we need to look for lighter things in order to be able to trap charge. The first candidate would be a superfluid made out of electrons, since these are 5000 times lighter than ^{3}He-atoms. However, as electrons are charged, charge effects will overwhelm the whimpy spin-orbit coupling effects. So we need a system made out of electrons, but having a huge charge gap, i.e., we need a spin superfluid made out of a Mott insulator. Does this exist?

In recent years, there have been many advances in the research on highly frustrated systems on triangular lattices [91], which are realised in organic compounds. In the last two years, Kanoda *et al.* have done specific heat measurements in the spin liquid phase of the organic superconductor κ-$(ET)_2Cu_2(CN)_3$, see Figure 9.1. Although the spin liquid state is known to be a featureless paramagnet, the specific heat showed a linear behaviour as a function of temperature [37, 38].

The linear behaviour has led theorist P.A. Lee to the idea that this might be caused by fermionic spinons forming a Fermi surface [92]. It is plausible that at low energy scales, a BCS-like instability might give rise to an $S = 1$ spinon condensate. This would then be the desired spin superfluid made out of a Mott insulator. The theoretical complication is that due to the $SU(2)$ slave theories developed by Lee and Wen [44], there will be transversal gauge degrees of freedom, blocking the triplet channel.

This should give rise to some scepticism about whether the organics are able to become a triplet superfluid. Whether or not this is the case, to our opinion, the idea of charge trapping provides a good motivation to pursue the BCS-instability towards a triplet state of the spinon metal further.

9.2 Emergence of the deconfined spin liquid

An example of the emergence of a full non-Abelian gauge theory, is the doped Mott insulator, important for understanding high-T_c superconductivity. The doped Mott insulator is defined by having either one spin or one charge degree of freedom per site, interpreting the electron as a composite particle of a spinon and a holon, as discussed in Chapter 7 . Although at a first glance this seems like just one constraint equation, leading to a $U(1)$ gauge theory, the extra local particle-hole symmetry of the MI gives rise to a non-Abelian gauge theory described by the group $SU(2)$. The dynamics of the gauge fields are generated by integrating out the matter fields, renormalising the effective gauge coupling to a finite value. These dynamics immediately make it necessary to address the issue of confinement: is there any reality to the assertion that electrons are not elementary, if in real life only the confined state is encountered? The question whether a deconfined spin liquid state exists, being made out of holons, spinons and deconfining $SU(2)$ gauge fields, is similar to the question of the existence of the quark-gluon plasma in the context of $SU(3)$ QCD. In this regard, the ideas of X.-G. Wen on spin liquids in the Mott insulator might give insights into the quark-gluon plasma as well.

To summarise Wen's ideas as discussed in Chapter 6, the main idea is that on *assuming* that vacuum expectation values of the spinon operators exist, some $SU(2)$ gauge fluctuations might become massive, leaving behind an effective gauge theory with lower symmetry. In the determination of what the effective gauge theory is, the observation that mean field states with different symmetries can be gauge equivalent, is important. The main example is provided by the staggered flux liquid, breaking translation symmetry. It is equivalent to a d-wave pairing state of the fermions after a gauge transformation. The notion of gauge equivalence of mean field states, leads to a classification scheme of the mean field $SU(2)$ gauge theory which is different from the Wilson loop classification scheme in pure gauge theories. The equivalence classes of mean field states, named projective symmetry groups, determine which effective gauge theory remains after integrating out the matter fields. Then the excitation spectrum of the matter fields will have to determine whether the remaining massless gauge interactions are confining or not. If the effective theory is deconfining, the spin liquid is real, being the condensed matter sibling of the quark-gluon plasma. All the states in the same projective symmetry group leading to deconfinement, are thus protected against gauge fluctuations. This leads to the idea that there is an ordering principle at work. Since states with *different* symmetries can lead to the *same* physics, it cannot be classical order. This justifies the name of quantum order, and is by definition classified by the projective symmetry group.

