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Lattice cryptography: isomorphisms & dual attacks

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Lattice Cryptography: Isomorphisms & Dual Attacks

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Universiteit
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Lattice Cryptography: Isomorphisms & Dual Attacks

Ludo Niels Pulles

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List of Publications

This is a comprehensive list of all the publications and preprints that the author has contributed to during their PhD programme. Each publication here lists authors alphabetically ordered. Each author has contributed equally to each publication.

Four chapters from this thesis are based on the following published papers, ordered chronologically.

- [DPPvW22a] Léo Ducas, Eamonn W. Postlethwaite, Ludo N. Pulles, and Wessel P. J. van Woerden. Hawk: Module LIP makes lattice signatures fast, compact and simple. In Shweta Agrawal and Dongdai Lin, editors, *ASIACRYPT 2022, Part IV*, volume 13794 of *LNCS*, pages 65–94. Springer, Cham, December 2022. DOI: 10.1007/978-3-031-22972-5_3.
- [DP23c] Léo Ducas and Ludo N. Pulles. Does the dual-sieve attack on learning with errors even work? In Helena Handschuh and Anna Lysyanskaya, editors, *CRYPTO 2023, Part III*, volume 14083 of *LNCS*, pages 37–69. Springer, Cham, August 2023. DOI: 10.1007/978-3-031-38548-3_2.
- [vGP25] Daniël M.H. van Gent and Ludo N. Pulles. HAWK: Having automorphisms weakens key. *IACR Communications in Cryptology*, 2(2), 2025. DOI: 10.62056/a3qjp2w9p.
- [DP26] Léo Ducas and Ludo N. Pulles. Accurate score prediction for Dual-Sieve attacks. *Journal of Cryptology*, 39(8), 2026. DOI: 10.1007/s00145-025-09560-7.

This thesis is also based on the following specification document submitted to NIST for the standardisation of additional signature schemes.

- [BBD⁺26] Joppe W. Bos, Olivier Bronchain, Léo Ducas, Serge Fehr, Yu-Hsuan Huang, Thomas Pornin, Eamonn W. Postlethwaite, Thomas Prest, Ludo N. Pulles, and Wessel van Woerden. HAWK. Technical report, National Institute of Standards and Technology, 2026. URL: <https://csrc.nist.gov/Projects/pqc-dig-sig/round-3-additional-signatures>.

In addition, the author has published the following papers that are not part of this thesis.

- [PT24] Ludo N. Pulles and Mehdi Tibouchi. Cryptanalysis of EagleSign. In Clemente Galdi and Duong Hieu Phan, editors, *SCN 24, Part II*, volume 14974 of *LNCS*, pages 165–186. Springer, Cham, September 2024. DOI: 10.1007/978-3-031-71073-5_8.

- [DPS25] Léo Ducas, Ludo N. Pulles, and Marc Stevens. Towards a modern LLL implementation. In Goichiro Hanaoka and Bo-Yin Yang, editors, *ASIACRYPT 2025, Part III*, volume 16247 of *LNCS*, pages 65–99. Springer, Singapore, December 2025. DOI: 10.1007/978-981-95-5099-9_3.

- [KKN⁺26] Alexander Karenin, Elena Kirshanova, Julian Nowakowski, Eamonn W. Postlethwaite, Ludo N. Pulles, Fernando Viridia, and Paul Vié. Cool + cruel = dual, and new benchmarks for sparse LWE. In Joan Daemen and Emmanuel Thomé, editors, *EUROCRYPT 2026, Part IV*, volume 16544 of *LNCS*, pages 304–333. Springer, Cham, May 2026. DOI: 10.1007/978-3-032-25327-9_11.

Finally, the author authored the following preprint that is not part of this thesis.

- [PV25] Ludo N. Pulles and Paul Vié. Accelerating the primal hybrid attack against sparse LWE using GPUs. Cryptology ePrint Archive, Paper 2025/1990, 2025. URL: <https://eprint.iacr.org/2025/1990>.