Probabilistic graph inspections through forests

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CHAPTER 3

Wilson's occupation field along coupled Kirchhoff forests

This chapter is based on joint work in progress with L. Avena and A. Gaudillière.

3.1 Introduction

Wilson's algorithm is a celebrated procedure, that uses loop-erased random walks to efficiently sample from the well known Uniform Spanning Tree measure, and from the Kirchhoff forest measure. The latter is a distribution on the spanning rooted forests of a given weighted directed graph, and can be seen as a parametric generalization of the Uniform Spanning Tree measure. The number of components (i.e. trees) in the resulting Kirchhoff forest can be tuned by adjusting the intensity parameter $q > 0$ of the Kirchhoff forest measure. For large values of q a Kirchhoff forest will consist of many small trees, while for small q the measure will concentrate on rooted forest with few components. In particular, the Uniform Spanning Tree is recovered in the limit $q \to 0$.

In [6] it is shown that Wilson's algorithm can be used to couple together a continuum of Kirchhoff forests for all possible intensities $q \in (0, \infty)$. By reparametrizing $t := 1/q$ this coupling constructs a Markov process indexed by time t, such that for each time t the time marginal is a Kirchhoff forest of intensity $1/t$. A partial trajectory of this process, with $t \in [0, t_{\max}]$, can be sampled with approximately the same efficiency as sampling a single Kirchhoff forest of intensity $1/t_{\max}$. Recently, this forest coupling has been applied for estimating the spectrum of a graph's Laplacian matrix [9].

This coupling of realizations of Wilson's algorithm also couples together the associated occupation fields, which are obtained from the loops that are removed during Wilson's procedure. Thus is constructed a stochastic process of occupation fields, which will be called *(Wilson's) occupation field process*, defined in section 3.2.1 below, and which is the object of interest in the present chapter.

The occupation field of Wilson's algorithm is of independent interest, as it shows remarkable connections to Poissonian loop-ensembles (or random walk loop soups) and to the discrete Gaussian free field. These connections were first explored by Le Jan [55].

Outline

This chapter is organized as follows. In section 3.1.1 we will introduce the setting of this chapter. A detailed description of Wilson's algorithm will be provided in section 3.1.2. This description is used in section 3.2 to define the Kirchhoff forest coupling and the associated occupation field process.

In section 3.3 we state the results of this chapter. The main result, theorem 3.6, will give a complete description of the law of the occupation field process. A direct consequence of this theorem, corollary 3.6.1, shows how the connections between the occupation field of Wilson's algorithm and random walk loop soup can be extended to this dynamic setting. The proofs of these results will be deferred to section 3.4.

3.1.1 Setting

Given are a finite set $\mathcal{X}$ of size $n := |\mathcal{X}|$, and an arbitrary *Laplacian matrix* $L = (L(x,y))_{x,y\in\mathcal{X}} \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$, i.e. diagonal entries of L are non-negative, its off-diagonal entries are non-positive, and all of its row sums equal zero. Together the set $\mathcal{X}$ and the Laplacian matrix L form the fixed inputs of our model.

On the set $\mathcal{X}$ we define a discrete-time Markov chain $X = (X_k)_{k\in\mathbb{N}_0}$ with transition matrix $I - \frac{1}{\delta}L$, where I is the identity matrix, and δ is an auxiliary parameter that needs to satisfy

$$\delta \geq \max_{x\in\mathcal{X}} L(x,x) \tag{3.1}$$

for the transition matrix to be well-defined. This parameter δ determines the 'lazyness' of the discrete-time Markov chain, so for larger values of δ the chain is more likely to stay in the same state at each time-step.

The transition matrix $I - \frac{1}{\delta}L$ defines a weighted directed graph $\mathcal{G} = (\mathcal{X}, \mathcal{E}, w)$ with vertices $\mathcal{X}$. The weight function $w : \mathcal{X}\times\mathcal{X} \to [0,\infty)$ is defined by

$$w(x,y) := (\delta I - L)(x,y), \tag{3.2}$$

and the directed edge set is given by

$$\mathcal{E} := \{(x,y) \in \mathcal{X}\times\mathcal{X} \colon\ w(x,y) > 0\}. \tag{3.3}$$

Note that the graph $\mathcal{G}$ has self-loops at vertices x with $\sum_{y\in\mathcal{X}\setminus\{x\}} w(x,y) < \delta$. We further introduce the notation $w(E) := \prod_{(x,y)\in E} w(x,y)$ for the weight of a set of edges $E \subseteq \mathcal{E}$. The Markov chain X will be called the *random walk* on $\mathcal{G}$.