In this thesis, the spin liquid state describing the d-wave superconductor in the doped Mott insulator is studied. Some important lessons were drawn from the empty limit, the opposite limit from half-filling. Although a deconfined state surely will not exist in that regime, it turned out that $SU(2)$ theory is able to describe the energy of that state correctly. On the other hand, dynamical properties are not rendered correctly, since confining gauge interactions are ignored. Hence, $SU(2)$ gauge theory can be trusted as a good energy functional. The ramification of the empty limit considerations is that the holons should be treated as hard-core bosons, making phase separation possible. Performing mean-field calculations in the grand canonical ensemble, indeed lead to the prediction of phase separation, reaching deep into the superconducting regime. The calculation of the compressibility agrees with experimental results of Fujimori and coworkers on the chemical potential shift in LSCO [19]. The phase separation tendencies are also not inconsistent with numerical results [68], thus corroborating the reliability of the $SU(2)$ slave boson theory as an energy density functional. Slave theories have up to now not been believed to show phase separation. Our results show, however, that the $SU(2)$ mean field theory forms a bridge between on the one hand slave boson theories, and on the other hand models predicting phase separation.

Secondly, the constraint structure implies a very interesting result: describing the holons as a superfluid condensate, leads to s-wave pairing of the spinons. This result is rooted in the description of a hole as a superposition of a vacancy and two spinons: inducing holes induces double spinon occupancy. Although deconfinement is destroyed in high doping regimes, for lower doping the particle-hole symmetry of the Mott insulator is still remembered. In particular, in the superconducting doping regimes, the $SU(2)$ order parameter insists on having a $d+s$ wave structure, where the s-wave component grows linearly with doping. The $d+s$ structure has acquired support from c-axis tunneling experiments [20, 21] and Raman scattering [22]. We strongly suggest an experiment measuring the s-wave admixture as a function of doping.

These results provide good news for high-energy physicists. If the deconfined spin-liquid lead to results which have been confirmed, deconfined states in high-energy physics might also be on the real axis. This does not hold for the phase separation, however, since in $SU(3)$ the quark electric charges are $+\frac{2}{3}$ and $-\frac{1}{3}$, whereas in the $SU(2)$ theory, there are only repulsive charges $-e$. More surprisingly, deconfinement can emerge for low energies in condensed matter systems.

9.2.1 Outlook: isospin spirals in cuprates

Having established the phase separation tendencies, new perspectives are opened as to what the role of stripes in the superconducting cuprates is. The understanding is that phase separation is a necessary condition for stripes to exist. On the other hand, phase separation has never been seen in the high-T_c's. In fact, there are other properties at work. The Mott insulator is an antiferromagnet, which makes it advantageous for holes to order in stripes instead of phase separated islands. In fact, stripes should be regarded as antiphase boundaries in the antiferromagnet [93].

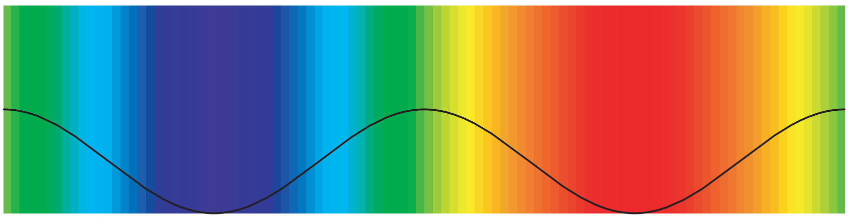

Figure 9.2: A pictorial representation of the unit cell of the isospin spiral. The green areas correspond with superconducting stripe regions (isospin angle $\theta_i = 0$), blue is the AB flux phase ($\theta = \frac{1}{2}\pi$) and red the BA flux phase ($\theta = -\frac{1}{2}\pi$). The drawn $\cos(2\theta_{i_x})$ profile shows the boson density as a function of the x-coordinate i_x in the unit cell. Here, θ_i is the isospin angle of the fermionic mean field state U_{ij}. Note that the bosonic isospin angle as defined in Chapter 8, Eq. 8.1.8, is in this case equal to $\theta_{bos} = \theta_i - \frac{1}{2}\pi$. Hence, $\langle b_1 \rangle = \langle b_2 \rangle$ still holds in the superconducting state. It is seen that the hole-rich region forms an antiphase boundary for the SF-liquid state in between the superconducting stripes.

