

Weak localization coexisting with a magnetic field in a normal-metal-superconductor microbridge

P. W. Brouwer and C. W. J. Beenakker

Instituut-Lorentz, University of Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands

(Received 12 April 1995; revised manuscript received 14 June 1995)

A random-matrix theory is presented which shows that breaking time-reversal symmetry by itself does *not* suppress the weak-localization correction to the conductance of a disordered metal wire attached to a superconductor. Suppression of weak localization requires applying a magnetic field as well as raising the voltage, to break both time-reversal symmetry and electron-hole degeneracy. A magnetic-field-dependent contact resistance obscured this anomaly in previous numerical simulations.

Weak localization is a quantum correction of order e^2/h to the classical conductance of a metal.¹ The word “localization” refers to the negative sign of the correction,² while the adjective “weak” indicates its smallness. In a wire geometry the weak-localization correction takes on the universal value³ $\delta G = -\frac{2}{3}e^2/h$ at zero temperature, independent of the wire length L or mean free path ℓ .⁴ The classical (Drude) conductance $G_0 \approx (N\ell/L)e^2/h$ is much greater than δG in the metallic regime, where the number of scattering channels $N \gg L/\ell$. Theoretically, the weak-localization correction is the term of order N^0 in an expansion of the average conductance $\langle G \rangle = G_0 + \delta G + \mathcal{O}(N^{-1})$ in powers of N . Experimentally, δG is measured by application of a weak magnetic field B , which suppresses the weak-localization correction but leaves the classical conductance unaffected.⁵ The suppression occurs because weak localization requires time-reversal symmetry (\mathcal{T}). In the absence of \mathcal{T} , quantum corrections to G_0 are of order N^{-1} and not of order N^0 . As a consequence, the magnetoconductance has a dip around $B=0$ of magnitude δG and width of order B_c (being the field at which one flux quantum penetrates the conductor).

What happens to weak localization if the normal-metal wire is attached at one end to a superconductor? This problem has been the subject of active research.⁶⁻¹² The term G_0 of order N is unaffected by the presence of the superconductor.⁶ The $\mathcal{O}(N^0)$ correction δG is increased but remains universal,^{9,10}

$$\delta G = -(2 - 8\pi^{-2})e^2/h \approx -1.19e^2/h. \quad (1)$$

In all previous analytical work zero magnetic field was assumed. It was surmised, either implicitly or explicitly,⁷ that $\delta G = 0$ in the absence of \mathcal{T} — but this was never actually calculated analytically. We have now succeeded in doing this calculation and would like to report the result, which was entirely unexpected.

We find that a magnetic field by itself is not sufficient to suppress the weak-localization correction, but only reduces δG by about a factor of 2. To achieve $\delta G = 0$ requires in addition the application of a sufficiently large voltage V to break the degeneracy in energy between the electrons (at energy eV above the Fermi energy E_F) and the Andreev-reflected holes (at energy eV below E_F). The electron-hole degeneracy (\mathcal{D}) is effectively broken when eV exceeds the

Thouless energy $E_c = \hbar v_F \ell / L^2$ (with v_F the Fermi velocity). Weak localization coexists with a magnetic field as long as $eV \ll E_c$. Our analytical results are summarized in Table I. These results disagree with the conclusions drawn in Ref. 7 on the basis of numerical simulations. We have found that the numerical data on the weak-localization effect were misinterpreted due to the presence of a magnetic-field-dependent contact resistance, which was not understood at that time.

The starting point of our calculation is the general relation between the differential conductance $G = dI/dV$ of the normal-metal-superconductor (NS) junction and the transmission and reflection matrices of the normal region,⁶

$$G = (4e^2/h) \text{tr } m(eV) m^\dagger(eV), \quad (2a)$$

$$m(\varepsilon) = t'(\varepsilon) [1 - \alpha(\varepsilon) r^*(-\varepsilon) r(\varepsilon)]^{-1} t^*(-\varepsilon), \quad (2b)$$

