On the renormalization of random network models

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CHAPTER 6

The Harmonic Oscillator of Random Graphs

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Alessio Catanzaro, Subodh Patil, and Diego Garlaschelli. *"A General Solution for Network Models with Pairwise Edge Coupling"*. arXiv e-prints (2025): arXiv-2503.

Abstract

Network Models with couplings between link pairs are the simplest models for a class of networks with Higher Order interactions. In this paper we give an analytic, general solution to this family of Random Graph Models extending previous results obtained for specific interaction structures. We use the Hubbard-Stratonovich transform to show that such higher order models can be decoupled into families of effectively separable random graph models, which might be enhanced with auxiliary, possibly multi-valued, parameters identifying the thermodynamical phases of the system. We study the diagonalizability of the matrix of the interactions between links, that is itself an adjacency matrix on the dual network whose vertices are the nodes pairs, that can be dealt with using tools of algebraic graph theory. By orthogonalizing the interaction, we carry out the full computation of the partition functions and discuss why in some cases interactions effectively reduce to their Mean-Field versions.

§6.1 Introduction

Complex system science is heavily reliant on networks to model a multitude of systems. Random networks are among the most used tools to encode both the random nature of such objects as well as their evident heterogeneity. A statistical mechanics approach to random networks has been widely used by the network science community in the last 20 years [48, 47, 90, 110]. Yet a fundamental hurdle in the modelling of such systems with the aforementioned approaches is that few among the innumerable conceivable models of random networks can be solved, and if so it is often through approximate techniques that are model-dependent.

First order networks are easily solved, as the Hamiltonian describing them has only link specific parameters, which gives rise to a separable partition function and hence independent, link-specific probabilities. Regretfully when any form of coupling among links are introduced one has often to resort to either trivial toy-models of the interaction, or to be satisfied with approximated solutions only [48, 50].

The purpose of this paper is to partially cover this knowledge gap by showing how a general solution for any second order unweighted, undirected random network can be written in an exact way, and to solve for some cases of interest by showing some non-trivial behaviour. We also note that the analytical techniques we use could be extended to provide for a solution of weighted and directed random networks, but we do not believe the physical behaviour of such models to be fundamentally different than the one we present here in terms of phase transitions.

Our approach relies on the implementation of the multidimensional Hubbard-Stratonovich transform to account for any coupling structure between links. This also implies an effective description of the model through independent links, conditional on a given order parameter that can be treated as a random variable (and being a function of the model given parameters). Our solution pipeline also extends some results already found in the literature in order to put them under a single umbrella of explicative framework.

A further motivation for obtaining such a general solution comes from network inference. Ordinarily, exponential random graph models with pairwise dependence between links have an intractable normalizing constant, which forces practitioners to resort to approximate estimationschemes such as Markov Chain Monte Carlo maximum likelihood or related curved exponential-family techniques, both of which are known to be computationally expensive and prone to model degeneracy. Because the solution we derive gives explicit access to the partition function and hence to the full likelihood of any second order model in this class, it removes this bottleneck: the log-likelihood of an observed network can be evaluated, and in many cases maximized, directly and exactly, rather than estimated through simulation. This places second order, pairwise-coupled models on the same operationally tractable footing that first order maximum-entropy models already enjoy for network reconstruction and null-model based inference [90].

The Chapter is organized as follows: in Section 6.2 we introduce the techniques to solve the general problem, the pipelines to compute the partition function and write the single link probability function in an exact, although not explicit, manner.

In Section 6.3 we show how the machinery works for some cases of second order Random Network Models (RNM) to give meaning to our formulae, and we give explicit solutions for a few models of interest. Some models for which we solve were actually already found in the literature, but as stand-alone works and without the general, explicative framework we provide. Moreover, in the case of the model of 6.3.4, we highlight a connection between the model already analyzed in [49**?**] to the field of Coding Theory [155]. Finally in Section 6.4 we comment on how to give a rule of thumb to determine the possible phase structure of the models at stake and if the mean field approximation can give a satifying picture of the whole phase space.

§6.2 Solution to II order Random Network

An Random Network Model (RNM) is a probability law $P(\mathbf{A})$ for the realizations of the adjacency matrix $\mathbf{A}$ of a network. We choose to deal with unweighted, undirected networks so that $\mathbf{A}$ is symmetric and its entries can be either 0 or 1. This latter fact implies the fundamental relation

$$A_{ij} = A_{ij}^m \quad \forall m \tag{6.1}$$

that simplifies enormously the computations that follow along the entire chapter.

For instance, we can use relation (6.1) to show that we can always represent the probability function describing the model in the exponential form

$$P(\mathbf{A}) = \frac{e^{-H(\mathbf{A})}}{Z} \quad Z = \sum_{\{\mathbf{A}\}} e^{-H(\mathbf{A})} \tag{6.2}$$

where the Hamiltonian is in the form to eventually enforce constraints

$$H(A) = \sum_l \theta_l f_l(\mathbf{A}). \tag{6.3}$$

If one desires to enforce the full ERG formalism [48, 47, 90, 110], then has to add the constraint equations such that

$$\sum_{\{\mathbf{A}\}} f_l(\mathbf{A}) P(\mathbf{A}) = f_l^* \quad \forall l, \tag{6.4}$$

where f_l^* is the 'target' expected value of the property f_l, which can be interpreted in two ways. The first interpretation is to take f_l^* as the empirical value $f_l^* = f_l(\mathbf{A}^*)$ from a real observed matrix $\mathbf{A}^*$. A second interpretation is one where the parameter space of the model is explored exhaustively and f_l^* is the expected value of the property over the ensemble implied by the chosen parameters. We will follow this latter interpretation throughout this chapter.

Moreover, again making use of relation (6.1) the Hamiltonian function can always be written as a multilinear function of the adjacency matrix elements, in the form

$$H(\mathbf{A}) = h^0 + \theta_1 \sum_{i,j} h^1_{ij} A_{ij} + \theta_2 \sum_{i,j,k,l} h^2_{ijkl} A_{ij} A_{kl} + \\ + \theta_3 \sum_{i,j,k,l,m,n} h^3_{ijklmn} A_{ij} A_{kl} A_{mn} + \dots . \quad (6.5)$$

The above expansion is analogous to similar computation performed in the context of QFT for Grassmannian variable, where the relation (6.1) is replaced by the anticommuting relation.

We now proceed to show how a solution to the second order approximation of a model defined by the Hamiltonian (6.5) can be found in general.

§6.2.1 Evaluation of the partition function

Our task is to compute the partition function or equivalently the single link probability function of the model at stake. We have to maipulate the second order term of equation (6.5). The term h^0 can obviously be ignored as it is just a multiplicative constant in front of the partition function. We are left with

$$H(\mathbf{A}) = -\sum_{i<j} h^1_{ij} A_{ij} - \sum_{i<j,k<l} h^2_{ijkl} A_{ij} A_{kl} \quad (6.6)$$

First and foremost we change the labels of the indices to get a more manageable expression, since we see that the sums can be expressed in the couples of indices (i, j) which we order with index a going from 1 to $V = \frac{N(N-1)}{2}$ (without self-loops). We then express the interaction coefficient in order to scale with V as $h^2_{ijkl} = \frac{\beta_{ijkl}}{2V}$. The new Hamiltonian becomes

$$H(\mathbf{A}) = -\sum_{a=1}^{V} h^1_a A_a - \frac{1}{2V} \sum_{a \neq b}^{V} \beta_{ab} A_a A_b \quad (6.7)$$

we further manipulate this object to have a compact term to operate with. First we see that

$$\frac{1}{2} \sum_{a \neq b}^{V} \beta_{ab} A_a A_b = \frac{1}{2V} \sum_{a,b}^{V} \beta_{ab} A_a A_b - \frac{1}{2V} \sum_{a} \beta_{aa} A_a = \frac{1}{2V} \langle A|\beta|A\rangle - \frac{1}{2V} \langle A|\vec{\beta}\rangle \quad (6.8)$$

where $\vec{\beta}^T = (\beta_{11}, \beta_{22}, \cdots, \beta_{VV})$. We can readily see that this term can be reabsorbed in the local field term (and moreover that it is vanishing in the large V limit).

Then we write also the first term in the braket notation so that the whole Hamiltonian is in the form

$$H(\mathbf{A}) = -\sum_{a}^{V} A_a \left(h_a^1 + \frac{\beta_{aa}}{2V} \right) - \frac{1}{2V} \sum_{a,b}^{V} \beta_{ab} A_a A_b = - \langle A|\epsilon\rangle - \frac{1}{2V} \langle \mathbf{A}|\beta|\mathbf{A}\rangle \quad (6.9)$$

where β is a matrix that couple the links between them selves. This matrix is an adjacency matrix (either expected or realized) in disguise, over the network of links of the actual graph (dual or line graph in the literature), and we will make use of the properties of such a matrix to derive the behaviour of the models.

We assume the β matrix is symmetric (as the correlation between links is). Then it admits a spectral representation

$$\beta = \sum_{k=1}^{r} \lambda_k \, |\omega_k\rangle \, \langle \omega_k| \quad (6.10)$$

where r is the rank of the matrix, λ_k are the eigenvalues and $|\omega_k\rangle$ their respective orthonormal eigenvectors. We also split the positive spectrum part from the negative one. Let then r_+ be the number of positive eigenvalues, (including their multiplicities) and $r - r_+$ the number of the negative ones.

The Hamiltonian can be hence written as

$$H(\mathbf{A}) = - \langle A|\epsilon\rangle - \sum_{k=1}^{r_+} \frac{\lambda_k}{2V} \langle \mathbf{A}|\omega_k\rangle^2 - \sum_{k=r_++1}^{r} \frac{\lambda_k}{2V} \langle \mathbf{A}|\omega_k\rangle^2 \quad (6.11)$$

where β is the interaction matrix in which all the coupling between links is encoded, and V is the total number of possible links. Note also that the local parameter is $\epsilon_{ij} = h_{ij}^1 + \frac{\beta_{ijij}}{2V}$, clearly reducing to h_{ij}^1 in the large N limit.