In the past decades, many approaches to understand the Hubbard model in some slave boson representation are made. One is the large-S expansion [48, 94], taking the limit of the spin value $S \to \infty$. The other is introducing more than two flavours of spin, such that an $SU(N)$ model is obtained . The large-N limit leads to dimerised states [52, 95], whereas the vacuum of the large-S limit is the antiferromagnet [48, 94]. The question arises if the group $SU(2)$ is able to describe the "anti-phase boundariness" of the antiferromagnet, the more so since the antiferromagnet and the dimerised state are incompressible, whereas the flux phases and the dSC spin liquids of the $SU(2)$ theory are compressible.

We propose a way in which the $SU(2)$ mean field theory can describe anti-phase boundaries, within the spin liquid states descending from the Mott insulator. The first observation is that for zero doping, the dSC or SFP state is just a gauge fix within the same projective symmetry group. Let us now consider a gauge in which the spinon mean field U_{ij} Eq. (6.2.2) rotates over the whole isospin sphere,

$$U_{ij} = \exp\left(i\theta_i \frac{\tau^1}{2}\right) \begin{pmatrix} -\chi & \Delta \\ \Delta & \chi \end{pmatrix} \exp\left(-i\theta_j \frac{\tau^1}{2}\right) \tag{9.2.1}$$

by a harmonically varying isospin angle

$$\theta_i = \mathbf{Q} \cdot i = q i_x, \quad \text{ordering vector } \mathbf{Q} \text{ in } x \text{ direction.} \tag{9.2.2}$$

In this way, the SFP state is smoothly connected to a dSC state. Observe that this state is in the same PSG for zero doping. Then the idea is that for underdoped samples this spiral state might be lower in energy than the phase separated state for the homogeneous $d+s$-wave superconductor. A cartoon representation is given in the figure 9.2.

The peculiar feature of the isospin spiral is that in the $SU(2)$ gauge theory the charge-density wave is made out of a superconductor, which is not the case in the large-S limit. The antiferromagnetic domains are now replaced by a spin liquid, carrying nodal fermions, a feature absent in the antiferromagnet. As the nodal fermions exist in both the dSC and SF phases, a very promising perspective is opened up, supported by experiments.

The first support comes from the results from Fujimori and coworkers [19] and Z.X. Shen and collaborators [96] for LaSCO. The chemical potential shift measurements of Fujimori in underdoped LaSCO are compatible with the existence of charge order, with Shen finding similar results. Furthermore, in the Nd-doped cuprates, clear features of static stripes are measured already in the nineties [70, 97]. On the other hand, existence of nodal fermions in underdoped cuprates is found as well [96]. The combination of these results seem to indicate the coexistence of striped charge order with nodal fermions. This idea is backed by recent results from the group of J.C. Davis [86], reporting that charge order and nodal fermions can coexist.

The $SU(2)$ gauge theory is able to capture 'Mottness', d-wave superconductivity and the protection of nodal fermions. The framework of the isospin spiral state in $SU(2)$ mean field theory forms an excellent explanation to explain the mystery why nodal fermions should exist in a strongly correlated background. This is a promising motivation to study the stability of the isospin spiral mean field states in underdoped cuprates.

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Samenvatting

Verschijnselen in de natuur te begrijpen door de samenstellende delen ervan te bestuderen is een belangrijk wetenschappelijk paradigma. In feite ligt hieraan dus de idee ten grondslag dat het geheel identiek is aan de som der delen. Voor veel systemen in het dagelijks leven gaat dit ook op. Denk hierbij bijvoorbeeld aan de thermodynamische gaswetten, die begrepen kunnen worden door te veronderstellen dat gassen zijn samengesteld uit moleculen. Verder kunnen chemische reacties tussen moleculen begrepen worden door ze te beschouwen als herschikkingen van atomen.

In de twintigste eeuw is deze ontwikkeling verder gegaan met het ontdekken van kleinere substructuren: atomen bestaan weer uit protonen, neutronen en electronen, en op hun beurt zijn protonen en neutronen samengesteld uit quarks. Deze laatste deeltjes kunnen alleen bij zeer hoge energieën en korte lengteschalen waargenomen worden, zodat er enorme deeltjesversnellers nodig zijn, zoals die van het CERN in Genève. De hoop is dat kennis van de allerkleinste deeltjes leidt tot onthulling van een Theorie van Alles, die dan de verklaring voor de hele natuur kan zijn.