where $\alpha(\varepsilon) \equiv \exp[-2i \arccos(\varepsilon/\Delta)]$. Equation (2) holds for subgap voltages $V \leq \Delta/e$, and requires also $\Delta \ll E_F$ (Δ being the excitation gap in \mathcal{S}). We assume that the length L of the disordered normal region is much greater than the superconducting coherence length $\xi = (\hbar v_F \ell / \Delta)^{1/2}$. This implies that the Thouless energy $E_c \ll \Delta$. In the voltage range $V \leq E_c/e$ we may therefore assume that $eV \ll \Delta$, hence $\alpha = -1$. The $N \times N$ transmission and reflection matrices t , t' , r , and r' form the scattering matrix $S(\varepsilon)$ of the disordered normal region (N being the number of propagating modes at the Fermi level, which corresponds to $\varepsilon = 0$). It is convenient to use the polar decomposition

$$\begin{pmatrix} r' & t' \\ t & r \end{pmatrix} = \begin{pmatrix} v_1 & 0 \\ 0 & w_1 \end{pmatrix} \begin{pmatrix} i\sqrt{R} & \sqrt{T} \\ \sqrt{T} & i\sqrt{R} \end{pmatrix} \begin{pmatrix} v_2 & 0 \\ 0 & w_2 \end{pmatrix}.$$

TABLE I. Dependence of the weak-localization correction δG of a normal-metal wire attached to a superconductor on the presence or absence of time-reversal symmetry (\mathcal{T}) and electron-hole-degeneracy (\mathcal{D}). The entry in the upper-left corner was computed in Refs. 9 and 10.

$-\delta G [e^2/h]$	\mathcal{T}	no \mathcal{T}
\mathcal{D}	$2 - 8/\pi^2$	$2/3$
no \mathcal{D}	$4/3$	0

Here v_1 , v_2 , w_1 , and w_2 are $N \times N$ unitary matrices, T is a diagonal matrix with the N transmission eigenvalues $T_i \in [0, 1]$ on the diagonal, and $R = 1 - T$. Using this decomposition, and substituting $\alpha = -1$, Eq. (2b) can be replaced by

$$m(\varepsilon) = \sqrt{T(\varepsilon)} [1 + u(\varepsilon) \sqrt{R(-\varepsilon)} u^*(-\varepsilon) \sqrt{R(\varepsilon)}]^{-1} \\ \times u(\varepsilon) \sqrt{T(-\varepsilon)}, \quad u(\varepsilon) \equiv w_2(\varepsilon) w_1^*(-\varepsilon). \quad (2c)$$

We perform our calculations in the general framework of random-matrix theory. The only assumption about the distribution of the scattering matrix that we make is that it is isotropic, i.e., that it depends only on the transmission eigenvalues.¹³ In the presence of \mathcal{T} (for $B \ll B_c$), $S = S^T$, hence $w_1 = w_2^T$. (The superscript T denotes the transpose of a matrix.) If \mathcal{T} is broken, w_1 and w_2 are independent. In the presence of \mathcal{D} (for $eV \ll E_c$), the difference between $S(eV)$ and $S(-eV)$ may be neglected. If \mathcal{D} is broken, $S(eV)$ and $S(-eV)$ are independent. Of the four entries in Table I, the case that both \mathcal{T} and \mathcal{D} are present is the easiest, because then $u = 1$ and Eq. (2a) simplifies to⁶

$$G = (4e^2/h) \sum_n T_n^2 (2 - T_n)^{-2}. \quad (3)$$

The conductance is of the form $G = \sum_n f(T_n)$, known as a linear statistic on the transmission eigenvalues. General formulas^{9,10} for the weak-localization correction to the average of a linear statistic lead directly to Eq. (1). The three other entries in Table I, where either \mathcal{T} or \mathcal{D} (or both) are broken, are more difficult because G is no longer a linear statistic. We consider these three cases in separate paragraphs.

(1) \mathcal{D} , no \mathcal{T} . Because of the isotropy assumption, w_1 and w_2 , and hence u , are uniformly distributed in the unitary group $\mathcal{U}(N)$. We may perform the average $\langle \dots \rangle$ over the ensemble of scattering matrices in two steps: $\langle \dots \rangle = \langle \langle \dots \rangle_u \rangle_T$, where $\langle \dots \rangle_u$ and $\langle \dots \rangle_T$ are, respectively, the average over the unitary matrix u and over the transmission eigenvalues T_i . We compute $\langle \dots \rangle_u$ by an expansion in powers of N^{-1} . To integrate the rational function (2) of u over $\mathcal{U}(N)$, we first expand it into a geometric series and then use the general rules for the integration of polynomials of u .^{14,15} The polynomials we need are

$$\langle G \rangle_u = \frac{4e^2}{h} \sum_{p,q=0}^{\infty} M_{pq}, \quad (4a)$$

$$M_{pq} = \langle \text{tr } T(u \sqrt{R} u^* \sqrt{R})^p u T u^\dagger (\sqrt{R} u^T \sqrt{R} u^\dagger)^q \rangle_u. \quad (4b)$$