All the ingredients are now in place to evaluate the partition function, as this latter form of equation (6.11) can be exponentiated as

$$e^{-H(\mathbf{A})} = e^{\langle \mathbf{A}|\epsilon\rangle + \sum_{k=1}^{r_+} \frac{\lambda_k}{2V} \langle \mathbf{A}|\omega_k\rangle^2 + \sum_{k=r_++1}^{r} \frac{\lambda_k}{2V} \langle \mathbf{A}|\omega_k\rangle^2} \quad (6.12)$$

$$= e^{\langle \mathbf{A}|\epsilon\rangle} \prod_{k=1}^{r_+} e^{\frac{\lambda_k}{2V} \langle \mathbf{A}|\omega_k\rangle^2} \prod_{k=r_++1}^{r} e^{\frac{\lambda_k}{2V} \langle \mathbf{A}|\omega_k\rangle^2} \quad (6.13)$$

and this form is very amenable to an Hubbard Stratonovich transformation. This integral representation allows us to disentangle the squares as

$$e^{a^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2} + \sqrt{2}ax} \quad (6.14)$$

if $a > 0$ and

$$e^{-a^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2} + i\sqrt{2}ax} \quad (6.15)$$

if $a < 0$. This allows to write

$$e^{-H(A)} = e^{\langle A|\epsilon\rangle} \prod_{k=1}^{r_+} \left[\frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{x^2}{2} + x\sqrt{\frac{\lambda_k}{V}}\langle A|\omega_k\rangle} \right] \prod_{k=r_++1}^{r} \left[\frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{x^2}{2} + ix\sqrt{\frac{\lambda_k}{V}}\langle A|\omega_k\rangle} \right] \tag{6.16}$$

which we want to compactly write as

$$\prod_{k=1}^{r} \left[\frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{x^2}{2} + x\sqrt{\frac{\lambda_k}{V}}\langle A|\omega_k\rangle + \frac{\langle A|\epsilon\rangle}{r}} \right] \tag{6.17}$$

by making sure we take the proper square root, so that

$$\sqrt{\lambda_k} = \begin{cases} |\sqrt{\lambda_k}| \text{ if } \lambda_k > 0 \\ i|\sqrt{|\lambda_k|}| \text{ if } \lambda_k < 0 \end{cases} \tag{6.18}$$

The last manipulation of this term is to realize that this product of integrals can be written as a repeated integral as

$$\begin{aligned} e^{-H(\mathbf{A})} &= \left(\frac{1}{\sqrt{2\pi}}\right)^r \int \cdots \int dx_1, \cdots dx_r e^{-\sum_{k=1}^r \frac{x_k^2}{2}} \prod_{k=1}^r e^{x_k\sqrt{\frac{\lambda_k}{V}}\langle A|\omega_k\rangle + \frac{\langle A|\epsilon\rangle}{r}} = \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^r \int \cdots \int dx_1, \cdots dx_r e^{-\sum_{k=1}^r \frac{x_k^2}{2}} \prod_{k=1}^r e^{\sum_{a=1}^V x_k\sqrt{\frac{\lambda_k}{V}} A_a(\omega_k)_a + \frac{A_a\epsilon_a}{r}} = \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^r \int \cdots \int dx_1, \cdots dx_r e^{-\sum_{k=1}^r \frac{x_k^2}{2}} \prod_{k=1}^r \prod_{a=1}^V e^{x_k\sqrt{\frac{\lambda_k}{V}} A_a(\omega_k)_a + \frac{A_a\epsilon_a}{r}} \end{aligned} \tag{6.19}$$

Now it is the time to integrate over the configurations

$$\begin{aligned} Z &= \sum_{\{A_a=0,1\}} e^{-H(\mathbf{A})} = \\ &= \sum_{\{A_a=0,1\}} \left(\frac{1}{\sqrt{2\pi}}\right)^r \int \cdots \int dx_1, \cdots dx_r e^{-\sum_{k=1}^r \frac{x_k^2}{2}} \prod_{k=1}^r \prod_{a=1}^V e^{x_k\sqrt{\frac{\lambda_k}{V}} A_a(\omega_k)_a + \frac{A_a\epsilon_a}{r}} = \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^r \int \cdots \int dx_1, \cdots dx_r e^{-\sum_{k=1}^r \frac{x_k^2}{2}} \prod_{k=1}^r \prod_{a=1}^V \sum_{\{A_a=0,1\}} e^{x_k\sqrt{\frac{\lambda_k}{V}} A_a(\omega_k)_a + \frac{A_a\epsilon_a}{r}} = \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^r \int \cdots \int dx_1, \cdots dx_r e^{-\sum_{k=1}^r \frac{x_k^2}{2}} \prod_{k=1}^r \prod_{a=1}^V \left[1 + e^{x_k\sqrt{\frac{\lambda_k}{V}}(\omega_k)_a + \frac{\epsilon_a}{r}}\right] \end{aligned} \tag{6.20}$$

and finally we can regroup the integrals in the form

$$Z = \prod_{k=1}^r \left[\sqrt{\frac{1}{2\pi}} \int dx e^{-\frac{x^2}{2}} \prod_{a=1}^V \left[1 + e^{x\sqrt{\frac{\lambda_k}{V}}(\omega_k)_a + \frac{\epsilon_a}{r}}\right] \right] \tag{6.21}$$

which allows for some further computation when the structure of the matrix β is known.

§6.2.2 Effective Separability of the partition function

We know from general Statistical Mechanics theory that the model defined by (6.6) might undergo phase transitions, with a phase structure dictated by the structure of the interaction matrix β. In the Hubbard-Stratonovich approach, the multiplicity of the possible phases of the model can be encoded in the vector of values $(x_1^*, x_2^*, \ldots)$ where each x_k^* can be either single-valued or multi-valued depending on the other parameters of the model. Conditionally on the values of $(x_1^*, x_2^*, \ldots)$, the model can be rewritten in an effectively noninteracting form where the full partition function (6.21) can be factorized.

A separable partition function in the form

$$Z = \prod_{a=1}^{V} z_a \tag{6.22}$$

is indeed what one would like to work with, since it has a straightforward connection to the probability of link formation. As a matter of facts, when the link variables are not coupled there is an identification between the terms in the product of Z and the bare link probabilities. This can be shown from

$$Z = \sum_{\{A_a=0,1\}} e^{-H(\mathbf{A})} = \sum_{\{A_a=0,1\}} \prod_{a=1}^{V} e^{h_a^{(1)} A_a} = \prod_{a=1}^{V} \left(1 + e^{h_a^{(1)}}\right) = \prod_{a=1}^{V} \frac{1}{1-p_a} \tag{6.23}$$

The formula (6.21) is amenable to show that this is the case also for second order interacting networks. As a matter of fact say that you isolate x_1 as

$$Z = \left(\frac{1}{\sqrt{2\pi}}\right)^{r-1} \int \cdots \int dx_2, \cdots dx_r e^{-\sum_{k=2}^{r} \frac{x_k^2}{2}} \prod_{k=2}^{r} \prod_{a=1}^{V} \left[1 + e^{x_k \sqrt{\frac{\lambda_k}{V}} (\omega_k)_a + \frac{\epsilon_a}{r}}\right] \cdot$$
$$\cdot \underbrace{\frac{1}{\sqrt{2\pi}} \int dx_1 e^{-\frac{x_1^2}{2}} \prod_{a=1}^{V} \left[1 + e^{x_1 \sqrt{\frac{\lambda_1}{V}} (\omega_1)_a + \frac{\epsilon_a}{r}}\right]}_{J_1}. \tag{6.24}$$

Now J_1 can be seen as a Gaussian expected value of a certain function, and it is clearly convergent, indeed

$$J_1 = \int dx_1 \mathcal{N}(0,1) \prod_{a=1}^{V} \left[1 + e^{x_1 \sqrt{\frac{\lambda_1}{V}} (\omega_1)_a + \frac{\epsilon_a}{r}}\right] = E_{\mathcal{N}(0,1)}\left[f(x_1)\right] = \mu \tag{6.25}$$

where we defined

$$f(x_1) \equiv \prod_{a=1}^{V} \left[1 + e^{x_1 \sqrt{\frac{\lambda_1}{V}} (\omega_1)_a + \frac{\epsilon_a}{r}}\right]. \tag{6.26}$$

Now we can subtract the expected value in the integral so that

$$0 = \int dx_1 \mathcal{N}(0,1) \left[\prod_{a=1}^{V} \left(1 + e^{x_1 \sqrt{\frac{\lambda_1}{V}} (\omega_1)_a + \frac{\epsilon_a}{r}} \right) - \mu \right]. \tag{6.27}$$

This last integral evaluates to 0, hence there must be at least one change of sign of the function

$$g(x_1) = \prod_{a=1}^{V} \left(1 + e^{x_1 \sqrt{\frac{\lambda_1}{V}} (\omega_1)_a + \frac{\epsilon_a}{r}} \right) - \mu \tag{6.28}$$

therefore, since the function is continuous, there must be at least a x_1^* in which $g(x_1^*) = 0$, so that there is at least one x_1^* in which

$$J_1 = f(x_1^*) = \prod_{a=1}^{V} \left[1 + e^{x_1^* \sqrt{\frac{\lambda_1}{V}} (\omega_1)_a + \frac{\epsilon_a}{r}} \right] \tag{6.29}$$

By repeating this argument for all the k, we can write the partition function in the form

$$Z = \prod_{k=1}^{r} \prod_{a=1}^{V} \left[1 + e^{x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_a + \frac{\epsilon_a}{r}} \right] \tag{6.30}$$

then by switching the products, we can identify the single link partition function as

$$z_a^{\text{eff}} = \prod_{k=1}^{r} \left[1 + e^{x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_a + \frac{\epsilon_a}{r}} \right]. \tag{6.31}$$

Finally, we can further manipulate this object to have a nicer representation

$$\begin{aligned} z_a^{\text{eff}} &= \sum_{A_a=0,1} \prod_{k=1}^{r} e^{A_a \left(x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_a + \frac{\epsilon_a}{r} \right)} = \sum_{A_a=0,1} e^{A_a \sum_{k=1}^{r} \left(x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_a + \frac{\epsilon_a}{r} \right)} \\ &= \sum_{A_a=0,1} e^{A_a \left(\epsilon_a + \sum_{k=1}^{r} x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_a \right)} = 1 + e^{\epsilon_a} \prod_{k=1}^{r} e^{x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_a} \equiv \frac{1}{1 - p_a^{\text{eff}}} \end{aligned} \tag{6.32}$$

or for the link probability itself

$$p_a^{\text{eff}} = \frac{e^{\epsilon_a + \sum_k x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_a}}{1 + e^{\epsilon_a + \sum_k x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_a}} \tag{6.33}$$

as

$$Z(x_1^*, x_2^*, \ldots) = \prod_{i<j} z_{ij}^{\text{eff}}(x_1^*, , x_2^*, \ldots) \tag{6.34}$$

where

$$z_{ij}^{\text{eff}}(x_1^*, x_2^*, \ldots) = 1 + e^{\epsilon_{ij}} \prod_{k=1}^{r} e^{x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_{ij}} \tag{6.35}$$

$$= 1 + e^{\epsilon_{ij} + \sum_{k=1}^{r} x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_{ij}} \tag{6.36}$$

in which the contribution of each link has been isolated. The last expression is equivalent to that corresponding to a modified, non-interacting ($\beta = 0$) version of the Hamiltonian in eq.(6.6), i.e.

$$H(\mathbf{A}|x_1^*, x_2^*, \ldots) = -\langle \mathbf{A} | \epsilon'(x_1^*, x_2^*, \ldots) \rangle \tag{6.37}$$

where we have introduced the modified parameters

$$|\epsilon'(x_1^*, x_2^*, \ldots)\rangle = |\epsilon\rangle + \sum_{k=1}^{r} x_k^* \sqrt{\frac{\lambda_k}{V}} \, |\omega_k\rangle \, . \tag{6.38}$$

for the partition function and

$$p_{ij}^{\text{eff}} = \frac{e^{\epsilon_{ij} + \sum_{k=1}^{r} x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_{ij}}}{1 + e^{\epsilon_{ij} + \sum_{k=1}^{r} x_k^* \sqrt{\frac{\lambda_k}{V}} (\omega_k)_{ij}}} \tag{6.39}$$

for the link probability expression. Formulae (6.36) and (6.39), together with equation (6.21) are the core results of this paper. These two latter formulae however need some explanation as a new key parameter x_k^* is introduced.

§6.2.3 Phase structure

The price to pay to write Equations (6.36) (6.39) with such a product over node couples and to have an effectively separable partition function over links is that a phase structure emerges. In our case we encode this behaviour in the parameter x_k^*. In the derivation of Equation (**??**) (see SI), we learn that x_k^* is the value(s) of the auxiliary field x_k for which $f(x_k^*)$ assumes the value of the Gaussian average of $f(x_k)$ over the real axis. This parameter is global and depends only on the specific structure of the interaction, combined with the local fields ϵ_{ij}, and most importantly its unicity is not guaranteed. Actually, when the parameters combination allows, x_k^* splits into multiple values (found numerically) that, when insterted in Equation (6.39), give rise to different values of the single link probability. A scheme of what happens is shown in Figure 6.1.

