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Complex pathways in multistable matter

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Citation

Meulblok, C. M. (2026, September 9). *Complex pathways in multistable matter*. Retrieved from <https://hdl.handle.net/1887/4309337>

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Complex pathways in multistable matter

Proefschrift

ter verkrijging van
de graad van doctor aan de Universiteit Leiden,
op gezag van de rector magnificus prof.dr. S. de Rijcke,
volgens besluit van het college voor promoties
te verdedigen op woensdag 9 september 2026
klokke 11:30 uur

door

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geboren te Haarlemmermeer, Nederland
in 1997

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Inspired by Swallowtail by Salvador Dalí, this cover presents an artistic representation of complex pathways by Colin M. Meulblok. The front highlights path dependency, while the back emphasizes multiperiodic orbits.

ISBN 978-94-6537-786-5

An electronic version of this dissertation is available on *openaccess.leidenuniv.nl*

The work presented in this dissertation was conducted at AMOLF, Amsterdam, an institute of the Netherlands Organisation for Scientific Research (NWO), and at the Leiden Institute of Physics, Leiden University. This work was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement no.01019474).

“We adore chaos because we love to produce order”

M.C. Escher

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Introduction

Massively multistable materials, such as crumpled sheets, display strikingly complex sequential responses when driven. These responses arise from their rugged energy landscapes, which emerge due to disorder and frustration. This makes the understanding, characterization and design of the sequential response of multistable materials both a fundamental challenge and a technological opportunity for sequential shapemorphing, soft robotics and *in-materia* computing.

In this thesis, we address three interrelated questions at this frontier. First, we explore how richer modes of driving fundamentally reshape material behavior. Second, we investigate the minimal building blocks that generate exotic sequential responses. Third, we develop a scalable driving strategy that utilizes dynamics to write many states in a single drive cycle to unlock the full potential of multistable materials.

Together, these contributions advance our understanding of the fundamental physics of frustrated and multistable systems, and build toward a framework for their characterization, control and rational design—paving the way toward programmable and adaptive materials.

1.1 Multistable materials

Describing and predicting how materials respond to external stimuli is a central question in physics. For many ordered systems, such responses can be characterized by global parameters such as elastic moduli. In contrast, driven disordered materials—including disordered magnets [1–4], granular media [5–9], and crumpled sheets [10–19] – feature highly non-linear responses for which we lack understanding and a coherent description.

The intrinsic disorder and (geometric) frustration in these materials give rise to complex energy landscapes and multistability [16, 20–23]. These, in turn, lead to hallmark phenomena such as *intermittency*, where smooth deformations are punctuated by sharp irreversible transitions, and *hysteresis*, where the response depends on the driving history [22–24]. Beyond these individual phenomena lies the broader importance of sequential responses: complex, multi-step deformation pathways that emerge as materials traverse their energy landscapes under driving [25, 26]. Such pathways are highly sensitive to driving protocols, and enable rich dynamical behaviors [27, 28]. Understanding and controlling these sequential responses is therefore essential, not only for advancing the fundamental physics of multistable and disordered systems, but also for unlocking their potential as programmable, adaptive materials.

1.1.1 Metamaterials

Throughout this thesis, we address fundamental questions about multistable materials with rugged energy landscapes. To do so, we employ *mechanical metamaterials* as model systems [29, 30]. Metamaterials are materials whose designed microscopic structures yield exotic macroscopic material properties [29, 30].

For example, one can design a metamaterial with a vanishing shear modulus, which behaves like a liquid in linear response - it resists compression but not shear - and can be tuned to mechanically cloak objects [31, 32] (Fig. 1.1a). Another popular example is a structure with a negative Poisson’s ratio; when compressed (stretched) along one axis, it contracts (expands) transversely [33, 34] (Fig. 1.1b-c). Such auxetic behavior enables dramatic geometric effects, including controlled changes in Gaussian curvature that allow flat sheets to deform into positively curved surfaces [35] (Fig. 1.1d)—an operation that, for example, underlies the classic challenge of projecting the globe onto a flat map.

Beyond linear response, elastic instabilities provide a principal strategy for nonlinear stress-strain responses [29, 36–46]. Two archetypal examples are *buckling*, where a slender element under compression undergoes a pitchfork bifurcation to one of two symmetry-broken states, and *snap-through*, a fold bifurcation in which a bistable element abruptly transitions between configurations [29, 36, 40, 41, 46]. By deliberately incorporating such instabilities into their archi-

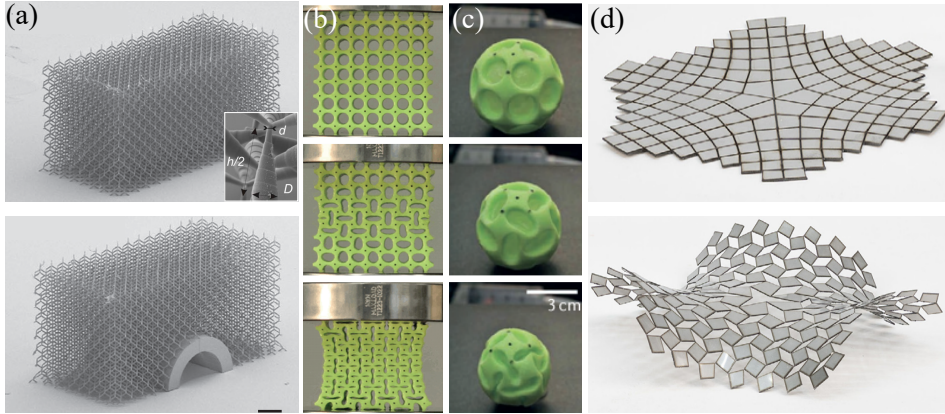


FIGURE 1.1: **Examples of mechanical metamaterials with exotic properties** (a) A metamaterial with vanishing shear modulus, allowing for mechanical cloaking. Adapted from [31, 32]. (b-c) Auxetic metamaterials, under uniaxial force in (b), and under uniform pressure in (c). (b) Adapted from [33]. (c) Adapted from [34]. (d) Such auxetic metamaterials can be designed to change their Gaussian curvature. Adapted from [35].

texture, mechanical metamaterials can be programmed to exhibit multistability and complex deformation pathways [38, 40–44]. Together, these features make metamaterials powerful model systems to study the physics of frustrated energy landscapes and sequential responses.

1.2 Sequential deformation pathways

Multistable (meta)materials are characterized by energy landscape, from which sequential deformation pathways emerge under external driving. Smooth deformations correspond to the continuous evolution of a local minimum, whereas irreversible transitions occur when a minimum vanishes at a critical threshold, forcing the system to jump to a distinct minimum (Fig. 1.2a-b). Graph-based frameworks provide a powerful tool to capture such sequential pathways [21, 23, 24, 47–49]. For example, in transition graphs, states are represented as nodes and irreversible transitions as directed edges (Fig. 1.2a). The color of each directed edge encodes the direction of the external driving (gold for increasing and red for decreasing; Fig. 1.2a), and together these graphs fully describe the quasistatic behavior of multistable materials under global driving [21, 23, 24, 47–49] (Fig. 1.2a). This representation naturally describes the intermittent dynamics of multistable materials and enables systematic analysis of their sequential response.

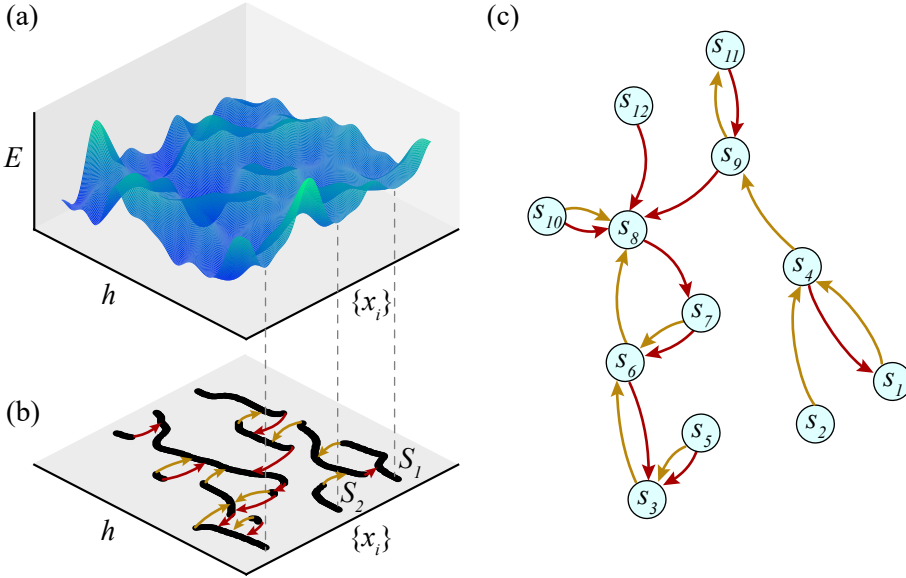


FIGURE 1.2: **Intermittent deformation pathway through a rugged energy landscape.** (a) A randomly generated energy landscape E , illustrating the intricate interplay between configurational degrees of freedom $\{x_i\}$ and an external driving parameter h . (b) The projected local energy minima (in black) trace the evolution of the system as h varies. Smooth deformations correspond to the continuous evolution of these minima as a function of h . Each collection of smoothly connected configurations defines a distinct state S_i . Irreversible transitions (arrows) occur when a local minimum vanishes at a critical value of h , forcing the system to jump to a distinct state. These transitions are annotated by their critical value of h and direction, where transitions induced by increasing (decreasing) h are shown in gold (red). (c) A graph-based framework that captures the multi-step deformation pathway of (a-b). This graph representation provides a systematic approach to analyze the intermittent deformation pathway through a the rugged energy landscape.

1.2.1 Memory effects

In turn, the structure of these graph encodes history-dependent responses, or *memory effects*, in multistable system [21, 23, 48]. A prominent example is return-point memory (RPM) where a material returns to a previous state once the driving retraces its extremal values [4, 21, 23, 48, 50]. RPM arises from a hierarchical organization of nested hysteresis loops with the no-passing property (Fig. 1.3a), which stipulates that two states, ordered by an energy or configurational metric, maintain their relative order as the external drive varies. The interplay between RPM and the no-passing property allows disordered systems to reliably encode and retrieve states under cyclic driving. Hence, RPM is a central organizing principle of the complex energy landscapes of multistable materials.

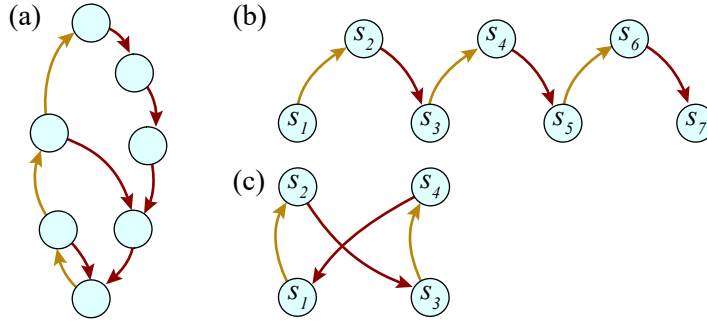


FIGURE 1.3: **Examples of more complex emergent responses** (a) Nested hysteresis loops exhibit return-point memory. (b-c) Breaking return-point memory and violating the no-passing property result in more complex responses, i.e., a transient response (b), and a subharmonic response (c).

When RPM is broken and no-passing is violated, more complex sequential responses emerge [51–60]. Here, upon cyclic driving, the state does not immediately return to its initial configuration when the driving retraces one of its extrema. Instead, under cyclic driving, the system can undergo a sequence of state changes before eventually settling into a limit cycle. The number of driving cycles required to reach the limit cycle, denoted by τ , characterizes the *transient* time (Fig. 1.3b). Moreover, the period of the limit cycle T need not equal the period of the driving; instead, the system can exhibit a *multiperiodic* response, requiring $T > 1$ driving cycles to complete one orbit (Fig. 1.3c). These examples reveal a broader spectrum of sequential responses beyond classical hysteresis and RPM; these more complex responses are studied in chapter three of this thesis.

1.2.2 External drive

Different driving protocols lead to different sequential deformation pathways. Even for the same material, varying the mode of driving may produce entirely distinct sequences of states. For example, a corrugated sheet subject to uniaxial compression will follow a pathway that strongly depends on the chosen compression axis [24]. So far, most studies have focused on sequential responses under scalar driving fields. An open question is how the response along a single driving axis constrains or predicts the response under distinct driving axis.

More generally, when multiple driving inputs are applied, one enters the regime of *vectorial driving protocols*. Such protocols introduce intrinsic non-commutativity, where the final state reached depends on the driving path. This path dependency fundamentally expands the space of possible responses, giving rise to responses without analogues in scalar driving. In addition, to what extend memory effects—such as return-point memory, transients, and multiperi-

odic responses—under vectorial driving remain, is largely unresolved. Addressing these questions requires the development of new frameworks capable of capturing both the combinatorial complexity of high-dimensional input spaces and the emergent memory encoded within them. This is addressed in chapter two.

1

1.3 A machine approach

Microscopically, intermittent jumps in multistable materials often originate from localized bistable elements. Mechanical systems provide striking illustrations: in compressed crumpled sheets, abrupt pops of local crumples generate intermittent jumps [61, 62], while in sheared granular packings, such responses arise from localized particle rearrangements [23, 63–66]. At the macroscopic scale, materials can thus be seen as networks of such localized elements which, when driven collectively, form these intricate sequential deformation pathways [22, 24, 47, 48, 67, 68]. In mechanical systems, these elements are embedded into a elastic matrix, which leads to interactions and enables more advanced memory effects beyond RPM.

This perspective—that sequential responses originate from interacting bistable elements—suggests a *machine-like approach* of multistable materials. In this machine approach, each localized element functions as a *material bit*, whose interactions govern the global response and underlie the memory effects. From this perspective, multistable materials can thus be understood, controlled, and designed as physical automata, capable of exhibiting, sequential responses, memory effects, and even computation.

In this thesis, we pursue this machine approach to multistability. Using mechanical metamaterials as model systems, we explore how sequential deformation pathways are shaped by their mode of driving and their material bits. Together, these investigations advance a framework for the characterization, control, and rational design of multistable materials, paving the way toward *in-materia* computing.

1.4 Outline

In this thesis, we advance the fundamental physics of multistable materials and build toward a framework for their characterization, control and design.

In **Chapter 2**, we investigate the responses of multistable materials under *vectorial* driving. Moving beyond scalar driving that dominate previous work, we uncover fundamentally new responses, including non-Abelian and mixed-mode responses. We trace these to fold singularities and identify the ancestor–descendant–sibling triplet as the elementary building block of path-dependent responses. To handle the complexity that emerges with many driving inputs, we introduce graph-based *motifs* that naturally connect to sequential computing. These methods suggest that the responses of multistable and disordered materials to high-dimensional inputs provide a framework for their classification.

In **Chapter 3**, we turn to the *building blocks* of sequential responses. We introduce *togglerons* as elementary multiperiodic units within a hierarchy of material bits. Through experiments, we demonstrate how togglerons and their interactions can be harnessed to control and design transients and multiperiodic orbits. This establishes a systematic route toward constructing sequential materials with programmable functionality.

In **Chapter 4**, we investigate the snap-through mechanism of the toggleron. Using a modal expansion approach, we capture and analyze this new class of snapping behavior. Our findings open new avenues for quantitatively and qualitatively controlling the snap-through of buckled beams, with promising applications in mechanical sensors, actuators, and metamaterials.

Finally, in **Chapter 5**, we introduce *dynamic driving* as a scalable strategy to address individual material bits with a single global driving function. This strategy leverages material sensitivity time derivatives of the drive to access any state of a multistable material with a single, simple driving cycle. This approach generalizes to other physical domains and establishes a powerful paradigm for controlling multistable systems without requiring local actuation.



Path-dependency and emergent computing under vectorial driving

The sequential response of frustrated materials reveals their memory effects and emergent computational potential. Despite their spatial extension, most studies rely on a single global stimulus effectively reducing the problem to scalar driving. Here, we introduce vectorial driving by applying multiple spatially localized stimuli, revealing a wealth of path-dependent phenomena, including non-Abelian responses, mixed-mode behavior, and chiral loop transients. We find that these path dependencies arise from recurring patterns linked to fold singularities connecting triplets of states—ancestor, descendant, and sibling. We introduce three levels of description of sequential, path-dependent responses. At the most detailed level, pt-graphs and strain maps capture arbitrary vectorial driving and link pathways to underlying singularities, providing a complete description analogous to t-graphs. For high-dimensional driving, we introduce b-graphs, a simpler, coarser representation focusing on binarized inputs, which naturally connect to sequential computing and Boolean circuits. Individual samples can encode multiple b-graphs, selectable by driving strength and reprogrammable via additional inputs. Finally, graph-based motifs enable systematic analysis of b-graphs and offer statistical measures of pathway complexity for very high-dimensional systems where full pt- or b-graphs are impractical. Our work paves the way for strategies to explore, harness, and understand complex materials and memory, while advancing embodied intelligence and in-material computing.

Adapted from *Path-dependency and emergent computing under vectorial driving*, Phys. Rev. X **16** 031021 (2026).

2.1 Introduction

A central goal of condensed matter physics is to describe and understand how materials respond to external stimuli. This is particularly challenging for frustrated, multistable materials, such as amorphous solids or crumpled sheets [13, 17, 19, 24, 25, 61, 64–66, 69, 70]. Their state does not only depend on the current driving, but also on the driving history, leading to a wealth of memory effects which classify materials and give insight into their underlying physics [20, 22]. Examples of such memory effects include measuring after how many driving cycles the response becomes periodic [24, 52–57, 59, 64, 71, 72], or whether the material returns to a previous state when the driving returns to an earlier extremum [24, 47, 58, 61, 73]. More complex memory effects can be leveraged to realize *in materia* computing, where the materials state represents the outcome of a calculation, and the initial state and driving history its input [47, 72].

So far, studies of memory and computing have focused on driving with a single driving field, such as global compression or shear. For such *scalar*, quasistatic driving of athermal materials, transition graphs are emerging as a powerful tool for characterizing memory effects [21, 23, 24, 47, 48, 53, 58, 59, 68, 73–77]. These represent states as nodes and irreversible transitions under sweeps of the driving parameter ε as edges. They encode the response to any sequential driving protocol, thus allowing to precisely characterize memory effects and emergent computational capabilities [47, 68].

However, the spatial extension of complex materials naturally leads to the question of spatially textured driving. For example, one can push a crumpled sheet at various locations in various orders. We refer to this as *vectorial* sequential driving, with each component of $\vec{\varepsilon}$ corresponding to, e.g., the compression or strain at a given location. For strongly nonlinear responses, associated with complex materials, superposition no longer describes the response. Instead the driving *path* from $\vec{\varepsilon}_1$ to $\vec{\varepsilon}_2$ plays a crucial role, leading to sequential responses, memory effects and computational capabilities that have no counterpart in scalar driving. We note that sequential vectorial driving itself is not new - think of the Carnot cycle - and path-dependencies have been studied extensively for materials that exhibit plasticity [78], here we focus on developing a coherent and practical framework to characterize the path-dependencies of multistable, athermal materials that are driven quasistatically.

We address the following broad questions: What new phenomena, memory effects and computational capabilities can be accessed with vectorial driving? How can path-dependent sequential responses be described? How to deal with the complexity associated with high-dimensional driving?

To address these questions, we combine experiments with simulations and theory to probe the path-dependent sequential response of metamaterials based on chains of coupled nonlinear units (Fig. 2.1). These chains naturally allow vec-

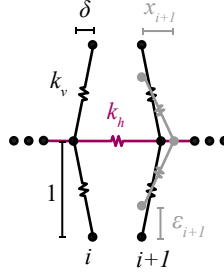


FIGURE 2.1: **Geometry.** Metamaterial consisting of a staggered chain of n left-right symmetry broken units (here i and $i + 1$). Each unit consists of two nearly vertical elastic elements (spring constant k_v), and neighboring units are coupled with elastic elements (spring constant k_h). The material is driven by a vectorial strain $\vec{\varepsilon}$, consisting of vertical strains ε_i applied independently by compressing each unit. We non-dimensionalize lengths by the distance between the endpoints of each unit at rest, $2\ell_0 = 2$, and specify the configuration by the (non-dimensional) horizontal displacements $\vec{x} = (x_1, x_2, \dots)$. The essential design parameters are the dimensionless symmetry breaking $\delta = x_i(\varepsilon_i = 0)$ and spring ratio $k := k_h/k_v$.

torial driving by compressing the individual units. By designing the units as oppositely biased elastic beams, the metamaterial forms a chain of tunable double-well elements with antagonistic nearest neighbor interactions which captures generic features of the path-dependent response of complex materials.

We start by investigating a short chain comprising two coupled units (Sec. 2.2). By comparing the evolution of a given initial state under different driving paths with the same initial and final driving, we uncover two hallmarks of path-dependency: a *non-Abelian* response, where the final state depends the path, and a *mixed-mode* responses, where the nature of the evolution - reversible, continuous or irreversible, discontinuous - depends on the path. We show that fold singularities govern path dependency, and that paths crossing them link three states - ancestor, descendant, and sibling - leading to ADS-triplets that form the elementary unit of path-dependency.

Based on these triplets, we present a general framework which allows to encode the sequential response to *any* vectorial driving protocol (Sec. 2.3). This framework uses strain maps to label the location of fold singularities in driving space, and path-transition graph (pt-graphs) which extend the notion of t-graphs developed for scalar driving. We show that while pt-graphs under scalar driving can be mapped to t-graphs, pt-graphs for higher dimensional driving have qualitatively distinct features.

To enable the experimental characterization of path dependencies under high-dimensional driving (i.e., many inputs), we introduce binarized on/off driving protocols. These protocols focus on paths connecting strains $\vec{\varepsilon}' := \varepsilon^M \vec{b}$,

where ε^M is the overall strain and $\vec{b}$ is a binary vector (Sec. 2.4). The response can then be captured by ‘b-graphs’ that are simpler than the pt-graphs, and facilitate a natural link between sequential computing with Boolean circuits. Strikingly, we show that a single sample embeds multiple circuits which can be selected by the strain scale ε^M . We then analyze larger samples and determine the b-graphs as a function of the driving scale ε^M for a four-unit chain (Sec. 2.4.2).

To address the combinatorial complexity of b-graphs under high-dimensional vectorial driving, we introduce motifs. These can be seen as low-dimensional building blocks of the b-graph, but also can be determined independently of the full b-graph, even in systems of arbitrary size. We first discuss edge motifs, which consider driving paths restricted to one-dimensional subspaces of $\vec{\varepsilon}$, and we show that these are composed of elementary ADS-patterns. Loop motifs, associated with loop-like paths in two-dimensional subspaces, probe essential aspects of the sequential response to vectorial driving and exhibit substantial diversity—our four-unit example shows 13 distinct types. They capture a broad range of path dependencies, including the elementary mixed-mode and non-Abelian responses.

We analyze the loop motifs to uncover emergent memory effects and computational capabilities of our samples (Sec. 2.5). We observe chiral loop transients where the transient time - i.e. the number of driving loops before the response becomes periodic - depends on the orientation of the loop. This exemplifies the notion of memory effects that have no counterpart for scaling driving. Finally, by splitting the binarized driving at the four inputs of our sample in two for computation, and two for ‘programming’, we show that a single sample can be reprogrammed to mimic a variety of sequential Boolean circuits - on top of the variety that is accessible by varying the magnitude of these drivings.

Finally, to illustrate that our framework is applicable to systems with disordered pathways, we study compressed corrugated sheets (Sec. 2.7). Despite the absence of a simple parametrization of the states, we show that being able to determine the location of irreversible transitions and compare different configurations is sufficient to describe the pathways in our framework. In particular, these systems exhibit complex motifs, including edge and loop transient structures not seen in the metamaterial chain. While obtaining the full pt-graph or set of b-graphs is challenging for such systems, this demonstrates that motifs are a powerful tool to systematically uncover and characterize the complex path dependencies under vectorial driving of multistable systems.

Together, our findings uncover the elementary motifs that underlie path-dependent behavior in multi-stable systems. While we focus on mechanics, our approach can be applied to any multi-stable system under athermal and quasi-static driving. Vectorial driving thus opens new avenues to characterize frustrated systems, to observe new classes of multi-dimensional responses and mem-

ories, and to develop strategies for powerful *in-materia* computing.

We address the following broad questions: What new phenomena, memory effects and computational capabilities can be accessed with vectorial driving? How can path-dependent sequential responses be described? How can the descriptions for scalar and vectorial driving be connected? How to deal with the complexity associated with high-dimensional driving?

To address these questions, we combine experiments with simulations and theory to probe the path-dependent sequential response of metamaterials based on chains of coupled nonlinear units (Fig. 2.1). These chains naturally allow vectorial driving by compressing the individual units. By designing the units as oppositely biased elastic beams, the metamaterial forms a chain of tunable double-well elements with antagonistic nearest neighbor interactions which captures generic features of the path-dependent response of complex materials.

2.2 System and Phenomenology

We introduce chain-like multistable metamaterials that readily enable compression at multiple locations, along with a corresponding minimal spring model (Sec. 2.2.1). We investigate a short chain and compare the response to pairs of strain paths that have the same initial and final strains (Sec. 2.2.2). We identify two archetypical scenarios for the path-dependent response under vectorial sequential driving: First, a *non-Abelian* response, where the final configuration depends on the strain path (Sec. 2.2.2); second, a *mixed-mode* response, where the response is either smooth or non-smooth, depending on the strain path (Sec. 2.2.2). Both can be understood to arrive from a fold bifurcation, which links three configurations that we term ancestor, descendant and sibling (Sec. 2.2.2).

2.2.1 Metamaterial, Strain Protocols and Model

Design.— We aim to realize a metamaterial that naturally allows compression at multiple locations, is scalable, can be multistable, and features path dependent responses. To meet these requirements, we investigate a chain-like geometry of n units that each have a strongly nonlinear response and can become bistable. We design such a geometry based on a spring chain, consisting of n vertical units that each contain two equal springs meeting in a central node that acts as a hinge. Neighboring units are coupled by a horizontal spring between these nodes. The horizontal positions of the ends of the units are fixed such that in their relaxed state, the units are not perfectly vertical but alternately kink left and right (Fig. 2.1). This configuration ensures a deterministic response when the ends of the units are compressed vertically. Moreover, if one unit is compressed sufficiently to flip the orientation of the kink of its neighbor, we anticipate that first compressing unit i and then $i + 1$ results in a different configuration that

first compressing unit $i + 1$ and then i : this multistable metamaterials has a path dependent response (Fig. 2.2).

Sample Geometry.— Experimentally, we realize metamaterials based on this design in flexible silicone rubber using standard 3D printing and molding techniques (see Sec. A2.1). Each unit i consists of two connected, nearly vertical elastic beams of stiffness k_v which are coupled by horizontal beams of spring constant k_h ; the ratio $k := k_h/k_v$ is an important parameter. The beams have tapered connections to minimize torques. As we are interested in the limit where $k \ll 1$, we realize the soft horizontal springs using an elliptical annulus (Fig. 2.2). The distance between the endpoints of each unit at rest, $2\ell_0$, defines the length-scale ℓ_0 that we use to non-dimensionalize all lengths. The units are pre-kinked over a (non-dimensional) distance δ , such that their rest positions (defined as the horizontal deviation from the units midpoint) are $x_i = (-1)^{i+1}\delta$ (this requires the beams rest length to be $1 + \delta^2/2$). We focus here on geometries with $\delta \approx 0.08$ and $k \approx 0.09$, and create samples with $n = 2$ and $n = 4$ units (see Sec. A2.1).

Strain protocols.— We use a custom-made device to control the spatially localized compressive strains ε_i with accuracy $\sim 10^{-3}$ by symmetrically shortening the end-to-end distances $2\ell = 2\ell_0(1 - \varepsilon_i)$ such that all nodes remain horizontally aligned. We use quasi-static driving (strain rate is $5 \cdot 10^{-4} \text{ s}^{-1}$) and track the real-space configuration of the structure given by $\vec{x} = (x_1, x_2, \dots)$ using a CCD camera, resulting in an accuracy in x_i of 0.002 (see Sec. A2.1). Together, our set-up allows for automated driving protocols and precise measurements of the configurational response.

We consider strain paths P that quasistatically ramp the input strains $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots)$ between sequences of specific points in strain-space. While we also consider general driving protocols, we often employ ‘binary’ paths that consists of sequences of elementary compressions c_i (increase of ε_i from a minimum value ε_i^m to a maximum value ε_i^M) and decompressions d_i (decrease of ε_i from ε_i^M to ε_i^m) while the other strains are kept fixed. In many cases we take $\varepsilon_i^m = 0$ and $\varepsilon_i^M = \varepsilon^M$, so that ε^M sets an overall strain scale. Finally, we denote sequences of compression and decompression steps using square brackets, e.g., $[c_1, c_2, d_1, d_2]$ represents a loop-like compression path (read from left to right).

Minimal Model.— In addition to the experiments, we numerically explore a spring model. We model each beam as a linear spring with freely rotating hinges, use equidistance vertical compression points, and break the left-right symmetry by taking horizontal springs with alternating rest length $1 + (-1)^i 2\delta$ (Fig. 2.1). Expanding the elastic energy to the lowest order in x_i yields:

$$E = \sum_{i=1}^n \left(\frac{x_i^2}{2} - \gamma_i \right)^2 + \sum_{i=1}^{n-1} \frac{k}{2} (x_i - x_{i+1} - (-1)^i 2\delta)^2, \quad (2.1)$$

where $\gamma_i := \varepsilon_i + \delta^2/2 - \varepsilon_i^2/2 \approx \varepsilon_i + \delta^2/2$. The first term describes the tuneable

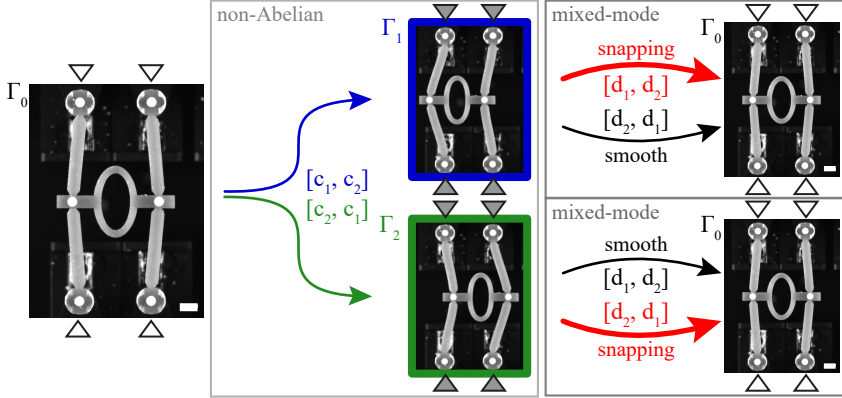


FIGURE 2.2: **Phenomenology.** Left: top view of the experimental sample A in its rest configuration Γ_0 ($n = 2$, $k \approx 0.09$ and $\delta \approx 0.08$, scalebar: 5 mm; see Sec. A2.1 for details). The central ellipse materializes a soft horizontal spring, and the top and bottom of the vertical beams connect to the strain apparatus, where white and gray triangles indicate relaxed and compressed boundaries. Middle and Right: illustration of two hall-marks of path dependency, under sequential compression respectively decompression of the first and second unit (c_1, c_2 resp. d_1, d_2). Middle: a non-Abelian response where the final configuration Γ_1 obtained by compression path $[c_1, c_2]$ (green) differs from configuration Γ_2 obtained by compression path $[c_2, c_1]$ (blue). Right: Mixed mode response where the response is smooth (black) or non-smooth (red) depending on the sequence of the decompression path.

quartic potential of each unit, and the second term describes the interactions and lifts the units degeneracy to make the response deterministic. In the limit of small δ , one can rescale the parameters to eliminate δ , leaving the effective interaction strength k as the main system-dependent parameter (see Sec. A2.2).

2.2.2 Path dependency

Here we explore path dependency for a two-unit sample (A), and demonstrate two paradigmatic forms of path-dependency which have no counterpart under scalar driving: a non-Abelian response and a mixed-mode response.

Non-Abelian Response

We consider a pair of sequential driving paths $P_1 : [c_1, c_2]$ and $P_2 : [c_2, c_1]$, with $\varepsilon_i^m = 0$ and $\varepsilon_i^M = \varepsilon^M = 0.03$. The sample is initially relaxed ($\vec{\varepsilon} = 0$) in the unique rest configuration Γ_0 , such that the driving paths P_1 and P_2 produce smooth deformations until the final configurations $\Gamma_1 = P_1(\Gamma_0)$ and $\Gamma_2 = P_2(\Gamma_0)$ are reached. We compare the final configurations Γ_1 and Γ_2 and find $\Gamma_1 \neq \Gamma_2$, demonstrating path dependence (Fig. 2.2). Since Γ_1 and Γ_2 experience

the same strain $\vec{\varepsilon} = (\varepsilon^M, \varepsilon^M)$, this non-Abelian response necessitates bistability. Similar non-Abelian behaviors have recently been observed in various other strongly nonlinear, multistable metamaterials under sequential compression [79–81].

In our system, which was designed with such behavior in mind, this non-Abelian response is observed for a large range of strains (see Sec. 2.4). This behavior can intuitively be understood from the left ($x_i < 0$) or right ($x_i > 0$) leaning orientations of the units only, although we track the position of all coordinates x_i . Consider the evolution of the real-space configuration $\{x_i\}$ during P_1 (Fig. 2.2). In the first stage (c_1), unit 1 compresses leftward, pulling unit 2 to lean left as well. Further compression of unit 2 in stage two (c_2) pushes both units left, making the resulting configuration Γ_1 left-leaning. By contrast, in P_2 , compression of unit 2 shifts unit 1 rightward, yielding a right-leaning Γ_2 (Fig. 2.2). We note, however, that such orientation changes are not required to observe a non-Abelian response: intermediate values of $\varepsilon^M \approx 0.011$ lead to a more subtle non-Abelian regime without changes in the units orientations (see Sec. A2.3).

Mixed-mode response

Next, we follow the configurations during the decompression paths $P_3 : [d_1, d_2]$ and $P_4 : [d_2, d_1]$. Starting from either of the configurations in the bistable regime at $\vec{\varepsilon} = (\varepsilon^M, \varepsilon^M)$, both paths P_3 and P_4 lead to the same configuration, i.e., $P_3(\Gamma_1) = P_4(\Gamma_1) = \Gamma_0$ and $P_3(\Gamma_2) = P_4(\Gamma_2) = \Gamma_0$. However, whereas $P_4(\Gamma_1)$ leads to a smooth deformation, $P_3(\Gamma_1)$ features a non-smooth, irreversible transition – and similarly for $P_3(\Gamma_2)$ and $P_4(\Gamma_2)$ (Fig. 2.2)¹. This mixed-mode response represents a different type of path-dependency than the non-Abelian response.