Neglecting terms of order N^{-1} , we find

$$M_{pq} = \begin{cases} N \tau_1^2 (1 - \tau_1)^{2p} & \text{if } p = q, \\ \tau_1 (\tau_1^2 + \tau_1 - 2\tau_2) (1 - \tau_1)^{p+q-1} - 2 \min(p, q) \\ \times \tau_1^2 (\tau_1^2 - \tau_2) (1 - \tau_1)^{p+q-2} & \text{if } |p - q| \text{ odd,} \\ 0 & \text{else,} \end{cases}$$

where we have defined the moment $\tau_k = N^{-1} \sum_i T_i^k$. The summation over p and q leads to

$$\frac{h}{4e^2} \langle G \rangle_u = \frac{N \tau_1}{2 - \tau_1} - \frac{4\tau_1 - 2\tau_1^2 + 2\tau_1^3 - 4\tau_2}{\tau_1(2 - \tau_1)^3}. \quad (5)$$

It remains to average over the transmission eigenvalues. Since τ_k is a linear statistic, we know that its sample-to-sample fluctuations $\delta \tau_k \equiv \tau_k - \langle \tau_k \rangle$ are an order $1/N$ smaller than the average.¹³ Hence

$$\langle f(\tau_k) \rangle_T = f(\langle \tau_k \rangle) [1 + \mathcal{O}(N^{-2})], \quad (6)$$

which implies that we may replace the average of the rational function (5) of the τ_k 's by the rational function of the average $\langle \tau_k \rangle$. This average has the $1/N$ expansion

$$\langle \tau_k \rangle = \langle \tau_k \rangle_0 + \mathcal{O}(N^{-2}), \quad (7)$$

where $\langle \tau_k \rangle_0$ is $\mathcal{O}(N^0)$. There is no term of order N^{-1} in the absence of \mathcal{T} . From Eqs. (5)–(7) we obtain the $1/N$ expansion of the average conductance,

$$\frac{h}{4e^2} \langle G \rangle = \frac{N \langle \tau_1 \rangle_0}{2 - \langle \tau_1 \rangle_0} - \frac{4\langle \tau_1 \rangle_0 - 2\langle \tau_1 \rangle_0^2 + 2\langle \tau_1 \rangle_0^3 - 4\langle \tau_2 \rangle_0}{\langle \tau_1 \rangle_0 (2 - \langle \tau_1 \rangle_0)^3} \\ + \mathcal{O}(N^{-1}). \quad (8)$$

Equation (8) is generally valid for any isotropic distribution of the scattering matrix. We apply it to the case of a disordered wire in the limit $N \rightarrow \infty$, $\ell/L \rightarrow 0$ at constant $N\ell/L$. The moments $\langle \tau_k \rangle_0$ are given by³

$$\langle \tau_1 \rangle_0 = \ell/L, \quad \langle \tau_2 \rangle_0 = \frac{2}{3} \ell/L. \quad (9)$$

Substitution into Eq. (8) yields the weak-localization correction $\delta G = -\frac{2}{3} e^2/h$, cf. Table I.

(2) \mathcal{T} , no \mathcal{D} . In this case one has $u^\dagger(-eV) = u(eV)$ and $u(eV)$ is uniformly distributed in $\mathcal{U}(N)$. A calculation similar to that in the previous paragraph yields for the average over u :

$$\frac{h}{4e^2} \langle G \rangle_u = N \tau_{1+} \tau_{1-} (\tau_{1+} + \tau_{1-} - \tau_{1+} \tau_{1-})^{-1} + (\tau_{1+} + \tau_{1-} \\ - \tau_{1+} \tau_{1-})^{-3} [2\tau_{1+}^2 \tau_{1-}^2 - \tau_{1+}^3 \tau_{1-}^2 - \tau_{1+}^2 \tau_{1-}^3 \\ - \tau_{2+} \tau_{1-}^2 - \tau_{2+}^2 \tau_{1-} + \tau_{2+} \tau_{1-}^3 + \tau_{1+}^3 \tau_{2-}], \quad (10)$$