In the cases in which the parameter splits, the actual value of the link probability changes, hence identifying different phases (possibly denser or sparser). When drawing a realization of the ensemble, one has to keep in mind that also the various values of x_k^* come each with its own probability and can be computed as we show in Section 6.3.1.

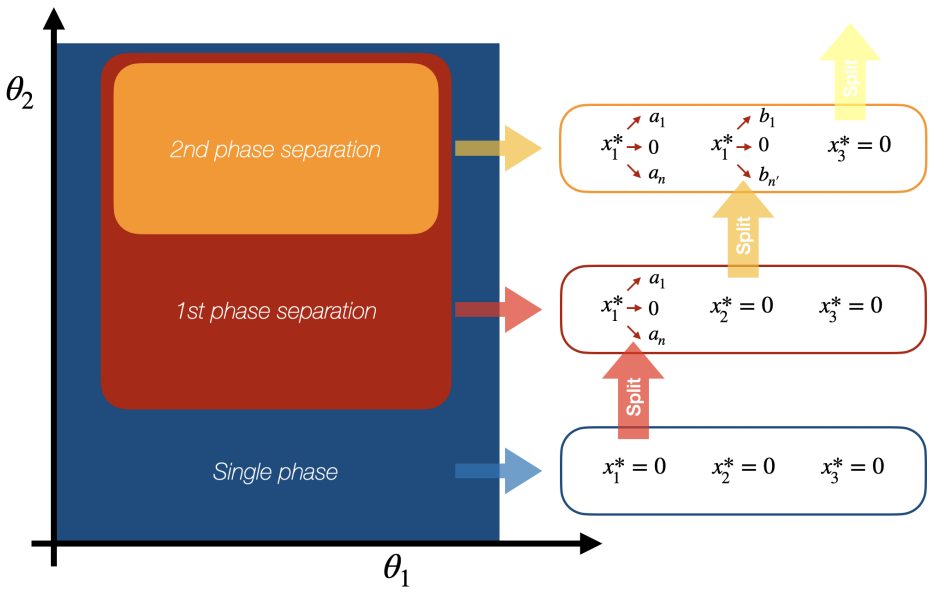

Figure 6.1: A schematic representation of the phase diagram of a II order RNM and the parameter x_k^* in the sparse case (large, negative ϵ_{ij} implying $x_k^* \simeq 0$). Firstly one has to look if the region of the parameter space whom is looking at may be of single phase or with phase separation. This can be done by actually computing the partition function but in general is done by looking at the multiplicity of solutions of the equation defined by putting Formula (6.28) *equal to zero.*

§6.3 II order interacting models

We will now show how equations (6.21), (6.36) and (6.39) can be used to actually solve some II order RNMs. We will start with a very simple, homogeneous one to use as a lamppost for the general behaviour and explore more complex models going on.

§6.3.1 RNM with Homogeneous interactions

Say that the Hamiltonian is in the form

$$H(\mathbf{A}) = -\theta_1 \langle \mathbf{A}|\epsilon\rangle - \frac{\theta_2}{2V} \langle \mathbf{A}|\beta|\mathbf{A}\rangle \text{ with } \beta = |1\rangle\langle 1| \tag{6.40}$$

so that every link interacts with any other link in the same way. The the eigenvalues of this matrix are all 0 except from 1 whose value is $\lambda = V$. The relative eigenvector is also easy to find as it is just $\frac{|1\rangle}{\sqrt{V}}$. Moreover we set ourselves in the case $\epsilon_a = \epsilon$, which can be further reabsorbed in the θ_1 parameter. The partition function then becomes easy to compute explicitly in the large N limit. By formula (6.21) we have

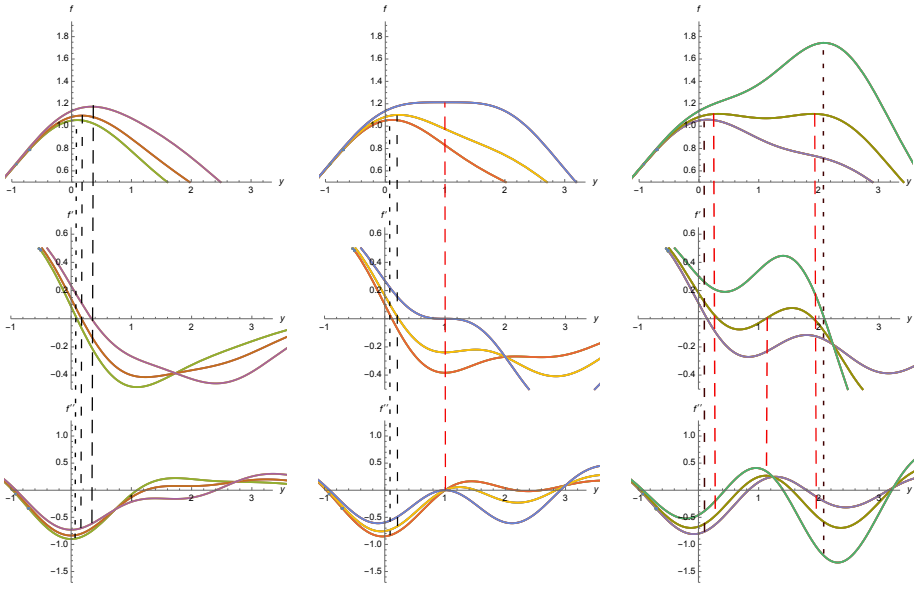

Figure 6.2: Plots of the function z^{eff} for different values of θ_1,θ_2. From top to bottom we plot z^{eff}, its first and second derivative to locate the maxima and show that there is a regime in which 2 maxima are present. The parameter θ_1 always takes the values $-3, -2.5, -2$, while θ_2 is, from left to right $3, 4, 5$ respectively.

$$z_a^{\text{eff}} = \lim_{V\to\infty}\left[\sqrt{\frac{1}{2\pi}}\int dx e^{-\frac{x^2}{2}}\left[1+e^{x\sqrt{\frac{\theta_2}{V}}+\theta_1}\right]^V\right]^{\frac{1}{V}} =$$

$$= \lim_{V\to\infty}\left[\sqrt{\frac{V}{2\pi}}\int dy\left[e^{-\frac{y^2}{2}}\left(1+e^{y\sqrt{\theta_2}+\theta_1}\right)\right]^V\right]^{\frac{1}{V}} =$$

$$= \max_{-\infty\le y\le\infty}\left\{e^{-\frac{y^2}{2}}(1+e^{y\sqrt{\theta_2}+\theta_1})\right\} \quad (6.41)$$

where the substitution $x = y\sqrt{V}$ has been made in the second row. The last step comes from a Laplace saddle point evaluation.

To evaluate the z^{eff} in the large limit we need to see where its derivative vanishes and its second derivative is negative, which can be done graphically and is shown in figure 6.2. We can see that for $\theta_2 \geq 4$ there is a range of θ_1 for which the partition function has more than one maximum. Indeed, in the parameter space we can locate a whole region in which the maxima are two and hence there are two possible values of the link probability. This is shown in figure 6.3.

Phase coexistence

In figure 6.4 we can see the effective link probability for two "slices" of the phase diagram, one across the single phase region (red, $\theta_1 = -4$) and another one across the single phase, then a bifurcation point, then again single phase (blue, $\theta_1 = -1$).

The key ingredients for the analysis of this simple, interacting model are now in place. The single link probability $p_{\text{eff}} = 1 - \frac{1}{z}$ is the probability by which links are assigned, yet we know that z can have two different values that will lead to two different phases. From figure 6.4 it is clear that when $\theta_2 >> 4$ and $\theta_2 < 4$ only one phase is present, the sparse one and the dense one respectively, while in the regime in which $\theta_2 \gtrsim 4$ the two phases coexist.

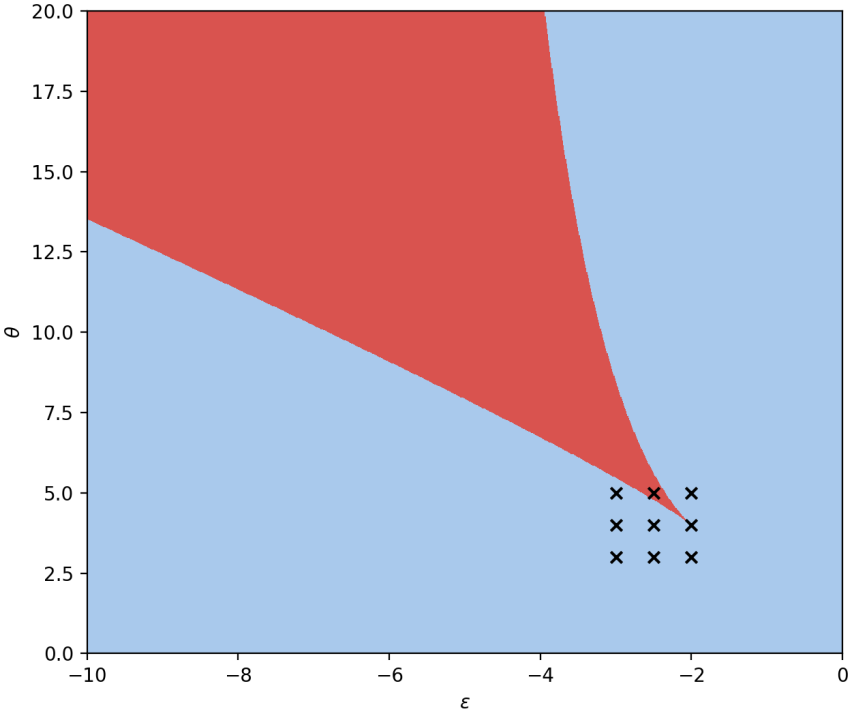

Figure 6.3: Phase diagram for the system. The blue region corresponds to the place where there is only one maximum, hence only one link probability. In the red region the maxima of equation (6.41) *are two and the link probabilities as well. The crosses indicate the coordinates of Fig. 6.2*

This behavior can be effectively described by the scheme in which, every time a realization of the networks has to be generated, a coin is tossed. This coin determines if the realization will be a dense or a sparse one, *i.e.* this first coin toss chooses the coin that will be tossed for each link in the realization. Then, when another realization has to be generated, this "phase coin" is tossed again and so on. For each instance of the network there will be one previous coin toss that will determine its phase.