In de natuurkunde van de gecondenseerde materie doet zich echter de omgekeerde situatie voor: het is vaak onmogelijk om het gedrag van een macroscopisch systeem te begrijpen vanuit het begrip van de individuele samenstellende delen. Dit uit zich al in het simpele voorbeeld van een metaal, bijvoorbeeld koper. Van één koperatoom kan men niet zeggen of het metallisch is of niet. Alleen een hele verzameling koperatomen kan de beslissing nemen om bijvoorbeeld een geleidend metaal te worden. Dit is een voorbeeld van hoe het geheel meer kan zijn dan de som der delen.

Hoe tegengesteld deze filosofieën ook mogen lijken, sommige veeldeeltjessystemen zoals die in de gecondenseerde materie bestudeerd worden, blijken een aantal begrippen uit de hoge-energiefysica te imiteren. Deze begrippen rusten dan ook vaak op één en het zelfde principe. Het belangrijkste concept dat door beide vakgebieden gedeeld wordt, is dat van symmetrie en symmetriebreking.

Symmetrieën zijn nauw verbonden met behoudswetten. Zo impliceert translatiesymmetrie het behoud van impuls, de tweede wet van Newton. Verder hangt het behoud van impulsmoment samen met rotatiesymmetrie. Dit zijn allemaal voorbeelden die vanuit de klassieke mechanica begrepen kunnen worden. Een voorbeeld dat begrepen kan worden vanuit quantumveldentheorie is het ladingsbehoud in de electrodynamica. Dit behoud wordt geïmpliceerd door een (globale) rotatiesymmetrie beschreven door de complexe groep $U(1)$, die de globale fase van de operatoren in de theorie beschrijft. Symmetrieën kunnen ook lokaal gemaakt worden: als de actie die het systeem beschrijft invariant is onder het toepassen van groepselementen die van plaats tot plaats van elkaar kunnen verschillen, dan spreekt men van een lokale ijktheorie. In het geval van de commutatieve groepen $U(1)$ of $\mathbb{Z}_2$, wordt een dergelijke theorie Abels genoemd.

Een voorbeeld van een begrip uit de gecondenseerde materie dat er zelfs eerder was dan de hoge-energie pendant, is het Anderson-Higgs mechanisme voor Abelse $U(1)$-ijktheorieën. In de theorie van supergeleiding kan met dit mechanisme het Meissner-effect begrepen worden, nl. het effect dat in supergeleiders stromen gaan lopen die electromagnetische velden buiten het supergeleidende materiaal houden. Dit komt doordat er een condensaat is dat massa geeft aan de electromagnetische velden, de $U(1)$-ijkvelden, zodat ze een korte dracht krijgen. Precies dit mechanisme wordt ook verondersteld te verklaren waarom elementaire deeltjes zoals die in CERN waargenomen zijn massa hebben. In dat laatste geval zijn de ijktheorieën echter niet-Abels, namelijk $U(1) \times SU(2) \times SU(3)$.

De verrassing is dat aspecten van niet-Abelse ijktheorieën ook hun tegenvoeters in de gecondenseerde materie kunnen hebben. Deze idee nu ligt ten grondslag aan dit proefschrift. De twee aspecten die hierin behandeld worden, zijn parallel transport, en *deconfined* ijktheorieën. Laten we hier wat dieper op ingaan.

Parallel transport is met name bekend geworden door de algemene relativiteitstheorie. Deze zegt dat massa de ruimte kromt, zodat de "kortste" weg (geodeet) niet per se de Euclidische rechte lijn hoeft te zijn. Deze kromming kan vertaald worden in parallel transport-velden: ze transporteren deeltjes dusdanig dat ze geodeten volgen. In ijktheorieën kunnen de ijkvelden ook worden opgevat als een parallel-transportstructuur. Zo kunnen $SU(2)$-ijkvelden worden gezien als velden die spins roteren. Dit voorbeeld vormt de ijktheoretische context van Deel I. van dit proefschrift.

Het blijkt dat spin-baan gekoppelde systemen beschreven kunnen worden als zijnde een theorie met $SU(2)$-parallel transport. De "ijkvelden" zijn echter geen ijkvelden met een eigen dynamica, maar worden éénduidig vastgelegd door de electromagnetische velden die op het systeem worden toegepast. Spin-baankoppeling is dus een voorbeeld van niet-Abels parallel transport.

