Complex pathways in multistable matter

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1 Introduction

Massively multistable materials, such as crumpled sheets, display strikingly complex sequential responses when driven. These responses arise from their rugged energy landscapes, which emerge due to disorder and frustration. This makes the understanding, characterization and design of the sequential response of multistable materials both a fundamental challenge and a technological opportunity for sequential shapemorphing, soft robotics and *in-materia* computing.

In this thesis, we address three interrelated questions at this frontier. First, we explore how richer modes of driving fundamentally reshape material behavior. Second, we investigate the minimal building blocks that generate exotic sequential responses. Third, we develop a scalable driving strategy that utilizes dynamics to write many states in a single drive cycle to unlock the full potential of multistable materials.

Together, these contributions advance our understanding of the fundamental physics of frustrated and multistable systems, and build toward a framework for their characterization, control and rational design–paving the way toward programmable and adaptive materials.

1.1 Multistable materials

Describing and predicting how materials respond to external stimuli is a central question in physics. For many ordered systems, such responses can be characterized by global parameters such as elastic moduli. In contrast, driven disordered materials–including disordered magnets [1–4], granular media [5–9], and crumpled sheets [10–19] – feature highly non-linear responses for which we lack understanding and a coherent description.

The intrinsic disorder and (geometric) frustration in these materials give rise to complex energy landscapes and multistability [16, 20–23]. These, in turn, lead to hallmark phenomena such as *intermittency*, where smooth deformations are punctuated by sharp irreversible transitions, and *hysteresis*, where the response depends on the driving history [22–24]. Beyond these individual phenomena lies the broader importance of sequential responses: complex, multi-step deformation pathways that emerge as materials traverse their energy landscapes under driving [25, 26]. Such pathways are highly sensitive to driving protocols, and enable rich dynamical behaviors [27, 28]. Understanding and controlling these sequential responses is therefore essential, not only for advancing the fundamental physics of multistable and disordered systems, but also for unlocking their potential as programmable, adaptive materials.

1.1.1 Metamaterials

Throughout this thesis, we address fundamental questions about multistable materials with rugged energy landscapes. To do so, we employ *mechanical metamaterials* as model systems [29, 30]. Metamaterials are materials whose designed microscopic structures yield exotic macroscopic material properties [29, 30].

For example, one can design a metamaterial with a vanishing shear modulus, which behaves like a liquid in linear response - it resists compression but not shear - and can be tuned to mechanically cloak objects [31, 32] (Fig. 1.1a). Another popular example is a structure with a negative Poisson's ratio; when compressed (stretched) along one axis, it contracts (expands) transversely [33, 34] (Fig. 1.1b-c). Such auxetic behavior enables dramatic geometric effects, including controlled changes in Gaussian curvature that allow flat sheets to deform into positively curved surfaces [35] (Fig. 1.1d)–an operation that, for example, underlies the classic challenge of projecting the globe onto a flat map.

Beyond linear response, elastic instabilities provide a principal strategy for nonlinear stress-strain responses [29, 36–46]. Two archetypal examples are *buckling*, where a slender element under compression undergoes a pitchfork bifurcation to one of two symmetry-broken states, and *snap-through*, a fold bifurcation in which a bistable element abruptly transitions between configurations [29, 36, 40, 41, 46]. By deliberately incorporating such instabilities into their archi-

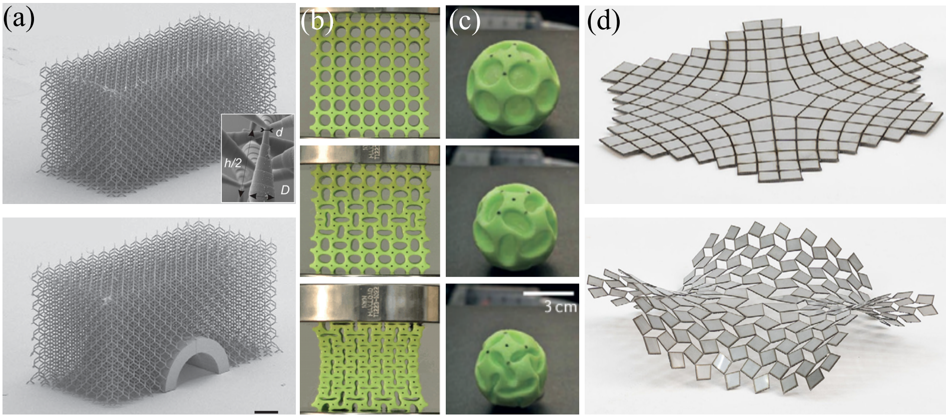

FIGURE 1.1: **Examples of mechanical metamaterials with exotic properties** (a) A metamaterial with vanishing shear modulus, allowing for mechanical cloaking. Adapted from [31, 32]. (b-c) Auxetic metamaterials, under uniaxial force in (b), and under uniform pressure in (c). (b) Adapted from [33]. (c) Adapted from [34]. (d) Such auxetic metamaterials can be designed to change their Gaussian curvature. Adapted from [35].

tecture, mechanical metamaterials can be programmed to exhibit multistability and complex deformation pathways [38, 40–44]. Together, these features make metamaterials powerful model systems to study the physics of frustrated energy landscapes and sequential responses.