Of interest are the *(spanning) rooted forests* of the graph $\mathcal{G}$, which are subsets $F \subseteq \mathcal{E}$ of directed edges such that:

(*i*) each vertex has at most one outgoing edge in F;

(*ii*) F does not contain any undirected cycles.

The *roots* of a rooted forests F are those vertices that do not have an outgoing edge in F. We denote by $\rho(F)$ the set of roots of F, and by $r(F) := |\rho(F)|$ the number of roots. The set of all rooted forests of the graph is denoted by $\mathcal{F}$.

A *Kirchhoff forest* of intensity $q > 0$ is an $\mathcal{F}$-valued random variable Φ_q with distribution

$$\mathbb{P}(\Phi_q = F) := \tfrac{1}{Z(q)} q^{r(F)} w(F), \tag{3.4}$$

where $Z(q) := \sum_{F\in\mathcal{F}} q^{r(F)} w(F)$ denotes the normalizing partition function.

3.1.2 Wilson's algorithm

Wilson's algorithm is a method for sampling Kirchhoff forests, that is not only of interest from an applied point of view, but is also a useful tool for the theoretical analysis of Kirchhoff forests.

A brief description of Wilson's procedure can be given as follows. Define a killed random walk on the graph. Pick an arbitrary vertex and run a killed random walk

starting from that vertex. Each time the random walk makes a cycle, the edges in the cycle are removed from the trajectory of the random walk, to obtain the loop-erased trajectory.
Then repeatedly pick a new vertex that has not been picked before and run a random walk until either is killed or it hits a vertex in the loop-erased trajectory of any of the previous random walks, and continue until all vertices have been picked. The union of all directed edges in the loop-erased trajectories of the random walks now form a Kirchhoff forest. The intensity of the Kirchhoff forest can be tuned by adjusting the killing time of the random walk.

To define the coupled forest process and the associated occupation field process we require a more detailed description of Wilson's procedure. In fact, we will give two distinct, albeit equivalent, such descriptions. The first is given in section 3.1.2.2 and utilizes loop-erased random walks, while the second, given in section 3.1.2.4, uses a 'cycle popping' procedure based on the Diaconis-Fulton stack representation of a random walk.

We note that Wilson's method depends only indirectly on the law of the used random walks, as only the law of the loop-erased trajectories are relevant. As there are multiple killed random walks of which the distributions of their loop-trajectories coincide, there is some freedom in the choice of random walk. The descriptions of Wilson's algorithm given below, will use the random walk X defined in section 3.1.1 above. This specific choice of random walk will be elucidated in section 3.2.1.1.

3.1.2.1 Notation for walks on graphs

Let $\mathcal{P}$ denote the set of all finite length *walks* in $\mathcal{G}$, i.e.

$$\mathcal{P} := \bigcup_{l\in\mathbb{N}_0} \{(x_0, x_1, \dots, x_l) \in \mathcal{X}^{l+1} \colon\ (x_{i-1}, x_i) \in \mathcal{E} \text{ for all } i \in [l]\}, \tag{3.5}$$

where we use the notation $[l] := \{1, 2, \dots, l\}$ for the set containing the first l positive integers. Elements of $\mathcal{P}$ will commonly be denoted by $\gamma = (x_0, x_1, \dots, x_l)$. For $\gamma \in \mathcal{P}$ its *length* is the unique non-negative integer $l \in \mathbb{N}_0$ for which $\gamma \in \mathcal{X}^{l+1}$. So in particular single vertices are walks of zero length. We introduce the notations

$$s(\gamma) := \{x_0, x_1, \dots, x_l\}, \text{ and } e(\gamma) := \{(x_{i-1}, x_i) \in \mathcal{E} \colon\ i \in [l]\} \tag{3.6}$$

for the support of γ, and the set of edges traversed by γ, respectively. If γ has length zero, then its set of traversed edges is empty. A walk $(x_0, x_1, \dots, x_l) \in \mathcal{P}$ with length $l \geq 1$ for which $x_0 = x_l$ is called a *closed walk* or *cyclic walk*. The set of closed walks is denoted by $\mathcal{P}^{\mathrm{cl}}$. A *self-avoiding walk* is a walk for which $x_i \neq x_j$ for all distinct $i, j \in [l]_0 := \{0, 1, \dots, l\}$.