where we have abbreviated $\tau_{k\pm} = \tau_k(\pm eV)$. The next step is the average over the transmission eigenvalues. We may still use Eq. (6), and we note that $\langle \tau_k(\varepsilon) \rangle \equiv \langle \tau_k \rangle$ is independent of ε . [The energy scale for variations in $\langle \tau_k(\varepsilon) \rangle$ is E_F , which is much greater than the energy scale of interest E_c .] Instead of Eq. (7) we now have the $1/N$ expansion

$$\langle \tau_k \rangle = \langle \tau_k \rangle_0 + N^{-1} \delta \tau_k + \mathcal{O}(N^{-2}), \quad (11)$$

which contains also a term of order N^{-1} because of the presence of \mathcal{T} . The $1/N$ expansion of $\langle G \rangle$ becomes

$$\frac{h}{4e^2} \langle G \rangle = \frac{N \langle \tau_1 \rangle_0}{2 - \langle \tau_1 \rangle_0} + \frac{2\delta\tau_1}{(2 - \langle \tau_1 \rangle_0)^2} \\ + \frac{2\langle \tau_1 \rangle_0^2 - 2\langle \tau_1 \rangle_0^3 - 2\langle \tau_2 \rangle_0 + 2\langle \tau_1 \rangle_0 \langle \tau_2 \rangle_0}{\langle \tau_1 \rangle_0 (2 - \langle \tau_1 \rangle_0)^3} \\ + \mathcal{O}(N^{-1}). \quad (12)$$

For the application to a disordered wire we use again Eq. (9) for the moments $\langle \tau_k \rangle_0$, which do not depend on whether \mathcal{T} is broken or not. We also need $\delta\tau_1$, which in the presence of \mathcal{T} is given by³ $\delta\tau_1 = -\frac{1}{3}$. Substitution into Eq. (12) yields $\delta G = -\frac{4}{3}e^2/h$, cf. Table I.

(3) *No \mathcal{T} , no \mathcal{D}* . Now $u(eV)$ and $u(-eV)$ are independent, each with a uniform distribution in $\mathcal{N}(N)$. Carrying out the two averages over $\mathcal{N}(N)$ we find

$$\frac{h}{4e^2} \langle G \rangle_u = \frac{N\tau_{1+}\tau_{1-}}{\tau_{1+} + \tau_{1-} - \tau_{1+}\tau_{1-}}. \quad (13)$$

The average over the transmission eigenvalues becomes

$$\frac{h}{4e^2} \langle G \rangle = \frac{N\langle \tau_1 \rangle_0}{2 - \langle \tau_1 \rangle_0} + \mathcal{O}(N^{-1}), \quad (14)$$

where we have used that $\delta\tau_1 = 0$ because of the absence of \mathcal{T} . We conclude that $\delta G = 0$ in this case, as indicated in Table I.

This completes the calculation of the weak-localization correction to the average conductance. Our results, summarized in Table I, imply a universal B and V dependence of the conductance of an NS microbridge. Raising first B and then V leads to two subsequent increases of the conductance, while raising first V and then B leads first to a decrease and then to an increase.

So far we have only considered the $\mathcal{O}(N^0)$ correction δG to $\langle G \rangle = G_0 + \delta G$. What about the $\mathcal{O}(N)$ term G_0 ? From Eqs. (8), (12), and (14) we see that if either \mathcal{T} or \mathcal{D} (or both) are broken,

$$G_0 = \frac{4e^2}{h} \frac{N\langle \tau_1 \rangle_0}{2 - \langle \tau_1 \rangle_0} = (2e^2/h)N(\frac{1}{2} + L/\ell)^{-1}. \quad (15)$$