Eerder hebben we het onderwerp van de hydrodynamica aangestipt, dat berust op massabehoud. In hoofdstuk 3 beantwoorden we een vraag die zich nu opdringt: bestaat er zoiets als niet-Abelse hydrodynamica? We laten zien dat voor niet-fasecoherente systemen er geen behoudswetten bestaan, laat staan dat men hydrodynamica daarmee kan bedrijven. In het geval van een fasecoherent spin condensaat treedt er echter een aangename verassing op. De quantummechanica eist dat de ordeparameter die het condensaat beschrijft éénwaardig is, en deze eis zorgt ervoor dat op mesoscopische schaal hydrodynamische behoudswetten opduiken. Dit is een mooi voorbeeld van emergentie in gecondenseerde-materie systemen: collectief gedrag kan tot rijkere verschijnselen aanleiding geven dan de samenstellende delen alleen.

Het vierde hoofdstuk vormt een uitwerking van het tweede hoofdstuk. We gebruiken de combinatie van spin-baan koppeling en het bestaan van een spin supervloeistof om een effect te bewijzen dat analoog is aan quantisatie van magnetische flux door supergeleidende ringen. We tonen aan dat een cylindrisch symmetrisch electrisch veld aangelegd op een spin-baan gekoppelde spin-supervloeistof een macroscopisch gequantiseerde ladingsdichtheid moet dragen.

Het vijfde hoofdstuk bespreekt ^{3}He, dat voorzover bekend het enige systeem is dat een spin supervloeistof kan vormen en spin-baan gekoppeld is. Dit is dus een goede kandidaat om het door ons voorspelde effect aan te tonen. We tonen aan dat dit door de dipolaire koppeling helaas onmogelijk is. Daar komt nog eens bij dat door het feit dat ^{3}Heveel zwaarder is dan een electron, de spin-baan koppeling zo klein is dat men een onpraktisch grote hoeveelheid ^{3}He nodig heeft.

In Deel II. bespreken we een systeem dat alle eigenschappen van een ijktheorie in zich draagt, de gedoteerde Mott isolator. Deze is belangrijk in de context van hoge-temperatuur supergeleiding, zoals waargenomen in de cupraten. In de ongedoteerde toestand zijn dit isolerende antiferromagneten met één electron per eenheidscel. Wanneer er electronen verwijderd worden (dotering), kunnen deze materialen echter supergeleidend worden. Hoewel deze ontdekking al twintig jaar geleden gedaan is, is het onderliggende mechanisme onbegrepen. Het feit dat het basismateriaal één electron per eenheidscel heeft, leidt tot een theoretisch idee dat veelbelovend is, zoals uitgelegd in hoofdstuk 6. Deze randvoorwaarde van enkele bezetting kan vertaald worden naar de introductie van een $SU(2)$-ijkveld, dat volledig dynamisch moet zijn om deze randvoorwaarde exact op te leggen. Dit ijkveld kan de vrijheidsgraden van spin en lading aan elkaar "lijmen" om het huis-tuin-en-keuken electron te vormen. In dat hoofdstuk wordt uitgelegd dat binnen de gemiddelde-veld benadering dit theoretisch idee van deze spin vloeistof werkelijkheid kan worden, analoog aan het bestaan van het quark-gluon plasma in deeltjesversnellers. Het begrip "projectieve symmetrie" speelt hierin een belangrijke rol.

In het zevende hoofdstuk beschouwen we de gedoteerde Mott isolator in de lege limiet, d.i., nul electronen per eenheidscel, om te laten zien dat in eerdere formuleringen van de $SU(2)$-ijktheorie het harde-kern karakter van de electronen onterecht niet is meegenomen. Wij tonen aan dat als dit wel gedaan wordt, zelfs de lege limiet binnen de $SU(2)$-ijktheorie correct beschreven kan worden.

Hoofdstuk 8 vormt de culminatie van dit proefschrift, waarin twee hoofdresultaten worden aangetoond. In de eerste plaats leidt het harde-kern gedrag van de ladingen tot fasescheiding, die ook kwantitatief overeenstemt met experimentele en eerdere numerieke resultaten. Het tweede resultaat is nieuw: als men de randvoorwaardestructuur en de harde-kern conditie correct behandelt, dan blijkt de supergeleidende ordeparameter een $d + s$-golf symmetrie te dragen. Deze symmetrie is in overeenstemming met Raman-verstrooiingsexperimenten en met c-as tunneling. Voorts groeit de s-golf component lineair met doping. Voorzover bekend is de $SU(2)$-ijktheorie de enige theorie die deze voorspelling doet.