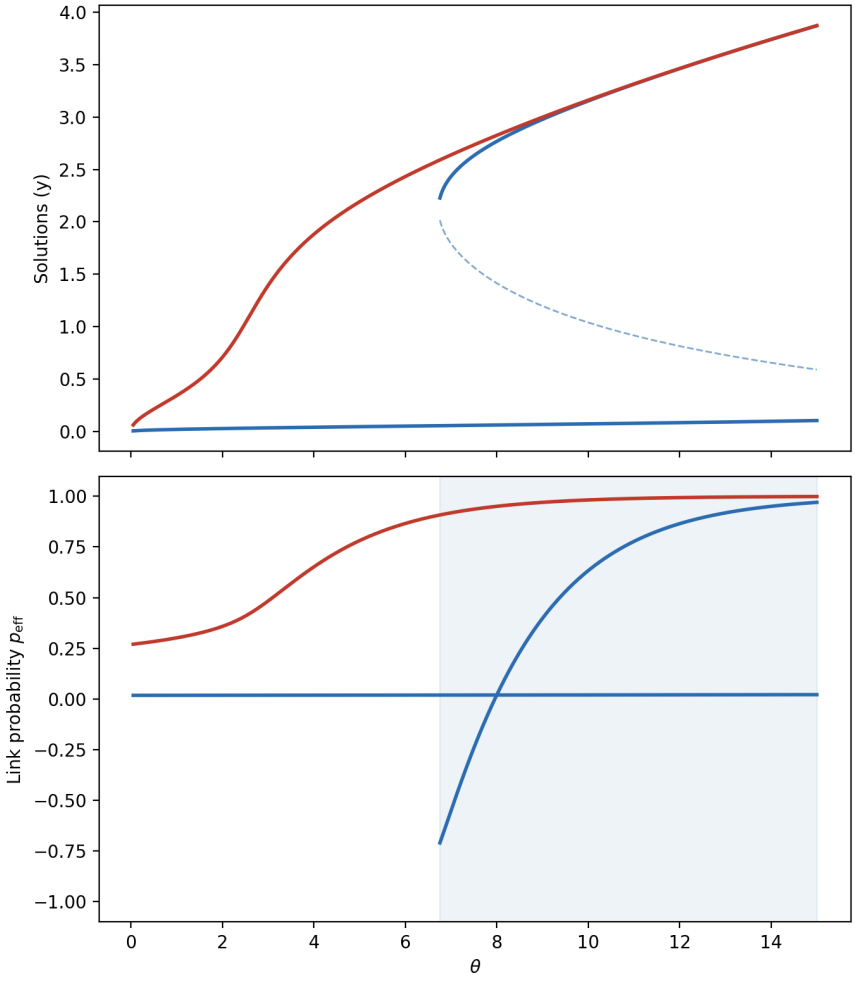

Figure 6.4: Possible solutions of equating to zero the derivative (top panel, dashed line indicate minima) of 6.41 and effective link probability (bottom panel, shaded region is where two solutions are possible) vs θ from 0 to 15 with $\epsilon = -4$ (blue) and $\epsilon = -1$ (red).

The relative abundance of the two phases (and hence the weights of the "phase coin") can be computed in the large V limit by plugging the respective values of p in the formula for the overall $P(\mathbf{A})$. For instance, for the dense phase in which p^{eff}

$$P\left\{\sum_{i<j} A_{ij} = Vp_d\right\} \propto \binom{V}{V\frac{1+p_d}{2}} e^{\theta_1 V p_d + \frac{\theta_2}{2} V p_d^2} \tag{6.42}$$

$$\simeq \exp\left\{V\left[\theta_1 p_d + \frac{\theta_2}{2}p_d^2 + + \frac{1-p_d}{2}\ln\frac{2}{1-p_d} + \frac{1+p_d}{2}\ln\frac{2}{1+p_d}\right]\right\} \tag{6.43}$$

and for the sparse phase the same with p_d replaced with $p_s \equiv p_{\text{eff}}|_{x_1^*=0}$, and the exact proportionality constant is immediately computed by imposing

$$P\left\{\sum_{i<j} A_{ij} = Vp_d\right\} + P\left\{\sum_{i\leq j} A_{ij} = Vp_s\right\} = 1. \tag{6.44}$$

§6.3.2 A square-like interaction

After the previous simple model, that is given as a lamppost to highlight the main features of the systems at stake, we examine a slightly more complicated 1-rank model. We will show that this interaction structure give rise to an interesting phenomena that captures the power of single link probability representation found in equation (6.39). Let us consider the Hamiltonian with the general rank-one interaction matrix $\beta = |v\rangle\langle v|$

$$H(\mathbf{A}) = -\sum_{i<j}\epsilon_{ij}A_{ij} - \frac{C}{2V}\sum_{\substack{i<j\\k<l}} v_{ij}v_{kl}A_{ij}A_{kl} \tag{6.45}$$

In this case the link probability would simply read

$$p_{ij} = \frac{e^{\epsilon_{ij}+x^*\sqrt{C}v_{ij}}}{1+e^{\epsilon_{ij}+x^*\sqrt{C}v_{ij}}} \tag{6.46}$$

Heterogeneity as a low temperature phenomenon

Let us assume then that the model is even simpler and the local variable $\epsilon_{ij} = \epsilon$ is constant. Then the possible values of p_{ij} are of two kinds: a constant/homogeneous one in the high temperature/non effectively interacting phase, and an heterogeneous value for the phases in which $x^* \neq 0$.

In such a simple case already we see that the two phases describe two very different models: one is a standard homogeneous Erdos Renyi with probability given by $p = \ln\frac{e^\epsilon}{1+e^\epsilon}$, and the other in which heterogeneity comes into place. We can also construct the interaction in such a way that $v_{ij} = v_i + v_j$. Then this reads as an effective configuration model in the low temperature phase as

$$p_{ij} = \frac{K x_i x_j}{1 + K x_i x_j} \tag{6.47}$$

where $K = e^{\epsilon + x^* \sqrt{C}}$ and $x_i = e^{v_i}$, $x_j = e^{v_j}$. If we bring back the heterogeneity also in the ϵ_{ij} parameter as well, assuming the same separation $\epsilon_{ij} = \epsilon_i + \epsilon_j$ we will have a family of configuration models, described by the probability

$$p_{ij} = \frac{x_i x_j}{1 + x_i x_j} \tag{6.48}$$

where $x_i = e^{\epsilon_i + \sqrt{C} x^* v_i}$.

§6.3.3 Community Induced Interactions

We now extend the complexity of the simple models we have solved to include interacting link communities. Such communities, involving link instead of nodes, can arise for different reasons and link community detection has been already a quite active field [152, 153].

We will solve for a model in which link communities are induced by the nodes' ones, but the computations can be readily generalized to the desired case where communites are defined by some other property.

Among the most used (simple) models in network theory there are the Stochastic Block Model (SBM) and Core-Periphery one (CP). They both posit the partition of nodes into communities, 2 in the CP case or more for the SBM. We want to show here that such partitions can induce quite naturally a correlation structure between links so that already a quite complex behaviour arises. The process to construct the correlation structure is the following:

- for simplicity, let us assume there are two communities of nodes, with N_1 and N_2 nodes in each;
- links between nodes of community 1 has a certain probability $\frac{e^{\epsilon_1}}{1+e^{\epsilon_1}}$, links between nodes of community 2 has a probability $\frac{e^{\epsilon_2}}{1+e^{\epsilon_2}}$ and links between the two communities has probability $\frac{e^{\epsilon_3}}{1+e^{\epsilon_3}}$; creating a block matrix of the form

$$\epsilon = \begin{pmatrix} \epsilon_1 \mathbf{1}_{N_1 \times N_1} & \epsilon_3 \mathbf{1}_{N_1 \times N_2} \\ \epsilon_3 \mathbf{1}_{N_2 \times N_1} & \epsilon_2 \mathbf{1}_{N_2 \times N_2} \end{pmatrix} \tag{6.49}$$

  where $1_{N \times M}$ is the all-ones rectuangular $N \times M$ matrix.
- this community structure for the nodes induces a community structure for the links: the links between nodes of community 1, that are $V_1 = N_1(N_1 - 1)/2$, the links between nodes of community 2 that are $V_2 = N_2(N_2 - 1)/2$, and the links between nodes belonging to different communities that are $V_3 = N_1 N_2$;
- this will result in a 3×3 block structure of the interaction matrix, as if there was a SBM for three communities of links in the form

$$\beta = \begin{pmatrix} \beta_1 \mathbf{1}_{V_1 \times V_1} & \beta_{12} \mathbf{1}_{V_1 \times V_2} & \beta_{13} \mathbf{1}_{V_1 \times V_3} \\ \beta_{12} \mathbf{1}_{V_2 \times V_1} & \beta_2 \mathbf{1}_{V_2 \times V_2} & \beta_{23} \mathbf{1}_{V_2 \times V_3} \\ \beta_{13} \mathbf{1}_{V_3 \times V_1} & \beta_{23} \mathbf{1}_{V_3 \times V_2} & \beta_3 \mathbf{1}_{V_3 \times V_3} \end{pmatrix} \tag{6.50}$$

A scheme of how this community induced interaction is created is shown in figure 6.5.

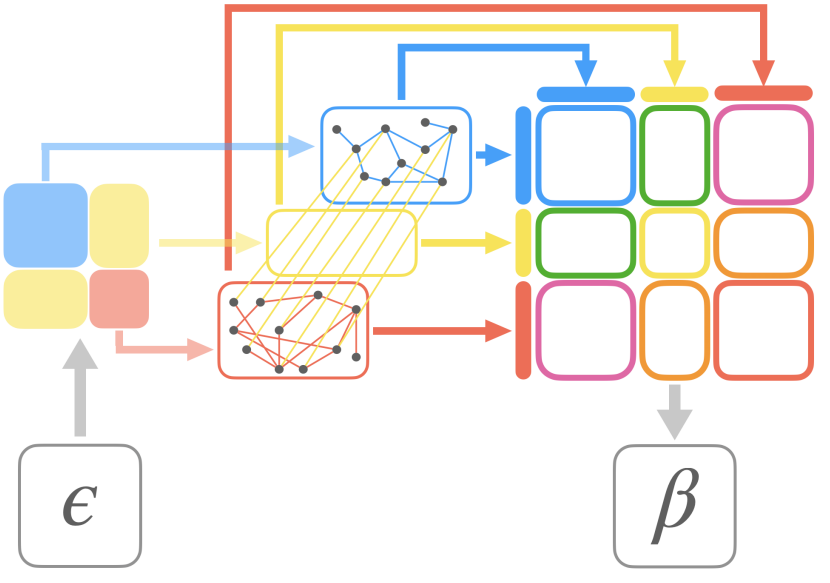

Figure 6.5: A pictorial representation of how the 3×3 correlation matrix is induced. The two nodes community structure allows for three values of link probability: in the first community (blue), in the second community (red), across the two communities (yellow). The possible interactions between these links are then blue/blue, blue/yellow (green), blue/red (purple), and so on.

This three by three matrix is the first example we can devise of phase space that has more than one region of phase coexistence, as indeed we will show that up to 27 phases can coexist. The rank three interaction defined by matrix β can be again decomposed with a basis of orthonormal vectors.

We now proceed to compute also the single link partition function for each one of the three kind of links that the community structure induces. The first big difference from the homogeneous interacting model is that there is a mixture of the eigenmodes and the equations that define the maxima are not the very same partition functions. This means that there are three distinct maximization procedure, each one returning a parameter that can have multiple values, and then the link-specific partition function is a function of all three parameters, hence possibly multifurcating in many different values.

Laplace large size approximation

The theorem we will make use of is the following [154]:

Let $g(x)$ and $f(x)$ be continuous and positive in the range $]\infty, \infty[$. Then

$$\lim_{N\to\infty}\left(\int dx g(x) f(x)^N\right)^{\frac{1}{N}} = \max_{-\infty\leq x\leq\infty}\{f(x)\} \tag{6.51}$$

The integrals we are interested in this work are of the kind

$$\frac{1}{\sqrt{2\pi}}\int dx\left[e^{-\frac{x^2}{2}}\prod_{a=1}^{V}\left(1+e^{x\sqrt{\frac{\lambda_k}{V}}(\omega_k)_a+\frac{\epsilon_a}{r}}\right)\right] = \sqrt{\frac{V}{2\pi}}\int dy\left[e^{-V\frac{y^2}{2}}\prod_{a=1}^{V}\left(1+e^{y\sqrt{\lambda_k}(\omega_k)_a+\frac{\epsilon_a}{r}}\right)\right] \tag{6.52}$$

where the substitution $x = y\sqrt{V}$ was made. We will assume for simplicity that all ϵ_a are the same[1]. Now we represent the eigenvector entries in the following way. Say that we are interested in the k^{th} eigenmode, we say that the corresponding eigenvector will be made of the following entries

$$(\omega_k)_1 \text{ with multiplicity } m_1 \tag{6.53}$$

$$(\omega_k)_2 \text{ with multiplicity } m_2 \tag{6.54}$$

$$(\omega_k)_3 \text{ with multiplicity } m_3 \tag{6.55}$$

and so on. Then we introduce the fraction of equal entries as $c_i = \frac{m_i}{V}$. Of course $\sum_{i=1}^{T} c_i = 1$.