Fold bifurcation

To understand the root cause of the non-Abelian and mixed mode response, we follow the evolution of the configurations Γ_1 and Γ_2 between strains starting from $(\varepsilon^M, \varepsilon^M)$ to $(0, \varepsilon^M)$ (Fig. 2.3a). For this two beam system, we can faithfully represent the configuration by the collective coordinate $x_1 + x_2$. However, the presence of collective coordinates are not required for our framework, and are merely convenient for this specific case. Our data shows that the irreversible transition is associated with a saddle-node, or fold bifurcation, and this is confirmed in our numerical model where we can follow both stable and unstable configurations (Fig. 2.3b). To understand the root cause of the non-Abelian and mixed mode response, we follow the evolution of the configurations Γ_1 and Γ_2 between strains starting from $(\varepsilon^M, \varepsilon^M)$ to $(0, \varepsilon^M)$ (Fig. 2.3a). Our data shows

¹While here the discontinuous transition is associated with decreasing strain, we will later also encounter discontinuous transitions under increasing strain.

that the irreversible transition is associated with a fold bifurcation, and this is confirmed in our numerical model where we can follow both stable and unstable configurations (Fig. 2.3b). As we argue below, this situation exemplifies the central motif that causes path dependencies in multistable materials: On one side of the fold curve, there are two types of configurations (ancestor and sibling). Moving across the fold curve, the ancestor vanishes, and the system then jumps to another configuration that we call the descendant. Note that while descendant and sibling are smoothly connected, they can be distinguished by the presence of the ancestor, or equivalently, their strain in comparison to the fold transition. Hence, three configurations - that we refer to as ancestor, descendant and sibling - are connected by an irreversible (fold) transition from the ancestor to the descendant and a reversible transition between the descendant and sibling (Fig. 2.3c-d).

As we argue below, this situation exemplifies the central motif that causes path dependencies in multistable materials: On one side of the fold bifurcation, there are two types of configurations (ancestor and sibling). Moving across the bifurcation, the ancestor vanishes, and the system then jumps to another configuration that we call the descendant. Note that while descendant and sibling are smoothly connected, they can be distinguished by the presence of the ancestor, or equivalently, their strain in comparison to the fold transition. Hence, each crossing of a fold singularity is associated with three configurations - that we refer to as ancestor, descendant and sibling. These are connected by an irreversible (fold) transition from the ancestor to the descendant and a reversible transition between the descendant and sibling (Fig. 2.3c-d).

2.3 General Approach For Sequential Vectorial Driving

We now present a general framework to describe the response for arbitrary driving paths, based on the central role that fold singularities and ADS-triplets play, and illustrate our approach using sample A. To describe the response to *any* generic driving path, we introduce strain maps and path-transition graphs (pt-graphs). Strain maps encode the location of the fold singularities in driving space, and pt-graphs describe transitions between states that occur when the driving path crosses one of these fold singularities. Hence, nodes and edges of the pt-graph describe states and their transition, and the edges are labeled by their respective fold singularity. The edges either are directed, corresponding to irreversible transitions (such as the path from ancestor to descendant) or undirected, corresponding to reversible, smooth transitions (such as path between descendant and sibling). Constructing this description requires three steps: first identify the strain map; second, use it to define states; and third determine the

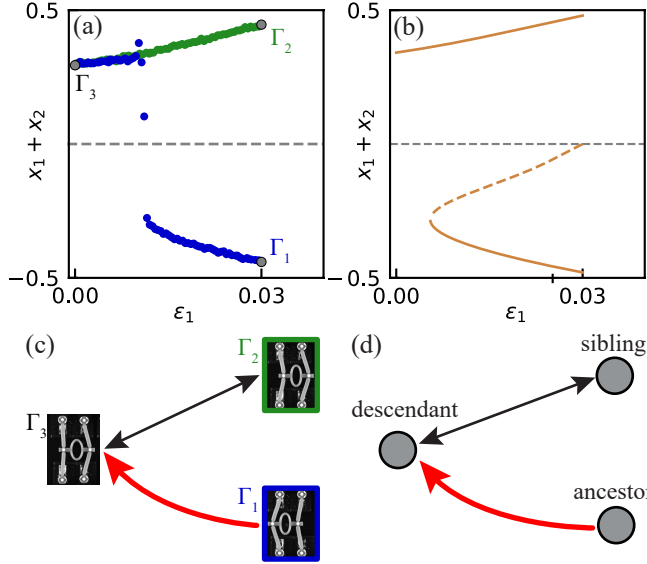


FIGURE 2.3: **Fold bifurcation and ADS-triplet.** (a-b) Evolution of the configuration $\vec{x}$ of sample A as a function of strain ε_1 which is lowered from 0.03 to 0, projected on the collective coordinate $x_1 + x_2$; here we fix $\varepsilon_2 = \varepsilon^M = 0.03$. (a) Experimental data for sample A; we initialize the sample in configurations Γ_1 and Γ_2 using the strain paths shown in Fig. 2.2. The error bars of the measured points are smaller than the symbol size. (b) Simulations of the corresponding spring model, where ε_1 is both increased and decreased, showing stable (full) and unstable (dashed) solutions, thus revealing a fold singularity. (c-d) Graph representation of the general structure underlying path dependency: three configurations (ancestor, descendant and sibling) are connected by reversible (black) and irreversible (red) transitions; Γ_1 is the ancestor, Γ_3 the descendant, and Γ_2 the sibling.

pt-graph.

We obtain the strain map for sample A by determining the locus in strain space of the fold singularities that govern the irreversible transitions for $0 \leq \varepsilon_i \leq 0.04$ (Fig. 2.4a). To do so, we explore CCW and CW loop-like driving protocols, $P_5 : [c_1, c_2, d_1, d_2]$ and $P_6 : [c_2, c_1, d_2, d_1]$; in P_5 (P_6) we fix $\varepsilon_1^M = 0.04$ ($\varepsilon_2^M = 0.04$) and sweep ε_2^M (ε_1^M). Using P_5 , we obtain the locus of the irreversible transitions (f) from 'left-leaning' to 'right-leaning' configurations; P_6 allows to define the irreversible transitions (g) from 'right-leaning' to 'left-leaning' configurations (Fig. 2.4a). The curves f and g thus specify the location of the fold singularities in strain space. For this specific example, f and g meet in a singular point Q that corresponds to a cusp singularity, associated with a pitchfork bifurcation from a symmetric configuration where $x_1 = -x_2$ to a pair of asymmetric configurations (Fig. 2.4a; Sec. A2.3).

We first obtain the strain map for sample A by determining the locus in

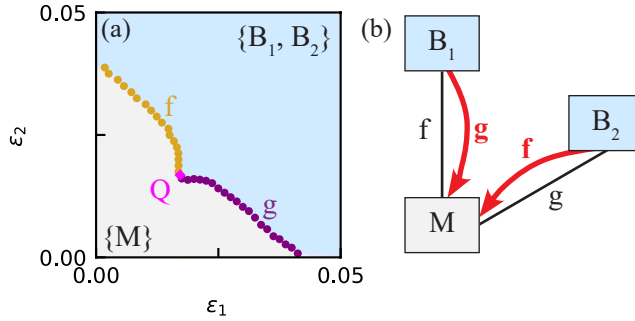


FIGURE 2.4: **Strain map and pt-graph.** The path dependent response described by a strain-map (a) and pt-graph (b), here for sample A ($n = 2, k \approx 0.09$). (a) The locations of the fold transitions f and g form curves of codimension one that separate the strain space into a monostable domain with state M (gray) and a bistable domain with states B_1 and B_2 (blue). The curves f and g meet in point Q which corresponds to a codimension-two cusp singularity, or pitchfork bifurcation (Sec. A2.3). The error bars of the measured points in f and g are smaller than the symbol size. (b) The pt-graph describes the smooth reversible connections (undirected black edges) and discontinuous irreversible transitions (directed red edges) for strain paths that cross the respective fold singularity curves of the strain map.

strain space of the fold singularities that govern the irreversible transitions for $0 \leq \epsilon_i \leq 0.04$ (Fig. 2.4a). To do so, we explore CCW and CW loop-like driving protocols, $P_5 : [c_1, c_2, d_1, d_2]$ and $P_6 : [c_2, c_1, d_2, d_1]$; in P_5 (P_6) we fix $\epsilon_1^M = 0.04$ ($\epsilon_2^M = 0.04$) and sweep ϵ_2^M (ϵ_1^M). Using P_5 , we obtain the locus of the irreversible transitions (f) from 'left-leaning' to 'right-leaning' configurations; P_6 allows to define the irreversible transitions (g) from 'right-leaning' to 'left-leaning' configurations (Fig. 2.4a). The curves f and g thus specify the location of the fold singularities in strain space. For this specific example, f and g meet in a singular point Q that corresponds to a cusp singularity, associated with a pitchfork bifurcation from a symmetric configuration where $x_1 = -x_2$ to a pair of asymmetric configurations (Fig. 2.4a; Sec. A2.3).

Second, for a given range of strains, the strain map partitions the strain space into domains. We define states as equivalence classes of configurations that are smoothly connected *within each domain*. For this example, the loci of the fold bifurcations form the fold curves f and g , which partition the strain space into a monostable domain and a bistable domain, leading to three states (Fig. 2.4a). The evolution of the configuration in the monostable domain is smooth and reversible, and we thus associate all these configurations with a state M . In the bistable domain, we find two stable configurations for any value of the strain. These two configurations smoothly deform along strain paths within the bistable domain, defining two states B_1 and B_2 - note that configurations associated with states B_1 and B_2 cannot smoothly deform into each other for strain paths within

the bistable domain. In passing, we note that our definition of states as configurations smoothly connected within domains explains why we distinguish between the descendant and sibling: these two states correspond to smoothly connected configurations but in two different domains. Hence, the locus of the fold bifurcations determines a strain map and defines the states; for the example, the strain map has two fold curves (Fig. 2.4) and three states (M , B_1 and B_2).

Third, to specify the full path dependent response, we construct a pt-graph. The nodes of this mixed graph correspond to the states M , B_1 and B_2 , and its edges correspond to paths that cross one of the fold singularity curves, and are labeled accordingly. The pt-graph combines directed and undirected edges. The undirected edges correspond to smooth, reversible paths between descendant and sibling states that, while smoothly connected, occur in different domains in strain space; the directed edges correspond to non-smooth, irreversible paths from an ancestor state to a descendant state. Hence, to construct the pt-graph we determine how states are connected when *crossing* the fold curves (in this chapter we ignore singular paths, such as paths crossing through Q). For our specific example, the situation is simple. State M smoothly transforms into state B_1 when we cross from the monostable to the bistable domain across f , and smoothly transforms into state B_2 across g ; these smooth paths are reversible, so that B_1 and B_2 smoothly connect to M across f and g , respectively. The irreversible transitions occur when starting in the bistable domain. For instance, beginning in state B_1 , crossing boundary g leads to an irreversible transition to M ($B_1 \rightarrow M$). Similarly, crossing boundary f results in the irreversible transition $B_2 \rightarrow M$ (Fig. 2.4).

The combination of the strain map and the pt-graph together captures the general features of the path dependent response in multistable systems for *any* generic strain path (i.e., not crossing singular points such as Q). This provides a unified framework to describe non-Abelian and mixed-mode responses, as well as transient and complex memory effects under vectorial sequential driving. In particular, they allow to determine precisely when certain types of responses occur: starting from any configuration associated with state M , a non-Abelian response occurs for two paths that either cross f or g and meet in the bistable domain. Similarly, starting from any configuration associated with B_1 or B_2 , a mixed-mode response occurs for pairs of strain paths that cross f or g and meet in the monostable domain.

This example illustrates general features of pt-graphs. First, each fold curve is associated with an ancestor, descendant, sibling (ADS) triplet. In particular, B_2 , M , and B_1 are the ancestor, descendant and sibling associated with curve f . Similarly, B_1 , M , and B_2 form the ADS-triplet of curve g . Second, the topology of the strain map constrains the number of edges in the pt-graph as each state and each adjacent fold-boundary correspond to an outgoing edge in the pt-graph. Hence, the number of outgoing edges is $\sum_{\text{domain}} n_s n_f$, where n_s and n_f

are the number of states and adjacent fold curves per domain, respectively; here $\Sigma_{\text{domain}} n_s n_f = 6$. As reversible edges in the pt-graph correspond to a pair of incoming and outgoing edges, this implies that $\Sigma_{\text{domain}} n_s n_f = 2n_r + n_i$, where n_r and n_i are the numbers of reversible and irreversible edges of the pt-graph, respectively. We will discuss pt-graphs for more complex singularities elsewhere [82–84].

The combination of the strain map and the pt-graph together captures the general features of the path dependent response in multistable systems for *any* generic strain path (i.e., not crossing singular points such as Q). This provides a unified framework to describe non-Abelian and mixed-mode responses, as well as transient and complex memory effects under vectorial sequential driving. In particular, they allow to determine precisely when certain types of responses occur: starting from any configuration associated with state M , a non-Abelian response occurs for two paths that either cross f or g and meet in the bistable domain. Similarly, starting from any configuration associated with B_1 or B_2 , a mixed-mode response occurs for pairs of strain paths that cross f or g and meet in the monostable domain.

The combination of a strain map and pt-graph provides a complete description of the sequential response to vectorial driving, analogous to transition graphs (t-graphs) that describe the sequential response under scalar driving [23, 25, 47–49, 58, 59, 68, 74, 75]. Determining such complete graphs can be challenging, in particular for systems with many states, and for high-dimensional driving. However, t-graphs have proven to be indispensable for understanding the sequential response of multistable systems subject to scalar driving [23, 25, 47–49, 58, 59, 68, 74, 75], and we anticipate that pt-graphs will play a similar role for systems under vectorial driving. Moreover, as we will show below, the strain map and pt-graph provide a solid conceptual foundation on which to build simplified, effective descriptions.

2.4 B-graphs and Boolean Circuits

Here we introduce *b-graphs*, which greatly simplify the description of vectorial path dependencies. The key idea is to binarize the strains as $\varepsilon_i = \varepsilon_i^M b_i$, where $\vec{b} = (b_1, b_2, \dots)$ with $b_i \in \{0, 1\}$. We then consider strain paths composed of elementary steps that change a single b_i and determine whether the system evolves continuously or discontinuously. This construction yields b-graphs and removes the need for strain maps, the most cumbersome part of the pt-graph framework. Note that the b-graph depends on the choice of ε_i^M — so that a single system described by one pt-graph/strain map may encompass multiple b-graphs.

B-graphs provide a direct connection from the sequential response of multistable systems to *in materia* computing [47]. We make this explicit by mapping

the b-graphs of sample A onto Boolean logic circuits. Because the beam configurations are near-binary (left/right), we project the real-space configuration $\vec{x}$ onto a binary output variable $\vec{Q} = (q_1, \dots)$, where $q_i = 0$ (1) when $x_i < 0$ ($x_i > 0$). We stress that this binarization is merely a convenient feature of the beam system and is not required for the b-graph description, which only relies on identifying whether configurations are distinct. Nevertheless, the vector $\vec{Q}$ enables a representation of the resulting sequential computations as Boolean logic circuits.

2

2.4.1 Boolean circuits

We determine the Boolean circuits that are embedded in Sample A. We study how the binary output $\vec{Q}(\vec{b})$ responds to binary strain paths composed of compression steps (c_i) and decompression steps (d_i), starting from the neutral configuration $\vec{Q} = (01)$ at $\vec{b} = (00)$ (see Sec. A2.4)².

Different values of ε_i^M can produce different Boolean circuits. In sample A, there are three relevant strain scales: at $\varepsilon_1 = \varepsilon_2 = \varepsilon_c \approx 0.017$ there is the cusp singularity; for $\varepsilon_i^M > \varepsilon_g \approx 0.018$, compressing a single beam changes the sign of x_i of the other; and the fold curves terminate at the axes for $\varepsilon = \varepsilon_{f,i} \approx 0.04$. In the narrow interval $\varepsilon_c < \varepsilon_i < \varepsilon_g$, the binary projection is not suitable (see Sec. A2.3), and we focus on the pathways for ε_i outside this interval.

Strikingly, we find five different sequential responses, depending on the strain scales ε_i^M (Fig. 2.5). First, for small $\varepsilon_i^M < \varepsilon_c$, the strains are too small to reach the bistable domain and the response is smooth and path independent, yielding a trivial circuit (Fig. 2.5a). For intermediate strains ($\varepsilon_g < \varepsilon_i^M < \varepsilon_{f,i}$), two stable configurations are reached for $\vec{b} = (11)$, resulting in non-Abelian pathways and irreversible transitions (Fig. 2.5b). This path dependent response is captured by an set-reset-latch (sr-latch), a simple circuit that can store one bit of information [85] (Fig. 2.5b; for details, see Sec. A2.4). For larger strains ($\varepsilon_i^M > \varepsilon_f$), every $\vec{b} \neq (00)$ produces two stable configurations, which can be represented by a logic circuit preceding the same one-bit memory element (Fig. 2.5c). Finally, asymmetric protocols where one of ε_1^M and ε_2^M is in between ε_g and ε_f , and the other is above ε_f can be represented variations of such circuits (Fig. 2.5d-e). Thus, by selecting the magnitude of the binary strains at either ‘input’, sample A can mimic four related yet distinct non-trivial Boolean circuits.

2.4.2 B-graphs

While the representation using Boolean circuits highlights the potential for *in materia* computing with multiple inputs, it relies on a binarization of the sys-

²For brevity of notation, we drop commas and write these binary quantities as (00) instead of (0, 0)

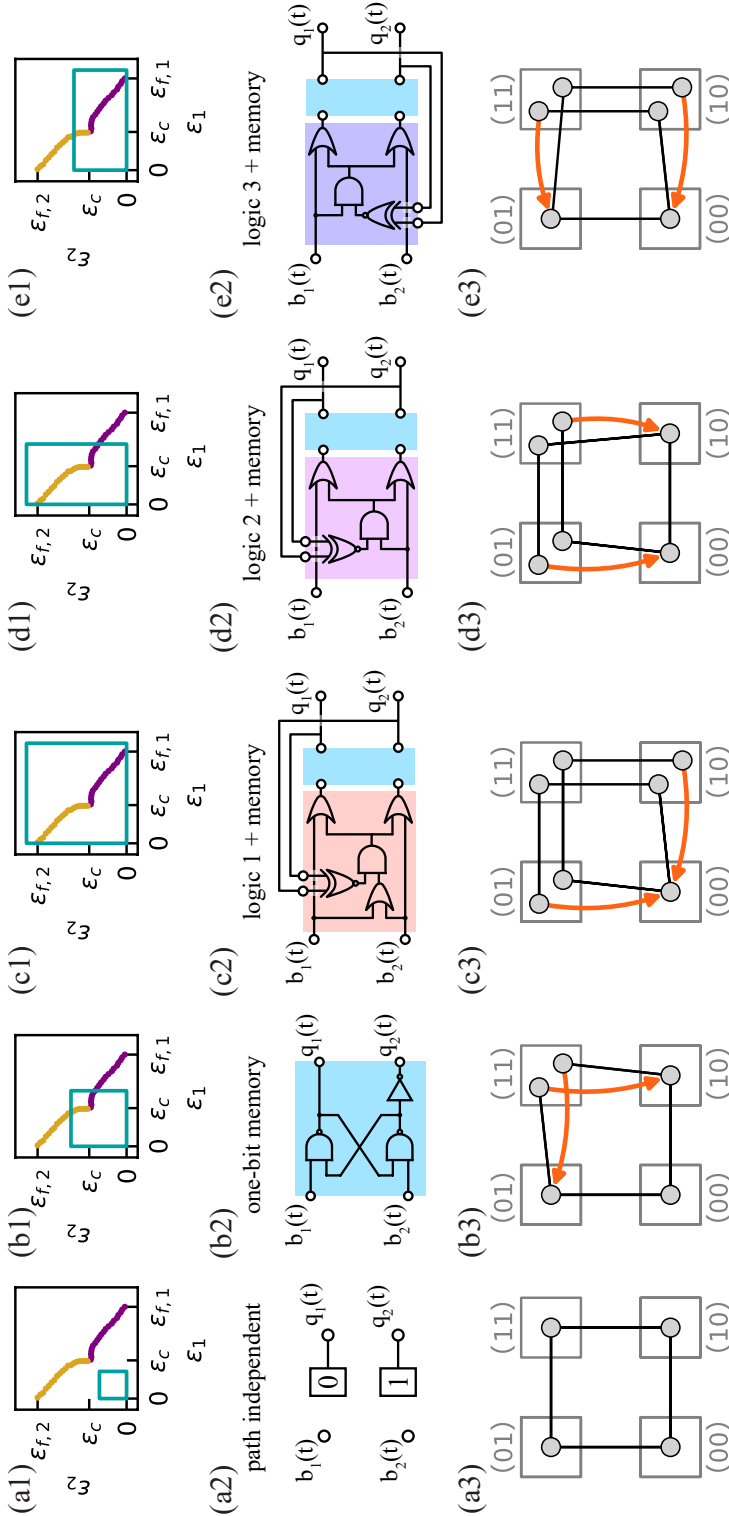


FIGURE 2.5: **Sequential response of sample A to binary strain protocols.** Top row: fold curves (yellow, purple) and binary strain protocols (green). For sample A, $\epsilon_c \approx 0.017$, $\epsilon_g \approx 0.018$, $\epsilon_{f,1} \approx 0.039$ and $\epsilon_{f,2} \approx 0.041$. In (a-c), we use symmetric strain scales $\epsilon^M = 0.01$, 0.025 , and 0.045 , respectively. In (d-e), the binary strain protocols are asymmetric with $(\epsilon_1^M, \epsilon_2^M) = (0.025, 0.045)$ and $(0.045, 0.025)$, respectively. Mid row: Corresponding Boolean circuits, where the blue blocks represent rs latches as in b2 (see Sec. A2.4 for details). Bottom row: Corresponding b-graphs. Here, each node represents a configuration grouped (in squares) by the binary strain b_i (labeled above or below the squares), and each edge represent a smooth (undirected and black) or irreversible (directed and orange) transition between two configurations.

tem's configurations $\vec{x} \rightarrow \vec{Q}$, which is not always possible. Moreover, Boolean circuits are hard to interpret, in particular for larger systems, and technical issues arise to the frequent occurrence of race-conditions in such circuits—During race conditions, the logic state flips at multiple locations in the circuit simultaneously, leading to sensitivities to tiny differences in their timing. These race conditions are a major reason for the widespread use of clocked, synchronous circuits in digital technology, rather than more frugal, faster and energy efficient asynchronous circuits.

We therefore introduce binary graphs (b-graphs) that retain the full output configuration descriptor $\vec{x}$ and capture the evolution to binarized strains for a specific fixed set of strain scales ε_i^M (Fig. 2.5a3-e3). In these b-graphs, each unique configuration corresponds to a node, and all nodes are grouped according to their respective binary strain b_i . The smooth or non-smooth evolution during each (de)compression step is represented by undirected and directed edges respectively. A b-graph is thus a mixed graph where each node has n outgoing edges corresponding to the n possible binary (de)compression steps, as we show for sample A (Fig. 2.5a3-e3).

B-graphs contain a subset of the information in pt-graphs and circumvent the challenges associated with obtaining complex strain maps, in particular for higher dimensional strains. B-graphs thus provide a practical, scalable and natural strategy to explore the saillant features of the path dependent response. In particular, we note that the non-Abelian and mixed-mode response, as well as the ADS-pattern, can be represented in b-graphs.

High-Dimensional Strain Space

For increasing number of units and inputs, the strain space becomes high dimensional and keeping track of all domains and singularity boundaries in a pt-graph becomes challenging. To demonstrate that b-graphs allow to characterize the path-dependencies of systems with moderately high-dimensional vectorial driving, we determine the b-graph for a four-unit ($n = 4$) sample B with the same geometrical parameters as sample A ($k = 0.09$, $\delta = 0.08$) at fixed $\varepsilon_i^M = \varepsilon^M = 0.035$ (Fig. 2.6a-b).

Starting from the initial monostable configuration at $\vec{b} = (0000)$, we obtain all reachable configurations by a recursive strategy: First, we apply all $4!$ compression sequences that reach $\vec{b} = (1111)$, yielding 21 distinct configurations—in Sec. A2.5 we list the corresponding real space configurations. Next, for each of these states, we apply any possible single-step decompression, which in this example yields one additional configuration. We repeat this procedure until no new configurations are discovered, and keep track of the smooth or non-smooth nature of all transitions (see Sec. A2.5). This yields a b-graph consisting of $n_c = 22$ configurations, $n_s = 35$ smooth transitions, and $n_i = 18$ irreversible transitions;

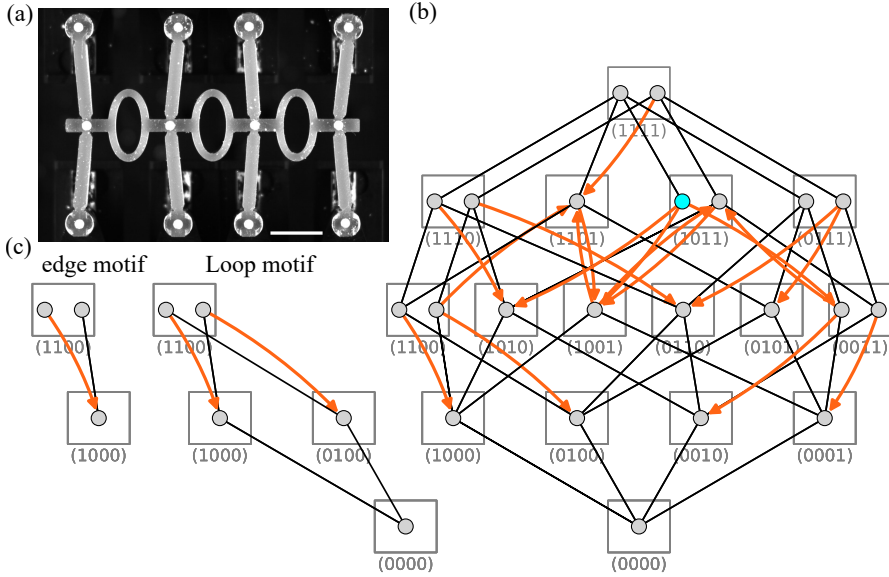


FIGURE 2.6: **B-graphs.** (a) Four-beam ($n = 4$) sample B (scalebar is 5 mm, see Sec. A2.1). (b) B-graph for sample B at $\varepsilon^M = 0.035$. Each rectangle represents a binary strain $\vec{b}$ as labeled. We organize all 2^4 strain states by $\sum b_i$ along the vertical axis, and by their lexicographical order along the horizontal axis. The grey configurations are found by sequential compression of the rest-configuration at $\vec{b} = (0000)$; the blue configuration requires one decompression step, e.g., $d_2(1111)$. (c) examples of low-dimensional building blocks (motifs) of the b-graph in (b). left: one-dimensional 'fold' (or 'ADS') edge motif; right: two-dimensional 'non-Abelian' loop motif.

note that in general, $2 \cdot n_s + n_r = n \cdot n_c$ (Fig. 2.6b). This b-graph provides a much simpler representation of the sequential response of this sample - albeit for fixed values of ε_i^M - than could be achieved using pt-graph together with a four dimensional strain map. We also have determined the b-graphs for sample B for other values of ε^M experimentally (see Sec. A2.5).

We close by summarizing the main points. Binarized input variables allow the sequential response of multistable systems under vectorial driving to be represented by b-graphs, which are straightforward to obtain and much easier to visualize than pt-graphs and high-dimensional strain maps. They therefore provide a useful framework for describing the response in terms of sequential logic. For a single sample, varying the strain magnitudes yields multiple b-graphs, demonstrating that even moderately sized systems are computationally rich. More generally, our example illustrates that b-graphs are well suited to systems with a moderate number of driving dimensions.

2.5 Motifs

How can b-graphs be systematically analyzed, and how can we deal with systems with much larger driving dimensions? To address both questions, we introduce motifs (Fig. 2.6c). These can be seen as low-dimensional building blocks of b-graphs, but also offer a practical tool for characterizing the statistical response of very high-dimensional systems, where a full description of the response or b-graph is neither feasible nor particularly insightful.

We illustrate the notion of motifs now first for the b-graph of sample B (Fig. 2.6b). Geometrically, binary strains can be represented as vertices of a n -dimensional hypercube, where the 2^n (here: 16) vertices correspond to $\vec{b}$, the $n \cdot 2^{n-1}$ (here: 32) edges represent the paths between adjacent strains, and the $n(n-1) \cdot 2^{n-3}$ (here: 24) faces correspond to loops involving (de)compressing two strains. We focus here on *edge motifs*, which capture the qualitative patterns of configurations and transitions between adjacent strains, and *loop motifs*, which describe these patterns associated with the two-dimensional faces (Fig. 2.6c). We deal with symmetries by considering motifs equivalent under flipping of the compression and decompression of each strain, i.e., $b_i \leftrightarrow (1 - b_i)$, and under relabeling of the configurations at a given strain. As we show below, motifs are powerful tools for identifying the main features and distinct path dependencies encoded by the b-graph: for example, sample B exhibits three edge and 10 loop motifs that are absent in the two-beam system, and loop motifs allow to detect the paradigmatic non-Abelian and mixed-mode path dependencies (Fig. 2.6c).

2.5.1 Edge motifs

We determined the six edge-motifs of sample B, three of which only occur once (Fig. 2.7a-f). We refer to the two simplest motifs as the smooth motif and the fold motif (Fig. 2.7a-b). These motifs are dominant in many b-graphs and, in particular, are the only edge motifs found in the two-beam sample A (Fig. 2.5). The smooth motif is characterized by a smooth reversible path between two single configurations and arises when there are no fold singularities along the strain path³ (Fig. 2.7a). The fold-motif contains a descendant for one strain and an ancestor-sibling pair for the other, and emerges when the path crosses a single fold singularity⁴ (Fig. 2.7b). We note that the descendant can be associated with either the compressed or decompressed strain — compare, e.g., the transitions between $\vec{\varepsilon}_b = (1100)$ and (1000) and between $\vec{\varepsilon}_b = (1100)$ and (1101) (Fig. 2.6)

³The reverse is not true: for example, the path between a descendant and sibling is also smooth even though it crosses a fold line, but if the corresponding ancestor is not observed, this path cannot be distinguished from a simple smooth motif.

⁴As before, the reverse is not necessarily true.

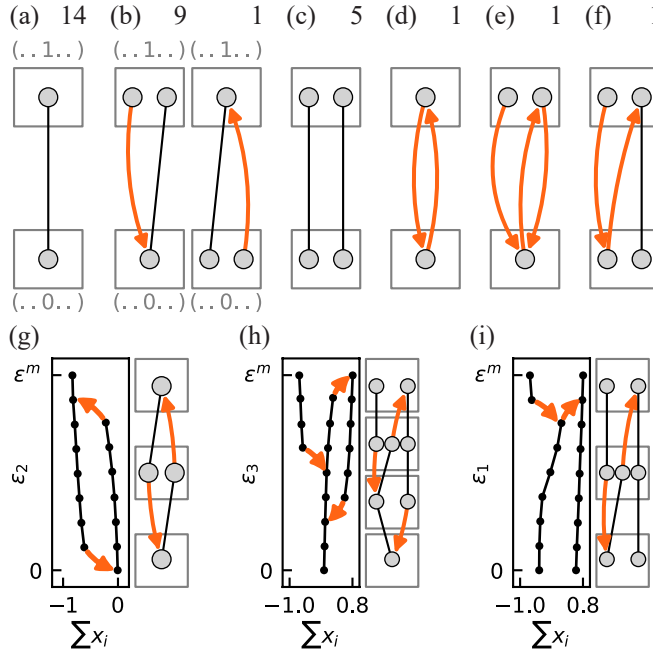


FIGURE 2.7: **Edge motifs.** (a-f) The edge motifs of sample B at $\epsilon^M = 0.035$. The top-right number indicates how often these are observed out of 32, showing that smooth (a) and fold (b) motifs are most frequent. Note that while in (a-b) we indicate the binary strains $(\dots 0 \dots)$ $(\dots 1 \dots)$, we omit these in further panels as we do not distinguish between increasing and decreasing strains (i.e., the two examples in (b) are equivalent). (c) The double smooth motif is a juxtaposition of two smooth motifs, and thus reducible. (d-f) Three more complex, reducible edge motifs occurring once in the b-graph. (g-i) We follow the evolution of the corresponding configurations along the strain paths of motifs (d-f) and characterize the configurations by collective coordinate $\sum x_i$ (strain paths between $[b_i = [(1001), (1101)]$ (g), $[(1001), (1011)]$ (h), and $[(0011), (1011)]$ (i)). In each case we find that the evolution along these paths combines two or three fold motifs (left panels), so that we can compose the motif from more elementary smooth and fold motifs (right panels). Note that, although the two fold singularities in (i) occur in the same experimental interval, these are independent events.

— and we do not distinguish between these as we considering motifs equivalent under flipping of compression and decompression (Fig. 2.7b).

The more complex edge-motifs are *reducible* – composed of multiple smooth and fold motifs via juxtaposition and/or concatenation (Fig. 2.7g-i). The simplest example of a reducible motif is a juxtaposition of motifs, such as the double smooth motif, that can be broken up into two disjointed smooth motifs (Fig. 2.7c). The other motifs (Fig. 2.7d-f) are associated with paths that cross multiple fold singularities and can be formed by concatenation (Fig. 2.7g-i). For instance, consider the pair of irreversible paths that is observed, e.g., on the path between

$\vec{\varepsilon}_b = (1001)$ and (1011) (Fig. 2.7d and Fig. 2.6). Following the configuration along this path reveals two fold singularities (Fig. 2.7g), so that this motif is composed from a pair of fold motifs. Similarly, other complex motifs (Fig. 2.7e-f) are similarly reducible and composed from smooth and fold motifs (Fig. 2.7h-i). These examples illustrate that, as generic paths only cross fold singularities, any edge motif observed is composed of smooth and fold motifs, highlighting their fundamental role.

2

2.5.2 Loop motifs

We now consider loop motifs, which probe non-trivial path dependencies unique to vectorial strain inputs by involving two 'active' beams and while keeping all other strains constant. As before, we consider two motifs equivalent under permutations of the two active beams and flipping of compressions and decompressions. By extracting all loop motifs from the b-graph of sample B, we identify 13 distinct loop motifs, which include those observed for two-beam sample A (Fig. 2.8). Note that six of these motifs feature no-return configurations. These configuration feature no incoming transitions within the loop, and can only be reached by strain protocols involving the passive beams (Fig. 2.8). We stress that each of these represent a distinct sequential logical operation, and discuss the emergent computational power of this system in section 2.6.

While all edge motifs can be constructed from just two irreducible motifs - smooth and fold motifs - the situation for loop motifs is significantly more complex. To identify the irreducible loop motifs, we focus on those that persist when the size of the loop in strain space is shrunk, and consider all different arrangements of fold curves that cross the four edges composing a loop. Apart from the trivial loop where all edges are smooth, there are three generic possibilities for the arrangements of the fold curves: either the loop is intersected by a single fold curve, the loop contains two fold curves that intersect within the loop, or the loop contains two fold curves that terminate in a higher order cusp singularity. All irreducible loop motifs can then be constructed starting from the combinatorics of the fold curves intersecting different edges — however, this construction is not trivial. Without going into too much detail, we note that fold curves have an orientation (on which side lies the ancestor node), cusp structures can correspond to both supercritical and subcritical bifurcation which lead to different motifs, and irreversible transitions can connect to multiple different descendants. We estimate that there are significantly more than 20 of such irreducible loop motifs.