Deze resultaten plaveien de weg voor een overkoepelende verklaring van experimentele resultaten die elkaar lijken uit te sluiten. Aan de ene kant blijken er in de supergeleidende fase nodale fermionen te bestaan, ook in het ondergedoteerde regime. Aan de andere kant blijken ladingsinhomogeniteiten (stripes) ook onvermijdelijk. Van deze stripes wordt tot nu toe echter gedacht dat ze het bestaan van nodale fermionen verbieden, hoewel dit niet in experimenten bevestigd wordt. Dit is een groot raadsel voor theoretici in het vakgebied. In het concluderende hoofdstuk stellen we een mogelijke oplossing voor. Het principe van projectieve symmetrie leidt tot het idee van de isospinspiraal. Dit is een gemiddelde-veld toestand die de landingsinhomogene supergeleider kan verenigen met nodale fermionen.

Deze resultaten en ideeën onderbouwen het nut van het beschouwen van niet-Abelse ijktheorieën in gecondenseerde-materie systemen, en kunnen instructief zijn voor de hoge-energie fysica.

Publications

1. R.A. Duine, H.T.C. Stoof and B.W.A. Leurs, *Noisy dynamics of a vortex in a partially Bose-Einstein condensed gas*, Phys. Rev. A **69**, 053623 (2004).

2. B.W.A. Leurs and J. Zaanen, *The* $SU(2)$*-Gauge Theory of the Substantially Doped Mott-Insulator: the phase separated* $d+s$ *superconductor.*, submitted to Phys. Rev. Lett.

3. B.W.A. Leurs, Z. Nazario, D.I. Santiago and J. Zaanen, *Non-Abelian hydrodynamics and the flow of spin in spin-orbit coupled substances*, accepted for publication in Annals of Physics.

4. B.W.A. Leurs, K.E. Luna and J. Zaanen, *Inhomogeneous states and Nodal Fermions in the* $SU(2)$ *Gauge Theory*, cond-mat/0707.3709.

Curriculum Vitae

Ik ben geboren op 26 februari 1979 te Sittard. Ik groeide op in Echt, en bezocht in Sittard het Gymnasium van het College Sittard. Daar behaalde ik in 1997 mijn diploma met de hoogste lof. In dat jaar ging ik aan de Universiteit Utrecht natuurkunde en wiskunde studeren. Mijn afstudeerscriptie schreef ik onder begeleiding van prof. dr. H.T.C. Stoof, en handelde over hoe vortices in Bose-Einstein-condensaten zich gedragen als men de invloed van de niet-gecondenseerde deeltjes niet kan verwaarlozen. In augustus 2002 behaalde ik mijn doctoraal natuurkunde cum laude, en in oktober rondde ik mijn wiskundeopleiding met een vrij doctoraal diploma af.

De daaropvolgende maand trad ik in dienst van de Stichting voor Fundamenteel Onderzoek der Materie (FOM), als onderzoeker in opleiding aan de Universiteit Leiden. Onder begeleiding van prof. dr. Jan Zaanen onderzocht ik verschillende ijktheoretische en topologische aspecten van hoge-temperatuur supergeleiders en supervloeistoffen. In dit proefschrift heb ik getracht de belangrijkste resultaten van dat onderzoek in een overkoepelende context onder te brengen.

Gedurende mijn promotietijd heb ik korte voorjaarsscholen van de Landelijke Onderzoeksschool Theoretische Natuurkunde in Nijmegen gevolgd. Daarnaast heb ik landelijke conferenties bezocht in Dalfsen, Eindhoven en Veldhoven, in welke laatste plaats ik een mondelinge voordracht heb gegeven over mijn werk. Ook heb ik in 2006 de internationale conferentie "Materials and Mechanisms of High-Temperature Superconductivity" in Dresden bijgewoond. Voorts heb ik in het voorjaar van 2005 een zes weken durend werkbezoek gebracht aan Stanford University, Palo Alto. Tenslotte heb ik gedurende deze vier jaar het werkcollege van het vak Statistische en Thermische Fysica I gegeven.

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