Now we can write the product in the integral as

$$\prod_{a=1}^{V}\left(1+e^{y\sqrt{\lambda_k}(\omega_k)_a+\frac{\epsilon_a}{r}}\right) = \prod_{i=1}^{T}\left(1+e^{y\sqrt{\lambda_k}(\omega_k)_i+\frac{\epsilon}{r}}\right)^{c_i V} \tag{6.56}$$

We are now ready to compute the limit

[1] Or at least are coupled to the respective entry of the eigenvector as in the model described in 6.3.3. However a generalization to the case in which they also have their respective multiplicity and heterogeneity can be done.

$$\lim_{V\to\infty}\left[\sqrt{\frac{V}{2\pi}}\int dy\left[e^{-\frac{y^2}{2}}\prod_{i=1}^{T}\left(1+e^{y\sqrt{\lambda_k}(\omega_k)_i+\frac{\epsilon}{r}}\right)^{c_i}\right]^V\right]^{\frac{1}{V}}$$
$$=\underbrace{\lim_{V\to\infty}\left[\sqrt{\frac{V}{2\pi}}\right]^{\frac{1}{V}}}_{=1}\lim_{V\to\infty}\left[\int dy\left[e^{-\frac{y^2}{2}}\prod_{i=1}^{T}\left(1+e^{y\sqrt{\lambda_k}(\omega_k)_i+\frac{\epsilon}{r}}\right)^{c_i}\right]^V\right]^{\frac{1}{V}}$$
$$=\max_{-\infty\le x\le\infty}\left\{e^{-\frac{y^2}{2}}\prod_{i=1}^{T}\left(1+e^{y\sqrt{\lambda_k}(\omega_k)_i+\frac{\epsilon}{r}}\right)^{c_i}\right\} \quad (6.57)$$

from which the exact computations performed for the various models follow. In particular we want to highlight the fact that if, there are entries whose multiplicities are extensive, *i.e.* grow as a fraction of V, and other sets with subextensive multiplicities, then the only surviving part is that of the extensive entries, as they clearly dominate the product. In a more formal way, let us have M different values of the entries, and let the large V structure be $c_1(V)\to_{V\to\infty} C_1>0, c_2(V)\to_{V\to\infty} C_2>0,\ldots,c_M(V)\to_{V\to\infty} C_M>0, c_{M+1}(V)\to_{V\to\infty} C_{M+1}=0, c_{M+2}(V)\to_{V\to\infty} C_{M+2}=0,\ldots c_T(V)\to_{V\to\infty} C_T=0$. Then

$$\lim_{V\to\infty}\left[\int dy\left[e^{-\frac{y^2}{2}}\prod_{i=1}^{T}\left(1+e^{y\sqrt{\lambda_k}(\omega_k)_i+\frac{\epsilon}{r}}\right)^{c_i}\right]^V\right]^{\frac{1}{V}}$$
$$=\max_{-\infty\le x\le\infty}\left\{e^{-\frac{y^2}{2}}\prod_{i=1}^{M}\left(1+e^{y\sqrt{\lambda_k}(\omega_k)_i+\frac{\epsilon}{r}}\right)^{c_i}\right\} \quad (6.58)$$

A basis for the community interaction matrix

We can write a rank three matrix in the following form

$$\beta=\lambda_1|\omega_1\rangle\langle\omega_1|+\lambda_2|\omega_2\rangle\langle\omega_2|+\lambda_3|\omega_3\rangle\langle\omega_3| \quad (6.59)$$

where the eigenvectors are

$$\langle\omega_1|=\frac{1}{\sqrt{n_1}}(\underbrace{(\omega_1)_1,\ldots,(\omega_1)_1}_{V_1},\underbrace{(\omega_1)_2,\ldots,(\omega_1)_2}_{V_2},\underbrace{(\omega_1)_3,\ldots,(\omega_1)_3}_{V_3}) \quad (6.60)$$

$$\langle\omega_2|=\frac{1}{\sqrt{n_2}}(\underbrace{(\omega_2)_1,\ldots,(\omega_2)_1}_{V_1},\underbrace{(\omega_2)_2,\ldots,(\omega_2)_2}_{V_2},\underbrace{(\omega_2)_3,\ldots,(\omega_2)_3}_{V_3}) \quad (6.61)$$

$$\langle\omega_3| = \frac{1}{\sqrt{n_3}}(\underbrace{(\omega_3)_1,\ldots,(\omega_3)_1}_{V_1},\underbrace{(\omega_3)_2,\ldots,(\omega_3)_2}_{V_2},\underbrace{(\omega_3)_3,\ldots,(\omega_3)_3}_{V_3}). \tag{6.62}$$

where the norms are

$$n_1 = V_1\,(\omega_1)_1^2 + V_2\,(\omega_1)_2^2 + V_3\,(\omega_1)_3^2 \tag{6.63}$$

$$n_2 = V_1\,(\omega_2)_1^2 + V_2\,(\omega_2)_2^2 + V_3\,(\omega_2)_3^2 \tag{6.64}$$

$$n_3 = V_1\,(\omega_3)_1^2 + V_2\,(\omega_3)_2^2 + V_3\,(\omega_3)_3^2 \tag{6.65}$$

The eigenvalues can be computed from the secular equation

$$\beta\,|\omega_i\rangle = \lambda_i\,|\omega_i\rangle \tag{6.66}$$

so that

$$\begin{cases} \lambda_1 = V_1\beta_1 + V_2\beta_{12}\frac{(\omega_1)_2}{(\omega_1)_1} + V_3\beta_{13}\frac{(\omega_1)_3}{(\omega_1)_3} \\ \lambda_2 = V_1\beta_1 + V_2\beta_{12}\frac{(\omega_2)_2}{(\omega_2)_1} + V_3\beta_{13}\frac{(\omega_2)_3}{(\omega_2)_1} \\ \lambda_3 = V_1\beta_1 + V_2\beta_{12}\frac{(\omega_3)_2}{(\omega_3)_1} + V_3\beta_{13}\frac{(\omega_3)_3}{(\omega_3)_1} \end{cases} \tag{6.67}$$

The equation for the spectral decomposition defines the relations between the eigenvector components and the β parameters

$$\begin{cases} \beta_1 = \lambda_1\,(\omega_1)_1^2 + \lambda_2\,(\omega_2)_1^2 + \lambda_3\,(\omega_3)_1^2 \\ \beta_2 = \lambda_1\,(\omega_1)_2^2 + \lambda_2\,(\omega_2)_2^2 + \lambda_3\,(\omega_3)_2^2 \\ \beta_3 = \lambda_1\,(\omega_1)_3^2 + \lambda_2\,(\omega_2)_3^2 + \lambda_3\,(\omega_3)_3^2 \\ \beta_{12} = \lambda_1\,(\omega_1)_1\,(\omega_1)_2 + \lambda_2\,(\omega_2)_1\,(\omega_2)_2 + \lambda_3\,(\omega_3)_1\,(\omega_3)_2 \\ \beta_{13} = \lambda_1\,(\omega_1)_1\,(\omega_1)_3 + \lambda_2\,(\omega_2)_1\,(\omega_2)_3 + \lambda_3\,(\omega_3)_1\,(\omega_3)_3 \\ \beta_{23} = \lambda_1\,(\omega_1)_2\,(\omega_1)_3 + \lambda_2\,(\omega_2)_2\,(\omega_2)_3 + \lambda_3\,(\omega_3)_2\,(\omega_3)_3 \end{cases} \tag{6.68}$$

and these are supplemented by the orthogonality equations so that

$$\begin{cases} V_1\,(\omega_1)_1\,(\omega_2)_1 + V_2\,(\omega_1)_2\,(\omega_2)_2 + V_3\,(\omega_1)_3\,(\omega_2)_3 = 0 \\ V_1\,(\omega_1)_1\,(\omega_3)_1 + V_2\,(\omega_1)_2\,(\omega_3)_2 + V_3\,(\omega_1)_3\,(\omega_3)_3 = 0 \\ V_1\,(\omega_2)_1\,(\omega_3)_1 + V_2\,(\omega_2)_2\,(\omega_3)_2 + V_3\,(\omega_2)_3\,(\omega_3)_3 = 0 \end{cases} \tag{6.69}$$

which completely defines the ω and the eigenvalues. To compute the partition function we also define the quantities $\alpha = \frac{V_1}{V}$, $\beta = \frac{V_2}{V}$ and $\gamma = \frac{V_3}{V}$. The partition function to be computed will be

$$Z = \tag{6.70}$$

$$\underbrace{\frac{1}{\sqrt{2\pi}}\int dx e^{-\frac{x^2}{2}}\left[1+e^{x\sqrt{\frac{\lambda_1}{Vn_1}}(\omega_1)_1+\frac{\epsilon_1}{3}}\right]^{\alpha V}\left[1+e^{x\sqrt{\frac{\lambda_1}{Vn_1}}(\omega_1)_2+\frac{\epsilon_2}{3}}\right]^{\beta V}\left[1+e^{x\sqrt{\frac{\lambda_1}{Vn_1}}(\omega_1)_3+\frac{\epsilon_3}{3}}\right]^{\gamma V}}_{I_1}. \tag{6.71}$$

$$\underbrace{\frac{1}{\sqrt{2\pi}}\int dx e^{-\frac{x^2}{2}}\left[1+e^{x\sqrt{\frac{\lambda_2}{Vn_2}}(\omega_2)_1+\frac{\epsilon_1}{3}}\right]^{\alpha V}\left[1+e^{x\sqrt{\frac{\lambda_2}{Vn_2}}(\omega_2)_2+\frac{\epsilon_2}{3}}\right]^{\beta V}\left[1+e^{x\sqrt{\frac{\lambda_2}{Vn_2}}(\omega_2)_3+\frac{\epsilon_3}{3}}\right]^{\gamma V}}_{I_2}. \tag{6.72}$$

$$\underbrace{\frac{1}{\sqrt{2\pi}}\int dx e^{-\frac{x^2}{2}}\left[1+e^{x\sqrt{\frac{\lambda_3}{Vn_3}}(\omega_3)_1+\frac{\epsilon_1}{3}}\right]^{\alpha V}\left[1+e^{x\sqrt{\frac{\lambda_3}{Vn_3}}(\omega_3)_2+\frac{\epsilon_2}{3}}\right]^{\beta V}\left[1+e^{x\sqrt{\frac{\lambda_3}{Vn_3}}(\omega_3)_3+\frac{\epsilon_3}{3}}\right]^{\gamma V}}_{I_3} \tag{6.73}$$

$$= (z_1)^{\alpha V}(z_2)^{\beta V}(z_3)^{\gamma V} \tag{6.74}$$

In the large V we have to compute

$$\lim_{V\to\infty}(Z)^{\frac{1}{V}} = (z_1)^{\alpha}(z_2)^{\beta}(z_3)^{\gamma} = \lim_{V\to\infty}(I_1)^{\frac{1}{V}}\lim_{V\to\infty}(I_2)^{\frac{1}{V}}\lim_{V\to\infty}(I_3)^{\frac{1}{V}} \tag{6.75}$$

that gives

$$\lim_{V\to\infty}(I_1)^{\frac{1}{V}}$$
$$= \max_{y_1}\left\{e^{-\frac{y_1^2}{2}}\left(1+e^{y_1\sqrt{\frac{\lambda_1}{n_1}}(\omega_1)_1+\frac{\epsilon_1}{3}}\right)^{\alpha}\left(1+e^{y_1\sqrt{\frac{\lambda_1}{n_1}}(\omega_1)_2+\frac{\epsilon_2}{3}}\right)^{\beta}\left(1+e^{y_1\sqrt{\frac{\lambda_1}{n_1}}(\omega_1)_3+\frac{\epsilon_3}{3}}\right)^{\gamma}\right\}$$
$$\equiv \max f(y_1) \tag{6.76}$$