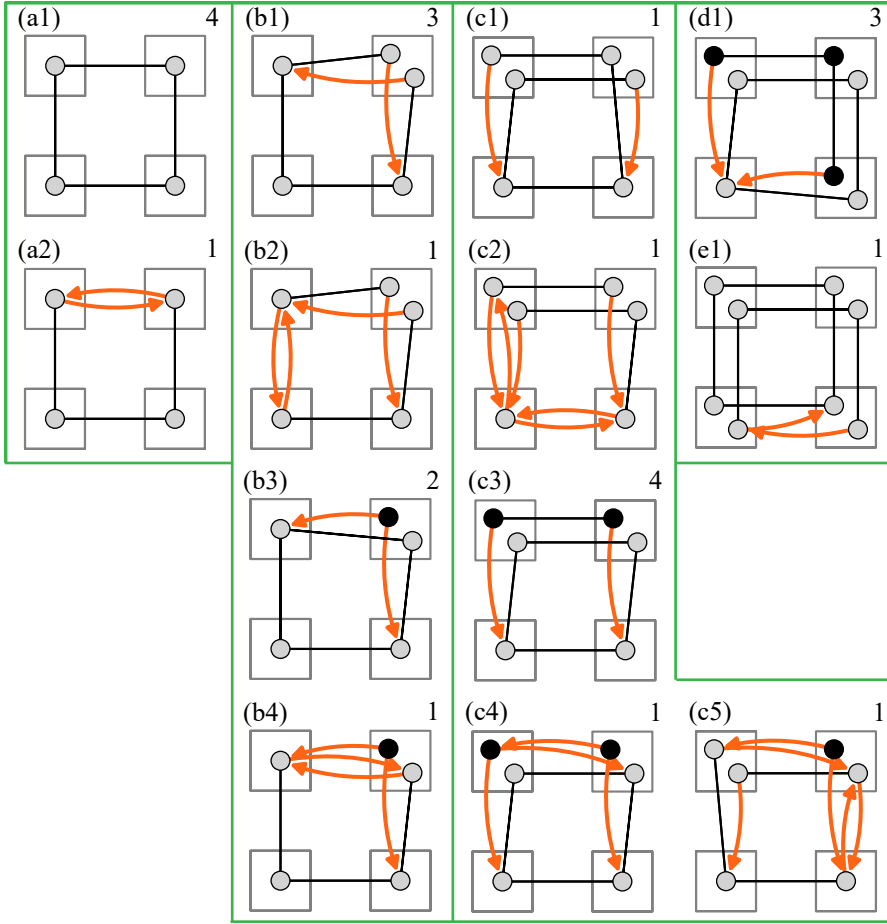


FIGURE 2.8: **Loop motifs.** The loop motifs of sample B at $\epsilon^M = 0.035$. The black nodes lack an incoming connection in the motif, and correspond to 'no-return' configurations in their specific motif. We group the motifs by their number of configurations, and orient these such that the larger number of configurations are to the right and top. Five of these motifs (a1,b1,b3,c3,d1) occur more than once in sample B, indicated by the top-right number.

2.6 Emergent Memory And Computing

We now discuss the path-dependent response under vectorial driving from two perspectives: as an emergent phenomenon that leads to novel memory effects that allows to understand and classify disordered media (Sec. 2.6.1), and as a new avenue towards *in materia* computing (Sec. 2.6.2).

2.6.1 Transient responses

The response to cyclic driving provides an important fingerprint of the memory effects of complex systems [20–24, 47, 52–59, 61, 64–66, 68, 71–73, 76, 77, 86, 87]. For scalar cyclic driving, deterministic, dissipative, multistable systems must settle in a periodic response after a finite number of cycles [20, 22, 24, 47, 52–59, 61, 68, 71, 72]. For a wide class of systems that can be seen as composed of weakly interacting elements, the response satisfies so-called Return Point Memory which forbids the presence of transients longer than one cycle [4, 21]. Recent observations of finite transients, in, e.g., disordered media and metamaterials, are thus a hallmark of complex memory [20, 22, 24, 47, 52–59, 61, 68, 71, 72].

Here we analyze the edge and loop motifs of sample B from the perspective of transient responses. We first discuss the transients captured in edge-motifs, which are equivalent to the transients under scalar driving, and then introduce loop-transients. We find that edge and loop transients can occur independently, and that the presence of loop transients may depend on the orientation of the driving loops. Vectorial or multi-input driving thus opens a new window on a wider diversity of complex memory effects.

First, edge motifs capture cyclic driving of a strain ε_k , and yield a sequence of configurations $c_0 \rightarrow c_1 \rightarrow c_2 \dots$ that must repeat. The index of the first repeating configuration, t_e , defines the *transient time* $\tau_e := t_e/2$ – for example, for a sequence of configurations $\Gamma_0 \rightarrow \Gamma_1 \rightarrow \Gamma_2 \rightarrow \Gamma_1 \rightarrow \dots$, $\tau_e = 1/2$ [20, 22, 24, 47, 52–59, 61, 68, 71, 72]. Clearly, the smooth motif leads to $\tau_e = 0$ and the fold motif can produce $\tau_e = 1/2$, provided the initial configuration is the ancestor (Fig. 2.7b). In composite motifs, which involve two or more folds, in our examples $\tau_e = 0, 1/2$, and 1 respectively (Fig. 2.7d-f).

Next, we turn to loop motifs and the response to loop-cyclic strain paths that involve two ‘active’ beams – notice that such loop-cyclic paths can be clockwise or counterclockwise. We define the loop transient time as $\tau_l := t_l/4$, where $t_l > 0$ is the index of the first repeating configuration. With the exception of two very simple loop-motifs (Fig. 2.8a), all others exhibit non-zero loop transients. Starting from a no-return configuration, $\tau_l \geq 1/4$, and the longest transient observed is ($\tau_l = 1$) (Fig. 2.8e).

Loop-transients exhibit properties that have no counterparts in transients under scalar driving, which we illustrate using the cusp motif (Fig. 2.8b1, Fig. 2.9a-b). The loop transient τ_l often depends on the loop-orientation: starting from the “green” configuration of the cusp motif, τ_l is $1/4$ for a CCW loop, and 0 for a CW loop (Fig. 2.9a-b). This example illustrates that a non-zero loop-transient can occur along a totally smooth and reversible path – something that is impossible along linear paths (Fig. 2.9a). Finally, the presence of loop and edge transients is decoupled. Starting from the top node in the bistable region, for the CCW loop-cyclic path $\tau_l = 1/4$ (Fig. 2.9a), while an edge-cyclic path starting in the

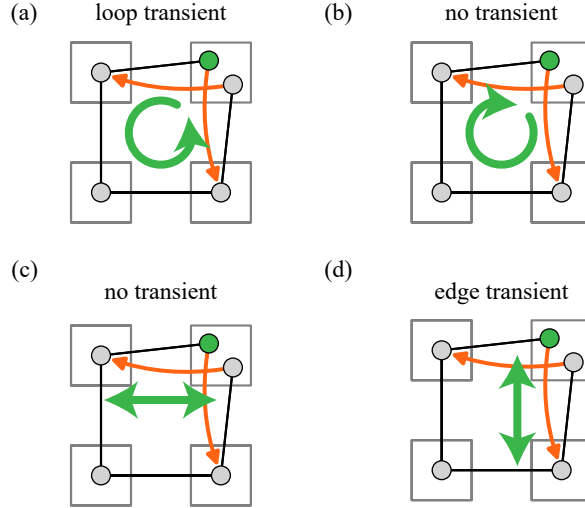


FIGURE 2.9: **Transients.** Illustration of the impact of orientation on the transient time. We focus on a single motif, and transient times under scalar and loop actuation (green arrow) starting from a given configuration (green circle). (a-b) The transient response of an edge-cyclic strain path depends on the edge; (in (a), $\tau_e = 0$; in (b) $\tau_e = 1/2$). (c-d) The transient response of a loop-cyclic strain path depends on the orientation of the loop. (in (c), $\tau_l = 1/4$; in (d) $\tau_l = 0$.)

same direction has $\tau_e = 0$ (Fig. 2.9c)— similarly, the CW loop-cyclic path leads to $\tau_l = 0$ (Fig. 2.9b), while the corresponding edge-cyclic path yields $\tau_e = 1/2$ (Fig. 2.9d).

2.6.2 Tunable emergent computation

The complex response of multistable materials with multiple inputs opens a route to tunable computation. In our example of sample B, the loop motifs already uncover a wide range of different sequential logic circuits, but here we focus on the aspects of reprogrammability, where we use two inputs, e.g. 1 and 3, to tune the sequential logical operations on two other inputs, e.g., 2 and 4 (Fig. 2.10). For this example, compressing the programming beams enables the selection of four distinct mechanical algorithms that operate on the input beams. Hence, high-dimensional input spaces allow for allocating specific input dimensions to tune the *in materia* computing properties of the remaining input dimensions. While a similar strategy has been used to obtain reprogrammable logical circuits in disordered electronic materials with multiple terminals, we believe that our example illustrates that multistable systems naturally allow for sequential operations that mix logic and memory [88–91]. In general, we expect that these computations can be captured by finite state machines [47], where it is an open question how

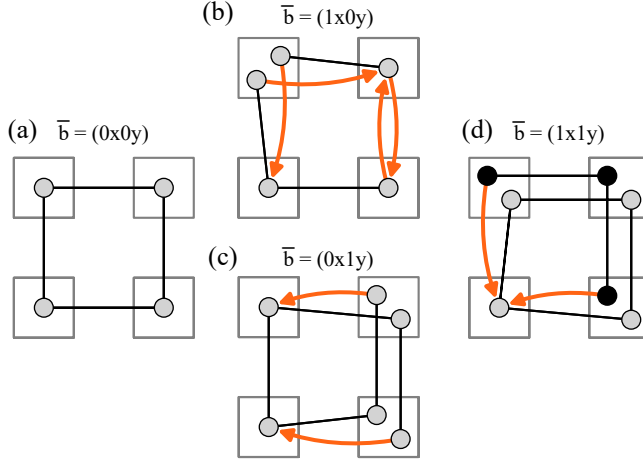


FIGURE 2.10: Loop motifs of sample B at $\varepsilon^M = 0.035$ with active beams 2 and 4 and distinct tuning beams. In this representation the strains on the active beams 1 and 3 correspond to the x and y direction in the motifs, and the binary strains on the passive beams b_1, b_3 is 0 or 1, as indicated by the labels.

the number of states, input characters and computational capabilities grow with the dimensionality of the driving input.

We point out that a wider variety of more powerful computations can be realized by using more complex driving protocols, samples with different interaction strength k , and more complex geometries with more inputs. First, relaxing the constraints that the minimal input strains, ε_i^m are zero and the maximal input strains, ε_i^M are equal, naturally expands the space of computations and reprogrammabilities. Similarly, more information can be encoded using non-binary driving sequences, pushing the computational power of our multistable samples in the direction of finite state machines with many states and input characters [47]. Second, in numerical simulations we have observed that for lower values of k , the number of stable configurations per strain can reach 2^n , and the number of reachable configurations is at least five. Hence, such samples would allow for longer and more varied transients and multi-bit sequential operations. Finally, our chain geometry encodes specific interactions, and planar or fully connected geometries may encode more complex computations.

2.6.3 Computational capabilities

To understand the computational capabilities of our systems, we note that there is a competition between the number of states that can be stored and the computational expressivity. This balance can be clearly illustrated using the beam

system. In the limit where the horizontal springs have a vanishing spring constant $k_h \rightarrow 0$, a system with n beams can in principle store 2^n distinct states. However, computational expressivity vanishes in this limit, since compressing the beams simply leads to independent and deterministic increases of the coordinates x_i . In the opposite limit of a large horizontal spring constant, the system only has two stable states with constant $x_i - x_{i-1}$, thus the expressivity again vanishes. While there is no universal measure of sequential expressivity, and its relation to formal finite-state-machine descriptions [47] or scaling with the number of beams n remains open, these limits suggest an optimum in design space where a large fraction of the 2^n states is reachable and the final state depends intricately on the driving sequence.

2.7 Disordered systems

To highlight the general applicability of our approach, we consider the response to vectorial driving of thin elastic sheets with parallel corrugations which form a disordered system (Appendix). In earlier work, we tracked the sequences of snapping transitions under compression along the direction of the grooves and showed that such "groovy sheets" are multistable (Fig. 2.11a) [24, 70]. Despite their regular appearance, the response of groovy sheets is comparable to those of disordered systems: the precise locations and critical strain associated with the snapping events are set by fabrication imperfections and are thus frozen yet essentially random [24]. In addition, and in contrast to our chain-like metamaterials, the different states of the sheet are not easily distinguished by a set of predetermined coordinates. Hence, groovy sheets allow to test our approach in the context of disordered systems.

To explore the motifs of this disordered system, we apply two-dimensional vectorial compression $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ to two separated locations along the corrugations (Fig. 2.11b) — note that in previous work compression was homogeneous [24]. Despite the absence of simple coordinates that characterize its evolution, we can easily determine experimentally whether (i) variations of $\vec{\varepsilon}$ lead to smooth deformations or snappy and irreversible transitions, and (ii) whether two configurations are distinct or not [24]. As discussed before, this is sufficient to determine b-graphs.

To explore the rich path dependence of these sheets, we first scan strain space to identify regions of interest (Appendix). From this survey, we highlight three motifs - not observed in the chain - that emerge under binary protocols where $\varepsilon_i = \varepsilon^m + b_i \varepsilon_i^\Delta$ (see Sec. 2.2.1). We first illustrate two non-trivial motifs observed for $\varepsilon^m = 0$ (Fig. 2.11c-d). The first motif is Abelian, but shows a combination of smooth, reversible deformations and non-smooth irreversible transitions (Fig. 2.11c). The second motif is non-Abelian, demonstrating that such path

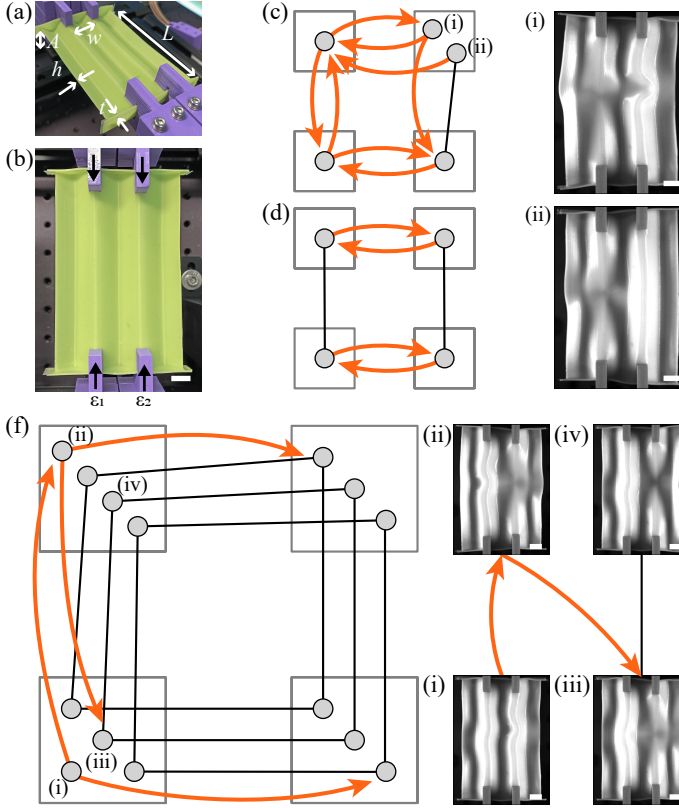


FIGURE 2.11: Path-dependent response of disordered material formed by a corrugated rubber sheet. (a) Sideview of the corrugated sheet clamped in our experimental device (for details see Sec. A2.5). (b) Topview of the corrugated sheet compressed by a vectorial drive $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ applied along the grooves at distinct locations as indicated. Scale bars in (b-e) represent 10 mm. (c-e) Loop motifs from binarized driving $\varepsilon_i = \varepsilon^m + b_i \varepsilon_i^\Delta$, where $b_i = 0, 1$. (c) Abelian loop motif with four irreversible transitions, for $(\varepsilon^m, \varepsilon_1^\Delta, \varepsilon_2^\Delta) = (0, 0.045, 0.020)$. (d) A non-Abelian motif with two distinct configurations (i)-(ii) at $\vec{b} = (11)$ as shown on the right. Here $(\varepsilon^m, \varepsilon_1^\Delta, \varepsilon_2^\Delta) = (0, 0.045, 0.045)$. (e) Complex motif containing both edge transients, and loop transients for $(\varepsilon^m, \varepsilon_1^\Delta, \varepsilon_2^\Delta) = (0.020, 0.025, 0.025)$. Panels (i)-(iv) show the evolution of an edge transient under cyclic driving. Other edge transients with $\tau_e = 0.5$, as well as loop transients $\tau_l = 0.25, 0.5$ also arise in this motif.

dependent responses are a general feature of multistable materials. This specific motif exhibits an intricate combination of mostly irreversible transitions not seen in the chain (Fig. 2.11d). For the third motif, we use pre-compression with strain $\varepsilon_m = 0.025$, where the energy landscape is expected to be more complex and contains more states [24]. We observe a complex motif that exhibits both edge transients ($\tau_e = 0.5, 1$) and loop transients $\tau_l = 0.25, 0.5$ (Fig. 2.11e).

We note that these measurements are not sufficient to determine the PT-graph – even though a given strain path between two configurations may show an irreversible transition, we cannot rule out that there are other paths where these configurations are smoothly connected. In contrast, determining the full set of b-graphs, lets say for each loop of the form $\varepsilon_i = \varepsilon_i^m + b_i \varepsilon_i^\Delta$ is possible although tedious. By focusing on the motifs one effectively can sample the complexity of the sequential response under vectorial driving.

This example illustrates the broad applicability of our framework, not only to model systems such as our metamaterial chain, but also to disordered materials that lack a simple parametrization of their states. Vectorial driving reveals new insights into their sequential response and poses experimental constraints on putative models of complex materials. In this context, motifs provide a particularly effective tool for characterizing complex behavior.

2.8 Conclusion And Outlook

Vectorial driving reveals a wealth of path-dependent responses and offers new insights into the complexity of frustrated materials. We motivated this paper with broad questions about new phenomena and computational capabilities, the description of vectorial responses, and how to address the complexity of high-dimensional driving. We now summarize our findings.

First, we observe striking new phenomena such as the non-Abelian response, mixed mode response and orientation-dependent loop transients (Secs. 2.2.2 and 2.5). These allow even small systems to act as memory latches which are programmed by the sequence of driving (Sec. 2.4); more general, they open a route to multi-input finite state machines [47]. Moreover, we offered three related frameworks for the description of the path-dependent sequential responses. At the most fundamental level, pt-graphs provide a complete description of the sequential response to any driving protocol and generalize t-graphs from scalar to vectorial driving. This description directly links the path-dependent response to the underlying structure of singularities. We show that co-dimension one fold singularities underpin all generic path dependencies and lead to ADS-triplets that form the basis of graph-based descriptions of the sequential response. Pt-graphs differ from t-graphs in that their edges can be reversible or irreversible and are labeled by boundaries in parameter space rather than single critical thresholds [21, 23–25, 47, 48, 73–76]. However, the two are closely related: restricting a pt-graph to a single one-dimensional path yields a t-graph, and pairs of ADS-triplets form elementary hysteresis loops (hysterons) [21, 23–25, 47, 48, 58, 59, 61, 73–76].

Obtaining a complete pt-graph is challenging for large systems with many states or high-dimensional driving spaces, compounding a problem already

present for scalar driving, where determining the full t-graph may not be possible or practical [21, 23–25, 47, 48, 58, 59, 61, 73–76]. We offer two strategies to address the most salient features of the response to vectorial driving. B-graphs simplify the description by binarizing the driving input, and we demonstrate that b-graphs can be obtained for a four-input systems. This approach provides an effective, simplified driving strategy that reveals new memory effects and computational capabilities (Sec. 2.4). For even larger systems, we introduce motifs as a general strategy to probe the sequence dependent sequential response for any system, regardless of its size, and demonstrate their applicability for complex chains and for a disordered system (Secs. 2.4.2 and 2.7).

More broadly, the historic increase in the complexity of measured data in physical experiments—from scalar measurements to images, and from static to dynamic data—is increasingly matched by a comparable increase in the complexity of driving protocols. For example, several recent studies have explored multistable systems driven along multiple directions, including shear applied in different orientations [77, 92–98], uniaxial compression applied along different axes [81], and sequential combinations of compression and shear [8, 99, 100]. While these works typically focus on specific protocols or responses, they nonetheless provide concrete settings in which vectorial driving naturally arises and complex path-dependent responses, including non-Abelian responses, emerge. Our work differs in that we explicitly consider spatially textured driving and provide a framework—based on pt-graphs, b-graphs, or motifs—to systematically organize the resulting sequential responses. In addition, recent work on rate-dependent responses naturally introduces a vectorial driving space and opens the door to systematic studies of memory effects under time-ordered inputs without requiring new experimental detection techniques [101–103]. More generally, any system with multiple controllable driving “directions” offers a natural platform in which the concepts developed here could be applied. While our paper focuses on mechanical systems, we believe our approach may be relevant for a wide range of platforms, including in optics [104, 105], chemistry [106, 107] and electronics [89]. Together, vectorial driving opens new avenues for characterizing frustrated systems, uncovering novel multi-dimensional responses and memories, and advancing strategies for powerful *in-materia* computing [91].

Finally, vectorial driving allows a new window on complex materials, as illustrated in section 2.7. Even though the internal state of such materials may be complex [108], as long as one can determine the discontinuities along driving paths and can compare whether two configurations are equal, our framework is applicable. One accessible and novel way to characterize extended disordered systems is to test whether their response to two-dimensional driving is non-Abelian as a function of the distance between driving locations and the driving strength, which may reveal characteristic length and actuation scales [109]. Moreover, we expect a wealth of new memory effects, such as multiperiodic re-

sponses under loop-like driving paths [20, 22, 24, 52–58, 64, 71], and higher-dimensional equivalences of return-point memory [20–24, 61, 73].

Appendix

A2.1 Experimental Details

A2.1.1 Geometry and Fabrication

Our samples A and B are fabricated by pouring degassed Mold Star 30 silicone rubber (Young's modulus of ≈ 1 MPa, Poisson's ratio of ≈ 0.5) into open face molds that are 3D printed on UltiMaker S3 printers. The samples are cured at room temperature. After curing, the samples are removed by breaking the molds carefully.

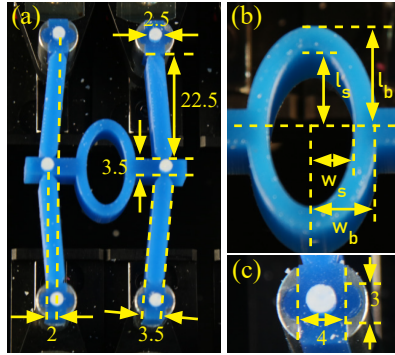


FIGURE A2.1: **Sample geometry**; all dimensions in millimeters, errorbar ± 0.1 mm. (a) Sample A with two units, (b) Zoom in on elliptical annulus. (c) Zoom in on terminating block.

Sample A features two units and sample B features four units; both have the same beam parameters ($t = 3.5$ mm, $\delta = 0.08$), a thickness of 10 mm to prevent out-of-plane buckling, and tapered connections to minimize torsional effects (Fig. A2.1a). We embedded an elliptical annulus ($w_b = 12$ mm, $l_b = 20$ mm, $w_s = 8$ mm and $l_s = 16$ mm) in the horizontal beam to lower its spring constant (Fig. A2.1b). Each unit is terminated by a rectangular block that allows to attach the sample to the compression device (Fig. A2.1c). Finally, we use 2.5 mm diameter protrusions that are painted white to track the real-space configuration

(Sec. A2.1.3).

A2.1.2 Measuring Effective Coupling

We measured the mechanical response of the vertical beam and horizontal spring to determine the effective interaction coefficient $k = k_h/k_v$ of our samples. We fabricate separated vertical units and elliptical spring units using the same 3D-printing and molding techniques, and probe their force-compression response $F(U)$ in an uniaxial testing device (Instron type 5965), which controls the compression U better than 10^{-3} mm and allows us to measure the compressive force F with an accuracy 10^{-4} N. We use linear fits to $F(U)$ and find that $k_v = 9.17 \pm 0.01 \cdot 10^2$ N/m and $k_h = 81 \pm 5$ N/m, leading to $k \approx 0.09$.

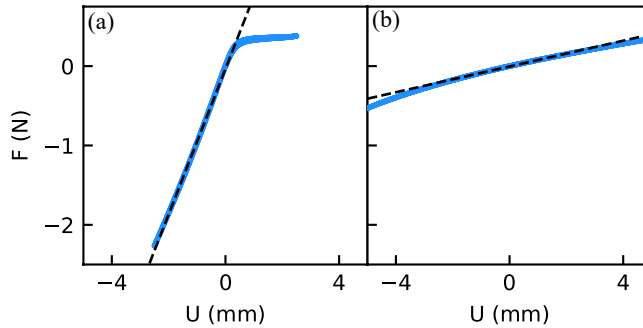


FIGURE A2.2: **Mechanical response** of a vertical beam (a) and horizontal spring (b) and linear fits (dashed lines). The vertical spring buckles at large compression ($U > 0.7$ mm) - in our path dependent experiments, we do not reach this limit - and we restrict the fit to the domain $U \in [-2 \text{ mm}, 0.5 \text{ mm}]$ - The dashed lines represent linear fits.

A2.1.3 Driving

We use a custom-made compression device to individually control the compression on each beam. The beams are symmetrically compressed from both top and bottom to ensure horizontal alignment of the middle nodes. The compression is controlled by a DC servo motor actuator (Thorlabs Z825B), providing an accuracy of 0.01 mm and a repeatability of 0.01 mm. To maintain operation within the quasi-static limit, the compression rate is set to 0.1 mm/s, with a maximal acceleration of 3 mm/s^2 . During each compression sequence, snapshots of the metamaterial configuration are captured using a CCD camera (Basler acA2040-90um, 12.9 pixel/mm). From the snapshots, the node positions are extracted using standard python image analysis packages, achieving an accuracy of 0.005 of the dimensionless quantities $\vec{x}$.

A2.1.4 Repeatability

All strain protocols for each experimental b-graph shown in Figs. 2.5, 2.6, and 2.11 (The data of Figs. 2.7-2.10 are derived from these figures) were repeated at least three times. In all cases, the measured responses were consistent across the repetitions.

A2.2 Numerical Details

Complementary to experiments, we use a numerical spring model. We define the elastic energy $E(x_i)$ by approximating each biased beam with two (nearly) vertical springs with spring constants k_v and rest length $l_0 = \sqrt{1 + \delta^2}$, where δ is the bias of the beam. The energy of each vertical spring reads:

$$E_{v_i} = \frac{1}{2}(l - l_0)^2 = \frac{1}{2} \left(\sqrt{(1 - \varepsilon_i)^2 + x_i^2} - \sqrt{1 + \delta^2} \right)^2, \quad (\text{A2.1})$$

where, x_i is the position of the middle of the beam with respect to the clamped-clamped boundaries and $2\varepsilon_i$ is the compressive strain. The horizontal spring has alternating rest length $1 - (-1)^i 2\delta$ and spring constant k_h and energy:

$$E_{h,i} = \frac{k}{2} (x_i - x_{i+1} - 2\delta)^2. \quad (\text{A2.2})$$

As each unit in our system consists of two vertical springs, we obtain a total energy:

$$E = \sum_{i=1}^n \left(\sqrt{(1 - \varepsilon_i)^2 + x_i^2} - \sqrt{1 + \delta^2} \right)^2 + \dots \sum_{i=1}^{n-1} \frac{k}{2} (x_i - x_{i+1} - (-1)^i 2\delta)^2. \quad (\text{A2.3})$$

where $k := k_h/k_v$ is the ratio between the horizontal and vertical spring constants.

We obtain equilibrium positions of $\{x_i\}$ given a strain $\vec{\varepsilon}$ by finding the roots of the forces $F_i = \partial_{x_i} E$ using the Newton-Raphson method. Each strain-driving step is divided into several small steps of 10^{-5} to mimic the quasi-static dynamics and to allow to distinguish between smooth and non-smooth deformations.

In the lowest order, the energy simplifies to:

$$E = \sum_{i=1}^n \left(\frac{x_i^2}{2} - \gamma_i \right)^2 + \sum_{i=1}^{n-1} \frac{k}{2} (x_i - x_{i+1} - (-1)^i 2\delta)^2, \quad (\text{A2.4})$$

where $\gamma_i := \varepsilon_i + \delta^2/2 - \varepsilon_i^2/2 \approx \varepsilon_i + \delta^2/2$. Hence, our model system essentially consists of coupled quartic potentials. In the small strain limit ($\varepsilon_i^2 \approx 0$), this approximation of E can be rescaled such that the bias δ drops out:

$$x_i \rightarrow \delta x'_i, \quad (\text{A2.5})$$

$$\varepsilon_i \rightarrow \delta^2 \varepsilon'_i, \quad (\text{A2.6})$$

$$k \rightarrow \delta^2 k, \quad (\text{A2.7})$$

$$E \rightarrow \delta^4 E', \quad (\text{A2.8})$$

resulting in:

$$E' = \sum_{i=1}^n \left(\frac{x_i'^2}{2} - \varepsilon_i - \frac{1}{2} \right)^2 + \sum_{i=1}^{n-1} \frac{k}{2} (x_i - x_{i+1} - (-1)^i 2)^2. \quad (\text{A2.9})$$

Hence, in lowest order, the coupling strength k is the only system parameter.

A2.3 Configurations Near Cusp

Here, we detail the real-space configurations in the bistable domain near the cusp/pitchfork Q . So far, we have associated the non-Abelian response with B_1 and B_2 being 'left-leaning' or 'right-leaning' for sample A (see Sec. 2.2.2). However, near the pitchfork bifurcation Q , the configurations associated with B_1 and B_2 both have $x_1 < 0$ and $x_2 > 0$, and are thus not 'left-leaning' or 'right-leaning' (Fig. A2.3). As a result, the binary projection of $\vec{x}$ does not capture the bistability in this (small) domain (Fig. A2.3a).

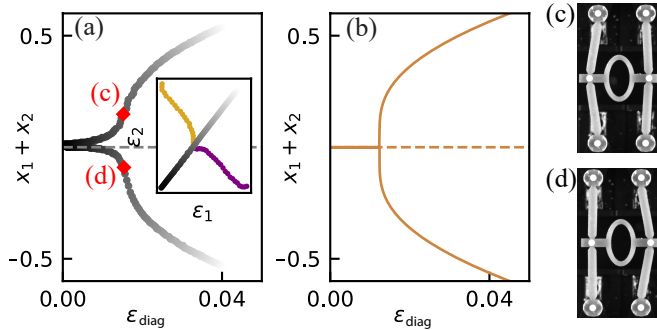


FIGURE A2.3: **Pitchfork bifurcation** observed along a non-generic strain path through a cusp singularity. (a) The collective coordinate $x_1 + x_2$ along a diagonal strain path ($\varepsilon_1 = \varepsilon_2 := \varepsilon_{\text{diag}}$, inset) starting from $\varepsilon_{\text{diag}} = 0.04$ and ending at $\varepsilon_{\text{diag}} = 0$ (sample A). (b) Results from spring model ($k = 0.09$ and $\delta = 0.08$) along the same path. (c-d) Snapshots of the system corresponding to the red dots indicated in panel (a); note that in both cases, the left beam is left leaning and the right beam is right leaning.

A2.4 Boolean Logic Details

Here, we detail the mapping from the five b-graphs, present in sample A, to their Boolean circuits. Recall that for this mapping we project the real-space configuration onto a binary output variable $\vec{Q} = (q_1 q_2)$, where $q_i = 0$ ($q_i = 1$) when $x_i < 0$ ($x_i > 0$). Furthermore, we introduced a binary input variable $\vec{b} = (b_1 b_2)$ with $b_i = 0, 1$, defining the strain as $\varepsilon_i = \varepsilon_i^M b_i$, and consider sequential protocols where $\vec{b}(t)$ and $\vec{b}(t-1)$ differ in precisely one location. As explained in the main text, we obtain the five different regimes by varying ε_i^M , avoiding the tiny interval $\varepsilon_c < \varepsilon_i^M < \varepsilon_g$ where the units are not clearly left or right leaning.

To describe the five distinct path dependent responses, we present the relation between $\vec{b}(t)$, $\vec{b}(t-1)$, $\vec{Q}(t)$ and $\vec{Q}(t-1)$ in truth tables (Fig. A2.4). We note that the truth table of the linear regime ($\varepsilon_i^M < \varepsilon_c$) is trivial ($\vec{Q}(t) = (01)$ irrespective of (previous) inputs). In the four non-trivial truth tables, we highlight cases where $\vec{Q}(t)$ depends on the previous state in blue (Fig. A2.4). In the main text, we map these truth tables to logic circuits, although we stress that these circuits are not unique. Finally, we stress a subtle point between ordinary SR latches and our circuits: since the outputs of SR latches usually are supposed to be connected (i.e. $Q_1 = \neg Q_2$, the input $\vec{b} = (00)$ is considered ill-defined - here the outputs are independent, and therefore all inputs are allowed.

A2.5 Experimental Four-Beam B-Graphs

Here, we detail the four-beam b-graph of sample B and provide four additional b-graphs for distinct ε^M . Different driving sequences that end in strain $\vec{b}$ can lead real-space configurations $\vec{x}$ that are only slightly different. We take two configurations as equivalent if $\sum_i |x_i(P_k(\vec{b})) - x_i(P_l(\vec{b}))| \leq 0.015$.

Next, we provide a schematic state description of the b-graph in Fig. 2.6b in Tab. A2.1. To do so, we represent $x_i < 0$ as $<$ and $x_i > 0$ as $>$, thus each state can be presented by four symbols such as ' $><><$ '.

In addition to the b-graph presented in Sec. 2.4.2 for $\varepsilon^M = 0.035$, we map the response of sample B for $\varepsilon^M = 0.03, 0.0325, 0.0375$, and 0.04 (Fig. A2.5). In these responses, we find the same edge motifs and the same loop motifs as in Sec. 2.4.2. Note that the qualitative response for $\varepsilon^M = 0.03$, and $\varepsilon^M = 0.0325$ is equivalent (Fig. A2.5a-b). Together, we conclude that high-dimensional b-graphs can be robustly mapped using our recursive strategy.