In the second equality we substituted³ $\langle \tau_1 \rangle_0 = (1 + L/\ell)^{-1}$, which in the limit $\ell/L \rightarrow 0$ reduces to Eq. (9). If both \mathcal{T} and \mathcal{D} are unbroken, then we have instead the result¹⁶

$$G_0 = (2e^2/h)N[1 + L/\ell + \mathcal{O}(\ell/L)]^{-1}. \quad (16)$$

The difference between Eqs. (15) and (16) is a contact resistance, which equals $h/4Ne^2$ in Eq. (15) but is twice as large in Eq. (16). In contrast, in a normal-metal wire the contact resistance is $h/2Ne^2$, independent of B or V . The B - and V -dependent contact resistance in an NS junction is superimposed on the B - and V -dependent weak-localization correction. Since the contribution to $\langle G \rangle$ from the contact resistance is of order $(e^2/h)N(\ell/L)^2$, while the weak-localization correction is of order e^2/h , the former can only be ignored if $N(\ell/L)^2 \ll 1$. This is an effective restriction to the diffusive metallic regime, where $\ell/L \ll 1$ and $N\ell/L \gg 1$. To measure the weak-localization effect without contamination from the contact resistance if $N(\ell/L)^2$ is not $\ll 1$, one has two options: (1) measure the B dependence at fixed $V \gg E_c/e$; (2) measure the V dependence at fixed $B \gg B_c$. In both cases we predict an increase of the conductance, by an amount $\frac{4}{3}e^2/h$ and $\frac{2}{3}e^2/h$, respectively. In contrast, in the normal-state weak localization leads to a B dependence, but not to a V dependence.

We performed numerical simulations similar to those of Ref. 7 in order to test the analytical predictions. The disordered normal region was modeled by a tight-binding

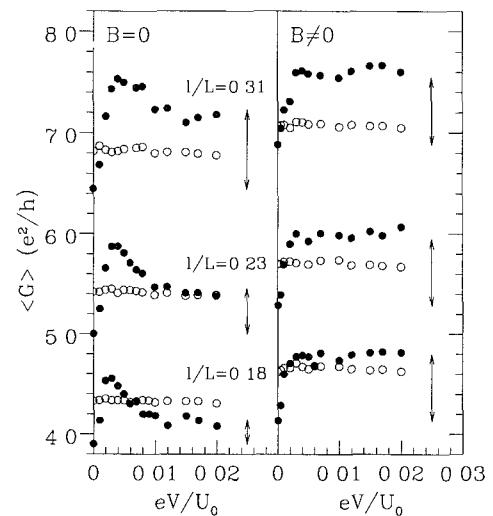


FIG. 1. Numerical simulation of the voltage dependence of the average differential conductance for $B=0$ (left panel) and for a flux $6 h/e$ through the disordered normal region (right panel). The filled circles are for an NS junction; the open circles represent the V -independent conductance in the normal state. The three sets of data points correspond, from top to bottom, to $\ell/L = 0.31, 0.23$, and 0.18 , respectively. The arrows indicate the theoretically predicted net increase of $\langle G \rangle$ between $V=0$ and $V \gg E_c/e$.

Hamiltonian on a square lattice (lattice constant a), with a random impurity potential at each site (uniformly distributed between $\pm \frac{1}{2}U_d$). The Fermi energy was chosen at $E_F = 1.57U_0$ from the band bottom ($U_0 = \hbar^2/2ma^2$). The length L and width W of the disordered region are $L = 167a$, $W = 35a$, corresponding to $N = 15$ propagating modes at E_F . The mean free path is obtained from the conductance $G = (2e^2/h)N(1 + L/\ell)^{-1}$ of the normal region in the absence of \mathcal{T} . The scattering matrix of the normal region was computed numerically at $\varepsilon = \pm eV$, and then substituted into Eq. (2a) to obtain the differential conductance.

In Fig. 1 we show the V dependence of G (averaged over some 10^3 impurity configurations) for three values of ℓ . The left panel is for $B=0$ and the right panel for a flux of $6 h/e$ through the disordered region. The V dependence for $B=0$ is mainly due to the contact resistance effect of order $N(\ell/L)^2$, and indeed one sees that the amount by which G increases depends significantly on ℓ .¹⁷ The V dependence in a \mathcal{T} -violating magnetic field is entirely due to the weak-localization effect, which should be insensitive to ℓ (as long as $\ell/L \ll 1 \ll N\ell/L$). This is indeed observed in the simulation. Quantitatively, we would expect that application of a voltage increases $\langle G \rangle$ by an amount $\frac{2}{3}e^2/h$ for the three