$$\lim_{V\to\infty}(I_2)^{\frac{1}{V}}$$
$$= \max_{y_2}\left\{e^{-\frac{y_2^2}{2}}\left(1+e^{y_2\sqrt{\frac{\lambda_2}{n_2}}(\omega_2)_1+\frac{\epsilon_1}{3}}\right)^{\alpha}\left(1+e^{y_2\sqrt{\frac{\lambda_2}{n_2}}(\omega_2)_2+\frac{\epsilon_2}{3}}\right)^{\beta}\left(1+e^{y_2\sqrt{\frac{\lambda_2}{n_2}}(\omega_2)_3+\frac{\epsilon_3}{3}}\right)^{\gamma}\right\}$$
$$\equiv \max g(y_2) \tag{6.77}$$

$$\lim_{V\to\infty}(I_1)^{\frac{1}{V}}$$
$$= \max_{y_3}\left\{e^{-\frac{y_3^2}{2}}\left(1+e^{y_3\sqrt{\frac{\lambda_3}{n_3}}(\omega_3)_1+\frac{\epsilon_1}{3}}\right)^{\alpha}\left(1+e^{y_3\sqrt{\frac{\lambda_3}{n_3}}(\omega_3)_2+\frac{\epsilon_2}{3}}\right)^{\beta}\left(1+e^{y_3\sqrt{\frac{\lambda_3}{n_3}}(\omega_3)_3+\frac{\epsilon_3}{3}}\right)^{\gamma}\right\}$$
$$\equiv \max h(y_3) \tag{6.78}$$

we now introduce the notation

$$y_1^* = \arg\max f(y_1) \tag{6.79}$$

$$y_2^* = \arg\max g(y_2) \tag{6.80}$$

$$y_3^* = \arg\max h(y_3) \tag{6.81}$$

$$\begin{aligned}
&(z_1)^{\alpha}(z_2)^{\beta}(z_3)^{\gamma} \\
&= \left[e^{-\frac{(y_1^*)^2}{2}}\left(1+e^{y_1^*\sqrt{\frac{\lambda_1}{n_1}}(\omega_1)_1+\frac{\epsilon_1}{3}}\right)^{\alpha}\left(1+e^{y_1^*\sqrt{\frac{\lambda_1}{n_1}}(\omega_1)_2+\frac{\epsilon_2}{3}}\right)^{\beta}\left(1+e^{y_1^*\sqrt{\frac{\lambda_1}{n_1}}(\omega_1)_3+\frac{\epsilon_3}{3}}\right)^{\gamma}\right] \\
&\left[e^{-\frac{(y_2^*)^2}{2}}\left(1+e^{y_2^*\sqrt{\frac{\lambda_2}{n_2}}(\omega_2)_1+\frac{\epsilon_1}{3}}\right)^{\alpha}\left(1+e^{y_2^*\sqrt{\frac{\lambda_2}{n_2}}(\omega_2)_2+\frac{\epsilon_2}{3}}\right)^{\beta}\left(1+e^{y_2^*\sqrt{\frac{\lambda_2}{n_2}}(\omega_2)_3+\frac{\epsilon_3}{3}}\right)^{\gamma}\right] \\
&\left[e^{-\frac{(y_3^*)^2}{2}}\left(1+e^{y_3^*\sqrt{\frac{\lambda_3}{n_3}}(\omega_3)_1+\frac{\epsilon_1}{3}}\right)^{\alpha}\left(1+e^{y_3^*\sqrt{\frac{\lambda_3}{n_3}}(\omega_3)_2+\frac{\epsilon_2}{3}}\right)^{\beta}\left(1+e^{y_3^*\sqrt{\frac{\lambda_3}{n_3}}(\omega_3)_3+\frac{\epsilon_3}{3}}\right)^{\gamma}\right]
\end{aligned} \tag{6.82}$$

by which we can recognize the terms

$$z_1 = \left[e^{-\frac{(y_1^*)^2}{2\alpha}}\left(1+e^{y_1^*\sqrt{\frac{\lambda_1}{n_1}}(\omega_1)_1+\frac{\epsilon_1}{3}}\right)\left(1+e^{y_2^*\sqrt{\frac{\lambda_2}{n_2}}(\omega_2)_1+\frac{\epsilon_1}{3}}\right)\left(1+e^{y_3^*\sqrt{\frac{\lambda_3}{n_3}}(\omega_3)_1+\frac{\epsilon_1}{3}}\right)\right] \tag{6.83}$$

$$z_2 = \left[e^{-\frac{(y_2^*)^2}{2\beta}}\left(1+e^{y_1^*\sqrt{\frac{\lambda_1}{n_1}}(\omega_1)_2+\frac{\epsilon_2}{3}}\right)\left(1+e^{y_2^*\sqrt{\frac{\lambda_2}{n_2}}(\omega_2)_2+\frac{\epsilon_2}{3}}\right)\left(1+e^{y_3^*\sqrt{\frac{\lambda_3}{n_3}}(\omega_3)_2+\frac{\epsilon_2}{3}}\right)\right] \tag{6.84}$$

$$z_3 = \left[e^{-\frac{(y_3^*)^2}{2\gamma}}\left(1+e^{y_1^*\sqrt{\frac{\lambda_1}{n_1}}(\omega_1)_3+\frac{\epsilon_3}{3}}\right)\left(1+e^{y_2^*\sqrt{\frac{\lambda_2}{n_2}}(\omega_2)_3+\frac{\epsilon_3}{3}}\right)\left(1+e^{y_3^*\sqrt{\frac{\lambda_3}{n_3}}(\omega_3)_3+\frac{\epsilon_3}{3}}\right)\right]. \tag{6.85}$$

In figure 6.6 we show an instance of interaction matrix value for which there is a complete separation of phases. Each one of the modes of the interaction give rise to a three-partite split in the corresponding part of the partition function.

§6.3.4 Node mediated interaction

Up until now we considered models in which some kind of community structure was induced between links, and this fact was the responsible for the phase transition. Now we consider a new kind of interaction, in which links are correlated via nodes' property, namely links are correlated if they share one node in common. This structure is highly

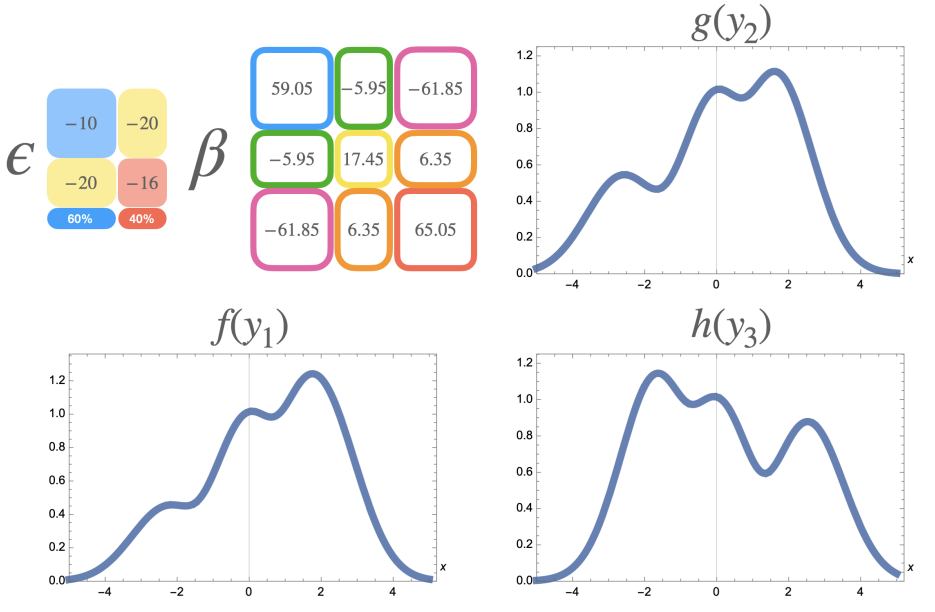

Figure 6.6: Instance of a crowded phase space: each one of the functions defined in (6.76), (6.77), (6.78) *has multiple maxima, hence the each link probability has 27 possible values.*

non-trivial and has been considered in the literature of ERGs as the "Two star" model [49]. In the following we will solve it in a different fashion, making us of some results coming from coding theory. A scheme of how this interaction works is depicted in figure 6.7.

The Johnson graph

Again we assume that all ϵ are the same for simplicity, but note that once one has diagonalized the matrix of the correlations, formula (6.39) allows to write the solution of a heterogeneous degree two-star model. Let us concentrate on the matrix then. The interaction we want to implement is such that links are correlated if they share one node. This kind of interaction can mimic for instance traffic jam propagation. We can define β as follows

$$\beta_{ijkl} = \begin{cases} 1 \text{ if } |(i,j) \cap (k,l)| = 1 \\ 0 \text{ otherwise.} \end{cases} \tag{6.86}$$

and this matrix is actually a known object in Algebraic Coding Theory known as the Johnson Graph $J(N,2)$ adjacency matrix. The Johnson Graph $J(N,k)$ is a graph in which the nodes are the subsets of $\{1,2,\ldots,N\}$ elements of dimension k. Edges between these nodes exist if the intersection of the two nodes has dimension $k-1$. If $k=2$ the adjacency matrix of $J(N,2)$ reduces exactly to the β we are looking for.

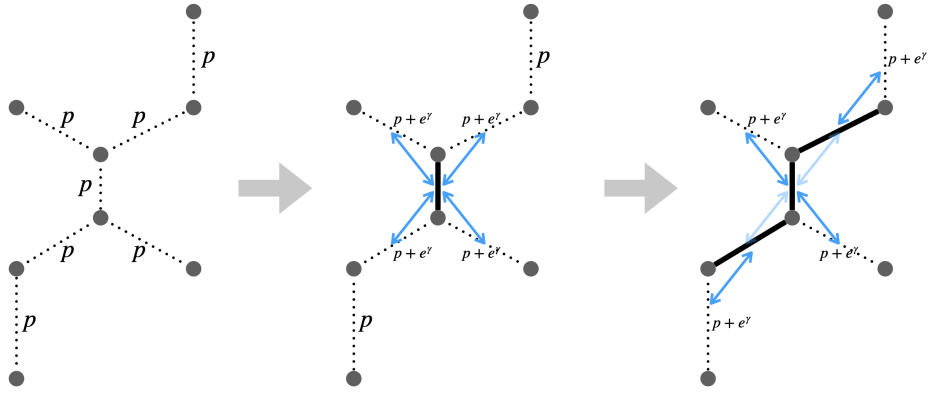

Figure 6.7: A scheme of how the interaction is mediated by nodes. At first, any link has the same (low) probability p to exist. If one flips up and connects two nodes, then it start interacting with the sites of the links that share one node with that link. Then more links might be promoted and propagate the interaction with their node-neighbours.