Groovy Sheets details

Our groovy sheets are fabricated in a three-step process. First, the corrugations are formed by casting degassed Zhermack Double Elite 32 silicone rubber

(a) $\vec{b}(t)$	$\vec{b}(t-1)$	$\vec{Q}(t-1)$	$\vec{Q}(t)$
(00)	(xx)	(xx)	(01)
(01)	(xx)	(xx)	(11)
(10)	(xx)	(xx)	(00)
(11)	(01)	(00)	(00)
(11)	(10)	(11)	(11)

(b) $\vec{b}(t)$	$\vec{b}(t-1)$	$\vec{Q}(t-1)$	$\vec{Q}(t)$
(00)	(xx)	(xx)	(01)
(01)	(00)	(01)	(11)
(01)	(11)	(00)	(00)
(01)	(11)	(11)	(11)
(10)	(00)	(01)	(00)
(10)	(11)	(00)	(00)
(10)	(11)	(11)	(11)
(11)	(01)	(00)	(00)
(11)	(01)	(11)	(11)
(11)	(10)	(00)	(00)
(11)	(10)	(11)	(11)

(c) $\vec{b}(t)$	$\vec{b}(t-1)$	$\vec{Q}(t-1)$	$\vec{Q}(t)$
(00)	(xx)	(xx)	(01)
(01)	(00)	(01)	(11)
(01)	(11)	(00)	(00)
(01)	(11)	(11)	(11)
(10)	(xx)	(xx)	(00)
(11)	(01)	(00)	(00)
(11)	(01)	(11)	(11)
(11)	(10)	(00)	(00)

(d) $\vec{b}(t)$	$\vec{b}(t-1)$	$\vec{Q}(t-1)$	$\vec{Q}(t)$
(00)	(xx)	(xx)	(01)
(01)	(xx)	(xx)	(11)
(10)	(00)	(01)	(00)
(10)	(11)	(00)	(00)
(10)	(11)	(11)	(11)
(11)	(01)	(11)	(11)
(11)	(10)	(00)	(00)
(11)	(10)	(11)	(11)

FIGURE A2.4: (a-d) Truth tables of sample A corresponding to the sequential Boolean circuits in Fig. 2.5b-e, respectively; the table for Fig. 2.5a is trivial and omitted here. Crosses indicate cases where the values of $\vec{b}(t-1)$ and $\vec{Q}(t-1)$ are irrelevant; blue boxes indicate cases where the output depends on the previous state.

(Young's modulus of ≈ 1 MPa, Poisson's ratio of ≈ 0.5) into a two-part mold fabricated using UltiMaker S3 printers. After curing at room temperature, the sample is demolded and mounted in a clamp featuring matching corrugations, leaving both ends of the sheet exposed. In the second and third step, a separate container is prepared with a thin layer of degassed silicone rubber. The exposed end of the sheet is then carefully lowered into this layer to create an end cap. After curing of the first end cap, this procedure is repeated for the opposite end. Together, these end caps stabilize the corrugations and facilitate coupling to the experimental device.

The sample shown in Fig. 2.11 has a thickness $h = 1.0 \pm 0.1$ mm, length $L = 100 \pm 3$ mm, and features sinusoidal corrugations with amplitude $A = 8.5 \pm 0.1$ mm and pitch $w = 22.5 \pm 0.5$ mm. The end caps have thickness $t = 1.0 \pm 0.2$ mm.

To explore the strain space, we determined the irreversible transitions for four different paths: $P_I : (0,0) \rightarrow (\varepsilon^m, 0) \rightarrow (\varepsilon^m, \varepsilon^M) \rightarrow (0, \varepsilon^M) \rightarrow (0, 0)$; $P_{II} : (0,0) \rightarrow (0, \varepsilon^m) \rightarrow (\varepsilon^M, \varepsilon^m) \rightarrow (\varepsilon^M, 0) \rightarrow (0, 0)$; $P_{III} : (0,0) \rightarrow (\varepsilon^M, 0) \rightarrow$

$\vec{b}$	$\vec{x}$
(0000)	(><><)
(1000)	(>>><)
(0100)	(<<<>)
(0010)	(><>>)
(0001)	(><><)
(1100)	(<<<<)
(1100)	(>>><)
(1010)	(>>>>)
(1001)	(><><)
(0110)	(<<<<)
(0101)	(<<<<)
(0011)	(><<<)
(0011)	(>>>>)
(1110)	(<<<<)
(1110)	(>>>>)
(1101)	(<<<<)
(1011)	(<<<<)
(1011)	(>>>>)
(0111)	(<<<<)
(0111)	(>>>>)
(1111)	(<<<<)
(1111)	(>>>>)

TABLE A2.1: States in the b-graph in Fig. 2.6b, where $x_i < 0$ is represented as < and $x_i > 0$ as >.

$(\varepsilon^m, 0) \rightarrow (\varepsilon^m, \varepsilon^M)$ $P_{IV} : (0, 0) \rightarrow (0, \varepsilon^M) \rightarrow (0, \varepsilon^m) \rightarrow (\varepsilon^M, \varepsilon^m)$, where we scan ε^m between 0.002 and 0.05, and we fix $\varepsilon^M = 0.054$. Based on these explorations, we selected the three strain paths corresponding to the motifs shown in Fig. 2.11.

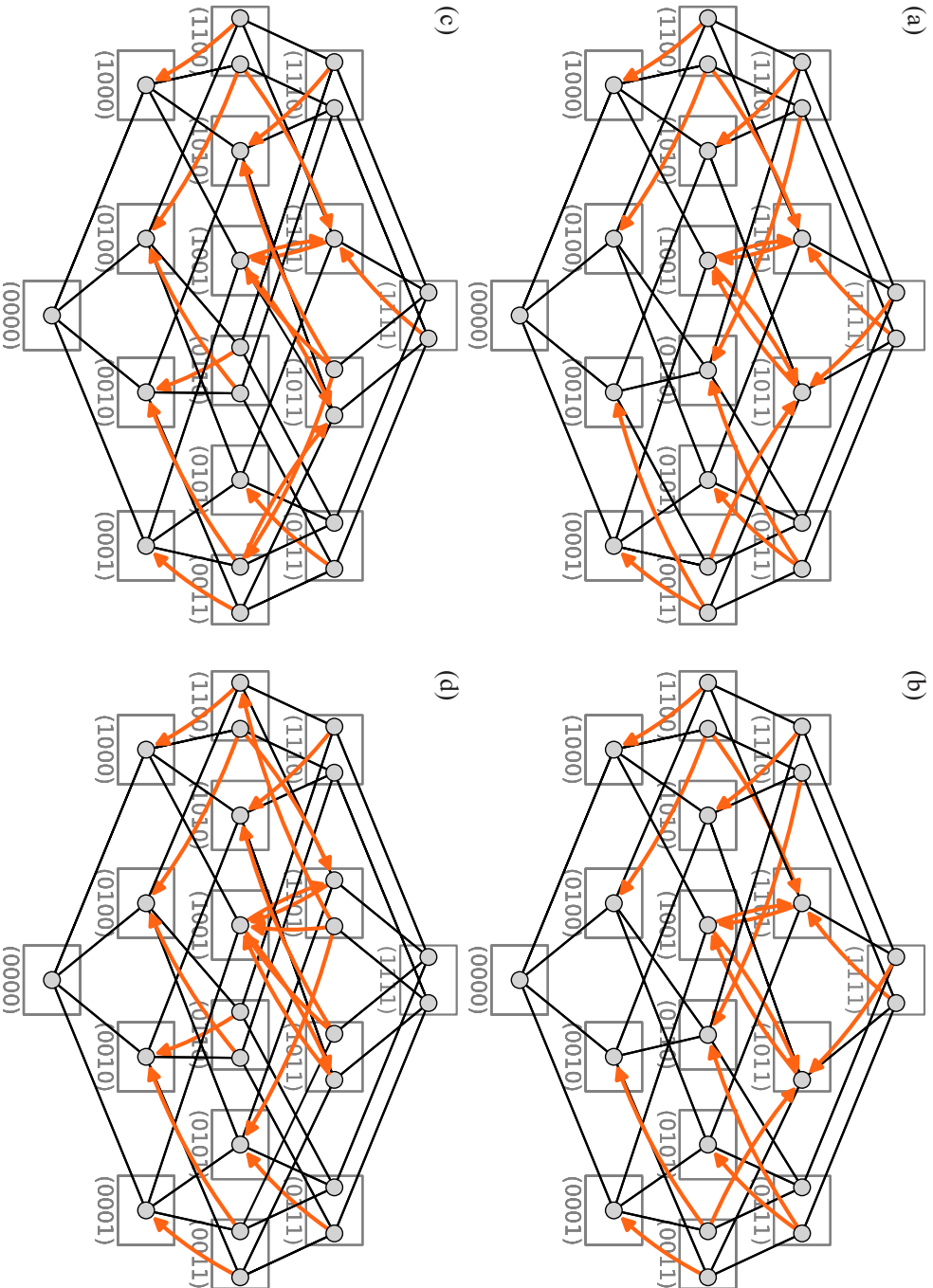
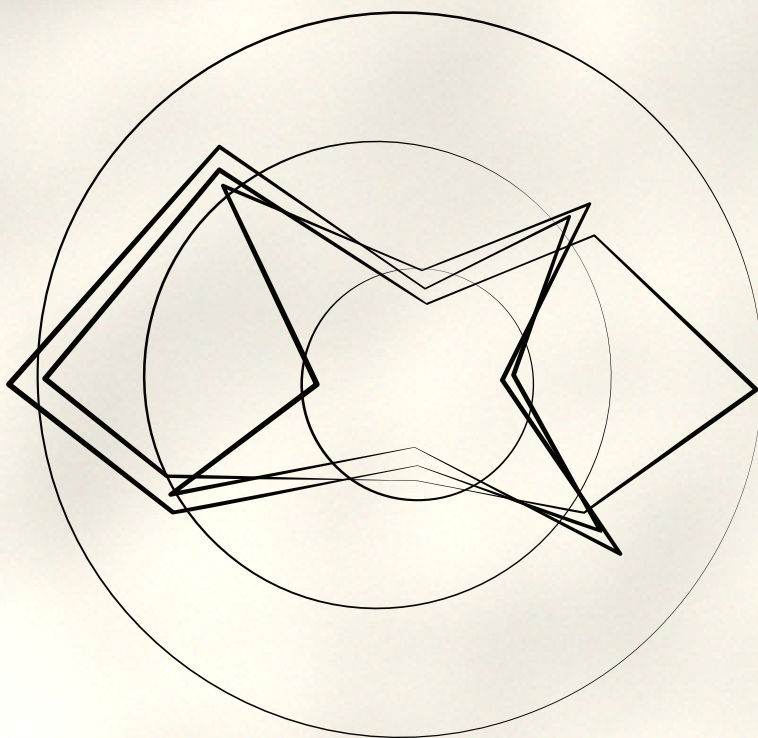


Figure A2.5: (a-d) b-graphs of sample B for $\epsilon^M = 0.03, 0.0325, 0.0375$, and 0.04 respectively. We note that the b-graph for $\epsilon^M = 0.03$, and 0.0325 have the same topology.



Transients and multiperiodic responses: a hierarchy of material bits

When cyclically driven, certain disordered materials exhibit transient and multiperiodic responses that are difficult to reproduce in synthetic materials. Here, we show that elementary multiperiodic elements with period $T = 2$ – togglerons – can serve as building blocks for such responses. We experimentally realize metamaterials composed of togglerons with tunable transients and periodic responses – including odd periods. Our approach suggests a hierarchy of increasingly complex elements in frustrated media, and opens a new strategy for rational design of sequential metamaterials.

3.1 Introduction

The sequential response of frustrated materials to cyclic driving reveals a wide range of memory effects [20, 22, 24, 61, 64–66, 68, 69, 71]. These memory effects range from return-point memory (RPM), where extrema of cyclic driving sweeps are encoded in the material and can be detected as kinks in the response function [4, 22, 23, 50], to more complex memory effects where, e.g., the material responds to the number of identical driving cycles [20, 22, 24, 47, 51, 68, 72, 77]. Understanding the emergence and organization of these memory effects provides key insight into the complex physics of disordered systems and the structure of their underlying energy landscapes [23, 25]. Moreover, reproducing such sequential responses in designer materials enables the realization of sequential shape morphing [27, 70, 83, 110–113], adaptive mechanical behavior [114], and *in-materia* computing such a counting of periodic signals or filtering specific driving sequences [47, 72, 88, 90, 91, 115–117].

At zero temperature and in the absence of plastic evolution, even massively multistable systems with a large number of states inevitably evolve toward a repeating pattern under purely cyclic, periodic driving [24, 52–60, 72]. The complete evolution of such a system is referred to as an *orbit*, which is the sequence of states under a specific driving protocol (Fig. 3.1a). For cyclic driving, orbits eventually repeat periodically. This allows defining the *transient* τ as the number of driving cycles required to reach a periodic response, and the *period* T as the number of driving cycles needed to complete one repetition of that response. Together, τ and T serve as fundamental descriptors of the material's sequential response under cyclic driving. Recently, non-trivial finite transients have been observed in metamaterials and simplified crumpled sheets [47, 72], whereas *multiperiodic* responses with $T > 0$ have been seen in a wide variety of simulations [22, 24, 52–60, 72]. Understanding and controlling such transients and multiperiodic responses gives insight into the rich physics of frustrated materials and, when harnessed through rational design, opens routes to smart materials with advanced sequential functionalities [47, 91, 113, 115, 118].

To facilitate the analysis and modeling of frustrated materials under cyclic drive, these systems can be described as collections of interacting multi-state elements, with binary spins and hysterons popular choices [21, 23–25, 48, 49, 58–61, 67]. For isolated spins, their internal state or phase is solely determined by whether the instantaneous value of the driving parameter ε is smaller or larger than a critical threshold value (Fig. 3.1b). In contrast, hysterons retain memory of past driving, and their phase is governed by both the current value and history of ε (Fig. 3.1c). We note that pairs of spins with appropriately chosen interactions can capture the hysteretic response of a single hysteron (see appendix A3.1). Hence, we view spins and hysterons as the lowest ranks in a hierarchy of increasingly complex elements, where elements of higher rank can be realized by

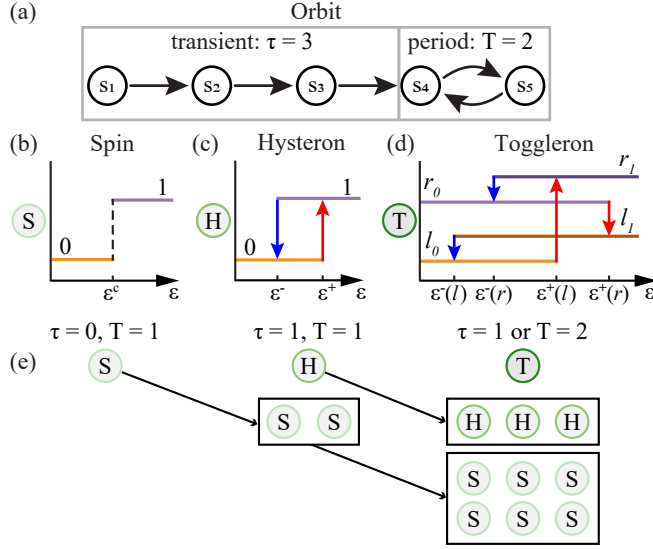


FIGURE 3.1: **Orbits and Material Bits.** (a) An example orbit under cyclic driving, evolving through five states $\{s_1, \dots, s_5\}$ with transient $\tau = 3$ and period $T = 2$. Each circle represents a state and each directed edge a single drive cycle. (b-d) Transitions and switching thresholds for spins, hysterons and togglerons. Even though togglerons have more than two states, we refer to these as material bits. (e) Hierarchy of material bits: Two interacting spins (S) can model a hysteron (H); six interacting spins or three interacting hysterons can model a toggleron (T).

combining multiple interacting elements of a lower rank (Fig. 3.1e).

Interactions between these elements are essential for producing nontrivial transients and multiperiodic behavior; for example, $T = 2$ orbits require six interacting spins or three interacting hysterons [51, 58] (Fig. 3.1e; see appendix A3.1). Hence, longer transients or periods require a larger number of interacting elements to be modeled. The growing dimensionality of the associated parameter space makes it increasingly difficult to navigate this space. Moreover, this makes designing and understanding long-period responses challenging. This challenge motivates an exploration of the trade-off between the complexity of individual elements and the number of elements required to achieve specific behaviors and orbits (Fig. 3.1b-e).

Here we introduce *togglerons*, elements that return to their initial state after two driving cycles, as effective building blocks to understand and realize both transient and multiperiodic behavior (Fig. 3.1d). Spins, hysterons and togglerons are part of a hierarchy, where elements of higher rank mimic multiple interacting elements of a lower rank; not only can two interacting spins form a single hysteron, togglerons can be formed from groups of spins or hysterons (Fig. 3.1e; see appendix A3.1). Such a hierarchical perspective naturally captures the balance

between the complexity of each elements and the number required for specific orbits, and suggests that studying higher-order responses through more complex building blocks—such as period-two units—offers a promising route toward understanding the origin of longer periods and designing metamaterials with programmable sequential dynamics.

We focus on mechanical togglerons and materialize these with cyclically compressed beams that buckle in a constrained environment. We investigate interacting togglerons and show how to design their properties and interactions to realize targeted orbits. We finally materialize a toggleron-metamaterial which, depending on the amplitude of the cyclic drive, exhibits orbits with $(\tau, T) = (2, 2), (0, 4)$ or $(1, 3)$. Our hierarchical framework, where clusters of simple interacting elements form aggregate units with complex sequential responses, enables the design of smart materials for sensing, soft robotics, and sequential *in materia* computing.

3.2 Togglerons

We introduce a *toggleron*, a single element that exhibits an intrinsic period-two response under cyclic driving. Its orbit consists of four distinct states connected by four irreversible transitions (Fig. 3.1d). Specifically, the toggleron undergoes one irreversible transition during each increase and decrease of the driving field, and returns to its initial state only after two full driving cycles.

To clarify the relation to hysterons, we label these states l_0, l_1, r_0 and r_1 (Fig. 3.1b-d). The subscript ‘0’ and ‘1’ denotes stability at low and high values of the driving field, respectively, while the labels l (left) and r (right) distinguish two symmetry-related states. The transition from the ‘0’ to ‘1’ states occur during increasing drive, while transitions from the ‘1’ to ‘0’ states occur during decreasing drive. In contrast, the transitions between l and r states occur only during increasing drive¹ (Fig. 3.1d). Thus, the usual hysteron states ‘0’ (low) and ‘1’ (high) are effectively doubled into left and right flavors; a toggleron cannot be reduced to a single bistable unit (Fig. 3.1b-c).

Over the course of two driving cycles, a toggleron follows a specific transition sequence: During the first up-sweep of the driving field ε , it switches from configuration l_0 to r_1 at switching threshold $\varepsilon^+(l)$. On the subsequent down-sweep, it transitions from r_1 to r_0 at $\varepsilon^-(r)$. In the next cycle, the configuration evolves from $r_0 \uparrow l_1 \downarrow l_0$, with switching events at $\varepsilon^+(r)$ and $\varepsilon^-(l)$, respectively (Fig. 3.1d).

The $T = 2$ behavior requires that both down switching thresholds ε^- are smaller than both up switching thresholds ε^+ , and that the cyclic protocol is

¹Other realizations where, e.g., such transitions occur under decreasing driving can be mapped to our representation by relabeling of the states.

chosen so that the driving passes through all thresholds $\varepsilon^\pm$ [119]. Importantly, multiperiodicity arises within a *single* element and does not rely on interactions between multiple hysterons or spins with finely tuned coupling strengths (Fig. 3.1b-d, see Appendix A3.1).

We materialize mechanical togglerons using slender beams ($L_0 = 100$ mm) constrained by lateral boundaries with two protrusions, or tips [120]. Under cyclic compression of the beam, these tips induce snapping between its left- and right-buckled configurations. We non-dimensionalize all lengths by L_0 , including the width t and center position x_b of the beam. The lateral boundaries are fixed to the bottom plate, and are parameterized by their heights z_b and z_t and horizontal position x_l and x_r (Fig. 3.2a; see appendix A3.2). The system is driven by quasistatic compression cycles, with strain ε varying between the minimal

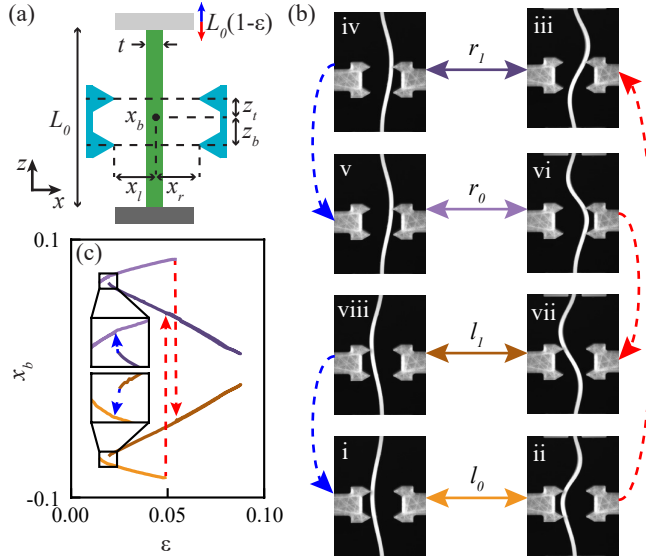


FIGURE 3.2: Geometry and phenomenology of togglerons. (a) We realize togglerons using a slender beam (green) constrained by fixed lateral boundaries consisting of dual tips (blue). The black circle at the center of the beam at rest indicates the origin of the (x, z) coordinate frame. (b-c) Evolution of a toggleron during two compression cycles $((\varepsilon_m, \varepsilon_M) = (0.015, 0.088), (t, z_t, z_b, x_l, x_r) = (0.025, 0.02, -0.16, -0.08, 0.08)$. Snapshots (b) show the beam configuration immediately before and after the irreversible transitions (dashed) during compression (red) and decompression (blue). Bidirectional arrows indicate reversible beam evolution and show the four states r_1, r_0, l_1 and l_0 ; colors match the four stable branches of the mid-point deflection x_b shown in panel (c). Irreversible transitions occur rapidly, effectively at fixed ε , and are indicated by (vertical) dashed arrows. (c) Evolution of x_b ; smooth evolution of the configuration within a given state are shown as colored curves, and the irreversible transitions as red and blue arrows. Insets highlight the irreversible transitions at $\varepsilon^-(0)$ and $\varepsilon^-(1)$.

value ε_m and a maximum value ε_M . Crucially, we take $\varepsilon_m > \varepsilon_b$, where ε_b is the buckling strain, which ensures that the beam remains buckled and history dependent [72].

To demonstrate the targeted sequential evolution, a toggleron is initialized in the left-buckled configuration l_0 and its beam's mid-point deflection x_b is tracked as the compression ε is cycled (Fig. 3.2b-c). The toggleron exhibits a reproducible sequence of configurations, undergoing four distinct irreversible transitions over two cycles—one during each up or down sweep of ε —and returns to its initial configuration at the end of the second cycle, as required (Fig. 3.2b).

To examine the sequence of configurations in more detail, we present snapshots of the real-space configuration immediately before and after each irreversible transitions (Fig. 3.2b), together with the full evolution of the mid-point x_b (Fig. 3.2c). During increasing ε , transitions between left-buckled l and right-buckled r configurations are triggered by the buckled beam coming into contact with the lateral tips (Fig. 3.2bi). This contact constrains the deformation and initially induces a smooth evolution into a higher-order buckling mode without loss of stability (Fig. 3.2bii,c). Upon further increase of ε , this constrained configuration loses stability, resulting in an irreversible snapping event in which the beam transitions to a right-leaning configuration r_1 (Fig. 3.2biii).

Under decreasing ε , the beam again initially deforms smoothly (Fig. 3.2biv,c) before undergoing a (subtle) unsnapping transition (Fig. 3.2bv). Although this unsnapping event involves only a small jump of x_b (Fig. 3.2c, inset), it is irreversible—under subsequent compression, the configurations r_1 and r_0 (and analogously l_1 and l_0) evolve along qualitatively different branches of the $x_b(\varepsilon)$ response (Fig. 3.1d, Fig. 3.2c, see appendix A3.2).

The rationale for the use of dual-tip boundaries is as follows [120]. To allow lateral deflection of the beam, the tips of the left and right boundaries must be placed to the left ($x_l < 0$) and right ($x_r > 0$) of the beam. Yet, these boundaries must induce snapping of the beam. Specifically, if a beam is initially buckled to the left, increasing compression should trigger a snap-through transition to the right. However, for a boundary with a single tip, compression induced snapping from left to right is impossible, since it requires $x_l > 0$ [116, 120–122]. In contrast, the dual-tip geometry allows to exert a moment on the beam, and when properly designed, induces snapping from l to r for $x_l < 0$, thus powering the toggling motion of the beam [120].

The locations of the tips directly control the critical thresholds. A wide range of vertical positions z_b, z_t induces snapping, and increasing the lateral distance between the tips and the beam—i.e., increasing $|x_l|$ or x_r —raises $\varepsilon^+(l)$ and $\varepsilon^+(r)$ [120]. In the example above, the toggleron is left-right symmetry ($x_l = -x_r$) so that $\varepsilon^+(l) = \varepsilon^+(r)$ and $\varepsilon^-(l) = \varepsilon^-(r)$. Breaking this symmetry—by introducing a difference between the horizontal positions x_l, x_r , or introducing differences between the vertical positions of left and right tips—allows for independent con-

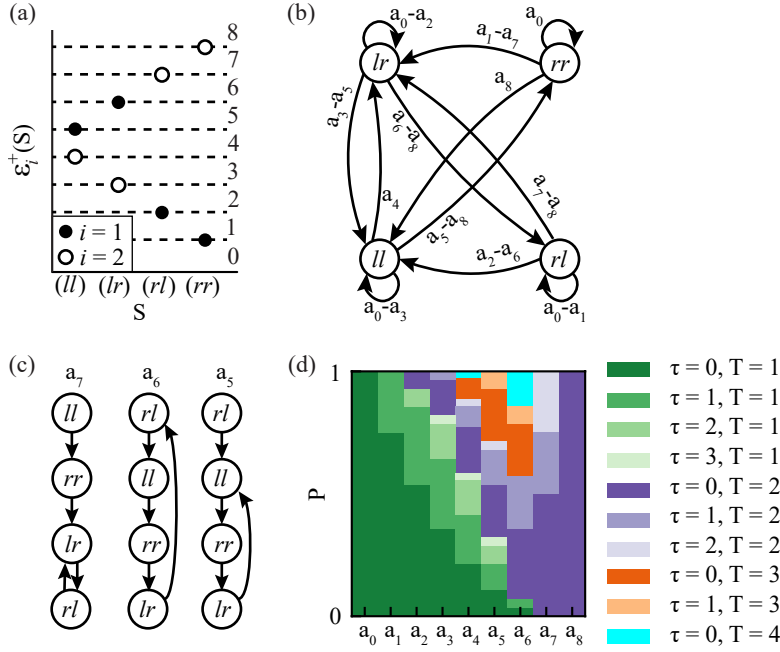


FIGURE 3.3: **Evolution of two interacting togglerons.** (a) Example of the ordering of the eight switching thresholds $\varepsilon_i^+(S)$ (symbols) which define nine strain intervals (0-8). (b) Corresponding FSM diagram. (c) Three example orbits under cyclic driving with a_7 , a_6 , and a_5 . (d) Normalized stacked bar chart for (τ, T) .

trol over $\varepsilon^\pm(l)$ and $\varepsilon^\pm(r)$. Hence, the dual-tip design robustly and controllably produces toggling behavior over a broad range of geometric parameters (for more details see [120]).

3.3 Elementary transients

Togglerons exhibit both multiperiodic and transient behavior. Consider periodic driving of (asymmetric) togglerons for $\varepsilon_m < \varepsilon^-$ so that the $l_1 \rightarrow l_0$ and $r_1 \rightarrow r_0$ transitions always occur. The (relative) values of $\varepsilon^+(l)$, $\varepsilon^+(r)$ and ε_M then control the evolution after each full driving cycle, allowing to drop the ‘0’ and ‘1’ indices and to characterize the toggleron as l or r . When $\varepsilon_M < \varepsilon^+(l)$ and $\varepsilon_M < \varepsilon^+(r)$, the toggleron remains stuck and $(\tau, T) = (0, 1)$. When ε_M exceeds both switching thresholds, $(\tau, T) = (0, 2)$. For $\varepsilon^+(l) < \varepsilon_M < \varepsilon^+(r)$, which can be realized by taking $|x_l| < |x_r|$, the evolution depends on the initial state: Starting from r , the toggleron remains stuck, while starting from l , it first evolves to r and then remains stuck, leading to an elementary transient: $(\tau, T) = (1, 1)$.

3.4 Interactions and complex orbits

To explore a broader range of transients and multiperiodic orbits [52–60, 72], we consider two interacting togglerons. We denote their collective states as $S = (ll), (lr), (rl)$ or (rr) , and note that these four states constrain the possible transients and periods to $\tau + T \leq 4$. We encode interactions in eight the state-dependent switching thresholds $\varepsilon_i^+(S)$, where $i = 1, 2$ labels the toggleron.

The language of finite state machines is useful to describe the response of multistable systems to sequential driving [47]. Finite state machines (FSMs) consist of a (finite) collection of states and input characters, and are defined by the transition table which specifies how an input character advances the machine from state to state [123, 124]. The collective states of a (finite) interacting toggleron system thus form the states of the corresponding FSM. To discretize the input, we note that the physical thresholds partition ε into nine intervals - we assume that the thresholds are non-degenerate - and that the response to a single driving cycle $\varepsilon : \varepsilon_m \uparrow \varepsilon_M \downarrow \varepsilon_m$ depends only on the interval containing the maximal driving value ε_M (Fig. 3.3a) [47, 49]. We choose one representative value ε_M^k in each interval $k = 0, 1, \dots, 8$ and define nine input characters a_k representing full driving cycles $\varepsilon : \varepsilon_m \uparrow \varepsilon_M^k \downarrow \varepsilon_m$ (Fig. 3.3a) [47]. Together, this allows to determine the evolution of any initial state S_0 after one driving cycle, $S_1 = a_k(S_0)$, thereby defining the transition table and enabling a graphical representation of the resulting FSM (Fig. 3.3b) [47].

Different orderings of the thresholds lead to different FSMs. We experimentally explored several orderings and verified that the FSM framework describes all cases; numerically we have explored all $8!$ ($\approx 4 \cdot 10^4$) possible orderings of the thresholds (see below). We believe that our physical design parameters are sufficiently diverse to realize a substantial fraction if not all orderings that respect that all up thresholds are larger than all down thresholds. Crucially, we realized the specific ordering shown in Fig. 3.3a that allows to vary τ and T by the maximal compression of the cyclic driving.

The FSM facilitates classification of transients and multiperiodic responses under cyclic driving. For example, for $\varepsilon_M = \varepsilon_M^7$, the response can be tracked by following the transitions in the FSM under repeated compression cycles a_7 . Starting from $S_0 = (ll)$, the system exhibits a transient of two before entering an orbit with $T = 2$: $(ll) \xrightarrow{a_7} (rr) \xrightarrow{a_7} (lr) \xrightarrow{a_7} (rl) \xrightarrow{a_7} (lr) \xrightarrow{a_7} \dots$ (Fig. 3.3b,c). Other initial states yield $(\tau = 1, T = 2)$ and $(\tau = 0, T = 2)$. Similarly, for $\varepsilon_M = \varepsilon_M^6$, a cycle of maximal length four ($\tau = 0, T = 4$) is observed. Strikingly, for $\varepsilon_M = \varepsilon_M^7$, a cycle of period *three* emerges (Fig. 3.3c).

Since the possible transients and periods only depend on the ordering of these thresholds, we determine (τ, T) for all $8!$ orderings and explore their statistics (Fig. 3.3d). The combination of non-symmetric elements ($\varepsilon^+(l) \neq \varepsilon^+(r)$) and interactions enables the realization of all allowed combinations of τ and T , in-

cluding multi-periodic orbits with odd periods. Together, our data and examples illustrate that interacting togglers exhibit a wide range of transients and multi-periodic orbits which depend on driving amplitude and initial state.

3.5 Design of orbits

For each orbit, conditions on the ordering of the switching thresholds can straightforwardly be designed. We illustrate this for the orbit $(rl) \xrightarrow{a_k} (ll) \xrightarrow{a_k} (rr) \xrightarrow{a_k} (lr) \xrightarrow{a_k} (ll) \xrightarrow{a_k} \dots$. Each transition, e.g., $(rl) \xrightarrow{a_k} (ll)$ produces two design inequalities, i.e., $\varepsilon_1^+(rl) < \varepsilon_M^k$ and $\varepsilon_2^+(rl) > \varepsilon_M^k$. These transitions collectively impose up to eight inequalities and define one or more admissible orderings of the switching thresholds and associated ranges for ε_M . Note that for the specific design problem of finding switching thresholds that exhibit the three orbits shown in Fig. 3.3c, we only require the specific ordering of the four largest thresholds, i.e., a partial order of the thresholds [49]. Hence, specific (combinations of) orbits, transients and periods can be designed by controlling the switching thresholds.

This simple design strategy highlights the utility of togglers. Realizing similar orbits with spins and hysterons require solving a larger, more intricate set of design inequalities [47–49, 58, 77]. This is primarily due to their limited response in isolation - hysterons can exhibit at most a $\tau = 1$ response, while spins lack any transient behavior. By employing a more versatile unit such as a toggler, which can exhibit both multi-periodic and transient responses, the design of more complex orbits becomes markedly more straightforward.

3.6 Materializing targeted orbits

To demonstrate the feasibility of this approach, we create a metamaterial that realizes the three orbits shown in Fig. 3.3c. This metamaterial consists of two asymmetric, interacting togglers, where flexible lateral boundaries mediate their interactions (Fig. 3.4a). We control the left-right symmetry by the spacing (at rest) between beam and boundaries (L_i, R_i) (Fig. 3.4a,b). Interactions are implemented by placing the lateral boundaries on a weakly deformable structure. Hence, when, e.g., beam one is in contact with its right pusher it displaces it, and by coupling the right pusher of beam one with, e.g., the left pusher of beam two, interactions between the togglers are realized.