Define an an equivalence relation $\simeq$ on the set of closed walks $\mathcal{P}^{\mathrm{cl}}$ as follows. For $\gamma_1, \gamma_2 \in \mathcal{P}^{\mathrm{cl}}$ with $\gamma_1 = (x_0, \dots, x_l)$ and $\gamma_2 = (y_0, \dots, y_l)$ we call γ_1 and γ_2 equivalent if there exists a cyclic permutation $\sigma : [l-1]_0 \to [l-1]_0$ such that $x_{\sigma(i)} = y_i$ for all $i \in [l-1]_0$.

We let γ° denote an element of the quotient set $\mathcal{P}^{\mathrm{cl}}/\simeq$. If γ° has a representative $\gamma = (x_0, \dots, x_l)$ for which $\gamma^- := (x_0, \dots, x_{l-1})$ is self-avoiding, then we call γ° an

(unbased) cycle. So, cycles can be seen as self-avoiding closed walks without a specified starting point. For any two representatives $\gamma_1, \gamma_2 \in \gamma^\circ$ it holds for their supports and traversed edges that $s(\gamma_1) = s(\gamma_2)$ and $e(\gamma_1) = e(\gamma_2)$. Hence, we can define the support of γ° by $s(\gamma^\circ) := s(\gamma_1)$ and the traversed edges of γ° by $e(\gamma^\circ) := e(\gamma_1)$.

The *occupation field* of a walk $\gamma = (x_0, x_1, \dots, x_l)$ is the map $\ell[\gamma] : \mathcal{X} \to \mathbb{N}_0$ defined by

$$\ell[\gamma](x) := \sum_{k=0}^{l} \mathbf{1}\{x_k = x\}, \tag{3.7}$$

where $\mathbf{1}$ denotes the indicator function. That is, $\ell[\gamma](x)$ denotes the *local time* spent by the walk γ at vertex x.

For each walk $\gamma = (x_0, x_1, \dots, x_l)$ we will define a self-avoiding walk called its *loop-erasure*. Iteratively define a sequence of self-avoiding walks $(\gamma_i)_{0 \le i \le l}$ as follows. Set $\gamma_0 := x_0$. For each $i \in [l]$, given $\gamma_{i-1} = (y_0, \dots, y_m)$ we define

$$\gamma_i := \begin{cases} (y_0, y_1, \dots, y_{m-1}, y_m, x_l) & \text{if } x_i \notin s(\gamma_{i-1}) \\ (y_0, y_1, \dots, y_{k_i - 1}, y_{k_i}) & \text{if } x_i \in s(\gamma_{i-1}), \end{cases} \tag{3.8}$$

where $k_i := \inf\{j \in [m]_0 \colon\ y_j = x_i\}$. The *loop-erasure* of γ is defined as

$$\mathrm{LE}[\gamma] := \gamma_l. \tag{3.9}$$

3.1.2.2 Wilson's algorithm using loop-erased random walks

Equip $\mathcal{X}$ with an arbitrary ordering, $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$. Let $X^{(1)}, X^{(2)}, \dots, X^{(n)}$ be independent copies of the random walk X, with each $X^{(i)}$ starting from vertex x_i. Let $T^{(1)}, T^{(2)}, \dots, T^{(n)}$ be i.i.d. $\mathbb{N}_0$-valued[1] geometric killing times with success parameter $\frac{q}{q+\delta}$, that is $\mathbb{P}(T^{(i)} = k) = \left(\frac{\delta}{q+\delta}\right)^k \frac{q}{q+\delta}$. Iteratively define a set of vertices $V^{(i)}$ and a loop-erased random walk $\Gamma^{(i)}$ as follows. Set $V^{(0)} := \varnothing$. For each $i \in [n]$ write

$$\tau^{(i)} := T^{(i)} \wedge \inf\{k \in \mathbb{N}_0 \colon\ X_k^{(i)} \in V^{(i-1)}\} \tag{3.10}$$

to denote the minimum of $T^{(i)}$ and the first hitting time of $V^{(i-1)}$ by $X^{(i)}$, and define

$$\Gamma^{(i)} := \mathrm{LE}\left[(X_k^{(i)})_{0 \le k \le \tau^{(i)}}\right], \text{ and } V^{(i)} := V^{(i-1)} \cup s(\Gamma^{(i)}). \tag{3.11}$$

Theorem 3.1 (Wilson [77]). *The set of edges*

$$\bigcup_{i \in [n]} e(\Gamma^{(i)}) \tag{3.12}$$

is a Kirchhoff forest of intensity q.