Moreover if one wants to treat this model as the Two-Star ERG, can see that the Hamiltonian can be readily parametrized and put in the more familiar form

$$\begin{aligned} H(\mathbf{A}) &= -\frac{\theta_1-\theta_2}{2}\sum_{i<j}A_{ij} - \frac{\theta_2}{2}\sum_{i<j}\sum_{k<l}\beta_{ijkl}A_{ij}A_{kl} = \\ &= -\frac{\theta_1-\theta_2}{2}\sum_{i<j}A_{ij} - \frac{\theta_2}{2}\sum_{i<j<k}A_{ij}A_{jk} = \\ &= -\theta_1 m(\mathbf{A}) - \theta_2 s(\mathbf{A}) \end{aligned} \tag{6.87}$$

From literature it is known that the Johnson Graph Adjacency matrix is full rank, and has known eigenspaces [155, 156, 157]. We choose to reabsorb the parameters so that $\frac{\theta_1-\theta_2}{2} = \epsilon$ and $\theta_2 = \gamma$. For our case of interest $k = 2$, the eigenvalues of β are

$$\begin{cases} \lambda = 2N-4 \text{ with multiplicity } \mu_1 = 1; \\ \mu = N-4 \text{ with multiplicity } \mu_2 = N-1; \\ \nu = -2 \text{ with multiplicity } \mu_3 = \frac{(N-2)(N-3)}{2}; \end{cases} \tag{6.88}$$

Before showing how the Partition function decouples in the eigenmodes, we must pay attention to the fact that the scaling of the largest eigenvalue is of the order $N \simeq \sqrt{V}$. This means that while in equation (5.2) the term to keep confined the interaction term was generically put to V, in this case that would kill the interaction

entirely. As a matter of fact, we have to rescale by a quantity of the same order of the largest eigenvalue of the interaction, *i.e.* in this case $\frac{1}{N}$.

Then partition function decouples as

$$Z = \underbrace{\left[\sqrt{\frac{1}{2\pi}}\int dx e^{-\frac{x^2}{2}} \prod_{a=1}^{V}\left[1+e^{x\sqrt{\frac{\gamma_\lambda}{N}}(\omega_k)_a+\frac{\epsilon}{r}}\right]\right]}_{I_1}\cdot$$
$$\cdot \underbrace{\prod_{k=2}^{N}\left[\sqrt{\frac{1}{2\pi}}\int dx e^{-\frac{x^2}{2}} \prod_{a=1}^{V}\left[1+e^{x\sqrt{\frac{\gamma_\mu}{N}}(\omega_k)_a+\frac{\epsilon}{r}}\right]\right]}_{I_2}\cdot$$
$$\cdot \underbrace{\prod_{k=N+1}^{V}\left[\sqrt{\frac{1}{2\pi}}\int dx e^{-\frac{x^2}{2}} \prod_{a=1}^{V}\left[1+e^{x\sqrt{\frac{\gamma_\nu}{N}}(\omega_k)_a+\frac{\epsilon}{r}}\right]\right]}_{I_3}. \quad (6.89)$$

The large N limit

We now want compute the terms of the partition function in the large N, V limit. The single-link partition function will be

$$z = \lim_{V\to\infty}(Z)^{\frac{1}{V}} = \lim_{V\to\infty}(I_1)^{\frac{1}{V}}\lim_{V\to\infty}(I_2)^{\frac{1}{V}}\lim_{V\to\infty}(I_3)^{\frac{1}{V}} \quad (6.90)$$

We have already seen the first term in previous cases, that is

$$\lim_{V\to\infty}(I_1)^{\frac{1}{V}} = \max_y\left\{e^{-\frac{y^2}{2}}\left(1+e^{y\sqrt{2\gamma}+\frac{\epsilon}{r}}\right)\right\} = \sum_{A_{ij}=0,1} e^{-\frac{y_*^2}{2}}\left(e^{\left[y_*\sqrt{2\gamma}+\frac{\epsilon}{r}\right]A_{ij}}\right) \quad (6.91)$$

where $y_* = \arg\max\left\{e^{-\frac{y^2}{2}}\left(1+e^{y\sqrt{2\gamma}+\frac{\epsilon}{r}}\right)\right\}$.

We see there is a phase transition if $\gamma > 2$ (and the proper ϵ is there).

For the third term we see that the scaling term N kills any effect coming from this eigenspace. As a matter of facts we can see that, when N is large, we can approximate

$$\left[\sqrt{\frac{1}{2\pi}}\int dx e^{-\frac{x^2}{2}} \prod_{a=1}^{V}\left[1+e^{x\sqrt{-\frac{2\gamma}{N}}(\omega_k)_a+\frac{\epsilon}{r}}\right]\right] \simeq$$
$$\simeq \max_{y_3}\left\{e^{-\frac{y^2}{2}}\left(1+e^{\frac{\epsilon}{r}}\right)\right\} = \left[1+e^{\frac{\epsilon}{r}}\right] \quad (6.92)$$

This term is a constant and does not depend on any parameter of the model, hence it does not give rise to any separation of phases. If one is interested in the correction that arise in the finite size case however, a orthogonal basis of the third eigenspace is needed, and we will briefly provide a method to do so.

Last we need to compute the second term I_2. For this term we need to be careful. As a matter of facts, the representation for which we are able to decompose the interaction matrix requires an orthonormal basis of eigenvectors to be performed. While in the all-to-all and link community we did not need to pay too much attention to this, since eigenvectors of different eigenvalues are orthogonal between them, in this case we need again to ensure that we find an orthonormal basis for the eigenspace of eigenvector μ.

The eigenvalue equation in this case becomes

$$\sum_{(k,l)\neq(i,j)} \beta_{(ij),(kl)} v^a_{kl} = \sum_{l\neq i,j} \left(v^a_{il} + v^a_{lj}\right) = \mu v^a_{ij} \tag{6.93}$$

We now introduce the following vector for $a = 2, 3, \cdots, N$

$$v^a_{ij} = \begin{cases} 0 \text{ if } (i,j) = (a, a+1) \\ 1 \text{ if } i = a \wedge j = a \\ -1 \text{ if } i = a+1 \wedge j = a+1 \\ 0 \text{ otherwise} \end{cases} \tag{6.94}$$

and can be checked that they are indeed eigenvectors and a basis. However they are not orthogonal, which can be seen by computing the scalar product between two of them

$$\langle v^a | v^b \rangle = 2(N-2)\delta_{ab} - (N-3)\delta_{ab\pm 1} \tag{6.95}$$

Orthogonalization of the basis

From a set of nonorthogonal basis vectors we can find a set of orthogonal ones by simultaneously orthogonalizing them according to the Lowdin transformations [158]. If we define the overlap matrix $\mathbf{S}$

$$S_{ab} = \langle v^a | v^b \rangle \tag{6.96}$$

we can get a set of orthogonal vectors as

$$|\omega^a\rangle = \sum_b S^{-\frac{1}{2}}_{ab} |v^b\rangle \tag{6.97}$$

Let us focus on the overlap matrix. Since it is symmetric, we can decompose it as

$$\mathbf{S} = \mathbf{U}\Lambda\mathbf{U}^T \tag{6.98}$$

and

$$\mathbf{S}^{-\frac{1}{2}} = \mathbf{U}\Lambda^{-\frac{1}{2}}\mathbf{U}^T. \tag{6.99}$$

This is a banded tridiagonal matrix. It is known that it can be diagonalized as

$$\Lambda_{cd} = \frac{\delta_{cd}}{\sqrt{2(N-2) - 2(N-3)\cos\left(\frac{c\pi}{N}\right)}} \tag{6.100}$$

$$U_{ac} = \sin\left(\frac{ac\pi}{N}\right) \tag{6.101}$$

so that we can compute

$$S_{ab}^{-\frac{1}{2}} = \sum_{cd} \sin\left(\frac{ac\pi}{N}\right)\sin\left(\frac{db\pi}{N}\right)\frac{\delta_{cd}}{\sqrt{2(N-2) - 2(N-3)\cos\left(\frac{c\pi}{N}\right)}} = \sum_{c} \frac{\sin\left(\frac{ac\pi}{N}\right)\sin\left(\frac{cb\pi}{N}\right)}{\sqrt{2(N-2) - 2(N-3)\cos\left(\frac{c\pi}{N}\right)}} \tag{6.102}$$

So that the orthogonalized vectors can be written as

$$|\omega^a\rangle = \sum_{a,c}^{N} \frac{\sin\left(\frac{ac\pi}{N}\right)\sin\left(\frac{cb\pi}{N}\right)}{\sqrt{2(N-2) - 2(N-3)\cos\left(\frac{c\pi}{N}\right)}} |v^b\rangle \tag{6.103}$$

One can realize that the structure of this vector is such that only $2N-2$ entries are nonzero. Hence we can apply the consideration at the end of section 6.3.3 to claim that we do not expect a phase separation coming from this term. If one is interested in the finite size system effects, an orthogonal basis for the third space is needed as well. In this case we refer to the work [156] that has a complete description of all the eigenvectors. Nonetheless they are not orthogonal but can be made so by following the same procedure of Lowdin transformations we just performed. This means that the second term can be evaluated as

$$\left[\sqrt{\frac{1}{2\pi}}\int dx e^{-\frac{x^2}{2}} \prod_{a=1}^{V}\left[1 + e^{x\sqrt{\frac{\gamma\mu}{N}}(\omega_k)_a + \frac{\epsilon}{r}}\right]\right] \simeq \simeq \max_y \left\{e^{-\frac{y^2}{2}}\left(1 + e^{\frac{\epsilon}{r}}\right)\right\} = \left[1 + e^{\frac{\epsilon}{r}}\right] \tag{6.104}$$

Again, this does not produce any phase separation.
Then the term resulting from the second and third term can then be evaluated as

$$\lim_{V\to\infty}(I_2)^{\frac{1}{V}} \lim_{V\to\infty}(I_3)^{\frac{1}{V}} \simeq \left[1 + e^{\frac{\epsilon}{r}}\right]^{\mu_2+\mu_3} = \sum_{A_{ij}=0,1} e^{\left[\frac{\mu_2+\mu_3}{r}\epsilon\right]A_{ij}}. \tag{6.105}$$

Bringing back all terms into a single one, and making p the subject of the formula, we get

$$p = 1 - \frac{1}{e^{-\frac{(y^*)^2}{2}}\left(1 + e^{y^*\sqrt{2\gamma}+\epsilon}\right)} \tag{6.106}$$

In figures 6.8 and 6.9 we show the Phase Diagram and the link probability of the model for a certain range of parameters, which both resemble the simple, homogeneous case.

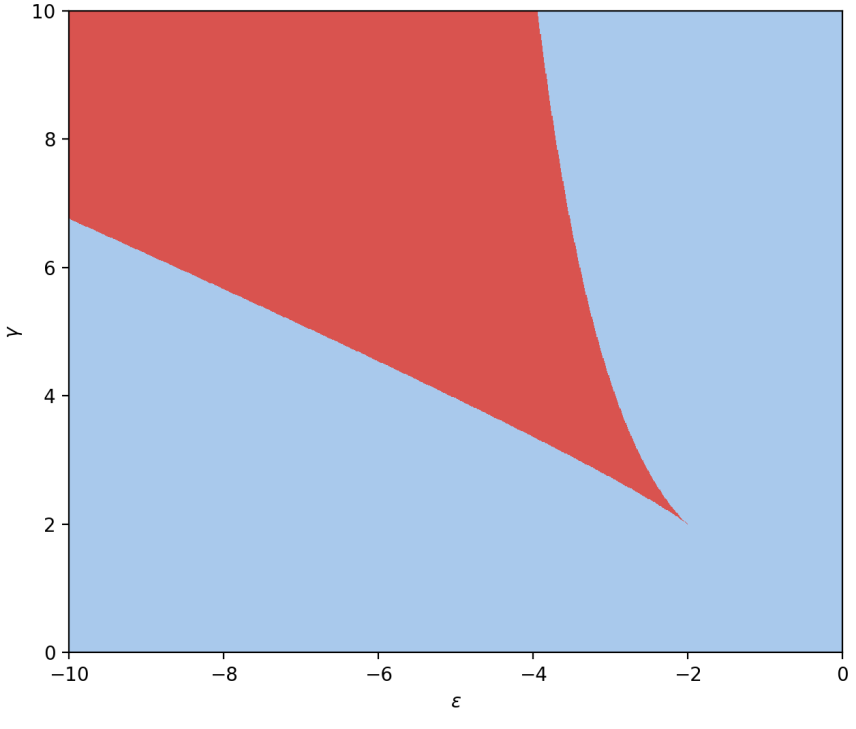

Figure 6.8: Phase Diagram of the Node-mediated Interaction model. In red the region where multiple phases are present, in blue the region where only one phase is possible.