We couple the boundaries via a rotating structure, which allows independent control over the magnitude and sign of the interactions (Fig. 3.4b). The sign is determined by the connection of the rotating structure. For example, when the right pusher of beam one and the left pusher of beam two are connected on opposite sides of the center of rotation, the pushers move outward relative to

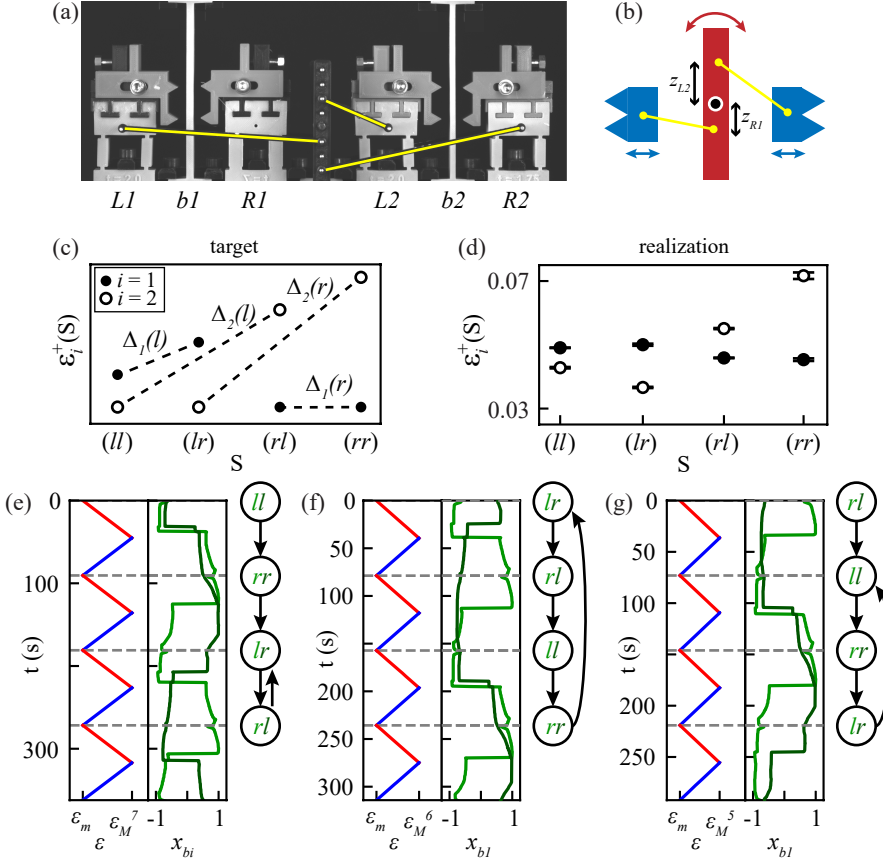


FIGURE 3.4: **Metamaterial consisting of two interacting togglerons.** (a) Geometry highlighting the connections from the flexible boundaries $L1$, $L2$ and $R2$ to a central rotating arm (yellow; see appendix A3.2). (b) Schematic representation of the coupling between boundaries $L2$ and $R1$. (c) Target ordering of the switching thresholds. (d) Realized switching thresholds for our specific design (for our specific design, see appendix A3.2). Error bars are included but are smaller than the marker size and therefore not clearly visible. The measured bare thresholds are $(e_1^+(l), e_1^+(r), e_2^+(l), e_2^+(r)) = (0.051, 0.44, 0.50, 0.53)$ and the interactions are $(\Delta_1^+(l), \Delta_1^+(r), \Delta_2^+(l), \Delta_2^+(r)) = (0.00, 0.001, 0.005, 0.019)$. (e-g) Evolution of the mid-point deflections x_{b1} and x_{b2} during four compression cycles with amplitude $\varepsilon_M^7 = 0.056$, $\varepsilon_M^6 = 0.050$, and $\varepsilon_M^5 = 0.047$, showing orbits with $\{\tau, T\} = \{2, 2\}$, $\{0, 4\}$ and $\{1, 3\}$.

their respective beam, inducing a negative interaction. In contrast, when both pushers are connected on the same side of the center of rotation, the outward movement of one results in an inward movement of the other, resulting in a positive interaction. The magnitude is controlled by the lever arm of the rotating structure and by the stiffness of the pushers (Fig. 3.4b). Together, these design parameters allow systematic tuning of both the strength and sign of the coupling

between togglers.

Materializing the target orbits requires a partial order of the thresholds, where the four largest thresholds are $\varepsilon_1^+(ll) < \varepsilon_1^+(lr) < \varepsilon_2^+(rl) < \varepsilon_2^+(rr)$ (Fig. 3.4c). The differences between, e.g., $\varepsilon_1^+(ll)$ and $\varepsilon_1^+(lr)$ (dashed lines in Fig. 3.4c) necessitate interactions between the togglers, which we model as

$$\varepsilon_i^+(S) = e_i^+(s_i) + \Delta_i(s_i)B(s_j). \quad (3.1)$$

Here $B(l) := -1$, $B(r) := 1$, and $e_1^+(l)$, $e_1^+(r)$, $e_2^+(l)$ and $e_2^+(r)$ denote the ‘bare’ thresholds of the isolated togglers. The interaction strengths are given by $\Delta_i(s_i)$, where, e.g., $\Delta_1(l) := (\varepsilon_1^+(lr) - \varepsilon_1^+(ll))/2$ denotes how strong toggler two modifies the threshold of phase l of toggler one. Hence, the targeted ordering implies three conditions: (i) $\Delta_1(r) \approx 0$; (ii) $\Delta_1(l) > 0$, $\Delta_2(l) > 0$ and $\Delta_2(r) > 0$; (iii) $\Delta_1(l) \ll \Delta_2(l) < \Delta_2(r)$ (Fig. 3.4c).

We implement these conditions by selecting appropriate design parameters. First, condition (i) is satisfied by boundary R_1 being uncoupled from other boundaries. Second, the sign of the interactions (condition (ii)) implies that the coupling makes boundaries L_1 , L_2 and R_1 all move in and outward together, which requires that $z_{L1}/z_{R2} > 0$ and $z_{L1}/z_{L2} < 0$. Third, the non-symmetric nature of the interactions (condition (iii)) is realized by choosing beam one thicker than beam two, and setting $z_{L1} < z_{L2} < z_{R2}$. Guided by these considerations, and iterative adjustments of x_{Li} and x_{Ri} to finetune the switching thresholds, we select metamaterial parameters that meet the target ordering (Fig. 3.4d; see appendix A3.2).

To demonstrate that this metamaterial exhibits the three target orbits, it is initialized in the appropriate initial state and its evolution is followed during cyclic sweeps of ε between $\varepsilon_m = 0.0125$ and ε_M^5 , ε_M^6 and ε_M^7 (Fig. 3.4efg). First, for the largest driving ($k = 7$), the metamaterial exhibits a transient of two and then locks in period-two orbit where both togglers toggle each driving cycle $(lr) \rightarrow (rl) \rightarrow (lr) \rightarrow \dots$ (Fig. 3.4c). Second, for intermediate driving ($k = 6$), the metamaterial exhibits the longest possible orbit of period $T = 4$ (Fig. 3.4d). Finally, pushing the limit of our experiment, when ε_M is chosen in the narrow interval corresponding to $k = 5$, the metamaterial exhibits a short $\tau = 1$ transient and then a period three response (Fig. 3.4e). Hence, a *single* metamaterial exhibits all three target orbits.

3.7 Discussion and Outlook

We introduced togglers, elements which under cyclic driving repeatedly evolve over two driving cycles, as a natural step in a hierarchy of elements that starts with binary spins and hysterons. Their intrinsic multiperiodic nature facilitates the construction of complex orbits with long transients or large

periods with few togglerons - features that require much larger assemblies of interacting spins or hysterons [58, 125]. As such, togglerons provide a compact and controllable route to encode rich sequential responses, opening avenues in sequential mechanisms, soft robotics and mechanical information processing [47, 72, 91, 115, 118]. Aggregate structures composed of togglerons further expand this design space and offer new routes to sequential or cyclic shape-morphing materials with potential applications ranging from complex gaits in soft-robots to *in-materia computing* [47, 83, 111, 113, 118, 119].

More broadly, in the context of amorphous and disordered systems, our work suggests that large periods and extended transients may arise from interactions between local clusters of particles which, in isolation, exhibit only short periods or transients [52–60]. This perspective provides a concrete mechanism by which complex memory effects may arise from comparatively simple microscopic constituents [77].

In extended or interacting networks of togglerons, multiple memories as well as nontrivial input sequence dependent responses could arise [77, 117]. The readout protocols required to recover these memories are therefore expected to be nontrivial, as the response can depend on both the number and ordering of training cycles—e.g., distinguishing between odd and even numbers of applied cycles. How systematic readout protocols—such as nested or competing cyclic drives—interact with hierarchies of composite elements remains an important open question.

Finally, we suggest that composite building blocks such as the togglerons, or the recently identified pairs of frustrated hysterons that capture non-trivial ‘latching’ memories [77] are not the end of the hierarchy. In particular for hierarchically organized materials, including glasses, exciting questions arise: which composite elements are formed, which are statistically dominant, how do they interact, and how far does this hierarchy extend? Addressing these questions may provide a fresh perspective on how complex orbits emerge in driven disordered matter.

Appendix

A3.1 Hierarchy of material bits

Here, we detail how higher-rank material bits can be constructed by composing interacting material bits of lower rank. We consider three cases: (i) Interacting binary spins that form a hysteron; (ii) Interacting hysterons that form a toggleron; (iii) Interacting spins that form a toggleron. We consider pairwise additive interactions [48, 49, 58, 59, 77]:

$$U_i^\pm(S) = u_i^\pm - \sum_{j \neq i} c_{ij} s_j, \quad (\text{A3.1})$$

where $U_i^\pm(S)$ denotes the switching thresholds of element i in the collective state $S = (s_1 s_2 \dots s_N)$. For the case of binary spins, $u_i^+ = u_i^-$ and $U_i^+(S) = U_i^-(S) = u_i^\pm$. The state-dependent switching thresholds are composed of the bare switching fields $u_i^\pm$ and the interaction term $-\sum_j c_{ij} s_j$, and we consider both symmetric ($c_{ij} = c_{ji}$) and asymmetric interactions ($c_{ij} \neq c_{ji}$).

We first show that two spins with symmetric or asymmetric interactions can form a hysteresis loop, and that this requires avalanches where multiple spins switch simultaneously. While the argument is conceptually simple, we are not aware of any prior work in which it has been explicitly presented. Hence, hysterons can be composed of two interacting spins. We then consider interacting hysterons that exhibit a multiperiodic ($T = 2$) loop, thus forming a toggleron. From numerical sampling it was known that this requires at least three hysterons, and that for symmetric interactions, four hysterons are needed [48, 58, 125]. Here we present a systematic exploration using recently developed techniques that does not rely on sampling and is able to find all possible orbits [49, 126]. We finally consider how interacting spins can form multiperiodic orbits, and explicitly construct a $T = 2$ orbit - toggleron - using six interacting spins.

A3.1.1 Interacting spins forming a hysteron

We now explore how binary spins can exhibit hysteretic behavior similar to a hysteron. First, we note that transitions in which one spin flips can not exhibit

hysteresis (this follows directly from Eq. (A3.1)). However, when avalanches are taken into account, pairs of interacting spins can produce hysteretic transitions, thus effectively forming a hysteron.

For symmetric interactions, a pair of interacting spins is characterized by three parameters: the bare switching fields u_1 and u_2 , and the interaction strength $c_{12} = c_{21} = c$. Without loss of generality, we assume $u_1 > u_2$, and consider four parameter regimes given by the pair of inequalities:

$$u_1 - u_2 \geq |c|, \quad (\text{A3.2})$$

$$c \geq 0. \quad (\text{A3.3})$$

We have verified that the only possible avalanches for symmetric interactions are between the states $S = (00)$ and (11) . These occur when $u_1 - u_2 < |c|$ and $c > 0$ [48, 49, 58, 59]. Then, by choosing $c > u_1 - u_2$, the spin-pair switches from state $S_1 = (00)$ to $S_2 = (11)$ when U exceeds $U^+(00) = u_2$ or $(11) \rightarrow (00)$ when U falls below $U^-(11) = u_1 - c$. Hence, two spins form a hysteron with states S_1 and S_2 , and bare hysteron switching fields $u_h^+ = u_2$ and $u_h^- = u_1 - c$. For asymmetric interactions ($c_{12} \neq c_{21}$), a spin pair can also exhibit hysteretic avalanches between $S = (01)$ and (10) , when $c_{ij} < 0$ and $|c_{21}| > u_1 - c_{12} - u_2$, and where, the spin-pair forms a hysteron with effective switching fields: $u^+ = u_1 - c_{12}$ and $u^- = u_1$. Hence, there are at least two routes for pair of spins to form a hysteron.

A3.1.2 Multiperiodic behavior with interacting hysterons

Interacting hysterons have been studied intensely in recent years using numerical sampling techniques, providing examples of $T = 2$ multiperiodic responses that thus form togglerons [48, 58, 59]. Here, we systematically explore multiperiodic orbits adopting methods from [49, 126]. We first construct, for N hysterons, all fundamental multiperiodic orbits, and then systematically check which of these are realizable. Fundamental multiperiodic orbits are repeating sequences of states connected by single hysteron flips containing multiple episodes of up or down transitions, specified by the starting state $S^0 = (s_1^0 s_2^0 \dots s_N^0)$ and the sequence of element flips $(\kappa^1, \kappa^2, \kappa^3, \dots)$ [126]. We deal with timeshift and re-labeling symmetries by indexing elements according to the order in which they are flipped, and only keep orbits with initially descending order [48, 49, 126]. In addition, we order the generated multiperiodic orbits by their minimum magnetization $m := \sum_i s_i$ and filter out isomorphic orbits [126]. Together, this procedure results in a complete set of fundamental multiperiodic orbits [49, 126].

To check whether a fundamental multiperiodic orbit is realizable, we first construct a set of linear inequalities on the design parameters that needs to be satisfied to realize the orbit using pairwise interactions (Eq. A3.1) [49]. Next,

to ensure that such an orbit is traversed for cyclic driving of U between U_m and U_M , we require that the appropriate (extremal) states are both reached and stable at U_m or U_M , yielding an additional set of linear inequalities. Solvability of both sets of inequalities is easily verified, and explicit solutions can be constructed if necessary [49].

For $N = 2$ hysterons, no fundamental multiperiodic orbits exist [48]. For $N = 3$ hysterons, asymmetric interactions can produce $T = 2$ and $T = 3$ orbits [48, 58](Fig. A3.1, Tab. A3.1). Symmetric interactions require at least $N = 4$ hysterons.

N	T	Realizable		Total
		$c_{ij} = c_{ji}$	$c_{ij} \neq c_{ji}$	
3	2	0	4	4
	3	0	1	1
4	2	62	205	205
	3	6	391	620
	4	0	76	322
	5	0	12	52

TABLE A3.1: Number of realizable multiperiodic orbits for symmetric and asymmetric interactions.

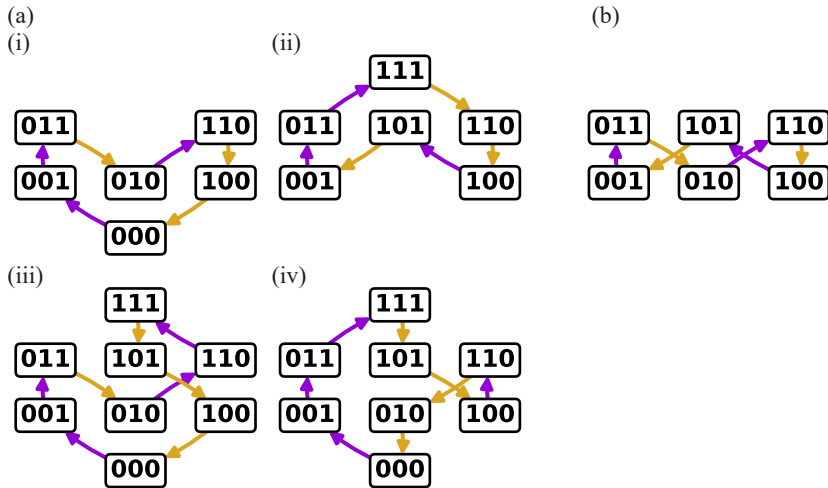


FIGURE A3.1: Realizable fundamental multiperiodic orbits with $N = 3$ hysterons and asymmetric interactions ($c_{ij} \neq c_{ji}$). (a) $T = 2$ orbits and (b) $T = 3$ orbits.

A3.1.3 Multiperiodic behavior with interacting spins

We now examine how binary spins can exhibit multiperiodic behavior similar to a toggleron. Prior numerical sampling studies have found that at least five interacting spins are required to generate period $T = 3$ orbits [51, 58]. Here, we focus on $T = 2$ behavior, and note that we are not aware (and have not been able) to create $T = 2$ orbits with five spins.

A systematic study of realizable fundamental multiperiodic orbits is not suited for spin systems, as, first, the number of orbits grows rapidly with system size, rendering exhaustive analysis impractical beyond $N > 4$ [126], and second, fundamental orbits exclude avalanches, which are essential for irreversible transitions in spin systems [49].

We construct multiperiodic orbits in spin systems by mapping spin pairs to single hysterons and using hysteron-based multi-periodic orbits. While not exhaustive, this approach offers a constructive method for generating multiperiodic behavior in systems with an even number of spins.

Our method starts with choosing a fundamental multiperiodic orbit (Fig. A3.2a). We generate a solution for the set of inequalities using numerical tools from [49]. The solution has bare switching fields:

$$(u_1^+, u_1^-) = (0.98, -0.49), \quad (\text{A3.4})$$

$$(u_2^+, u_2^-) = (0.66, -1.00), \quad (\text{A3.5})$$

$$(u_3^+, u_3^-) = (0.17, -0.92), \quad (\text{A3.6})$$

$$(\text{A3.7})$$

and interaction matrix:

$$c_{ij} = \begin{pmatrix} 0 & 0.43 & -0.68 \\ -1.00 & 0 & -0.1 \\ -0.64 & -0.94 & 0 \end{pmatrix}. \quad (\text{A3.8})$$

For $-1.00 < U_m < -0.49$ and $0.76 < U_M < 1.23$ the orbit has $T = 2$. We then map each hysteron i to a symmetrically interacting spin pair, with bare switching field of spin one $u_{1,i} = u_i^- - c$ and spin two $u_{2,i} = u_i^+$. The interactions between the spin pairs must satisfy $c_i > 0$ and $|c_i| > u_i^+ - u_i^-$. Therefore, we take $u_{1,i} = u_{2,1} + \delta$ and $c_i = u_i^+ - u_i^- + \delta$, with $\delta = 0.01$. Furthermore, we choose to spread the interactions between each hysteron equally over the interacting spin pairs, such that each spin shifts the same amount and the same distance as the hysteron. This yields:

$$(u_{1,1}, u_{2,1}) = (0.99, 0.98), \quad (\text{A3.9})$$

$$(u_{1,2}, u_{2,2}) = (0.67, 0.66), \quad (\text{A3.10})$$

$$(u_{1,3}, u_{2,3}) = (0.18, 0.17), \quad (\text{A3.11})$$

$$(\text{A3.12})$$

with interaction matrix (in block form):

$$c_{ij} = \begin{pmatrix} 0 & 1.48 & 0.215 & 0.215 & -0.34 & -0.34 \\ 1.48 & 0 & 0.215 & 0.215 & -0.34 & -0.34 \\ -0.50 & -0.50 & 0 & 1.67 & -0.05 & -0.05 \\ -0.50 & -0.50 & 1.67 & 0 & -0.05 & -0.05 \\ -0.32 & -0.32 & -0.47 & -0.47 & 0 & 1.10 \\ -0.32 & -0.32 & -0.47 & -0.47 & 1.10 & 0 \end{pmatrix} \quad (\text{A3.13})$$

Together, this produces a $T = 2$ orbit (Fig. A3.2b-c).

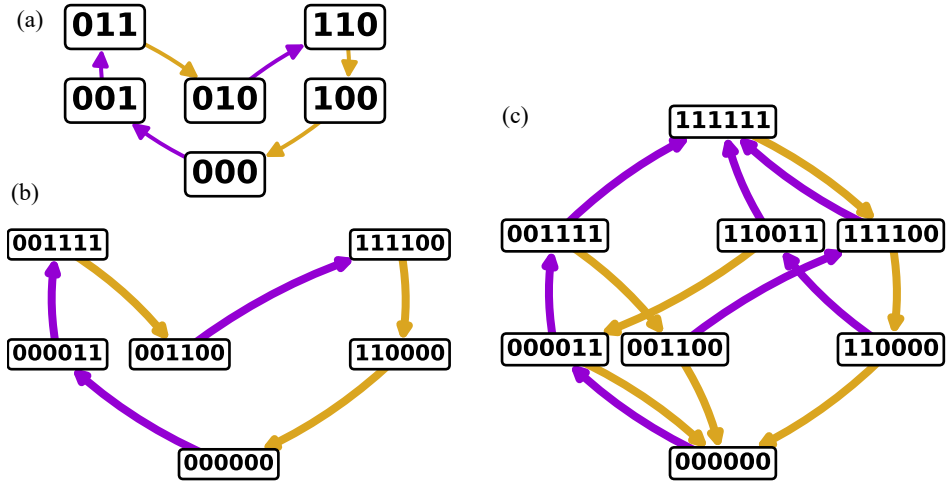


FIGURE A3.2: **Construction of multiperiodic orbits in spin system.** (a) Fundamental multiperiodic orbit with $T = 2$, realizable with $N = 3$ asymmetric interacting hysterons. (b) Resulting multiperiodic orbit with $T = 2$ with $N = 6$ interacting spins. (c) All transitions of the $N = 6$ interacting spin system.

A3.2 Experimental details

A3.2.1 Sample fabrication, Setup and design

We now discuss the fabrication of the togglers. Experiments are performed with beams of length $L_0 = 100 \pm 0.5$ mm, out-of-plane width of 20 ± 0.5 mm and various in-plane thickness (see below). Beams are fabricated by casting two-component polyvinyl siloxane elastomer (Zhermack Elite double 22 with Young's modules 0.8 MPa and Poisson's ratio ≈ 0.5) into 3D-printed molds. After curing for seven days at room temperature, the beams are removed and dusted with talc powder to minimize friction and prevent sticking. The resulting beams

exhibit predominantly elastic behavior, with minimal viscoelasticity, negligible rate-dependent effects, and no observable fatigue over repeated loading cycles.

In experiments on a single toggleron (Fig. 3.2), the lateral pushers are 3D printed and rigidly mounted on precision linear stages for accurate positioning and alignment. In the experiments on pairs of togglerons (Fig. 3.4), the lateral pushers are composed of three parts: 1) flexible legs, 2) pusher base, and 3) protrusions. The flexible legs are fabricated similarly to the beams, while the pusher base and protrusions are rigid and 3D printed. The locations x_l, x_r can be adjusted by sliding motion.

Cyclic driving is performed using a custom build compression device that allows for accurate parallel compression of wide structures (top and bottom plates remain parallel within a slope of < 0.6 mm/m [72]). The compressive strain ε is controlled by a stepper motor yielding an accuracy ± 0.01 mm. We remain in the quasi-static regime during compression, to avoid inertial and viscoelastic effects. During each cyclic compression protocol, a ccd camera captures the evolution of the beams at a rate of 60 Hz with ≈ 11 pixels/mm. During each cyclic compression protocol, a ccd camera captures the evolution of the beams at a rate of 60 Hz with ≈ 11 pixels/mm. The compression speed used is $d\varepsilon/dt = 0.1$ mm/s. We note that, while we remain in the quasi-static loading regime, inertial and rate-dependent effects—such as a delayed bifurcation and inertial overshoot—can influence the snap-through behavior [120, 121, 127, 128]; a detailed quantitative treatment of these effects is left for future work.

To control the initial buckling direction of each beam, we deliberately break the symmetry by briefly contacting the beams. After contact, we wait 30 seconds to avoid potential viscoelastic effects induced by this interaction. Each experiment in Fig. 3.2 and Fig. 3.4 is repeated at least three times and we report no significant deviations.

A3.2.2 Precise definition of phases and transitions

We now discuss how the toggleron phases, $s = l_0, l_1, r_0$, and r_1 , can be detected in the real-space configuration of the beam. We consider the evolution of the toggleron shown in Fig. 3.2. We extract the dimensional beam shape $X(Z)$ from the images, define rescaled coordinates $x = X/(L_0(1 - \varepsilon))$ and $z = Z/(L_0(1 - \varepsilon))$, and focus on the normalized beam shape $x(z)$. We expand the normalized shape using the first six Euler buckling modes [120]:

$$x(z) \approx \sum_{i=1}^6 c_i \xi_i(z) , \quad (\text{A3.14})$$

where $\xi_i(z)$ is the i -th mode [117, 121]. The irreversible transitions are associated to changes of the signs of the amplitudes of the first two modes; c_1 changes

sign during the transition at increasing ε , and c_2 changes sign for decreasing ε (Fig. A3.3). Hence, these signs allow to extract the toggleron phase directly from the beam shape (Table. A3.2).

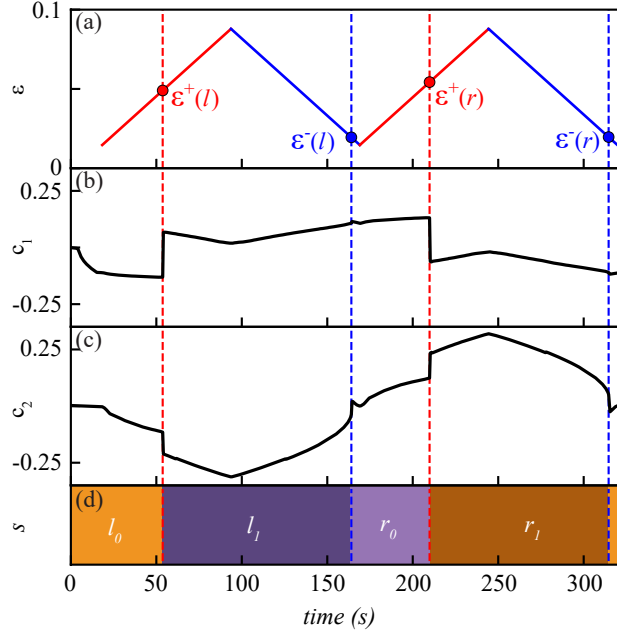


FIGURE A3.3: **Modal analysis** of the beam shape of the toggleron of Fig. 3.2. (a) Cyclic driving protocol with irreversible transitions during compression (red) and decompression (blue). (b-c) Evolution of the amplitudes of the first and second Euler modes, c_1 , c_2 . (d) Corresponding toggleron phases.

State	c_1	c_2
l_0	< 0	≤ 0
l_1	> 0	> 0
r_0	> 0	≥ 0
r_1	< 0	< 0

TABLE A3.2: Relation between the signs of c_1 and c_2 and toggleron phase.

A3.2.3 Design parameters

We summarize the design parameters of the two coupled togglerons presented in Fig. 3.4 of the main text. The geometry of the beams is given by $(t_{b1}, t_{b2}) = (4.0 \pm .1 \text{ mm}, 3.0 \pm .1 \text{ mm})$, and the hinge width of the flexible legs of each pusher base is $(w_{L1}, w_{R1}, w_{L2}, w_{R2}) = (2.0 \pm .1 \text{ mm}, 7.0 \pm .1 \text{ mm}, 2.0 \pm .1 \text{ mm}, 1.8 \pm .1 \text{ mm})$

(Fig. A3.4). The thickness of the flexible legs $w_p = 6.0 \pm 0.1$ mm for all boundaries except for $R2$ which is $w_p = 8.0 \pm 0.1$ mm.

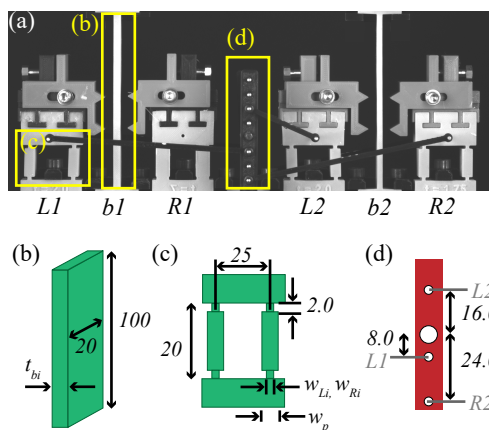
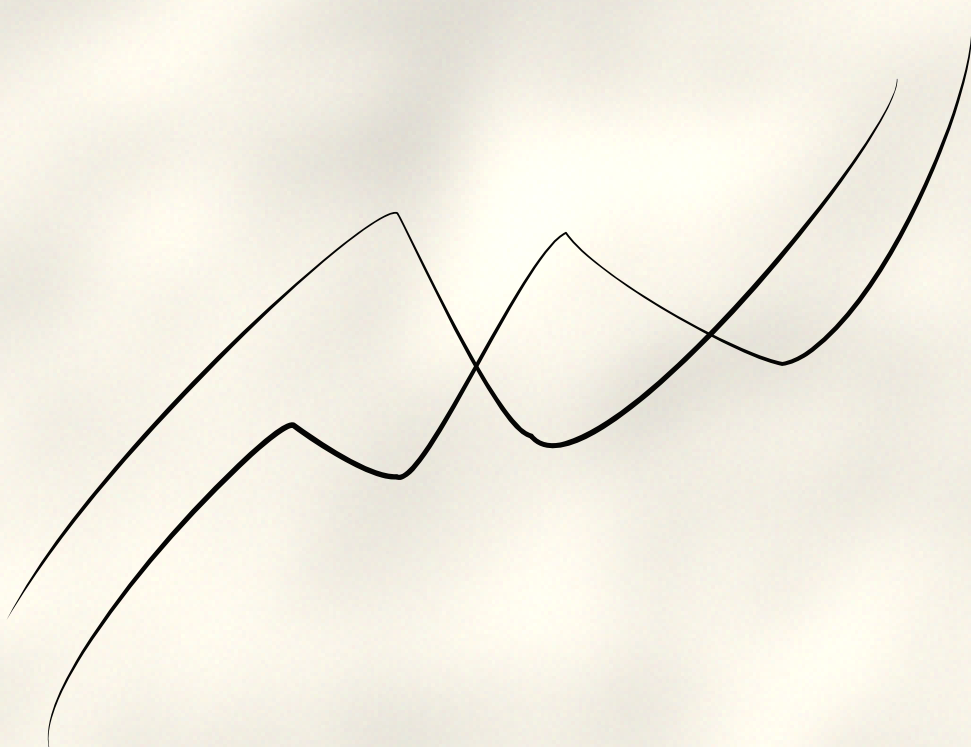


FIGURE A3.4: **Geometry.** (a) System indicating the left beam (b), the flexible pusher legs of boundary L1 (c) and central pivot of the coupling mechanism (d). (b-d) All dimensions in mm with error 0.1 mm; in (d), gray lines denote the connection to the specified pusher.



Early onset of snapping of slender beams under transverse forcing

The hysteretic snapping under transverse forcing of a compressed, buckled beam is fundamental for many devices and mechanical metamaterials. For a single-tip transverse pusher, an important limitation is that snapping requires the pusher to cross the longitudinal axis of the beam. Here, we show that dual-tip pushers allow *early-onset snapping*, where the beam snaps before the pusher reaches the longitudinal axis. As a consequence, we show that when a buckled beam under increased compression comes in contact with a dual-tip pusher, it can snap to the opposite direction — this is impossible with a single-tip pusher. We demonstrate how to maximize the distance between the pusher and the beam's longitudinal axis at the moment of snapping. Together, our work opens up a new avenue for quantitatively and qualitatively controlling and modifying the snapping of buckled beams, with potential applications in mechanical sensors, actuators, and metamaterials.

4.1 Introduction

Snap-through instabilities are widely observed in nature. The Venus flytrap and carnivorous waterwheel plants capture prey through snapping leaves [129–131]. Hummingbirds exploit a snap-through mechanism in their beak to catch insects mid-flight [132]. Grasshoppers and click beetles leverage similar mechanisms to power their leaps [133–135]. Similar snap-through instabilities have been harnessed in mechanical metamaterials to realize a wide range of functionalities [29, 136], including microfluidic passive valves [137], microlens shells [138], soft actuators [83], components of soft robots [139, 140] and counter-snapping [141]. Often, snap-through is triggered by a transverse pusher that forces the beam into an S-shaped configuration at the onset of snapping [121, 122, 142]. As a consequence, the pusher has to cross the beam’s longitudinal axis to induce snapping. This inherent geometric constraint limits design flexibility and hinders potential applications in fields such as *in-material* computing [24, 40, 47, 72, 76, 143, 144].

4

Here, we introduce a second pusher tip and demonstrate that dual-tipped pushers enable *early-onset snapping*, where snapping occurs before the pusher would cross the longitudinal axis. This enhances design flexibility and permits snapping under compressive loading with a fixed dual-tip pusher (Sect. 4.2). While rate-dependent effects—such as inertial overshoot—might cause the onset of instability shift, we focus on quasistatic driving conditions with a dual-tip pusher to achieve precise control of the onset threshold and isolate the underlying mechanisms. For example, for such pushers, we uncover a novel *two-step snap-through* mechanism, in which the beam passes through a stable intermediate state before fully snapping (Sect. 4.2). To elucidate the emergence of these behaviors, we develop a reduced-order numerical model based on a modal expansion of the beam shape ([121, 122, 145], Sect. 4.3). We use this model to visualize the deformation energy of the beam in a two-dimensional landscape, where transverse pushers translate to excluded zones, and where beam configurations in contact with the pusher reside precisely on the boundary of these zones. This allows a geometric interpretation of the stability and snap-through behavior of the beam. In particular, it reveals that the second pusher tip introduces additional local extrema in the energy landscape, which underlie both the early-onset and two-step snapping behavior (Sect. 4.4). Finally, we systematically map the snapping threshold as a function of the geometry of the dual-tip pusher, allowing precise control over the onset and nature of the instability (Sect. 4.4). Together, our work uncovers and characterizes a new class of snap-through behavior with potential applications in soft robots, smart sensors and actuators, metamaterials and *in materia* computing.

4.2 Phenomenology

We consider a slender beam under quasistatic compression in the presence of a dual-tip transverse pusher. The beams have rest length L_0 with rectangular cross-sections (in-plane thickness T , out-of-plane thickness W) in the slender beam limit $T \ll W \ll L_0$. Compression is applied by controlling the end-to-end distance L , defining the axial strain $\varepsilon = (L_0 - L)/L_0$. The pusher consists of two vertically spaced tips located at vertical coordinates Z_t and Z_b , both sharing a common horizontal coordinate X_b (Fig. 4.1a).

We begin by examining the qualitative evolution of a buckled beam as it makes contact with a fixed dual-tip pusher, rigidly connected to the bottom plate, under increasing compression (Fig. 4.1). The beam is initialized in its left buckled state without contact with the pusher tips. As ε increases, the beam and pusher tips make contact, resulting in higher-order deformation modes. For $Z_b = -16$ mm, the beam deforms smoothly toward a critical configuration (Fig. 4.1bi-iii), and then experiences a single, abrupt snap-through instability (Fig. 4.1biii-iv).

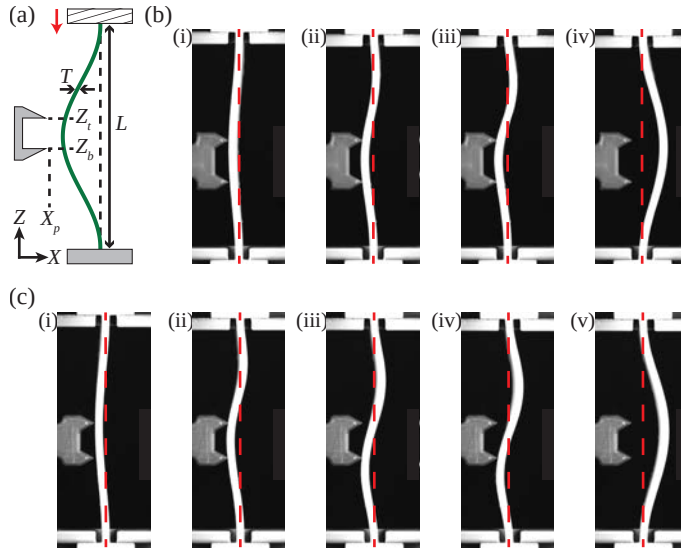


FIGURE 4.1: **Phenomenology.** (a) Schematic of a slender beam with rest length L_0 , compressed to end-to-end distance L . The dual-tip pusher is defined by the vertical location of its tips, Z_t and Z_b , and their horizontal offset from the beam's longitudinal axis, X_p . (b-c) Experimentally observed snap-through transitions induced by increasing compression using fixed dual-tip pushers and two values of Z_b . ($L_0 = 100$ mm, $T = 3$ mm, $X_p = -5$ mm, $Z_t = 2$ mm; see appendix for details). (b) One-step snapping at for $Z_b = -16$ mm, (i-iv: $\varepsilon = 0.011, 0.022, 0.027$, and 0.028). (c) Two-step snapping for $Z_b = -13$ mm: here the beam loses contact with the tips sequentially (i-v: $\varepsilon = 0.011, 0.025, 0.026, 0.028$, and 0.029).