While theorem 3.1 shows that the law of the rooted forest obtained by Wilson's procedure does not depend on the chosen vertex ordering, conditionally on the random walks $X^{(1)}, X^{(2)}, \dots, X^n$ the realization of the rooted forest does.

[1] In this chapter we adopt the convention that all geometric random variables are supported on $\mathbb{N}_0$.

3.1.2.3 Diaconis-Fulton stack representation

Diaconis and Fulton introduced an alternative method to index the randomness of a Markov chain [21]. Rather than using time as the index variable, they showed how a Markov chain can be constructed from space indexed randomness. It will be useful to use their representation for the random walks employed in Wilson's procedure, as doing so provides us with an alternative perspective on the entire procedure.

For each $x \in \mathcal{X}$ let $(A_i(x))_{i\in\mathbb{N}_0}$ be an independent sequence of i.i.d. $\mathcal{X}$-valued random variables with law

$$\mathbb{P}(A_i(x) = y) := \tfrac{1}{\delta} w(x, y), \quad \text{for all } y \in \mathcal{X}. \tag{3.13}$$

The random walk X can be constructed from the collection $\{(A_i(x))_{i\in\mathbb{N}_0} \colon\ x \in \mathcal{X}\}$, by iteratively setting

$$X_{k+1} := A_{i_k}(X_k), \quad \text{with } i_k := \ell[(X_j)_{0\le j\le k-1}](X_k), \tag{3.14}$$

where we use the convention that $\ell[\varnothing] := \underline{0}$, so that $i_0 = 0$.

One can imagine that an infinite stack of arrows $(A_i(x))_{i\in\mathbb{N}_0}$ is attached underneath each vertex x, with arrow $A_i(x)$ directly on top of arrow $A_{i+1}(x)$. Whenever the random walk visits a vertex, it reads the current top arrow from the stack of that vertex to determine its next step, after which that arrow is deleted from the stack.

We further introduce a collection $\{(B_i(x))_{i\in\mathbb{N}_0} \colon\ x \in \mathcal{X}\}$, where all $B_i(x)$ are i.i.d $\mathrm{Ber}(\frac{q}{q+\delta})$ random variables independent of the arrows. The variable $B_i(x)$ should be interpreted as the random color of arrow $A_i(x)$, where outcome 1 represents that the arrow is red and outcome 0 represents that the arrow is green. These colors are used to define the geometric killing time of the random walk. Whenever the random walk reads a green arrow, it makes the jump indicated by the arrow. But if it reads a red arrow, then it is killed instead.

3.1.2.4 Wilson's algorithm using cycle popping

Equip the set of unbased cycles with an arbitrary well-ordering. Set $\mathbf{d}_0 := \underline{0}$, where $\underline{0} \in \mathbb{N}_0^{\mathcal{X}}$ denotes the all-0 vector. For $i \in \mathbb{N}_0$ we iteratively define a random set of edges $E_i := \{(x, A_{\mathbf{d}_i(x)}(x)) \in \mathcal{E} \colon\ x \in \mathcal{X},\ B_{\mathbf{d}_i(x)}(x) = 0\}$ and let $\mathcal{P}_i^{\mathrm{cl}} := \{\gamma \in \mathcal{P}^{\mathrm{cl}} \colon\ e(\gamma) \subseteq E_i\}$ denote the set of closed walks whose edges are contained in E_i. Define the random cycle Γ_i° to be the minimal element of $\mathcal{P}_i^{\mathrm{cl}}/\simeq$ whenever $\mathcal{P}_i^{\mathrm{cl}}$ is non-empty, and set $\Gamma_i^\circ := \varnothing$ otherwise. Then we define $\mathbf{d}_{i+1} := \mathbf{d}_i + \mathbf{1}_{s(\Gamma_i^\circ)}$, where $\mathbf{1}_{s(\Gamma_i^\circ)} \in \{0,1\}^{\mathcal{X}}$ denotes the indicator of the support of Γ_i°. That is, at each iteration i one cycle is deleted (or 'popped') from the top of the DF-stacks, and $\mathbf{d}_i(x)$ denotes the number of deleted arrows from the stack at vertex x until iteration i.