Lastly, we want to comment on the unexpected similarity between this model and the simple, homogeneous one. This is due to the fact that while the mean field approximation is far from exact, since a large number of eigenvalues of the same size as the largest one exist, nonetheless they localize in a few nodes, and the respective eigenvectors are basically everywhere zero. This is not true if we want to consider the finite size effects on the system. In that case one would need to resort to the general single-link probability formula (6.39) with the eigenvector entries that are found using the method of appendix 6.3.4. The phase structure would have to be found with the equation described in appendix 6.2.2, by putting equation (6.25) equal to zero with the found entries. After that, it could also be possible to correct the model for an heterogeneous ϵ term.

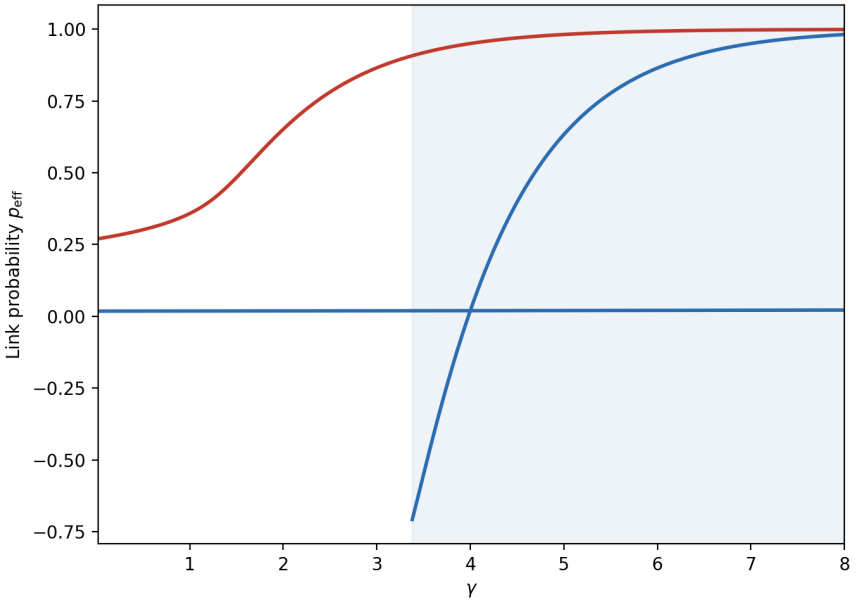

Figure 6.9: Single-link probability function, in the range $\gamma = [0, 15]$, with a large negative ϵ in blue, shaded region indicate where two solutions are possible.

§6.4 On the Mean Field Approximation and Phase Structure

Before the final remarks, we want to comment on the unexpected similarity between the node-mediated interaction model 6.3.4 and the simple, homogeneous one 6.3.1. One would expect that with such a non-trivial corrrelation structure, the mean field computation, which is essentialy the model of Section 6.3.1 with the right scalings for the parameters, would be a wrong approximation of the system. However that is not the case, and they basically coincide in the large N limit. Explicitly, the correspondence follows from Equation (6.91): the node-mediated model's mean-field mode carries effective coupling $\theta_{\text{eff}} = 2\gamma$, so that the homogeneous model of Section 6.3.1 with $\theta = 2\gamma$ reproduces the leading behavior of the node-mediated model of Sec. 6.3.4 exactly in this limit. This is due to the fact that while the mean field approximation is far from exact, since a large number of eigenvalues of the same size as the largest one exist, nonetheless they localize in a subextensive set of links, and the respective eigenvectors are zero everywhere else.

We can state this more precisely as a two-part criterion, already implicit in the derivation of Equation (6.91). An eigenmode k of β, with eigenvalue λ_k and eigenvector $|\omega_k\rangle$, drives genuine non-mean-field phase structure only if both of the following hold in the $V \to \infty$ limit:

(i) *Extensive eigenvalue:* λ_k/V tends to a finite, nonzero constant;

(ii) *Delocalized eigenvector:* the number of entries a for which $(\omega_k)_a$ is bounded away from zero is a macroscopic (non-vanishing) fraction of all V links participates in the mode.

If either condition fails, the corresponding auxiliary field is pinned to its mean-field/trivial value in the thermodynamic limit, and the mode contributes at most a finite-size correction rather than an independent order parameter. This is exactly what happens for the non-principal eigenspaces of the Johnson graph coupling in Sec.6.3.4: the ν-eigenspace fails condition (i) outright ($\nu/V \to 0$), while the μ-eigenspace satisfies (i) ($\mu/V \to \gamma$, finite) but fails (ii), since its eigenvector has only $2N-2$ nonzero entries out of $V = N(N-1)/2$, a fraction that vanishes as $N \to \infty$. Both mechanisms independently force the corresponding auxiliary field to its trivial value, which is why only the mean-field mode survives to shape the phase diagram of Fig. 6.8.

This is not true if we want to consider the finite size effects on the system, or the sparse limit in which the scaling of the parameters is different from the one we used. In both such cases one would need to resort to the general single-link probability Formula (6.39) with the eigenvector entries that are found using the method of SI. The phase structure would have to be found with the equation described in SI, by putting Equation (6.25) equal to zero with the found entries. After that, it could also be possible to correct the model for an heterogeneous ϵ term.

We also want to highlight the connection between our approach and Algebraic Graph Theory. When we recast the computation of the Partition functions in terms of the spectral decomposition of the coupling structure we are actually looking for the diagonalization of link coupling matrix. Given that this is a matrix describing binary relations in a set of V elements, it is itself an adjacency matrix and all theorems of Algebraic Graph Theory apply.

In this respect, a rough procedure to determine the possible phase separation structure of a model can be outlined by generalizing the argument done for the goodness of the mean field approximation. In the thermodynamic limit, only eigenvalues with the right scaling, *i.e.* outliers of the link-link adjacency matrix, will be relevant to the the computation of the partition function. From literature it is known that in most cases outliers other than the mean field one are symptoms of either a community structure or a hub presence. One would indeed be interested in those generated by the community structure, as they are the one that does not localize and create the long-range correlations needed for the phase transition to appear.

This is consistent with results on the detectability of community structure via spectral methods, where a community-induced eigenvalue must separate from the bulk of the spectrum by an amount related to the community strength before the corresponding (delocalized) eigenvector becomes informative, exactly mirroring the criterion above.

This heuristics for the outliers to be responsible for the phase transitions can be further argued for in the case of the regular random graph, as we hinted at in Section 6.3.1. Indeed looking at the spectra of regular random graphs, the first one in which a separation of the spectrum appears is when the degree is 3, as can be seen in Figure 6.10

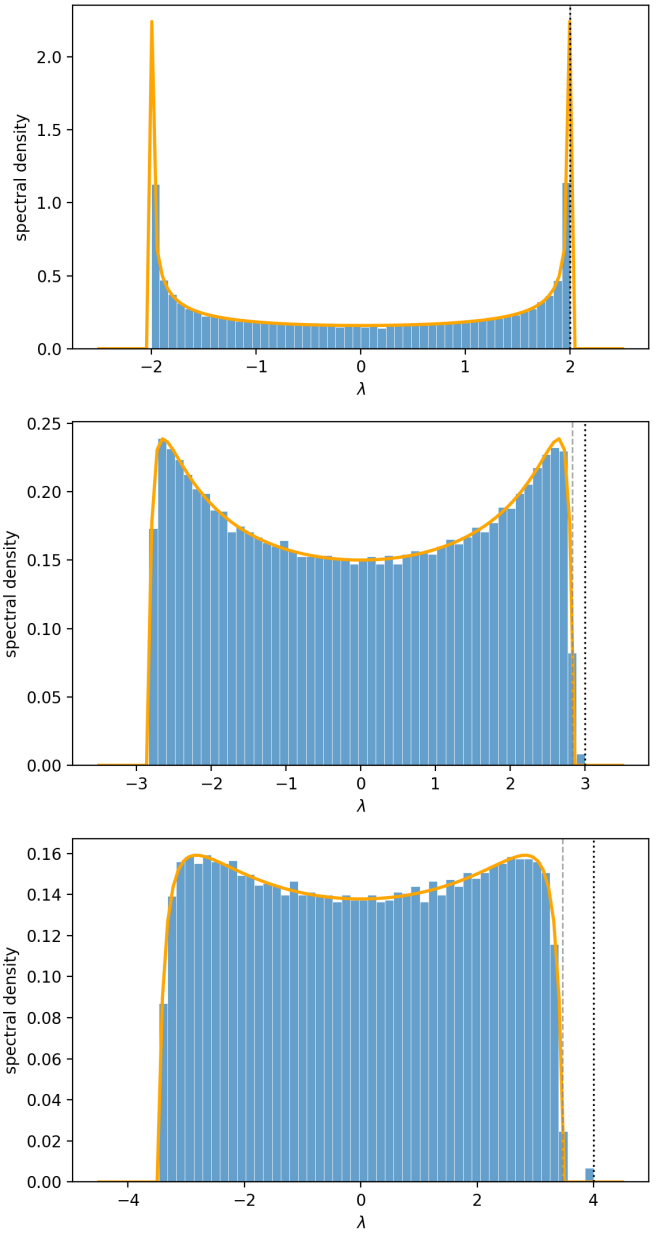

Figure 6.10: Spectrum of the adjacency matrix of a random regular graph with 1000 *nodes. In orange the Kensten-McKay law describing the distribution of the bulk of the spectrum. From top to bottom the degree k is* 2, 3, 4, *and we can see that the outlier starts to appear from the second figure only.*

§6.5 Final Remarks

The solution of Random Network Models is a fundamental, analytical tool to tackle many among the problems in the science of Complex Systems. In this paper we have shown how to solve the models that mimic a link-link coupling of real world systems. The power of our approach lies in the fact that is quite universal. Using the general Formula (6.39), heterogeneity in local features can be added at will while still retaining the same coupling structure. On the condition of having a viable way to decompose the interaction structure, we have also shown that Equation (6.21) can provide an exact, explicit solution of the partition function and the phase structure as well. An immediate practical consequence of this explicit solution is that it makes maximum likelihood estimation directly accessible for this class of second order models. Since the partition function of Equation (6.21) and the single-link probability of Equation (6.39) are available in closed, or at least saddle-point-exact, form, the likelihood of an observed network under any of the models we solve can be written down and maximized without resorting to the Markov Chain Monte Carlo or pseudo-likelihood approximations ordinarily required for pairwise-dependent exponential random graph models, and without incurring the model-degeneracy pathologies such approximate schemes are known to suffer from. We regard this as one of the more promising practical directions opened up by the present work: using the class of second order models solved here as tractable, exactly-normalizable null models for network reconstruction and hypothesis testing, in the same spirit already established for first order maximum-entropy models [90].

Moreover, in the various cases that we explored, we have seen that the phase structure can become easily crowded as the phase landscape can be quite hilly. Higher order features of networks are increasingly thought to be key features of complex systems, and the present work constitutes a little brick in the wall of understanding and solving coupled, complex structures. But we have also shown in Section 6.3.2 how seemingly higher order structures can be effectively reduced to lower order network models and be, in principle, indistinguishable from them. Lastly, the study of even higher order structure has still many more steps to undertake, and we believe some of the methods developed here, and properly generalized, could be of help in that direction of research, for example as null models to distinguish between true higher order interaction and the one just described by link-link coupling that is extensively dealt with in the present work.