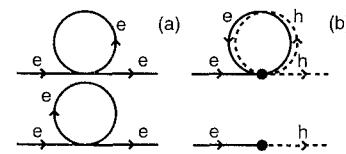


FIG. 2. (a) Two Feynman paths interfering constructively in the presence of \mathcal{T} . (b) Two paths involving Andreev reflection (solid dot), which interfere destructively both in the presence and absence of \mathcal{T} .

curves in the right panel, which agrees very well with what is observed. In the absence of a magnetic field the analytical calculation predicts a net increase in $\langle G \rangle$ by 0.79, 0.46, and $0.25 \times e^2/h$ (from top to bottom), which is again in good agreement with the simulation.

In normal metals, the weak-localization correction δG is explained in terms of constructive interference of pairs of time-reversed Feynman paths [Fig. 2(a)].⁵ This interference is destroyed by a magnetic field. One might wonder what kind of interfering paths are responsible for δG in an NS junction without \mathcal{T} . Although our theory is not formulated in terms of Feynman paths, an interpretation of the quantity M_{pq} in Eq. (4b) using Feynman paths is possible. The two

simplest interfering paths are shown in Fig. 2(b). Regardless of whether \mathcal{T} is broken or not, there is an exact cancellation of the phase shifts accumulated by the electron and the hole which traverse the loop in the same direction. What remains is a phase shift of π due to the double Andreev reflection. As a consequence, the path with the double loop interferes destructively with the path without a loop, giving rise to a negative δG .

We thank M. J. M. de Jong and J. A. Melsen for help with the numerical simulations and A. Altland for discussions on the Feynman-path interpretation. This work was supported by the Dutch Science Foundation NWO/FOM.

¹P. W. Anderson, E. Abrahams, and T. V. Ramakrishnan, Phys. Rev. Lett. **43**, 718 (1979); L. P. Gor'kov, A. I. Larkin, and D. E. Khmel'nitskii, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 248 (1979) [JETP Lett. **30**, 228 1979].

²Spin-orbit scattering is assumed to be negligible, otherwise a positive quantum correction appears.

³P. A. Mello and A. D. Stone, Phys. Rev. B **44**, 3559 (1991).

⁴The restriction to a wire geometry is essential for the universality. In a square or cube geometry, the weak-localization correction is L and ℓ dependent.

⁵G. Bergmann, Phys. Rep. **107**, 1 (1984).

⁶C. W. J. Beenakker, Phys. Rev. B **46**, 12 841 (1992).

⁷I. K. Marmorkos, C. W. J. Beenakker, and R. A. Jalabert, Phys. Rev. B **48**, 2811 (1993).

⁸Y. Takane and H. Otani, J. Phys. Soc. Jpn. **63**, 3361 (1994).

⁹C. W. J. Beenakker, Phys. Rev. B **49**, 2205 (1994).

¹¹K.-M. H. Lenssen *et al.*, in *Coulomb and Interference Effects in Small Electronic Structures*, edited by D. C. Glattli, M. Sanquer, and J. Trần Thanh Vân (Editions Frontières, Gif-sur-Yvette, 1994).

¹²Yu. V. Nazarov (unpublished).

¹³A. D. Stone, P. A. Mello, K. A. Muttalib, and J.-L. Pichard, in *Mesoscopic Phenomena in Solids*, edited by B. L. Al'tshuler, P. A. Lee, and R. A. Webb (North-Holland, Amsterdam, 1991).

¹⁴M. Creutz, J. Math. Phys. **19**, 2043 (1978).

¹⁵W. A. Friedman and P. A. Mello, Ann. Phys. (N.Y.) **161**, 276 (1985); P. A. Mello, J. Phys. A **23**, 4061 (1990).

¹⁶C. W. J. Beenakker, B. Rejaei, and J. A. Melsen, Phys. Rev. Lett. **72**, 2470 (1994).

¹⁷Our analytical calculation refers to the net increase of $\langle G \rangle$ between $V=0$ and $V \gg E_c/e$. We cannot describe the nonmonotonic V dependence at intermediate V , observed in the simulation for $B=0$.

¹⁰A. M. S. Macêdo and J. T. Chalker, Phys. Rev. B **49**, 4695 (1994).