For $Z_b = -13$ mm, the snap-through occurs in two-steps: first the beam loses contact with one tip and transitions to an intermediate state tip (Fig 4.1ci-iii), and then it loses contact with both tips and snaps to the right-buckled configuration (Fig 4.1civ-v).

We note that the emergence of snapping without the pusher crossing the beam's longitudinal axis is linked to the moment applied by the pair of tips which forces the beam into a configuration unattainable with a single-tip pusher. A precise theoretical understanding of how these snap-through scenarios arise is the focus of the remainder of this chapter.

4.3 Theoretical modeling

Our strategy to model snapping induced by dual-tip pushers starts from a standard expansion of the beam shape. First, we observe and explain why the first three modes in this expansion are dominant and focus on these. Then, using the constraint of fixed beam-length, we map our beams to a two-dimensional energy landscape. The transverse pusher constraints translate to excluded zones, and beam configurations in contact with the pusher exactly lie on the boundary of these zones. As we show below, this facilitates a geometric interpretation of both one-step and two-step snapping, offering a clear physical understanding of these phenomena and their dependence of the pusher parameters.

In this section, we first derive the modal expansion (Sec. 4.3.1). We then introduce a two-dimensional visualization of the energy landscape (Sec. 4.3.2). We end this section by presenting a geometric picture for the evolution of the beam configuration, and illustrate this for a classical single-tip pusher (Sec. 4.3.3).

4.3.1 Modal expansion and beam model

We derive the modal expansion for the beam shape. We parametrize this shape with coordinates $X(Z)$, and compress the beam via boundary conditions $X(\pm L/2) = 0$ and $X'(\pm L/2) = 0$. We focus on small deformations, neglect friction between beam and tips, and consider the beam as inextensible. The bending energy per unit width then reads:

$$\mathcal{E}_b = \frac{B}{2} \int_{-L/2}^{L/2} C(Z)^2 dZ, \quad (4.1)$$

with bending modulus $B = Et^3/12(1 - \nu^2)$, Young's modulus E , Poisson ratio ν and curvature of the beam $C = X''(Z)/(1 + X'(Z))^3$. Under the small strain approximation, the curvature can be written as $C(Z) \approx X''(Z)$.

We now impose two sets of constraints on this problem. First, inextensibility imposes that the length of the beam is conserved:

$$\int_{-L/2}^{L/2} \sqrt{1 + X'(Z)^2} dZ = L_0. \quad (4.2)$$

Under the small strain approximation, Eq. (4.2) can be expanded to yield:

$$L + \frac{1}{2} \int_{-L/2}^{L/2} X'(Z)^2 dZ = L_0. \quad (4.3)$$

Second, the pusher tips constrains the local maximum deflection of the beam:

$$\begin{cases} X(Z_t) < X_p, \\ X(Z_b) < X_p. \end{cases} \quad (4.4)$$

Combining the energy and constraints, we obtain the Lagrangian of this system:

$$\begin{aligned} \mathcal{L}[X] = & \frac{B}{2} \int_{-L/2}^{L/2} X''(Z)^2 dZ \\ & - P \left(L - L_0 + \frac{1}{2} \int_{-L/2}^{L/2} X'(Z)^2 dZ \right) \\ & - F_t (X_p - X(Z_t)) - F_b (X_p - X(Z_b)), \end{aligned} \quad (4.5)$$

where P, F_t, F_b are the Lagrange multipliers and can be interpreted, respectively, as the axial force and top and bottom transverse forces per unit width. Importantly, we note that Eq. (4.4) are *inequalities*, and we deal with these using the Karush-Kuhn-Tucker conditions [145].

We scale out the compressive strain [121], introducing the rescaled parameters:

$$\begin{cases} z = Z/L = Z/(L_0(1 - \epsilon)), \\ x = X/(L\sqrt{\epsilon}), \\ p = \frac{P}{B/L^2}, \\ f = \frac{F}{B/L^2} \frac{1}{\sqrt{\epsilon}}, \end{cases} \quad (4.6)$$

leading to:

$$\begin{aligned} \mathcal{L}[x] = & \frac{1}{2} \int_{-1/2}^{1/2} x''(z)^2 dz \\ & - p \left(\frac{1}{2} \int_{-1/2}^{1/2} x'(z)^2 dz - 1 \right) \\ & - f_t (x_p - x(z_t)) - f_b (x_p - x(z_b)). \end{aligned} \quad (4.7)$$

The advantage of this rescaling is that solutions for the buckled beam shape at different strains can be mapped onto each other using the rescaling. The price we pay is that the rescaled location of a fixed transverse pusher becomes strain dependent.

Finally, we perform an expansion of the beam shape on its first m modes, i.e., write $x(z) \approx \sum_1^m a_i \xi_i(z)$, with ξ_i the i^{th} order solution of the linearized Elastica problem with clamped-clamped boundary conditions[122]:

$$\begin{cases} \xi_i = -\frac{2}{n} \left(1 - \frac{\cos(nz)}{\cos(n/2)}\right), & \text{for } \sin(n/2) = 0 \\ \xi_i = -\frac{2}{n} \left(\frac{\sin(nz)}{\sin(n/2)} - 2z\right), & \text{for } \tan(n/2) = n/2, \end{cases} \quad (4.8)$$

where the values of n are ordered in an increasing manner, and are assigned an index i starting at 1. This projection significantly simplifies Eq. (4.7):

$$\begin{aligned} \mathcal{L}[a_1, a_i \dots a_m] &= \sum_1^m n_i^2 a_i^2 - p \left(\sum_1^m a_i^2 - 1 \right) \\ &- f_t \left(x_p - \sum_1^m a_i \xi_i(z_t) \right) - f_b \left(x_p - \sum_1^m a_i \xi_i(z_b) \right). \end{aligned} \quad (4.9)$$

The problem now consists in finding the set of coefficients a_i that minimizes $\sum_1^m n_i^2 a_i^2$, under the constraints $\sum_1^m a_i^2 - 1 = 0$ and $\sum_1^m a_i \xi_i(z_k) - x_p \leq 0$ (with $k = t, b$).

4.3.2 Energy landscape

To visualize the elastic energy landscape, we truncate the modal expansion to include only the dominant modes. The modes are ordered by their energy contribution (Eq. 4.8) and the minimal number of modes can be understood through a simple constraint-counting argument: the fixed length imposes one constraint, and each pusher tip adds another. Thus, a single-tip pusher requires at least two modes, while a dual-tip pusher requires at least *three* to satisfy the equations of motion. We confirm both numerically and experimentally that three modes are sufficient to capture the beam's evolution and that, when using more modes, the first three modes dominate the expansion (see appendix).

We leverage the energy concentration in the first three modes, together with the inextensibility (length) constraint, to construct a strain-independent two-dimensional energy landscape for our beams. For a three-mode truncation, the length constraint imposes the normalization condition: $a_1^2 + a_2^2 + a_3^2 = 1$. This naturally motivates a parametrization using spherical coordinates (θ, ϕ) :

$$\begin{aligned} a_1 &= \cos \theta \cos \phi, \\ a_2 &= \sin \theta \cos \phi, \\ a_3 &= \sin \phi. \end{aligned}$$

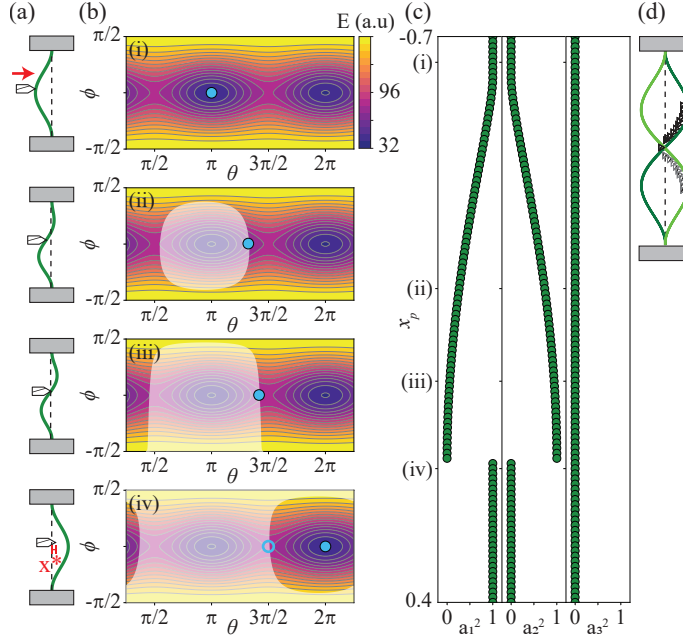


FIGURE 4.2: **Single-tip pusher.** (a) Numerically computed beam configurations $x(z)$ and corresponding (b) energy landscapes for a beam under fixed ε , as a single-tip pusher moves transversely from left to right at fixed height $z_t = 0.05$. (i-iv) Snapshots are shown for $x_p = -0.65, -0.21, -0.03$, and 0.14 . In the energy landscape, the cyan dot marks the current beam configuration and the white domain indicate the excluded zone imposed by the pusher. When the configuration is pushed passed the saddle point, the beam snaps to a new configuration (iv). (c) Squared mode amplitudes as a function of pusher distance x_p for $z_t = 0.05$, showing that most energy is concentrated in the first two modes. The snapshots of (a-b) are indicated on the x_p -axis. (d) Beam shapes and corresponding pusher position at the onset of snap-through for a range of values of z_t , which shows that in each case, the configuration just before snapping is purely S-shaped (light green beam) or Z-shaped (dark green beam), independent of z_t . The sign of a_2 is negative ($a_2 = -1$) if $z_t > 0$ (black triangles, dark green beam), and positive ($a_2 = 1$) if $z_t < 0$ (grey triangles, light green beam).

Within this coordinate system, we construct a two-dimensional energy landscape (Fig. 4.2b) that captures essential features of beam configurations independent of the compressive strain ε .

First, the left and right buckled states of the beam have mode amplitudes $(a_1, a_2, a_3) = (\pm 1, 0, 0)$, which map to the stable fixed points at $(\theta, \phi) = (0, 0)$ and $(\theta, \phi) = (\pi, 0)$. We stress that, due to the rescaling (Eq. 4.6), these fixed points are invariant under variations of ε . Second, the local energy minima are separated by an energy barrier that must be surmounted to induce a snap-through between the two buckled states; these correspond to saddle points that

represent purely antisymmetric 'S'-configurations ($(a_1, a_2, a_3) = (0, \pm 1, 0)$, $(\theta, \phi) = (\pm\pi/2, 0)$) (Fig. 4.2b). Third, the presence of transverse pushers imposes geometric constraints that restrict the beam's accessible configurations, giving rise to *excluded zones* and, as we will show later, additional stable equilibria.

4.3.3 Evolution

We now show how the landscape reveals the evolution of beam shapes under increasing compression or transverse motion of the pusher. We use a combination of gradient descent and sequential least squares programming methods to solve the model. First, the beam is in one of its free, buckled configurations. Then, as the pusher moves or ε is increased, the excluded zones evolve accordingly and eventually reach this configuration; the beam then makes contact with the pusher. Further increasing the driving leads to an overlap between the excluded zone and the free, buckled configuration. The system evolves, and the beam deforms due to its contact with the pusher. In such cases, the configuration lies either on the boundary of an excluded zone (single contact) or at the intersection of two such boundaries (dual contact). As long as these configurations represent local minima in the allowed part of the energy landscape, the beam is stable, but upon further pushing or compressing, such minima become unstable and the beam configuration snaps to another stable configurations.

We now illustrate this picture for a beam at fixed compression, transversely pushed by a single-tip pusher (Fig. 4.2a-b). We initialize the beam in the left buckled state (Fig. 4.2ai-bi). This configuration becomes inaccessible when the tip makes contact with the beam, creating an excluded zone that increases as x_p increases (Fig. 4.2aii-bii). Pushing further, the configuration reaches the saddle point corresponding to a mirrored S-shaped (or Z-shaped) beam (Fig. 4.2aiii-biii), after which the beam discontinuously snaps and evolves to the oppositely buckled configuration (Fig. 4.2aiv-biv) [121, 122, 142].

Our simulations show that most of the elastic energy is stored in the first two modes of deformation: as the beam is transversely pushed (x_p increases) the amplitude of the first mode decreases, while the amplitude of the second mode increases, the third mode remaining essentially at zero (Fig. 4.2c). When $a_2^2 = 1$, the configuration of the beam suddenly changes and a_2 jumps to zero, a_1 to -1 : snap-through occurred and $x_p = x^*$ (Fig. 4.2c). Varying the z_t position of the pusher does not qualitatively change this picture: with a single tip pusher the beam evolve along the $\phi = 0$ line, such that the unstable point lies at $(\theta = \pi/2, \phi = 0)$. Hence, with a single tip pusher, snap-through occurs when the beam reaches either a S-shape or Z-shape ($a_1 = 0$). This requires the pusher-tip to reach the longitudinal axis of beam (when $z_t = 0$)¹ or push beyond this

¹For the symmetric situation where $z_t = 0$, whether the energy minimum ends up the S-shape

longitudinal axis (when $z_t \neq 0$; see Fig. 4.2d) [121, 122, 142]. Hence, using a single tip pusher, snapping can be retarded, but can not have an early-onset.

4.4 Early-onset and two-step snapping with dual-tip pushers

We now use our model to explore the beam evolution for dual-tip pushers. We consider three different driving scenarios. In the first two, the vertical tip positions are fixed in the rescaled beam coordinates, which is theoretically simpler; we consider snapping induced by transverse pusher motion (scenario one), and by increased beam compression in the presence of a fixed pusher (scenario two). We also consider the experimentally more realistic third scenario, where the vertical tip positions are fixed in the lab-frame. We systematically vary the pusher parameters to obtain a detailed overview of the acceleration of the snapping, and where one-step and two-step snapping occur. We then perform the same analysis for scenario two and three (Sec. 4.4.3). While differing in details, we find that the overall mechanisms and even characteristic parameter ranges in all scenarios are closely related.

4.4.1 Three driving scenarios

In the first scenario, the beam is kept under constant compressive strain ε , while the pusher moves horizontally towards the beam (Fig. 4.3a). In this case, z_b and z_t remain constant during driving, and the problem is symmetric under swapping $z_t \leftrightarrow -z_b$. In the second and third scenario, the snapping is driven by increasing the axial compression of the beam, and the *transverse* position of the pusher is fixed in the labframe, so that x_p varies as $X_p/(L\sqrt{\varepsilon}) = (X_p/L_0) \cdot 1/((1-\varepsilon)\sqrt{\varepsilon})$ (Eq. 4.6).

Furthermore, in the second scenario, we consider the theoretical simple case that z_b and z_t have fixed values. This implies that, in the labframe, the vertical pusher tip positions vary with the compression as $Z_{t,b} = (1-\varepsilon)L_0 z_{t,b}$. In this scenario the swapping symmetry ($z_t \leftrightarrow -z_b$) is preserved (Fig. 4.3b). In the third scenario, we consider the experimental situation that the bottom plate and pusher positions are fixed in the labframe, so that Z_t and Z_p are constants. Identifying the bottom and top vertical coordinate of the beam as $-L_0/2$ and $-L_0/2 + L$ in the labframe, and $-1/2$ and $1/2$ in the rescaled frame, we find that $z_t = Z_t/L + \varepsilon/(2(1-\varepsilon))$ and $z_b = Z_b/L + \varepsilon/(2(1-\varepsilon))$. Hence, in this scenario the swapping symmetry is broken (Fig. 4.3c).

or 2-shape is set by spontaneous symmetry breaking, and the energy minimum remains at the line $\phi = 0$; whereas for $z_t \neq 0$ the shape is deterministically selected [121, 122, 142].

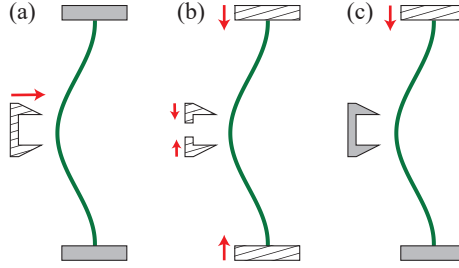


FIGURE 4.3: **Three scenarios.** (a) In scenario one, the beam is kept at constant compression ε while the pusher moves transversely. (b) In scenario two, the beam is compressed, the horizontal position of the pusher tips is fixed, and the vertical positions of the pusher tips are varied in the labframe so that their rescaled heights z_b and z_t are fixed. (c) In scenario three, the beam is compressed while the horizontal and vertical positions of the pusher tips are fixed in the labframe.

4.4.2 Transversely moving pusher: Driving scenario one

We now discuss the snap-through behavior when the beam is under constant compression and the pusher is moved transversely (Fig. 4.3a). When both tips are above (or below) $z = 0$, only the tip closest to $z = 0$ makes contact with the beam, yielding a behavior analogous to that of a single-tip pusher (Fig. 4.2). We therefore focus on the case where $z_t > 0$ and $z_b < 0$, so that both tips contact the beam. We consider a left-buckled beam, and denote the minimal horizontal location of a pusher, x_p^* , that allows for snapping. For appropriate tip positions, we observe early-onset snapping ($x_p^* < 0$). Concomitant with this novel behavior, the beam shapes are more complex and require a third mode to accurately describe the deformation.

One-step and two-step snapping

As function of the locations of the pusher tips, two qualitatively distinct types of snapping emerge. The snap-through consist of either *one step*, where both tips lose contact simultaneously (Fig. 4.4a), or *two steps*, where contact between tips and beam is lost sequentially (Fig. 4.4b). Each scenario may occur with the beam adopting either an S-shaped configuration (with $a_2 > 0$) or a $\mathcal{Z}$ -shaped configuration (with $a_2 < 0$; Fig. 4.4a-d).

The S and $\mathcal{Z}$ configurations are related by top-down symmetry, and are selected by the (broken) symmetry of the pusher (i.e., when $|z_b| \neq z_t$)². In these asymmetric cases, the pusher tip closest to the center ($z = 0$) first comes into contact with the beam and selects the sign of a_2 ; for $|z_b| > z_t$, the beam adopts

²When $|z_b| = z_t$, the top-down symmetry is preserved, and the beam assumes a symmetric "W"-shape, snapping only after reaching a near-pure mode-three configuration [121].

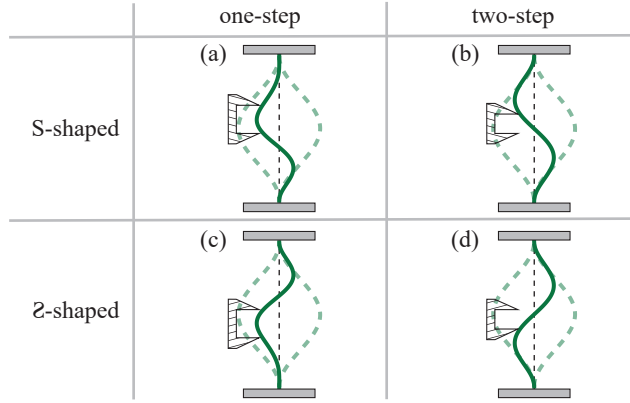


FIGURE 4.4: Examples of the initial, final (dashed) and critical configuration just before snapping (full). (a) For $(z_t, z_b) = (0.16, -0.03)$ the snapping occurs in one step at $x_p^* = -0.26$ – both tips lose contact simultaneously. In addition, as $z_t > |z_b|$, the top pusher tip contacts the beam first, forcing a S-shaped beam. (b) For $(z_t, z_b) = (0.10, -0.03)$ the snapping occurs in two steps, with the second step at $x_p^* = -0.22$ – the tips lose contact sequentially. Similar to (a) the top pusher tip contacts the beam first yielding a S-shaped beam. (c-d) When we swap $z_t \leftrightarrow -z_b$, we observe Z-shaped beams.

an S-shape with $a_2 > 0$ (Fig. 4.4a-b), while for $|z_b| < z_t$ it assumes an Z-shape with $a_2 < 0$ (Fig. 4.4c-d).

One-step snapping: energy landscape interpretation

We first analyze the emergence of the one-step scenario. We examine the energy landscape associated with the Z-shaped beam³ (Fig. 4.4c and 4.5a). Starting from the left buckled state at $(\theta, \phi) = (0, 0)$, the dual-tip pusher is transversely moved from left to right. Upon contact, the two tips create excluded zones in the energy landscape (Fig. 4.5ai).

In the absence of the excluded zones, all energy minima lie along the $\phi = 0$ line ($\partial_\phi E(\theta, \phi)|_{\theta=0} = 0$ at $\phi = 0$). However, in the presence of the excluded zones, additional stable points can be found along the boundary, and their stability is determined by the energy variation along these edges. Hence, all stable configurations reside along a curve that combines the relevant edges of the excluded zones and the $\phi = 0$ line; we refer to this curve as the *curve of inherent minima* (dashed line in Fig. 4.5ai, 4.5bi).

When the beam makes contact with both tips of the pusher, the configuration becomes trapped in a local energy minimum located at the intersection of two excluded zones, which—unlike the single tip case—enables the beam to explore configurations beyond the $\phi = 0$ line (Fig. 4.5ai). As x_p increases, the

³The evolution of an S-shaped beam follows by symmetry.

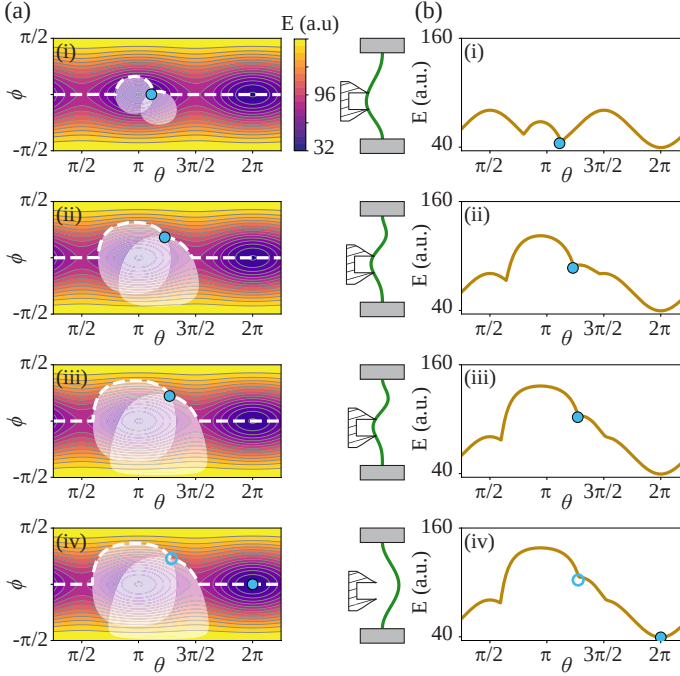


FIGURE 4.5: One-step snapping beam configuration evolution for driving scenario one. (a-b) *One-step snapping* for $(z_t, z_b) = (0.03, -0.16)$ with $x_p = [-0.55, -0.34, -0.26, -0.25]$ in (i-iv), respectively; (a) Energy landscape (left) with configuration denoted as blue dots and excluded zones as gray areas, and corresponding real-space configuration (right). (b) The energy along the curve of inherent minima. When the solution becomes unstable (iii-iv), the nearby solution with $\phi = 0$ is unstable, so that the beam loses contact with both pushers and evolves to the right-buckled configuration.

excluded zones expand and the beam initially remains trapped at their intersection (Fig. 4.5aii-aiii, 4.5bii-biii). However, this minimum eventually becomes unstable as the gradient turns negative (open circles in Fig. 4.5aiv, 4.5biv). This triggers the snap-through transition to the right-buckled configuration, where we note that in this case $x_p^* < 0$, i.e. snapping occurs before the pusher reaches the longitudinal axis of the beam (Fig. 4.5aiv, 4.5biv).

Two-step snapping: energy landscape interpretation

In the two-step snapping scenario, the initial evolution is similar to the one-step case: the two tips trap the beam in a local energy minimum situated at the intersection of the two excluded zones (Fig. 4.6ai, 4.6bi). The key difference is that when this local minimum loses stability, another local minimum exists that traps the beam at the intersection of one excluded zone and the $\phi = 0$

line (Fig. 4.6a_{ii}-a_{iii}, 4.6b_{ii}-b_{iii}). Here, the beam is still in contact with one of the pusher tips, and only by increasing x_p , the configuration reaches the saddle point at $(\theta, \phi) = (0, \pi/2)$ and the beam snaps (Fig. 4.6a_{iv}, 4.6b_{iv}).

Crucially, this second step is governed entirely by the pusher tip last in contact with the beam which is the tip located farthest from the middle (largest $|z|$). As a result, varying the position of the other pusher tip has no influence on the onset of this transition. Despite being driven by a single pusher tip, this second transition occurs *before* the tip crosses the beam's longitudinal axis. This behavior is markedly different from the standard single-tip scenario and arises from the reversed shape of the beam and the corresponding reversed sign of a_2 . This reversed shape is caused by the tip that is closest to the center, and that, at this point of the evolution, is no longer in contact with the beam. Hence, although only the bottom tip is in contact with the beam at the onset of snapping, the beam is effectively constrained in the 'unnatural' 2-shaped configuration throughout this last stage of the evolution (see Fig. 4.2a_{ii}-a_{iii}, Fig. 4.4d and Fig. 4.6a-b).

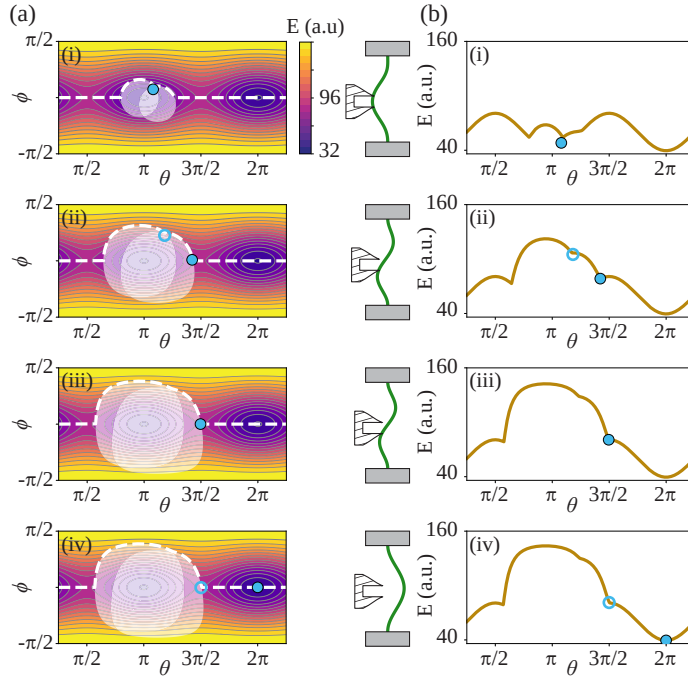


FIGURE 4.6: Two-step snapping beam configuration evolution for driving scenario one. (a-b) *Two-step snapping* for $(z_t, z_b) = (0.03, -0.10)$ with $x_p = [-0.55, -0.34, -0.22, -0.21]$ in (i-iv), respectively. When the solution becomes unstable (ii), the nearby solution with $\phi = 0$ is also stable, leading to the beam losing contact with only one tip, and a subsequent snapping at larger compression (iii-iv).

Bifurcation diagrams

We can further represent the difference between one-step and two-steps snapping, by tracking the extrema along the curve of inherent minima as a function of x_p for both scenarios (Fig. 4.7a-b). The snapping in the one-step scenario and the first step of the two-step scenario correspond to the same instability. However, the key difference is that in the one-step scenario this initial transition evolves to the $\theta = \pi$ configuration, whereas in the two-step scenario an intermediate state is reached (Fig. 4.7b). Together, these scenarios underscore that one-step and two-step snapping arise from the presence of multiple local energy extrema introduced by the second pusher tip.

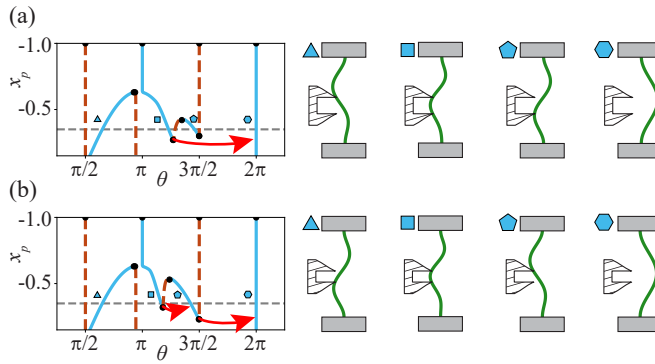


FIGURE 4.7: **Bifurcation diagrams corresponding** to Fig. 4.5 and Fig. 4.6. (a-b) Bifurcation diagrams illustrating the minima and maxima of the curve of inherent minima in blue and red dashed lines. Fold bifurcations are marked with black dots, and snapping events with red arrows. The real-space configuration of each stable branch at the gray dashed line ($x_p = -0.35$) are shown on the right. Each branch (labeled by the polygons) is identified by which pusher tip has contact with the beam.

Snapping threshold

Finally, we systematically investigate the influence of the tip heights, z_t and z_b , on the snapping phenomenology and the critical snapping threshold to the right-buckled state x_p^* (Fig. 4.8). As expected, the data obeys mirror symmetry along the line $z_t = -z_b$, which delimits S-shaped and Z-shaped scenarios (red and blue, respectively). We also observe that two-step snapping occurs within a small region of z_t , z_b parameters, where both are small; for larger z_t , z_b , one-step snapping is observed. The threshold x_p^* is most negative near the boundary of these two regions and can reach values of the order of -0.3 (Fig. 4.8(b)). Finally, while $x_p^* < 0$ for a large range of z parameters, pusher parameters with a large gap between their tips (e.g., $z_t = 0.2$, $z_b = -0.21$) may lead to single step snapping with $x_p^* > 0$. Nevertheless, we note that x_p^* for a dual-tip pusher is always less

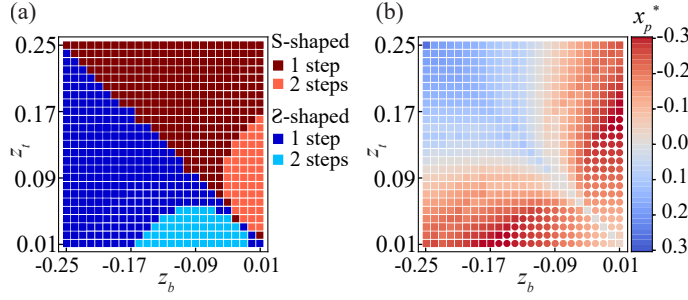


FIGURE 4.8: **Snapping in driving scenario one.** (a) State diagram for the four different scenarios as a function of (z_t, z_b) . S-shaped (blue) and Z-shaped (red) configurations obey a mirror symmetry along the line $z_t = -z_b$. One-step snapping is indicated in dark blue or red, two-step in light color. (b) Critical distance to snapping x_p^* as a function of (z_t, z_b) (squares denote one-step snapping, circles two-step snapping).

than for a single-tip pusher at either similar z_t or similar z_b . We conclude that early-onset snapping is a robust and tuneable feature when buckled beams are transversely pushed by a dual-tip pusher.

4.4.3 Compression-driven snapping

We now consider driving scenarios two and three, in which the transverse position of the pusher is fixed (i.e., X_p is constant), and snapping is induced by increasing the axial compression ε of the beam. In both scenarios, we consider transverse positions $X_p/L_0 = [-0.01, -0.03, -0.05]$ and quasistatically increase ε until $\varepsilon < 0.12$ to remain in the small strain approximation. In both cases, early-onset snap-through can be triggered, following similar one-step and two-step responses.

Driving scenario two

We first consider the case in which we fix the rescaled vertical pusher positions, and systematically explore snapping for a range of z_t and z_b (Fig. 4.9). Both one-step and two-step snapping responses are observed, and the results respect mirror (swap) symmetry (Fig. 4.9b-d). In contrast to driving scenario one, a region without snapping emerges (gray zones in Fig. 4.9); when the fixed horizontal position is too far away from the longitudinal axis of the beam, no snapping can occur. This non-snapping region can be inferred from the data from scenario one (Fig. 4.8b); a necessary condition for snapping in scenario two is that $(|X_p|/L_0) < (1 - \varepsilon)\sqrt{\varepsilon}|x_p^*|$, where x_p^* is deduced in scenario one. Note that this expression implies that the farther the pusher is from the beam, the greater the compressive strain ε required to trigger snap-through — consistent with the trends observed in Fig. 4.9.

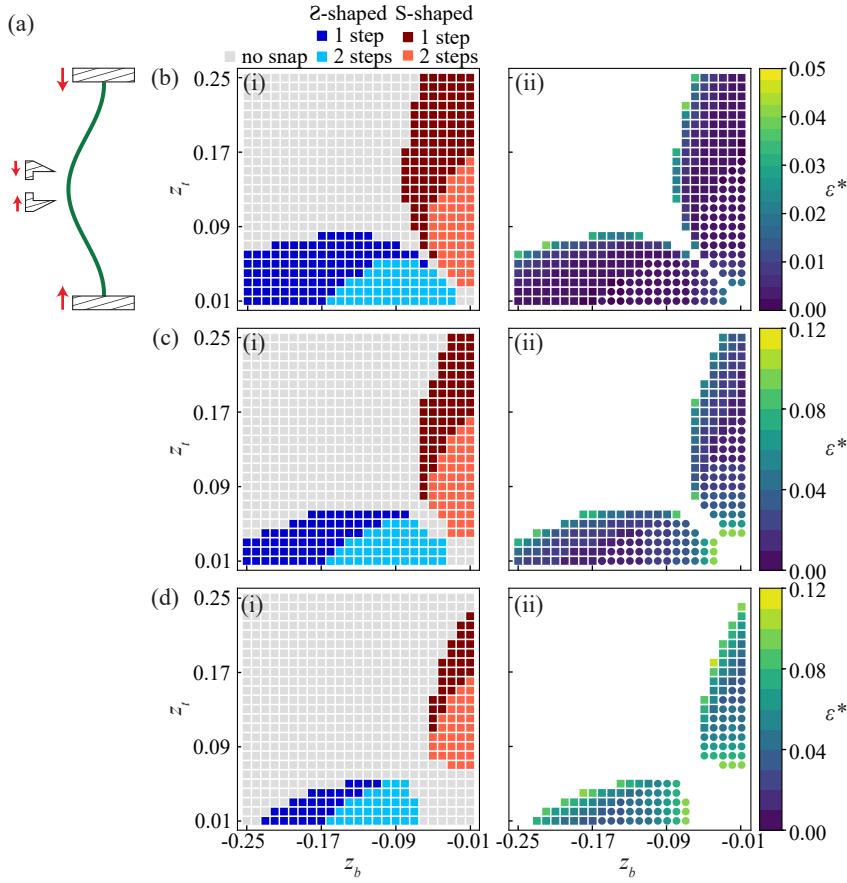


FIGURE 4.9: **Snap-through characterization for strain-driven scenario two.** (a-d) Driving scenario two, in which the rescaled vertical positions of the pusher tips, z_t and z_b , are constant throughout compression. (b-d) State diagrams (i) and critical strains ε^* (ii), for $X_P/L_0 = -0.01, -0.03$, and -0.05 , respectively. (e-h) Driving scenario three, in which the dimensional tip heights are fixed in the labframe. (f-h) Corresponding state diagrams (i) and critical snap-through strains ε^* (ii).