Write $i^* := \min\{i \in \mathbb{N}_0 \colon\ \Gamma_i^\circ = \varnothing\}$ to denote the time-step when the procedure terminates.

Theorem 3.2 (Wilson, [77]). *The set of edges E_{i^*} is a Kirchhoff forest of intensity q.*

While the cycle popping procedure makes use of an ordering to choose a cycle, the realization of the Kirchhoff forest does not depend on the chosen ordering. We denote the rooted forest E_{i^*} obtained from the stacks $\{(A_i(x), B_i(x))_{i\in\mathbb{N}_0} : \ x \in \mathcal{X}\}$ by

$$\text{CyclePopping}(\{(A_i(x), B_i(x))_{i\in\mathbb{N}_0} : \ x \in \mathcal{X}\}) := E_{i^*}. \tag{3.15}$$

3.2 Coupled forests and their occupation fields

The intensity parameter q of the Kirchhoff forest obtained by cycle popping the stacks $\{(A_i(x), B_i(x))_{i\in\mathbb{N}_0} : \ x \in \mathcal{X}\}$ is determined solely by the parameter of the Bernoulli distribution of the colors. Rather than giving each arrow a single color, we can give a dynamic color $(B_i^t(x))_{t\geq 0}$ to each arrow that changes as time progresses. We will use a common method to couple together Bernoulli variables with different parameters, by making them depend on a single uniform random variable.

Consider a collection $\{(U_i(x))_{i\in\mathbb{N}_0} : \ x \in \mathcal{X}\}$, where all $U_i(x) \sim \text{Unif}(0,1)$ are i.i.d random variables, independent of the arrows $\{(A_i(x))_{i\in\mathbb{N}_0} : \ x \in \mathcal{X}\}$. For each $x \in \mathcal{X}$, $i \in \mathbb{N}_0$ and $t \geq 0$ we define a Bernoulli random variable

$$B_i^t(x) := \mathbf{1}\{U_i(x) < \tfrac{1}{1+\delta t}\}. \tag{3.16}$$

Definition 3.2.1. The *coupled forest process* $(\Phi_{1/t})_{t\geq 0}$ is the $\mathcal{F}$-valued stochastic process that is obtained from $\{(A_i(x), U_i(x))_{i\in\mathbb{N}_0} : \ x \in \mathcal{X}\}$ by defining

$$\Phi_{1/t} := \text{CyclePopping}(\{(A_i(x), B_i^t(x))_{i\in\mathbb{N}_0} : \ x \in \mathcal{X}\}). \tag{3.17}$$

■

Theorem 3.3 (Avena & Gaudillière [6]). *The coupled forest process $(\Phi_{1/t})_{t\geq 0}$ has the following properties:*

(*i*) *for fixed $t > 0$ the marginal $\Phi_{1/t}$ is a Kirchhoff forest of intensity $1/t$;*

(*ii*) *the process $(\Phi_{1/t})_{t\geq 0}$ has càdlàg and piece-wise constant trajectories;*

(*iii*) *the process $(\Phi_{1/t})_{t\geq 0}$ satisfies the Markov property.*

The coupled forest process starts, at time $t = 0$, consisting of only isolated vertices. At random times, exactly one of the current roots 'wakes up'. When that happens either its tree coalesces onto another tree, or its tree fragments into smaller trees, that each have the possibility to coalesce onto other trees.

The coalescing dynamic is more prominent than the fragmentation, so that as time progresses in expectation the number of trees decreasses.

3.2.1 The occupation field process

Let Φ_q be a Kirchhoff forest of intensity q obtained using the cycle popping procedure. Using the notation of section 3.1.2.4, we define the *occupation field (of Wilson's algorithm)* $\ell[\Phi_q]$ as the $\mathbb{N}^{\mathcal{X}}$-valued random variable

$$\ell[\Phi_q](x) := \mathbf{d}_{i^*}(x) + 1. \tag{3.18}$$

If Φ_q is constructed using loop-erased random walks, i.e. $\Phi_q := \bigcup_{i\in[n]} e\left(\Gamma^{(i)}\right)$, then we equivalently have that

$$\ell[\Phi_q](x) = \mathbf{1}\{x \in \rho(\Phi_q)\} + \sum_{i=1}^{n} \ell[(X_k^{(i)})_{0\le k\le \tau^{(i)}-1}](x). \tag{3.19}$$

That is, the Wilson occupation field equals the sum of the occupation fields of the stopped random walks used in the procedure, where we only count the local time contribution of the final step of random walk $X^{(i)}$ if $X^{(i)}$ is killed before it hits any of the trajectories of the previous random walks.