1.2 Sequential deformation pathways

Multistable (meta)materials are characterized by energy landscape, from which sequential deformation pathways emerge under external driving. Smooth deformations correspond to the continuous evolution of a local minimum, whereas irreversible transitions occur when a minimum vanishes at a critical threshold, forcing the system to jump to a distinct minimum (Fig. 1.2a-b). Graph-based frameworks provide a powerful tool to capture such sequential pathways [21, 23, 24, 47–49]. For example, in transition graphs, states are represented as nodes and irreversible transitions as directed edges (Fig. 1.2a). The color of each directed edge encodes the direction of the external driving (gold for increasing and red for decreasing; Fig. 1.2a), and together these graphs fully describe the quasistatic behavior of multistable materials under global driving [21, 23, 24, 47–49] (Fig. 1.2a). This representation naturally describes the intermittent dynamics of multistable materials and enables systematic analysis of their sequential response.

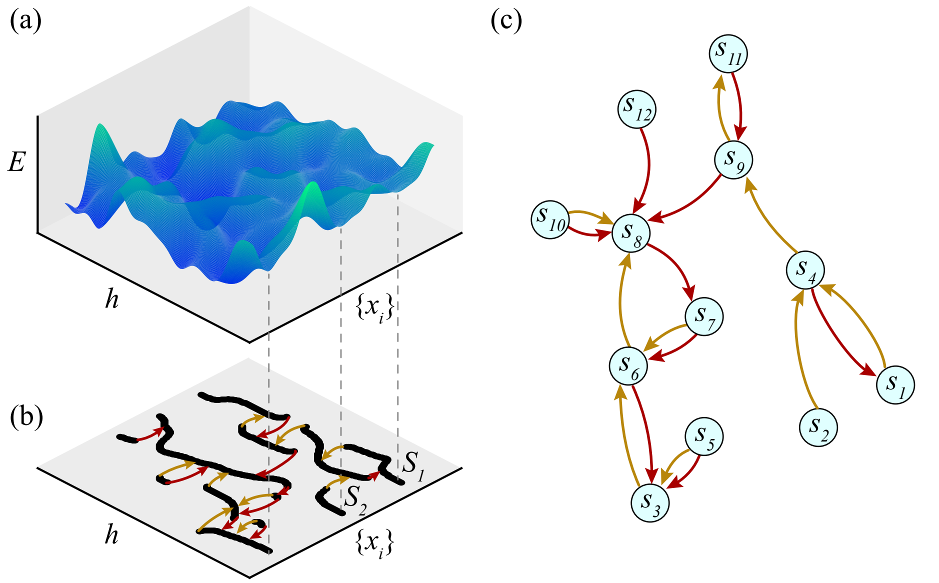

FIGURE 1.2: **Intermittent deformation pathway through a rugged energy landscape.** (a) A randomly generated energy landscape E, illustrating the intricate interplay between configurational degrees of freedom $\{x_i\}$ and an external driving parameter h. (b) The projected local energy minima (in black) trace the evolution of the system as h varies. Smooth deformations correspond to the continuous evolution of these minima as a function of h. Each collection of smoothly connected configurations defines a distinct state S_i. Irreversible transitions (arrows) occur when a local minimum vanishes at a critical value of h, forcing the system to jump to a distinct state. These transitions are annotated by their critical value of h and direction, where transitions induced by increasing (decreasing) h are shown in gold (red). (c) A graph-based framework that captures the multi-step deformation pathway of (a-b). This graph representation provides a systematic approach to analyze the intermittent deformation pathway through a the rugged energy landscape.

1.2.1 Memory effects

In turn, the structure of these graph encodes history-dependent responses, or *memory effects*, in multistable system [21, 23, 48]. A prominent example is return-point memory (RPM) where a material returns to a previous state once the driving retraces its extremal values [4, 21, 23, 48, 50]. RPM arises from a hierarchical organization of nested hysteresis loops with the no-passing property (Fig. 1.3a), which stipulates that two states, ordered by an energy or configurational metric, maintain their relative order as the external drive varies. The interplay between RPM and the no-passing property allows disordered systems to reliably encode and retrieve states under cyclic driving. Hence, RPM is a central organizing principle of the complex energy landscapes of multistable materials.

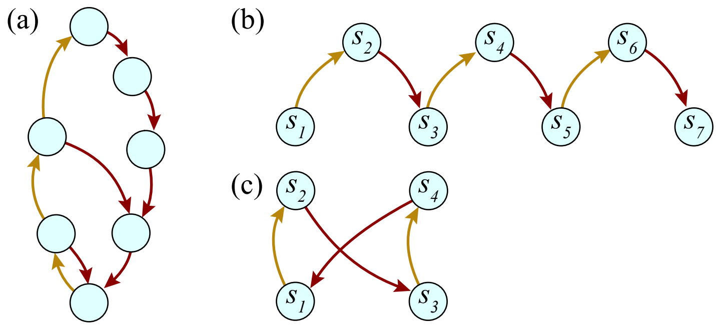

FIGURE 1.3: **Examples of more complex emergent responses** (a) Nested hysteresis loops exhibit return-point memory. (b-c) Breaking return-point memory and violating the no-passing property result in more complex responses, i.e., a transient response (b), and a subharmonic response (c).