Driving scenario three

Finally, we examine the geometry where the pusher heights are rigidly connected to the stationary bottom plate. We again find a wide range of configurations that trigger early-onset one-step and two-step snapping (Fig. 4.10b-d). As expected, the swap/mirror symmetry is broken and its skewness increases with X_P . Consequently, compression often leads to snapping configurations where the bottom tip comes closer to the center of the beam than the top tip.

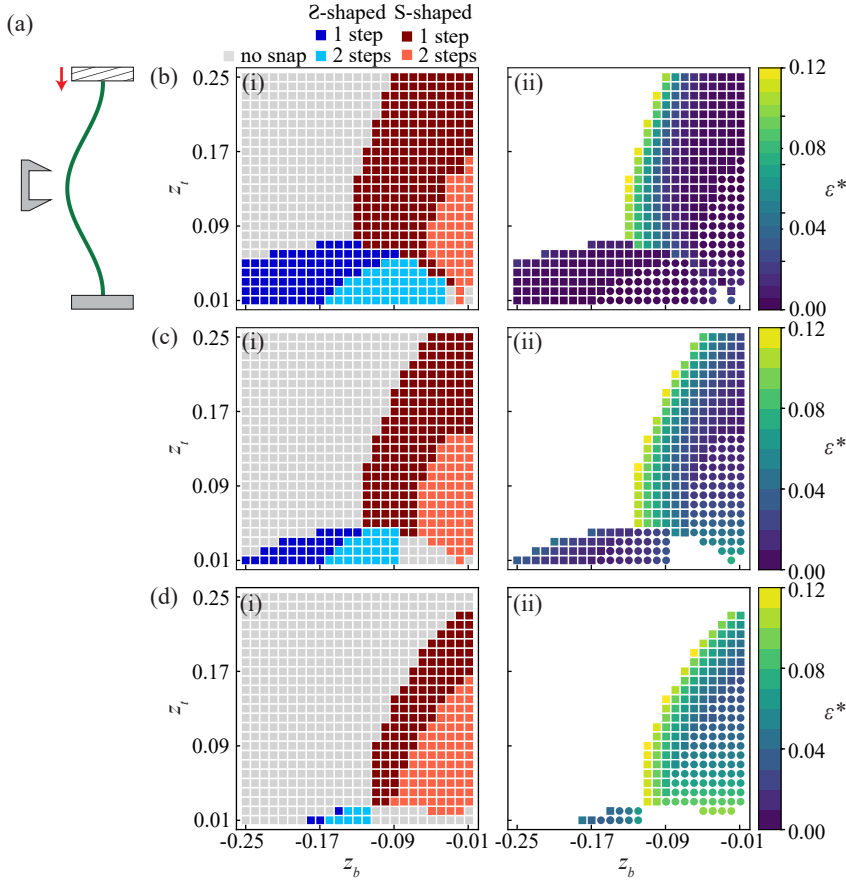


FIGURE 4.10: **Snap-through characterization for strain-driven scenario three.** (a-d) Driving scenario three, in which the dimensional tip heights are fixed in the labframe. (b-d) Corresponding state diagrams (i) and critical strains ε^* (ii), for $X_P/L_0 = -0.01$, -0.03 , and -0.05 , respectively.

Furthermore, we experimentally verified the numerical predictions and find good qualitative agreement (see appendix). Hence, despite the added complexity of this geometry, the general trends of snapping are similar to those seen in scenario one and two.

4.5 Conclusion

We investigated the snapping behavior of slender buckled beams due to transverse forcing with two-tipped pushers. Crucially, we showed that the introduction of a second pusher tip significantly enriches the deformation space, enabling both early-onset snapping and two-step snapping. Our strategy opens a new route to advanced snapping, including in parameter regimes that were hitherto inaccessible, with applications in soft robots, smart sensors and *in materia* computing [146].

While our analysis is quasistatic, dynamic effects can influence snap-through behavior. Inertial and rate-dependent effects—such as a delayed bifurcation and inertial overshoot—may shift the onset of snapping or alter the sequence of configurations [121, 127, 128], potentially enhancing or suppressing early-onset and two-step snapping. A detailed quantitative treatment of these effects for dual-tip actuation is left for future work.

Appendix

A4.1 Experimental mode decomposition

To validate our assertion that the first three modes are dominant in describing the beam shape, we perform a modal decomposition of the beam configuration in the representative experiments shown in Fig. 4.1.

Experiments are performed with a beam of length $L_0 = 100 \pm 0.5$ mm, in-plane thickness of 3 ± 0.5 mm and out-of-plane width 20 ± 0.5 mm. The beam is fabricated by casting two-component polyvinyl siloxane elastomer (Zhermack Elite double 22 with Young's modules 0.8 MPa and Poisson's ratio ≈ 0.5) into a 3D-printed mold. After curing, the beam is removed and dusted with talc powder to minimize friction and prevent sticking. The transverse pushers are 3D printed and mounted on precision linear stages for accurate positioning and alignment. Compression is applied using a custom-built apparatus designed for precise uniaxial loading, featuring high parallelism between the top and bottom plates, with a misalignment of less than 6×10^{-4} radians [72]. The axial compression is controlled by a stepper motor with an accuracy ± 0.01 mm and operated via an in-house LabView VI. Finally, the deformation of the beam is recorded using a CCD camera at 60 Hz with a resolution of ≈ 11 pixels/mm. The beam shape is extracted using open source software (ImageJ), and fitted to obtain the amplitudes a_i of the first six modes:

$$x_b(z_b) \approx \sum_{i=1}^6 a_i \xi_i(a_b). \quad (\text{A4.1})$$

We investigate both the one-step and two-step snapping behavior, by initializing the beam in the left-buckled state at $\varepsilon = 0.011$ and quasistatically (rate 10^{-3} min^{-1}) increasing the strain to $\varepsilon = 0.035$. We find that the deformation energy is concentrated in the first three modes (Fig. A4.1), and note that after the first snapping event in the two-step scenario the energy is concentrated in the first two modes (Fig. A4.1b). For other pusher geometries we have observed a similar dominance of the first three modes. Hence, the experiments confirm the validity of truncating the modal expansion after the third mode.

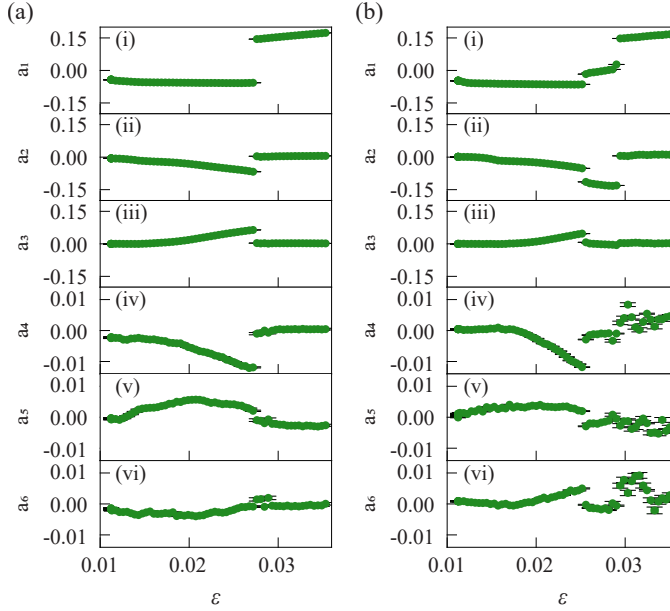


FIGURE A4.1: **Modal evolution of the beam.** (a) Amplitude of the first six modes a_1 - a_6 for the one-step scenario shown in Fig. 4.1b ($x_p^0 = 0.05$, $z_t^0 = 0.02$, and $z_b^0 = -0.16$). Note the difference in vertical scale between panels (i-iii) and (iv-vi). (b) Similar, for the two-step scenario shown in Fig. 4.1c ($x_p^0 = 0.05$, $z_t^0 = 0.02$, and $z_b^0 = -0.13$).

A4.2 Experimental verification

To verify the numerical results, we compare the experiments presented in Fig. 4.1 with the corresponding numerical predictions. We show the normalized amplitudes of the first three modes (green circles, Fig. A4.2) overlaid with the numerical evolution (red dashed line, Fig. A4.2). This shows good qualitative agreement for both the one-step and two-step scenario. The quantitative discrepancies can be attributed to: (i) friction between the pusher tips and the beam, (ii) the finite thickness of the beam, and (iii) the high sensitivity of the system to small positioning errors of the pusher tips.

In addition, we experimentally explore various pusher configurations and reproducibly find the two-step scenario across a wide range of pusher configurations. Even near the boundary between the one-step and two-step regimes, the two-step behavior persists, indicating that these intermediate states are stable, robust, and readily accessible.

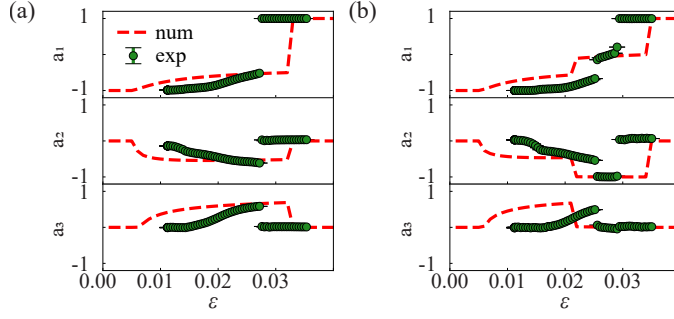


FIGURE A4.2: **Experimental verification.** (a-b) Normalized amplitude of the first three modes, a_1 - a_3 , from experiments (green circles) and numerical predictions (red dashed line). (a) One-step scenario from Fig. 4.1b ($x_p^0 = 0.05$, $z_t^0 = 0.02$, and $z_b^0 = -0.16$). (b) Two-step scenario From Fig. 4.1c ($x_p^0 = 0.05$, $z_t^0 = 0.02$, and $z_b^0 = -0.13$)

A4.3 Pusher boundary calculation

Here, we provide details regarding the calculation of the excluded zone(s) in the two-dimensional energy landscape of Fig. 4.5-4.6. These excluded zones emerge due to the transverse pusher. The boundary of such a zone is given by:

$$x_p = \sum_{i=1}^m a_i \xi_i(z_p), \quad (\text{A4.2})$$

where x_p and z_p are the transverse and longitudinal position of the pusher tip and a_i is the amplitude of mode i with shape ξ_i . Since $\sum_{i=1}^m a_i^2 = 1$ and we take $m = 3$, we parameterize a_i using spherical coordinates (θ, ϕ) :

$$\begin{aligned} a_1 &= \cos \theta \cos \phi, \\ a_2 &= \sin \theta \cos \phi, \\ a_3 &= \sin \phi. \end{aligned}$$

The boundary of the excluded zone thus follows:

$$\begin{aligned} x_p &= \xi_1(z_p) \cos(\theta) \cos(\phi) + \dots \\ &\quad \xi_2(z_p) \sin(\theta) \cos(\phi) + \xi_3(z_p) \sin(\phi). \end{aligned} \quad (\text{A4.3})$$

Solving this for ϕ gives:

$$\tan(\phi_{\pm}) = \frac{\xi_3(z_p)x_p \pm A \frac{\xi_3(z_p)}{|\xi_3(z_p)|} \sqrt{A^2 + B}}{\xi_3(z_p)x_p \mp |\xi_3(z_p)| \sqrt{A^2 + B}}, \quad (\text{A4.4})$$

with $A = \xi_1(z_p) \cos(\theta) + \xi_1(z_p) \sin(\theta)$ and $B = (\xi_3(z_p) + x_p)(\xi_3(z_p) - x_p)$.



Independent control of material bits for targeted memory

Mechanical metamaterials with bistable elements can store vast amounts of information, but writing these memories requires impractical local control or lengthy multi-cycle protocols. We overcome this limitation with a dynamic control strategy that accesses any configuration in a single global drive cycle by leveraging the system's sensitivity to the drive and its time derivatives. We realize this strategy with bistable beams on a rotating platform, where drive cycles become orbits in a control space of angular velocity and acceleration. State changes occur when these orbits cross switching thresholds, which we rationally design so that each state can be accessed by a single drive orbit. We construct a five-bit system and demonstrate its full addressability by selecting drive orbits that write all 26 uppercase letters of the alphabet in ASCII representation. This dynamic control paradigm offers a general route towards smart, remotely operated devices across various physical domains.

This work was carried out in collaboration with E. Gutierrez-Prieto and P.M. Reis of EPFL, and is adapted from *Dynamic drives allow independent control of material bits for targeted memory*, Sci. Adv. **12**, eaec1606 (2026). CMM developed the conceptualization, theory and assisted with the experiments and data analysis.

5.1 Introduction

Mechanical structures made from bistable elements can store configurational memories [23, 24, 47, 48, 58, 59, 61, 72, 74–77] to enable tunable stiffness [110, 116, 147], sequential shape morphing [70, 148, 149], and computation [24, 47, 72, 80, 90, 91, 115, 117, 150]. While control over configurations is essential, it typically requires local and individual access to all elements [110, 116, 147]. Here, we introduce a dynamic control strategy that enables transitions between any pair of configurations within a single global drive cycle by exploiting sensitivity to different (orders of) derivatives of the driving. We illustrate this approach using bistable beams that snap through based on their collective rotation rate and acceleration. Through rational design and selection of drive cycles, we demonstrate targeted transitions along controllable pathways, including the writing of five-bit memories that encode all alphabetic characters. This form of dynamical control can be generalized to inertial, fluidic, electromagnetic, and electronic systems, thus providing a powerful method for writing memories and enabling smart, remotely controllable devices for applications in microfluidics, implants, smart infrastructure, and underwater or medical robotics.

Developing driving protocols that access all configurations of mechanical systems of material bits (mbits), comprising bistable elements, is essential for the functionality of multistable (meta)materials [23, 24, 47, 48, 58, 59, 61, 70, 72, 74–77, 80, 90, 91, 110, 115–117, 147–150]. Unfortunately, individually controlling each mbit becomes impractical as their number N grows large [110, 116, 147], and, even though it is possible to design mbits such that a single global driving field can access all states, such strategies require baroque driving protocols featuring up to N drive cycles [21, 23]. However, there are several examples of physical systems that also respond to the driving rate: in mechanics, driving forces depend on the configuration and motion (inertia); in fluids, on static pressure and flow (viscosity); and in electronics, currents depend on voltage (resistance) and its rate of change (capacitance). This poses the question of how and whether dynamic control strategies can be developed that allow access to all 2^N states using a single global drive cycle.

To address this challenge, we introduce a rational design principle to select dynamic driving protocols that together enable a global driving signal to steer transitions between arbitrary mbit states along selectable pathways. As we show, at the core of our strategy lies the fact that dynamic driving can be naturally represented as driving orbits in a multi-dimensional space, where the different axes represent the relevant derivatives of the drive. The ability to connect regions in this space to different stable configurations – unconstrained by the well-ordering of one-dimensional space that governs quasistatic driving – is crucial for enabling any-to-any transitions and simplifies both system design and driving cycle selection.

5.2 Phenomenology

While the strategy introduced above should be general for a wide range of physical systems, we focus on a mechanical implementation where each mbit i is a pre-buckled, double-clamped elastic beam mounted on a rotating disk, with its buckling direction defining the binary state s_i (Fig. 5.1a; see appendix) [72, 146, 151]. The angular velocity $\Omega(t)$ and time-derivative $\dot{\Omega}(t)$ control the radial centrifugal forces $f_\Omega \sim \Omega^2$ and azimuthal Euler forces $f_E \sim \dot{\Omega}$ that jointly control the switching of the mbits [152] (Fig. 5.1b). This interplay enables $\Omega(t)$ to serve as a dynamic control input for selectable transition pathways.

We demonstrate this selectivity with system I (Fig. 5.1), which comprises two mbits that differ in compressive strain ε and tilt angle θ (see appendix, table A5.1). We consider the transition pathways under two dynamic (harmonic) driving protocols $\Omega(t) = -a_j \sin(2\pi f_j t)$, where a_j is fixed and f_j is varied, to induce transitions between collective states $S = (s_1 s_2) = (00)$ and (11) . Strikingly, while for f_1 the system goes through intermediate state (10) , for f_2 the pathway from (00) to (11) exhibits a different intermediate state (01) (Fig. 5.1e). To understand how dynamic driving shapes transition pathways, we map driving protocols to orbits in the Ω – $\dot{\Omega}$ plane (Fig. 5.2). Each protocol traces a distinct elliptical orbit, along which we plot the observed states and the up ($s_i : 0 \rightarrow 1$) and down ($s_i : 1 \rightarrow 0$) transitions of both mbits, revealing that switching thresholds depend on both Ω and $\dot{\Omega}$ (Fig. 5.1c–e; Fig. 5.2). A central insight is that transitions occur along *up* curves u_i ($s_i : 0 \rightarrow 1$) and *down* curves d_i ($s_i : 1 \rightarrow 0$) in the Ω – $\dot{\Omega}$ plane, which can be identified by aggregating data from many elliptical driving orbits (Fig. 5.2; see appendix). The up and down transition curves for each element partition the plane into domains corresponding to distinct stable states. Note that all states are stable at the origin (*i.e.*, without driving), and that only crossings oriented away from the origin can produce transitions (Fig. 5.2). Intersections of transition curves are key; for example, the two elliptical driving orbits cross u_1 and u_2 on opposite sides of their intersection, thus producing the distinct pathways: $(00) \rightarrow (10) \rightarrow (11)$ and $(00) \rightarrow (01) \rightarrow (11)$ (Fig. 5.2).

5.3 Orbit selection

We leverage the principles elucidated above to selected driving orbits for system I so that any of its four binary states can be written (Fig. 5.3a–d). We focus on orbits that start and end at the origin, where all states are stable. Note that any orbit with appropriate orientation (Fig. 5.2) can be mapped to a physically realizable input function $\Omega(t)$ (see appendix). Reaching a target state requires changing mbits from $0 \rightarrow 1$ or $1 \rightarrow 0$, which demands appropriate crossings of the corresponding u_i and d_i curves. For example, an orbit that crosses only d_1

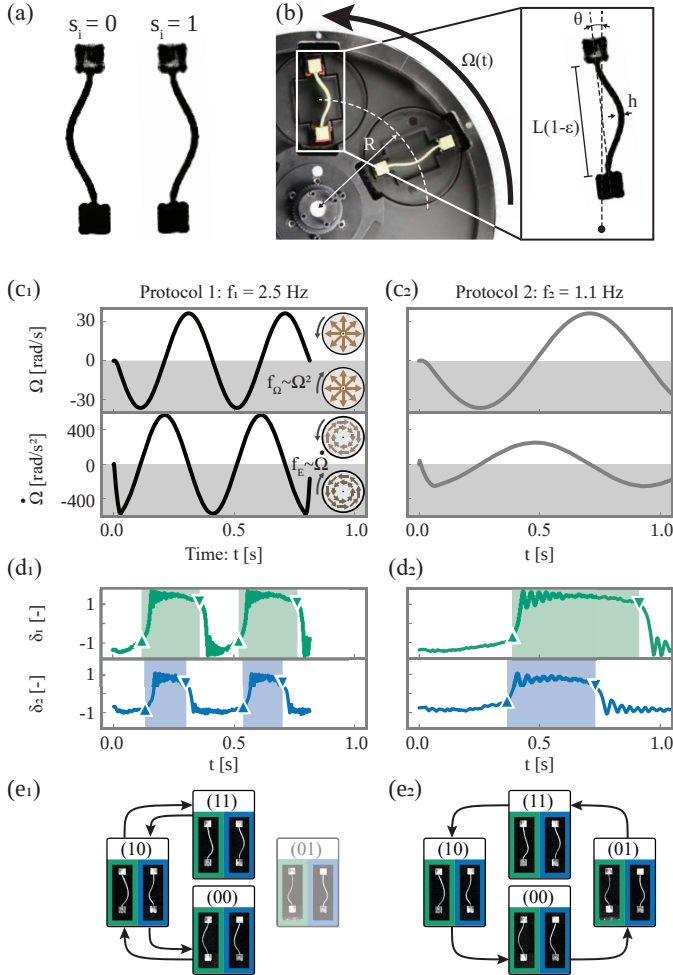


FIGURE 5.1: **Dynamic driving of mbits.** (a) Buckled beams form mbits whose buckling direction defines their state: $s_i = 0$ for counterclockwise and $s_i = 1$ for clockwise deflection (b) Rotating-disk apparatus showing system I and design parameters (see appendix). (c) Driving protocols $\Omega(t) = -a_j \sin(2\pi f_j t)$, where $a_1 = a_2 = 36.5$ rad/s and frequencies f_i as indicated. (d) Evolution of beam midspan δ_i showing distinct switching sequences connecting $S : (00) \rightarrow (11)$ via either (01) or (10). (e) Corresponding pathways of states.

and d_2 zeroes both mbits and produces $S = (00)$, regardless of the initial state (Fig. 5.3a). Note that specific switching curve topologies complicate orbit selection. Naively, one might expect that the target state $S = (01)$ can be written by simply crossing d_1 and u_2 , but any such orbit also crosses d_2 (Fig. 5.3b). In system I, writing the target state $S = (01)$ requires an orbit that crosses d_2 , d_1 and then u_2 (Fig. 5.3b). Similarly, all other target states of system I can be reached

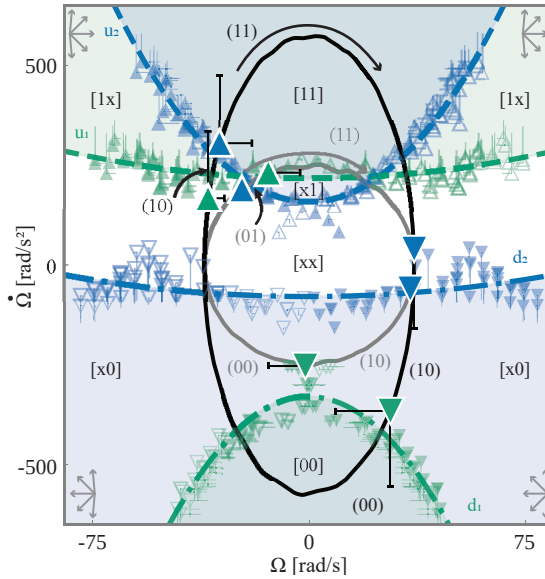


FIGURE 5.2: **Orbits, transitions and states.** Data for System I in the $(\Omega, \dot{\Omega})$ -plane, where green and blue correspond to mbits one and two, respectively. The ellipses correspond to protocol 1 (black) and 2 (grey), with arrows indicating the required clockwise orientation. The observed up transitions ($\triangle$), down transitions (∇), and states highlight the difference in intermediate states (bold). The switching onset for harmonic protocols with (frequency $f_j = 1.5$ Hz, drive amplitude $a_j \in [5.0, 80.0]$ rad/s) are shown as faint symbols with associated uncertainties (see appendix). The dashed and dot-dashed lines are quadratic fits serving as guides to the eye for the up- and down-switching curves, respectively (see appendix). These lead to partitions (shaded regions) where one or more collective states (square brackets) are stable, where ‘x’ indicates mbits that can be both 0 and 1. The arrows indicate the allowed orbit directions in each quadrant (open intervals of size π).

by selecting appropriate orbits (Fig. 5.3c,d). These examples demonstrate that dynamic control enables the rational selection of targeted orbits capable of writing any combination of mbits in system I, robust to external vibrations and the manufacturing tolerances of laser-cut fabrication.

To be able to write any possible state in a system with an arbitrary number of mbits, we first introduce hierarchically organized switching curves as a design target and then show how the physical mbits can be designed to realize such curves. As a schematic example, consider three up curves that intersect such that there is a range in driving space where each is closest to the origin (Fig. 5.4a). Initializing the system in state (000), it is then possible to select driving orbits $O(s_i)$ that set mbit i to 1 (Fig. 5.4a). Setting multiple mbits to 1 can be achieved by chaining such orbits or by selecting more complex multi-crossing driving orbits

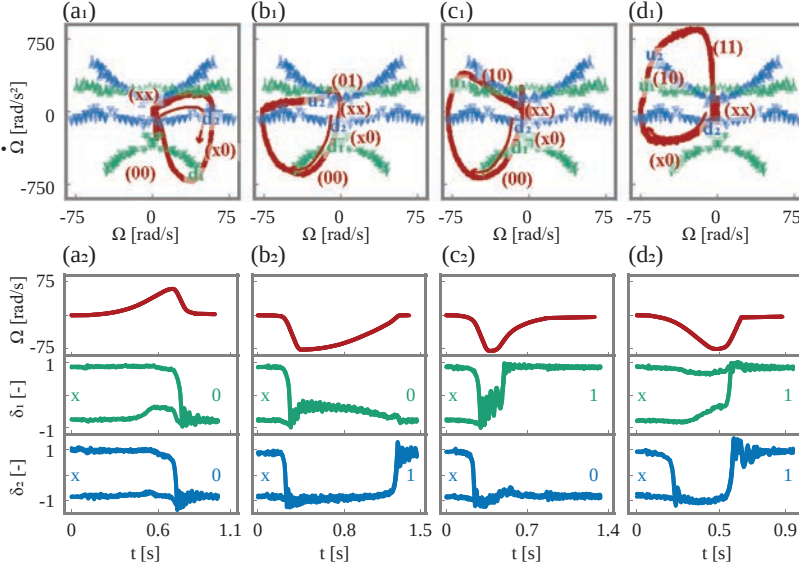


FIGURE 5.3: **Selected orbits** that, for System I, write specific targeted states: (a) (00), (b) (01), (d) (10) and (e) (11). (a₂-d₂) Angular velocity $\Omega(t)$ corresponding to selected orbits (top panels) and corresponding evolution of the midspan displacement of beams one (middle panels) and two (bottom panels), starting from any of four initial states, demonstrating how the final state is independent of the initial configuration.

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such as $O(s_3, s_1)$; this freedom allows control of intermediate states (Fig. 5.4a). Our proposed hierarchical design can be generalized to encompass more mbits, and is the prime design target to realize systems with highly targeted switching behavior, as we demonstrate next.

5.4 System design

We explore the real-space design of the mbits by investigating the dependence of the switching curve on the tilt angle θ_i and the compressive strain ε_i of the corresponding mbits. To guide us with a rational design, we use a von Mises-truss model comprising a mass connected to two linear springs (see S.M.). This model system captures the bistability of the mbits and, when subjected to centrifugal and Euler forces, allows to study the properties of the switching curves as a function of θ_i and ε_i [153–155] (Fig. 5.4b). We first fix $\theta = 0$ and find that the switching curves can be moved away from the origin by increasing ε , which controls the energy barrier between the $s = 0$ and $s = 1$ states (Fig. 5.4c₁). Second, non-zero tilt angles ($\theta \neq 0$) break the symmetry between the up and down switching curves, allowing widening of the u curves and narrowing of the d curves by increasing θ (Fig. 5.4c₂). Systematically varying the design param-

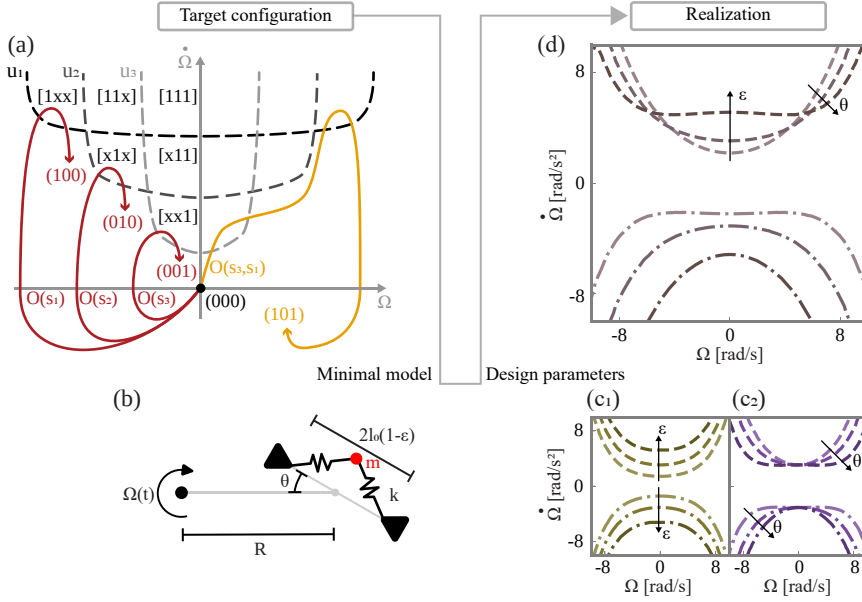


FIGURE 5.4: **System design.** (a) Schematic set of hierarchical switching curves (u_1, u_2, u_3) with selected orbits that enable the writing of arbitrary states: starting from state (000) , the orbits $O(s_1)$, $O(s_2)$, $O(s_3)$ and $O(s_3, s_1)$ write states (100) , (010) , (001) and (101) . (b) Minimal model for mbits, comprising two linear springs (spring constant k and rest length ℓ_0), each connected to a central mass m ; the key design parameters are the compressive strain ε and tilt angle θ . (c) Representative trends for the transition curves obtained from the minimal model with parameters ε and θ (arrows). Top: $\varepsilon = (0.03, 0.05, 0.07)$ with $\theta = 0$; Bottom: $\theta = (-3^\circ, 0^\circ, 3^\circ)$ with $\varepsilon = 0.05$. Experimental curves show similar trends (see S.M. Fig. A5.3). (d) Hierarchical up transition curves for $(\varepsilon, \theta) = [(0.04, -3^\circ), (0.05, 0^\circ), (0.07, 4^\circ)]$.

eters (θ_i and ε_i) enables the construction of the hierarchical switching curves: *up-curves* require increasing strain and tilt, while *down-curves* combine increasing strain with decreasing tilt (Fig. 5.4d). Hence, geometric control of the mbits facilitates designing systems with targeted hierarchical switching curves.

5.5 Sequential memory writing

Finally, we design a five-mbits system, whose collective state can be written with a single clockwise driving orbit starting and terminating at $(\Omega, \dot{\Omega}) = (0, 0)$. As a first step, we design the switching curves such that a target state $(t_1 t_2 t_3 t_4 t_5)$ can be reached from an initial arbitrary state $(xxxxx)$. Hierarchical design of either all up or all down switching curves is sufficient to do so in principle, but, instead, to enhance robustness, it is advantageous to leverage hierarchical order-

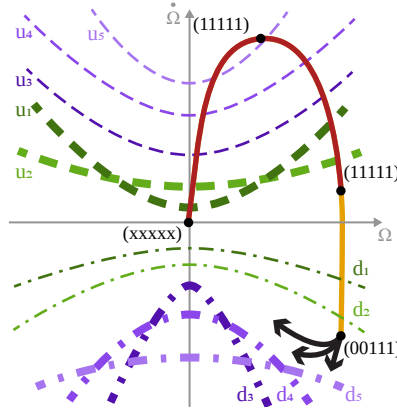


FIGURE 5.5: **Writing strategy.** Target hierarchical organization of the curves (u_1, u_2) and (d_3, d_4, d_5) (bold), together with appropriate placement of the other switching curves. Starting out from arbitrary state $(xxxxx)$, the first phase of the erase orbit (red) sets all bits, reaching state (11111) , whereas the second phase (gold) resets the first two bits reaching state (00111) , after which the writing phase commences (black).

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ing of subsets of the up and down curves, so that less critical intersections are required. Here, we opt for a design where (u_1, u_2) and (d_3, d_4, d_5) are hierarchically ordered, but note that other hierarchical orderings are valid too. We then use orbits that consist of a first erase phase, where the state evolves to (00111) via the intermediate state (11111) , and, then, a write phase, where the first two bits can be switched $0 \rightarrow 1$ and the last three bits can be switched $1 \rightarrow 0$ (Fig. 5.5). In such a scenario, the precise locations of the non-hierarchical curves are less critical, provided that (u_1, u_2) can be crossed without crossing (u_3, u_4, u_5) , and (d_1, d_2) are crossed before crossing (d_3, d_4, d_5) , so that in the erase phase we can reach state (00111) (Fig. 5.5). Using our numerical toolbox and some experimental fine-tuning, we realize the target ordering of the switching curves experimentally in System II (Fig. 5.6a; see appendix).

To demonstrate the expressive capabilities of our approach, we construct a library of driving orbits capable of writing all 26 binary states S that encode the Latin upper-case characters of the alphabet (see appendix, Fig. A5.4). Selecting appropriate orbits, as an example, we sequentially drive our five-mbit system to write the word “ORBIT” (Fig. 5.6b-c; see appendix). This capability of writing arbitrary words using Latin characters highlights that dynamic control of a rationally designed mbit system offers a powerful strategy to write and erase information.