The notation $\ell[\Phi_q]$ might incorrectly suggest that the occupation field is an observable of the Kirchhoff forest Φ_q. Therefore, we emphasize that the occupation field is constructed from a realization of Wilson's algorithm and not just from the resulting forest.

Although not explicitly mentioned as a result, in [77, proof of Thm 1] Wilson used the following observation.

Lemma 3.4 (Wilson [77], *proof of Thm. 1 therein*). *The Kirchhoff forest Φ_q is independent of the occupation field $\ell[\Phi_q]$.*

The construction of the coupled forest process, as given in definition 3.2.1, does not only couple together a family of Kirchhoff forests, but also their Wilson occupation fields. For notational brevity, we denote the occupation field of $\Phi_{1/t}$ by

$$N_t := \ell[\Phi_{1/t}], \tag{3.20}$$

thus defining an *occupation field process* $(N_t)_{t\ge 0}$.

The occupation field is closely related to the *running time* M_t of Wilson's algorithm. The latter being obtained by taking a sum of the occupation field over all vertices,

$$M_t := \sum_{x\in\mathcal{X}} N_t(x). \tag{3.21}$$

Equivalently, the running time equals the total number of edges traversed by the n random walks used in the construction plus the number of roots of the obtained forest. That is, using the notation of section 3.1.2.2

$$M_{1/q} = r(\Phi_q) - n + \sum_{i\in[n]} \tau^{(i)}. \tag{3.22}$$

In [61, prop. 1] Marchal expresses the probability generating function (pgf) of the Wilson running time in terms of the determinant of the transition matrix of the random walk used in Wilson's procedure. Here we apply this result to our specific choice of killed random walk with transition matrix $(I - \frac{1}{\delta}L)$ and killing rate $\frac{q}{q+\delta}$. We denote by $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ the spectrum of the Laplacian L.

Proposition 3.5 (Marchal [61]). *For $z \in (0,1)$ it holds that*

$$\mathbb{E}\left[z^{M_{1/q}}\right] = \frac{\det\left[\frac{q}{q+\delta}I + \frac{1}{q+\delta}L\right] z^n}{\det\left[I - \left(I - \frac{q}{q+\delta}I - \frac{1}{q+\delta}L\right) z\right]} = \prod_{j<n} \frac{\frac{q+\lambda_j}{q+\delta} z}{1 - \left(1 - \frac{q+\lambda_j}{q+\delta}\right) z}. \tag{3.23}$$

If $\lambda_j \in \mathbb{R}$ and $\delta \geq \lambda_j$ for all j, then we recognize each of the n factors as the pgf of an $\mathbb{N}$-valued geometric random variable with success parameter $\frac{q+\lambda_j}{q+\delta}$. Hence, in this case we find that $M_{1/q}$ is the sum of n independent geometrics.

3.2.1.1 Different random walks

As mentioned in section 3.1.2, Wilson's algorithm has some freedom in the choice of the transition matrix of the employed random walks. While the choice of random walk does not affect the law of the resulting rooted forest, the distribution of the occupation field does depend on the choice of random walk. In particular, since the transition matrix $(I - \frac{1}{\delta}L)$ used in this chapter depends on a parameter δ, the occupation field will depend on the parameter δ as well.

It is possible to construct the coupled forest process in a parameter free manner, e.g. by employing the simple random walk with transition matrix $D^{-1}A$, where A denotes the (weighted) adjacency matrix of $\mathcal{G}$ and D is the diagonal matrix such that $L = D - A$. However, the results presented here treat the δ dependent occupation field process.

The choice for the transition matrix $(I - \frac{1}{\delta}L)$ has several advantages. Firstly, the killing time of the random walk can be taken independent of the trajectory of the random walk, which simplifies some of the computations. More importantly, the spectrum of the transition matrix is obtained directly from the spectrum of the Laplacian L, which is relevant since several observables of the occupation field can be expressed in terms of the spectrum of the transition matrix, see e.g. proposition 3.5 above. Hence, using our choice of transition matrix ensures that these observables are expressible in terms of the Laplacian spectrum $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. This enables the possibility of future work to develop Laplacian spectrum estimation procedures based on observables of the occupation field process. However, development of such procedures lies outside of the scope of the present chapter.