When RPM is broken and no-passing is violated, more complex sequential responses emerge [51–60]. Here, upon cyclic driving, the state does not immediately return to its initial configuration when the driving retraces one of its extrema. Instead, under cyclic driving, the system can undergo a sequence of state changes before eventually settling into a limit cycle. The number of driving cycles required to reach the limit cycle, denoted by τ, characterizes the *transient* time (Fig. 1.3b). Moreover, the period of the limit cycle T need not equal the period of the driving; instead, the system can exhibit a *multiperiodic* response, requiring $T > 1$ driving cycles to complete one orbit (Fig. 1.3c). These examples reveal a broader spectrum of sequential responses beyond classical hysteresis and RPM; these more complex responses are studied in chapter three of this thesis.

1.2.2 External drive

Different driving protocols lead to different sequential deformation pathways. Even for the same material, varying the mode of driving may produce entirely distinct sequences of states. For example, a corrugated sheet subject to uniaxial compression will follow a pathway that strongly depends on the chosen compression axis [24]. So far, most studies have focused on sequential responses under scalar driving fields. An open question is how the response along a single driving axis constrains or predicts the response under distinct driving axis.

More generally, when multiple driving inputs are applied, one enters the regime of *vectorial driving protocols*. Such protocols introduce intrinsic noncommutativity, where the final state reached depends on the driving path. This path dependency fundamentally expands the space of possible responses, giving rise to responses without analogues in scalar driving. In addition, to what extend memory effects–such as return-point memory, transients, and multiperiodic responses–under vectorial driving remain, is largely unresolved. Addressing these questions requires the development of new frameworks capable of capturing both the combinatorial complexity of high-dimensional input spaces and the emergent memory encoded within them. This is addressed in chapter two.

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1.3 A machine approach

Microscopically, intermittent jumps in multistable materials often originate from localized bistable elements. Mechanical systems provide striking illustrations: in compressed crumpled sheets, abrupt pops of local crumples generate intermittent jumps [61, 62], while in sheared granular packings, such responses arise from localized particle rearrangements [23, 63–66]. At the macroscopic scale, materials can thus be seen as networks of such localized elements which, when driven collectively, form these intricate sequential deformation pathways [22, 24, 47, 48, 67, 68]. In mechanical systems, these elements are embedded into a elastic matrix, which leads to interactions and enables more advanced memory effects beyond RPM.

This perspective—that sequential responses originate from interacting bistable elements—suggests a *machine-like approach* of multistable materials. In this machine approach, each localized element functions as a *material bit*, whose interactions govern the global response and underlie the memory effects. From this perspective, multistable materials can thus be understood, controlled, and designed as physical automata, capable of exhibiting, sequential responses, memory effects, and even computation.

In this thesis, we pursue this machine approach to multistability. Using mechanical metamaterials as model systems, we explore how sequential deformation pathways are shaped by their mode of driving and their material bits. Together, these investigations advance a framework for the characterization, control, and rational design of multistable materials, paving the way toward *in-materia* computing.

1.4 Outline

In this thesis, we advance the fundamental physics of multistable materials and build toward a framework for their characterization, control and design.

In **Chapter 2**, we investigate the responses of multistable materials under *vectorial* driving. Moving beyond scalar driving that dominate previous work, we uncover fundamentally new responses, including non-Abelian and mixed-mode responses. We trace these to fold singularities and identify the ancestor–descendant–sibling triplet as the elementary building block of path-dependent responses. To handle the complexity that emerges with many driving inputs, we introduce graph-based *motifs* that naturally connect to sequential computing. These methods suggest that the responses of multistable and disordered materials to high-dimensional inputs provide a framework for their classification.

In **Chapter 3**, we turn to the *building blocks* of sequential responses. We introduce *togglerons* as elementary multiperiodic units within a hierarchy of material bits. Through experiments, we demonstrate how togglerons and their interactions can be harnessed to control and design transients and multiperiodic orbits. This establishes a systematic route toward constructing sequential materials with programmable functionality.

In **Chapter 4**, we investigate the snap-through mechanism of the toggleron. Using a modal expansion approach, we capture and analyze this new class of snapping behavior. Our findings open new avenues for quantitatively and qualitatively controlling the snap-through of buckled beams, with promising applications in mechanical sensors, actuators, and metamaterials.

Finally, in **Chapter 5**, we introduce *dynamic driving* as a scalable strategy to address individual material bits with a single global driving function. This strategy leverages material sensitivity time derivatives of the drive to access any state of a multistable material with a single, simple driving cycle. This approach generalizes to other physical domains and establishes a powerful paradigm for controlling multistable systems without requiring local actuation.