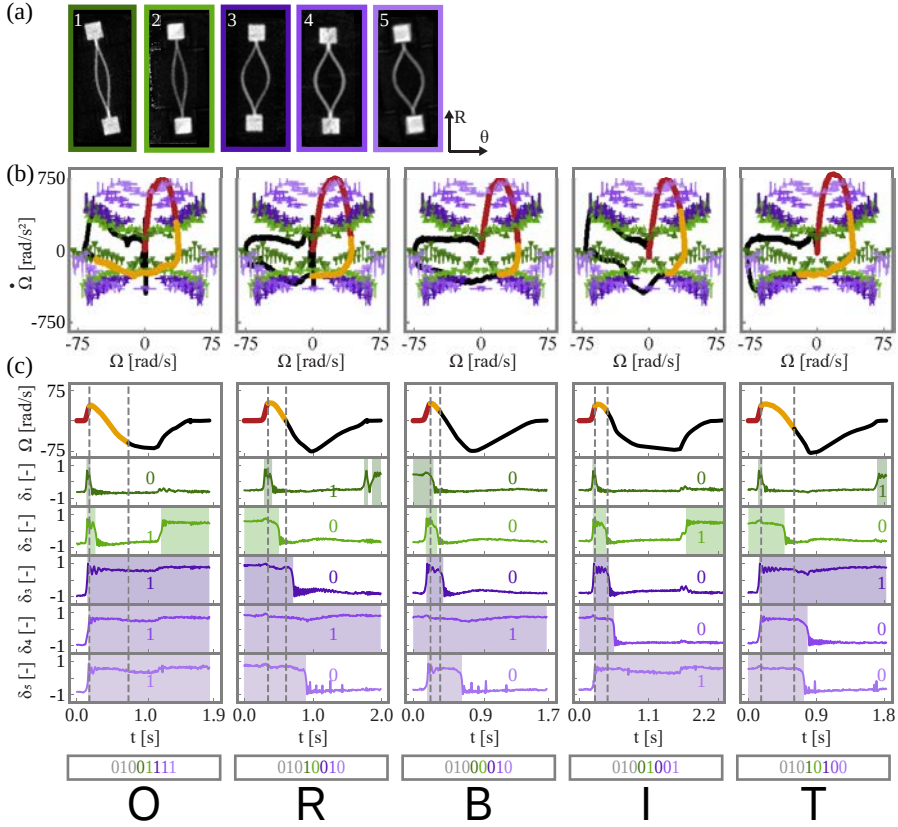


FIGURE 5.6: **Writing arbitrary states in System II.** (a) System II is comprised of five mbits with rationally designed tilt angle and compression that lead to the targeted ordering of switching curves. (b) Driving orbits corresponding to specific characters (red: erase phase one, gold: erase phase two, black: writing phase). (c) Experimental driving protocol and beam evolution demonstrate how dynamic control enables writing “OR-BIT” (see appendix). The ASCII encoding standard is used to map the bit strings to the Latin characters [156]. Although ASCII requires eight bits, the first three bits are identical for all uppercase characters; hence, we write the remaining five bits. Below each orbit, the full bit string is shown, with colors labeling the mbits, and gray indicating the three shared bits.

5.6 Discussion

We stress that dynamic driving extends beyond rotation, encompassing inertial or viscous effects in mechanics [26, 157–160], and, more broadly, dynamic effects in fluid flow [161, 162], chemical reactions [106], and electronic circuits [163]. For instance, mbits could function as bistable valves to enable global reconfiguration of centrifugal microfluidic circuits [137]. Similarly, electronic implementations of mbits can easily be made time-dependent using standard RC circuits, where volt-

age and its time derivative (analogous to Ω and $\dot{\Omega}$), allow a global time-dependent signal to control several mbits independently. Finally, in soft robotics, bistable elements with distinct intrinsic timescales could be differentially triggered by varying the rate of pneumatic or hydraulic actuation, enabling gait control that adapts to ambient viscosity [160].

The systematic exploration of diverse dynamically driven mbit platforms will enable agile information encoding for future smart and multifunctional devices. Although the current implementation is not optimized for energy efficiency, dynamic global driving is compatible with energy-efficient bit switching via appropriate material and design choices. Furthermore, the use of high-precision MEMS or NEMS fabrication methods could reduce geometric tolerances by orders of magnitude, while metallic materials in optimized geometries could reduce snapping timescales to microseconds, both significantly tightening threshold uncertainty. Energy dissipation per switching event is minimal and does not accumulate across elements due to sequential switching along the orbit. When combined with hybrid (global and local) driving strategies, in which multiple groups of mbits are dynamically driven independently, our approach provides a rational and scalable pathway toward much larger arrays.

Appendix

A5.1 Methods

A5.1.1 Experimental Methodology

Beams

The beams are fabricated by casting a silicone-based elastomer (VPS32, Zhermack Elite Double 32, with Young's modulus $E = 1.22 \pm 0.05$ MPa, Poisson's ratio $\nu \approx 0.5$, and density $\rho = (1.17 \pm 0.01) \cdot 10^3 \text{ kg m}^{-3}$ [164, 165]) using a fully enclosed mold. VPS32 exhibits a highly elastic mechanical response with minimal viscoelasticity, rate-dependence, and fatigue during mechanical cycling. Each mold consists of multiple layers of laser-cut acrylic components to produce uniform rectangular beams with thickness $h \in [1.5, 3.0]$ mm, length $L = 80$ mm, and width $b = 8$ mm. The extremities of the beams are cast to blocks with a larger width ($b \times h \times l = 15 \times 12 \times 15$ mm) to emulate clamped boundary conditions.

Placement

Each beam is aligned such that its width points out of the rotary plane and is placed in a laser-cut acrylic holder that can set the compressive strain in the range $\varepsilon \in [0.01, 0.1]$ with a resolution of ± 0.001 . These holders are mounted on a rigid disk in a way that the tilt angle can be varied in the range $\theta \in [-15^\circ, 15^\circ]$ with a resolution of $\pm 0.5^\circ$. Finally, the holders are enclosed by a transparent acrylic plate to minimize air drag during rotation (see Fig. 5.1).

System parameters

In Table A5.1, we present the geometric parameters of the mbits used in the two experimental systems (I and II) investigated in the main text.

	System I		System II				
mbit	1	2	1	2	3	4	5
L [mm]	80	80	80	80	80	80	80
h [mm]	1.9	1.9	1.6	1.9	2.1	2.3	2.6
b [mm]	8.0	8.0	8.0	8.0	8.0	8.0	8.0
R [mm]	150	150	150	150	150	150	150
ε [-]	0.050	0.015	0.015	0.026	0.037	0.055	0.060
θ [°]	-4.5	4.5	4.5	1.5	0.0	4.5	7.5

TABLE A5.1: **Parameters of the mbits in systems I and II.** The length L , in-plane thickness h , and out-of-plane width b have an experimental uncertainty of ± 0.1 mm. The radial position R of the beams has an uncertainty of ± 1 mm. The uncertainties in compression ε and angle θ are, respectively, ± 0.001 and $\pm 0.5^\circ$.

Rotational apparatus

The rigid disk holding all beams is mounted to a torque motor (ETEL RTMBi140-050A) capable of delivering up to 122 Nm of peak torque. The motor is driven by a Modular AccurET 600V/40A controller, which we program with a time series of angular positions to generate the desired velocity profile $\Omega(t)$. This setup enables precise trajectory execution, with maximum angular velocity and acceleration of $\pm 85 \text{ rad} \cdot \text{s}^{-1}$ and $\pm 10^3 \text{ rad} \cdot \text{s}^{-2}$, respectively. An encoder records the disk's angular position at 20 kHz, with an uncertainty of $\pm 8 \cdot 10^{-4}$ rad. To determine the angular velocity $\Omega(t)$ and acceleration $\dot{\Omega}(t)$, we apply a third-order Savitzky–Golay filter (15-point window) to the position data and perform numerical differentiation using fourth-order central finite differences.

Image analysis

The real-space configuration of each beam is imaged using a global shutter camera (IDS U3-3040SE-M-GL, with lens Vision-Lens 3.5mm EFL, F/1.4) is placed in the lab frame, above the rotating disk and aligned with its center of rotation. The shutter time is set to $100 \mu\text{s}$ to minimize image blur during rotation. We set a region of interest of 346×346 pixels, with 1.2 mm/pixel , to image the entire disk, setting a frame rate of 743 fps.

To quantify the configuration and dynamics of the beams, we developed an in-house image analysis algorithm with support from the EPFL Imaging Hub (Dr. Florian Aymanns). First, we locate two white markers, one at the center of rotation and the other at $R = 50 \text{ mm}$ from the center of rotation (see Fig. A5.1). Next, for every frame and all measured beams, the image analysis involves several key steps (see Fig. A5.1b): (1) We detect the two white markers to establish the angular position, φ , of the disk, and find that it is in good agreement with the

output of the encoder. (2) A neural network, trained by manually labeling clamp corners, identifies the corners of each clamp. (3) To approximate the neutral line of each beam, we use Hermite exponential splines from the Python library `splinebox` [166]. (4) We strategically place five spline knots: two at the identified beam edges and three additional knots equally distributed along the beam's longitudinal axis. (5) To precisely locate these central knots, we define three equally spaced lines normal to the clamp-to-clamp line. (6) Finally, on each line, we identify the point of maximum brightness by fitting a Gaussian function to the brightness distribution along the line. We locate the knots at the maximum of the fitted Gaussian. This image processing method achieves sub-pixel resolution (with ± 0.2 mm uncertainty) in the midpoint deflection of the centerline δ and generates a time series of the centerline of each beam represented in splines (see Fig. A5.1c).

Definition of snap-through events

To identify the snap-through of each beam, we start from the detected midpoint deflection $\delta(t)$ and follow a four-step approach: (1) We identify the times at which the midpoint of beam i crosses its longitudinal axis $\delta_i(t_i^0) = 0$ (orange crosses, Extended Data Fig. A5.2a-b). (2) We define the temporal window W_i centered about t_i^0 , $W_i = [t_i^0 - 25 \text{ ms}, t_i^0 + 25 \text{ ms}]$, and determine the maximum absolute velocity $|\dot{\delta}_i|_{\max}$ achieved within W_i (shaded area, Extended Data Fig. A5.2a). (3) We characterize the time of switching-onset t_i^* as $|\dot{\delta}_i(t_i^*)| = 0.15 |\dot{\delta}_i|_{\max}$ (purple triangles, Extended Data Fig. A5.2a). The threshold value 0.15 was empirically optimized to provide reliable switching-onset detection across our experiments. (4) We specify $(\Omega(t_i^*), \dot{\Omega}(t_i^*))$ as the switching-onset in the $(\Omega, \dot{\Omega})$ -plane. We take $(\Omega(t_i^*) - \Omega(t_i^0), \dot{\Omega}(t_i^*) - \dot{\Omega}(t_i^0))$ as an estimate of the uncertainty of the switching onsets, and indicate these with faint lines.

Switching curves

We experimentally map the switching curves of each mbit by applying a set of harmonic driving protocols $\Omega(t) = a_j \sin(2\pi f_j t + \phi)$, where we fix the frequency at $f = 1.5$ Hz, and systematically vary the amplitude $a_j \in [5.0, 80.0]$ rad/s with steps of 5.0 rad/s, and the phase $\phi = \{0, \pi\}$ rad (see Fig. A5.2b-c). Prior to each driving protocol, we set all mbits to $s_i = 0$ for $\phi = 0$ and to $s_i = 1$ for $\phi = \pi$. This collection of driving protocols enables a precise characterization of the transition curves within the $\Omega/\dot{\Omega} < 0$ domain. To complete the transition curves across the full domain, the experimental data is mirrored about $\Omega = 0$; this symmetry operation is justified since the centrifugal force $f_\Omega \sim \Omega^2$ is independent of the sign of the velocity. As a guide to the eye for the switching-curves data in Fig. 5.2, we fit a quadratic polynomial function to the measured

snap-through points. To ensure a mirror-symmetry about $\Omega = 0$, we consider a function without odd terms: $p(\Omega) = c_0 + c_2\Omega^2$ (dashed and dot-dashed curves in Fig. A5.2c, Fig. 5.2). We note that although the fitted curves give a good estimation of the location of the switching curves, selected orbits near these curves may trigger switching. Nevertheless, the selected orbits presented in this chapter were repeated five times and consistently demonstrated the reported switching behavior. A comprehensive characterization of the switching-curves is available in Ref. [167].

A5.1.2 Reduced-order model

Computational spring model

In parallel with the experiments, we use a reduced-order model to guide the rational design of the mbits switching curves. This model has a configuration analogous to a von Mises truss [153–155], comprising a central point (mass m , position $\mathbf{r}$) connected via two linear springs (spring constant k , rest length l_0) with end points $\mathbf{r}_{s1}$ and $\mathbf{r}_{s2}$ (Fig. 3b). We take $\mathbf{r}_{s1} = (1 - \varepsilon)(\sin(\theta), \cos(\theta))$, and $\mathbf{r}_{s2} = -\mathbf{r}_{s1}$, to ensure bistability for $\varepsilon > 0$. Similar to experiments, the tilt angle θ and compressive strain ε are the key design parameters. A total of four forces act on the central mass: First, the two springs give rise to the two forces $\mathbf{F}_{si} = k(|\Delta\mathbf{r}_{si}| - l_0)\Delta\mathbf{r}_{si}/|\Delta\mathbf{r}_{si}|$, where $\Delta\mathbf{r}_{si} = \mathbf{r} - \mathbf{r}_{si}$. Second, rotation gives rise to the centrifugal force $\mathbf{F}_\Omega = -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\mathbf{r} + R\mathbf{e}_y))$, and Euler force $\mathbf{F}_E = -m\dot{\boldsymbol{\Omega}} \times (\mathbf{r} + R\mathbf{e}_y)$, where R is the distance between the center of rotation and the position of the point mass $\mathbf{r}$ at rest, when $(\Omega, \dot{\Omega}) = (0, 0)$. We determine the equilibrium position(s) of the mass for a given angular velocity Ω and acceleration $\dot{\Omega}$ by numerically solving the force balance using gradient descent. To find the transition curve of the reduced-order model, we determine the monostable and bistable regions of each truss.

A5.1.3 Orbit selection

Realizing selected orbits

A target orbit can be described as $(\Omega(\tau), \dot{\Omega}(\tau))$, where τ is a pseudo-time. Such an orbit can be realized if and only if one can find a mapping between physical time t and τ , such that

$$\dot{\Omega}(\tau) = d\Omega(\tau)/dt = (d\Omega(\tau)/d\tau) \cdot (d\tau/dt). \quad (\text{A5.1})$$

Since τ is a monotonically increasing function of the physical time t , this equation implies the *directionality constraint* on each orbit, i.e., $d\Omega(\tau)/d\tau$ has the same sign as $\dot{\Omega}(\tau)$. To construct a specific orbit, we start from a given diskrete

sequence of points in the $(\Omega, \dot{\Omega})$ -plane, i.e., $(\Omega(\tau), \dot{\Omega}(\tau))$, and define τ as the trajectory index $\tau = (0, 1, 2, \dots) \cdot \Delta\tau$. We then use Eq. (A5.1) to find $\Delta t(\tau)$:

$$\Delta t(\tau) = \frac{\Omega(\tau + \Delta\tau) - \Omega(\tau)}{\dot{\Omega}(\tau)}, \quad (\text{A5.2})$$

and map the trajectory index τ to real time as $t(\tau) = \sum_0^\tau \Delta t(\tau)$. This procedure yields a time-parametrized angular velocity profile $\Omega(t)$ that matches the selected orbit in $(\Omega(\tau), \dot{\Omega}(\tau))$ -space and can be used as input for angular velocity-controlled experiments.

A5.2 Supplementary figures

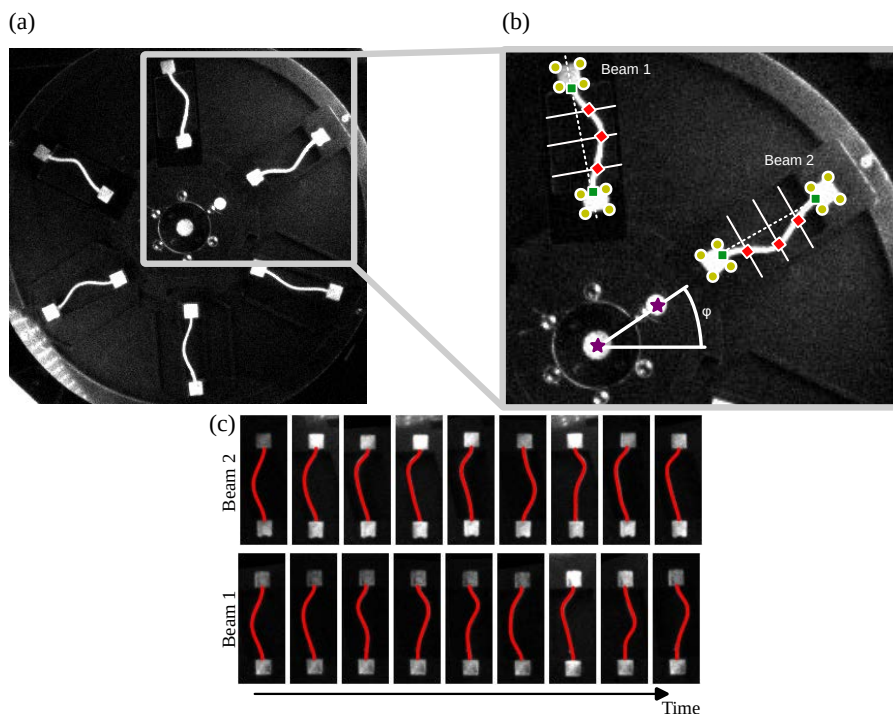


FIGURE A5.1: **Automated image-processing pipeline.** (a) Snapshot of the rotating device containing six flexible beams. (b) Schematic of the detection and spline-fitting workflow. White circles mark fiducial markers whose centroids (stars) define the instantaneous rotation angle φ . A neural network locates the clamp corners (light green circles), from which the two outer nodes (dark green squares) are established to determine the beam's neutral axis (dashed line). Three search lines, drawn perpendicular to this axis, detect intensity maxima corresponding to the remaining nodes used for cubic-spline interpolation (red diamonds). (c) Time-sequence montage of the fitted splines (red solid lines) overlaid on raw beam images (white), demonstrating sub-pixel accuracy of image analysis.

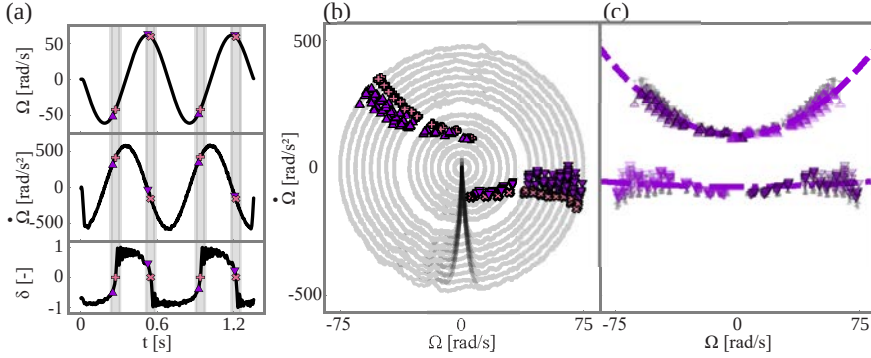


FIGURE A5.2: **Switching-curve extraction from snap-through dynamics.** (a) From top to bottom: angular velocity Ω , angular acceleration $\dot{\Omega}$, and midspan deflection δ of beam two from system I subject to harmonic driving protocol $\Omega(t) = a_j \sin(2\pi f_j t + \phi)$ with $a_j = 63.9$ rad/s, $f = 1.5$ Hz and $\phi = \pi$ rad. The orange crosses and purple triangles represent the point at which $\delta = 0$ and the onset of switching, respectively (see main text). The shaded region depict W_i . (b) all measured responses harmonic driving protocols with $a \in [5.0, 80.0]$ rad s⁻¹. The gray circles indicate the driving protocols. (c) The switching events represented with uncertainty. The solid and hollow symbols indicate the measured and mirrored data, respectively. The dashed and dot-dashed lines are the quadratic polynomial fits estimating the up- and down-curves, respectively.

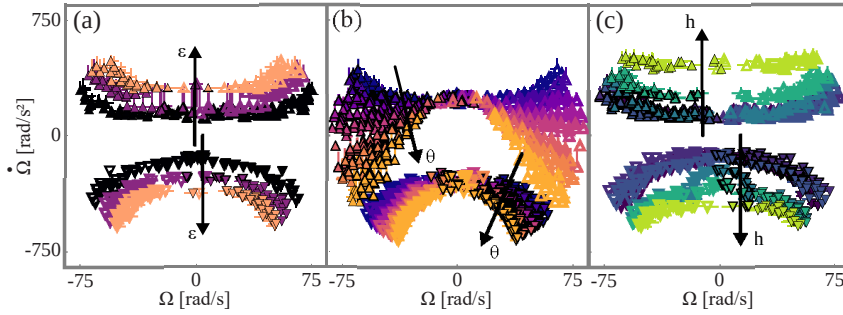


FIGURE A5.3: **Experimental mapping of transition curves across geometric parameters.** Each panel shows the observed up transitions ($\triangle$) and down transitions (∇) as a function of one of the design parameters. (a) Strain $\varepsilon \in \{0.013, 0.037, 0.062\}$, with fixed tilt angle $\theta = 1.5^\circ$ and beam thickness $h = 1.9$ mm. (b) tilt angle $\theta = \{1.5, 3.0, 4.5, 6.0, 7.5, 9.0, 10.5, 12.0\}^\circ$, with fixed strain $\varepsilon = 0.037$ and beam thickness $h = 1.9$ mm. (c) Beam thickness $h = \{1.6, 1.8, 1.9, 2.1, 2.4, 2.7\}$ mm, with fixed strain $\varepsilon = 0.037$ and tilt angle $\theta = 1.5^\circ$.

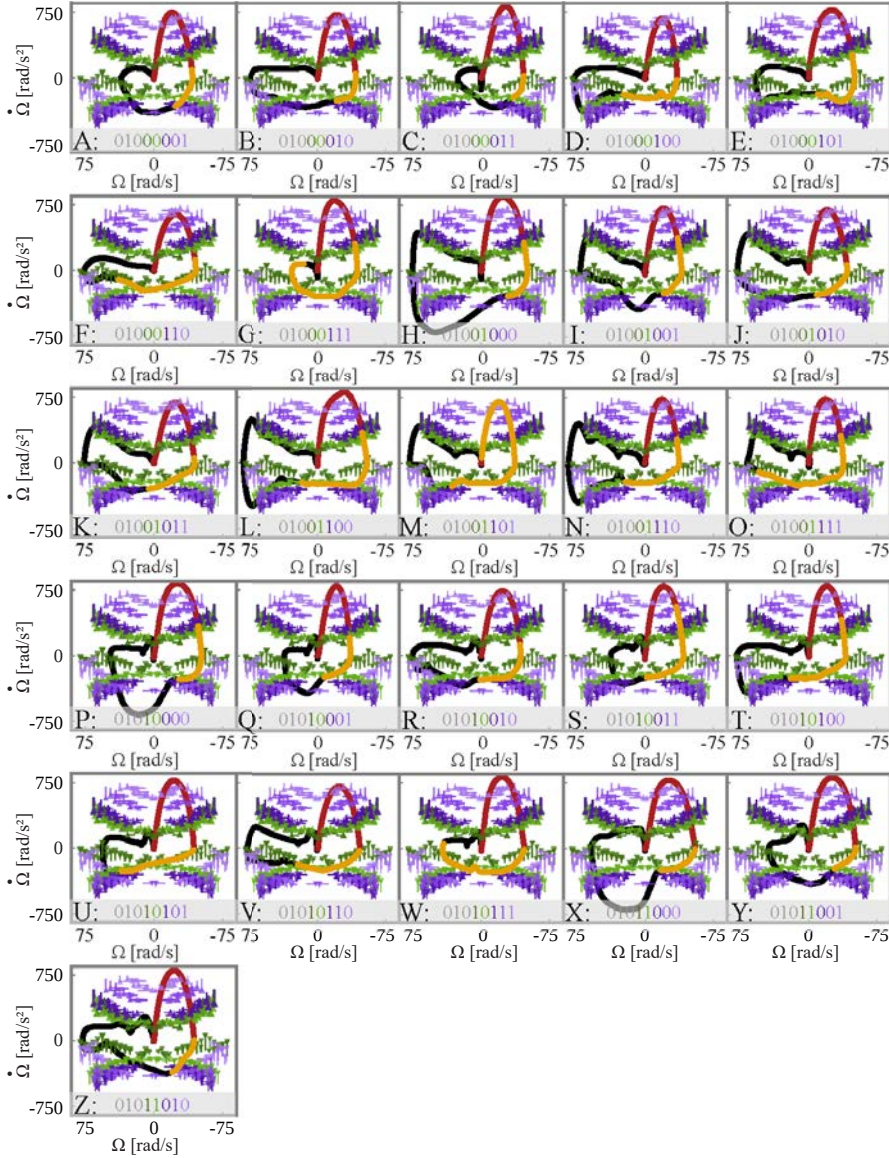


FIGURE A5.4: **Library of selected writing orbits.** All 26 selected orbits of system II encode the uppercase Latin character (A–Z) using the ASCII encoding standard [156]. Although ASCII requires eight bits, the first three bits are identical for all uppercase characters; hence we write the remaining five bits. Below each orbit, the full bit string is shown, with colors labeling the mbits, and gray indicating the three shared bits. The orbits are color-coded, with red indicating erase phase one; gold, erase phase two; and black the writing phase.

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Summary

Chapter one

Materials are often thought of as passive objects that simply deform when pressed or pulled. Yet certain complex materials behave in far more surprising ways: they not only react to applied forces but can also *remember* the history of those forces. Such memory effects set them apart from ordinary matter and make them fundamentally different. Even familiar materials like rubber or certain types of metals can display such history-dependent behavior, hinting at a much broader and richer class of physics.

Chapter two

Certain complex materials do not just react to being pushed or squeezed, but remember the order and location of those pushes. In chapter two, we explore happens when such materials are pressed at multiple locations in different sequences. Using a chain-like metamaterial as a model system, we show that the response depends not only on where and how much strain is applied, but also on the precise sequence of actions. For example, pressing two spots in one order can lead to a completely different final shape than pressing them in the reverse order (a “non-Abelian” response). Or the material respond smoothly in one sequence but suddenly snaps along another (“mixed-mode” response). To explain this, we identify basic building blocks of these responses, called *ancestor–descendant–sibling* triplets, and use graph-based maps to capture and explain these path-dependent memory effects. In addition, we introduce a method to simplify and tame the overwhelming complexity that arises when many inputs are applied, making the system easier to interpret, classify, and potentially design.

Beyond physics, we show that this path-dependent behavior can act as a form of natural computation. By treating pushes as “inputs” and the resulting states as “outputs”, the material can perform basic logic operations and store information, much like a computer circuit. Remarkably, a single sample can host multiple possible “circuits” that can be switched by adjusting the strength or location of the inputs. This suggests that complex materials could be used for new forms of computing directly inside matter (*in-materia computing*) while

also shedding light on how the nonlinear response of complex materials.

Chapter three

When materials are repeatedly and cyclically deformed, they typically settle into a simple repeating pattern known as a *limit cycle*. Yet some disordered or frustrated materials exhibit more exotic behaviors, such as long *transients* (in which many deformation cycles occur before a limit cycle is reached) or *multi-periodic* responses (where the repeating pattern itself spans multiple deformation cycles). These behaviors are poorly understood, remain difficult to model, and are challenging to reproduce in designed materials. To address this, we introduce *togglerons*—a fundamental unit that inherently produces multi-periodic responses with period two. We show that ensembles of interacting togglerons can produce long transients and large periods including odd ones. We argue that togglerons belong to a broader hierarchy of building blocks, alongside spins and hysterons, which together provide a framework for understanding, modeling, and designing these exotic behaviors. This hierarchical approach opens new strategies for creating smart materials, with potential applications in soft robotics, adaptive structures, and mechanical computing.

Chapter four

Snap-through mechanisms are fundamental in nature, technology, and even for the togglerons. For buckled beams, snapping is often triggered by a single transverse pusher that must cross the beam's longitudinal axis, limiting how and when the instability occurs. In this chapter, we introduce a dual-tip transverse pusher that enables *early-onset snapping*, where snap-through is initiated before the pusher reaches the longitudinal axis. We employ a modal expansion approach to capture and analyze this new class of snap-through behavior that is crucial for the togglerons of chapter three. Finally, we demonstrate how to maximize the distance between the pusher and the beam's centerline at the moment of snapping. Together, our work opens new avenues for qualitative and quantitative control over the snapping of buckled beams, with potential applications in mechanical sensors, actuators, and metamaterials.

Chapter five

Massively multistable materials can store vast amounts of information and are often represented as collections of bistable elements, or *material bits*. Independent control over these bits is crucial, yet current methods rely on either local, individual access to each bit or slow, multi-step global protocols. In this chapter, we introduce *dynamic control*, a strategy that achieves independent bit manipulation through a single global drive, using one overall input and the rate at which that input changes. We demonstrate this on a rotating mechanical platform, where five material bits are sufficient to encode all 26 letters of

the alphabet through simple single-cycle protocols. Beyond mechanics, this approach extends to fluids, electronics, and soft robotics, paving the way for smart, remotely programmable devices.

Samenvatting

Hoofdstuk één

Materialen worden vaak gezien als passieve objecten die eenvoudig vervormen wanneer ze worden ingedrukt of uitgerekt. Toch gedragen sommige complexe materialen zich op veel verrassendere manieren: ze reageren niet alleen op krachten, maar kunnen ook op de *geschiedenis* van die krachten. Zulke geheugeneffecten onderscheiden ze fundamenteel van gewone materialen. Zelfs vertrouwde materialen zoals rubber of bepaalde metalen kunnen dergelijk geschiedenisafhankelijk gedrag vertonen, wat laat zien dat de fysica heel rijk en breed is.

Hoofdstuk twee

Sommige complexe materialen reageren niet alleen op duwen of samendrukken, maar onthouden ook de volgorde en locatie van die actuaties. In dit hoofdstuk onderzoeken we wat er gebeurt wanneer zulke materialen op meerdere plaatsen en in verschillende volgordes worden geactueerd. Met behulp van een chain-like metamateriaal als modelsysteem laten we zien dat de respons niet alleen afhangt van waar en hoeveel vervorming wordt aangebracht, maar ook van de precieze volgorde. Het indrukken van bijvoorbeeld twee punten in een bepaalde volgorde kan leiden tot een volledig andere vorm dan wanneer dezelfde actuaties in omgekeerde volgorde wordt uitgevoerd (een *non-Abelian response*). Daarnaast kan het materiaal in de ene volgorde geleidelijk vervormen, terwijl het in een andere volgorde plotseling ‘popt’ (een *mixed-mode response*).

Om zulk actuaties-padafhankelijk gedrag te verklaren, identificeren we fundamentele bouwstenen van deze responsen, de zogenoemde *ancestor-descendant-sibling* triplets. Daarnaast maken we gebruik van graafgebaseerde representaties om deze padafhankelijke geheugeneffecten te beschrijven. Verder introduceren we een methode om de overweldigende complexiteit die ontstaat bij veel inputs te vereenvoudigen en beheersbaar te maken, waardoor het systeem beter te interpreteren, classificeren en ontwerpen is.

Naast de fysica laten we zien dat dit padafhankelijke gedrag kan functioneren als een vorm van natuurlijke berekening. Door actuaties als “inputs”

en de resulterende toestanden als “outputs” te beschouwen, kan het materiaal elementaire logische operaties uitvoeren en informatie opslaan, vergelijkbaar met een computercircuit. Opmerkelijk is dat één enkel materiaal meerdere mogelijke “circuits” kan bevatten, die kunnen worden omgeschakeld door de sterkte of locatie van de input aan te passen. Dit suggereert dat complexe materialen gebruikt kunnen worden voor nieuwe vormen van berekening direct in materie (*in-materia computing*) en werpt tegelijkertijd licht op hoe zulke materialen informatie opslaan en verwerken.

Hoofdstuk drie

Wanneer materialen herhaaldelijk en cyclisch worden vervormd, komen ze doorgaans terecht in een eenvoudig herhalend patroon, bekend als een *limit cycle*. Sommige wanordelijke of gefrustreerde materialen vertonen echter exotischer gedrag, zoals lange *transients* (waarbij vele vervormingscycli plaatsvinden voordat een *limit cycle* wordt bereikt) of *multi-periodic limit cycles* (waarbij het herhalende patroon zelf uit meerdere cycli bestaat). Deze gedragingen zijn nog slecht begrepen, moeilijk te modelleren en lastig te reproduceren in ontworpen materialen. Om dit aan te pakken introduceren we *togglers*—een fundamentele eenheid die intrinsiek multi-periodiek is met periode twee. We laten zien dat collecties van interacterende *togglers* lange transients en grotere periodes kunnen vertonen (ook periode drie). We betogen dat *togglers* deel uitmaken van een bredere hiërarchie van bouwstenen, naast spins en hystérons, die samen een kader vormen voor het begrijpen, modelleren en ontwerpen van deze exotische gedragingen in gefrustreerde materialen. Deze hiërarchische benadering opent nieuwe strategieën voor het creëren van slimme materialen, met mogelijke toepassingen in zachte robotica, adaptieve structuren en mechanische berekening.

Hoofdstuk vier

Snap-through-mechanismen zijn fundamenteel in de natuur, technologie en ook voor *togglers*. Bij geknikte balken wordt *snap-through* vaak veroorzaakt door een enkele duwer die de lengteas van de balk moet kruisen, wat beperkt hoe en wanneer de instabiliteit optreedt. In dit hoofdstuk introduceren we een duwer met twee contactpunten die *snap-through* eerder mogelijk maakt, waarbij de instabiliteit optreedt voordat de duwer de lengteas bereikt. Deze eigenschap is cruciaal voor de *togglers* van hoofdstuk drie. We gebruiken een modale-expansiebenadering om deze nieuwe klasse van *snap-through*-mechanismen te analyseren. Vervolgens laten we zien hoe de afstand tussen de duwer en de middellijn van de balk op het moment van *snap-through* kan worden gemaximaliseerd. Samen openen deze resultaten nieuwe mogelijkheden voor kwalitatieve en kwantitatieve controle over het *snap*-gedrag van geknikte balken, met potentiële toepassingen in mechanische sensoren, actuatoren en

metamaterialen.

Hoofdstuk vijf

Sterk multistabiele materialen kunnen enorme hoeveelheden informatie opslaan en worden vaak gemodelleerd als een verzameling bistabiele elementen, oftewel *material bits*. Onafhankelijke controle over deze bits is cruciaal, maar bestaande methoden vereisen ofwel lokale en individuele toegang tot elk bit, of trage, meerstaps globale protocollen. In dit hoofdstuk introduceren we daarom *dynamic control*, een strategie die onafhankelijke manipulatie van bits mogelijk maakt via één enkele globale aandrijving, waarbij zowel de invoer als de snelheid waarmee deze verandert worden benut. We demonstreren dit op een roterend mechanisch platform, waarbij vijf material bits voldoende zijn om alle 26 letters van het alfabet te coderen met behulp van eenvoudige protocollen van één cyclus. Naast mechanische systemen is deze aanpak ook toepasbaar op vloeistoffen, elektronica en zachte robotica, en biedt daarmee mogelijkheden voor slimme, op afstand programmeerbare systemen.

Publications

C.M. Meulblok, A. Singh, M. Labousse, M. van Hecke, *Path-dependency and emergent computing under vectorial driving*, Physical Review X, **16**, 031021 (2026)

C.M. Meulblok, M. van Hecke, *Transient and multiperiodic responses: a hierarchy of material bits*, New Journal of Physics **28**, 025002 (2026)

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Curriculum Vitae

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Acknowledgments

After four transformative years, this dissertation is complete. A journey marked by lessons drawn from challenges and experiences. Therefore, to whom it may concern, to those seen and unseen – *thank you*.

Martin, you have been a true mentor. Your ability to conceptualize complex ideas and translate them into clear, elegant insights has been both inspiring and formative. It has shaped the way I think, question, and approach problems. Thank you for being a listener, source of humor, and trusted advisor throughout.

I extend my appreciation for my collaborators, peers, and friends for their team spirit and support.

Finally, to my deer, I couldn't have done it without you. Your love, patience and support kept me going. Thank you for standing by my crazy side, celebrating the successes, and comforting me through moments of doubt.