3.3 Result: Law of the occupation field process

The result of this chapter is a complete description of the law of the occupation field process $(N_t)_{t\geq 0}$, of which theorem 3.6 and corollary 3.6.1 give two equivalent formulations.

For stating the result it will be convenient to consider three distinct probability spaces. On one probability space we define the DF-stacks $\{(A_i(x), U_i(x))_{i\in\mathbb{N}_0} : x \in \mathcal{X}\}$, consisting of arrows and uniform killing marks, and all random variables coupled thereto, e.g. the occupation field process $(N_t)_{t\geq 0}$. The law of the DF-stacks we denote by $\mathbb{P}$. We will further consider, on a second probability space, the killed random walk $(X_k)_{0\leq k\leq T_t}$, which is killed at an $\mathbb{N}_0$-supported geometric time $T_t \sim \mathrm{Geom}_{\mathbb{N}_0}\left(\frac{1}{1+\delta t}\right)$ that is independent of X, for some parameter $t \geq 0$. Denote by $\mathbf{P}_{\mathrm{unif}}$ the joint law of X and T_t, with X starting from a uniformly chosen vertex, and write $\mathbf{P}_x(\cdot) := \mathbf{P}_{\mathrm{unif}}(\cdot \mid X_0 = x)$. Any other auxiliary random variables, that are coupled to neither the DF-stacks nor to the killed random walk, will be defined on a third space

with law P and accompanying expectation E.

We define the matrix $K_{1/t} := \frac{1}{t}(\frac{1}{t}I + L)^{-1}$, and let $\mathrm{Tr}[\cdot]$ denote the trace operator.

Theorem 3.6. *The occupation field process $(N_t)_{t\geq 0}$ is a Markov process with piecewise constant càdlàg trajectories and distribution*

$$(N_t)_{t\geq 0} \stackrel{d}{=} \left(\underline{1} + \int_0^t \Lambda_s \ \Psi(\mathrm{d}s)\right)_{t\geq 0}, \tag{3.24}$$

where for all $t > 0$ the variables Λ_t are independent $\mathbb{N}_0^{\mathcal{X}}$-valued random variables with law

$$P(\Lambda_t = \mathbf{m}) := \mathbf{P}_{\mathrm{unif}}(\ell[(X_k)_{0\leq k\leq T_t}] = \mathbf{m} \mid X_{T_t+1} = X_0), \quad \textit{for all } \mathbf{m} \in \mathbb{N}_0^{\mathcal{X}}, \tag{3.25}$$

and Ψ is an inhomogeneous Poisson point process on $(0,\infty)$ with intensity measure

$$\mu((a,b]) := \int_a^b \mathrm{Tr}\left[\tfrac{1}{t}(K_{1/t} - \tfrac{1}{1+\delta t}I)\right] \ \mathrm{d}t, \tag{3.26}$$

that is independent of all Λ_t.

According to theorem 3.6, at time $t = 0$ the occupation field N_0 equals 1 at each vertex. As t increases the jump-times of the occupation field process are a Poisson point process with explicit rate

$$\kappa_t := \mathrm{Tr}\left[\tfrac{1}{t}(K_{1/t} - \tfrac{1}{1+\delta t}I)\right]. \tag{3.27}$$

At each of its jump times the occupation field is increased by a random amount whose distribution is that of the occupation field of a killed random walk that is conditioned to make at least one step and to be killed at its uniformly chosen starting point.

Conditional on the event $T_t \geq 1$ the distribution of T_t is the same as the unconditional distribution of $T_t + 1$. This fact is used in theorem 3.6 to simplify notation.

3.3.0.1 Closed walk decomposition

In [55] Le Jan showed that the occupation field of Wilson's algorithm has the same distribution as the occupation field of a Poissonian loop-ensemble. The statement of corollary 3.6.1 is a rephrasing of theorem 3.6, that clarifies how the connection between Wilson's occupation field and the loop-ensemble occupation field extends to our dynamical setting.

Define the random set of *jump times* of the occupation field process by

$$\mathcal{T} := \{t \in (0,\infty) \colon \quad N_t \neq \lim_{s\uparrow t} N_s\}. \tag{3.28}$$

So, $\mathcal{T}$ is distributed as the support of the Poisson point process Ψ.

Rather then seeing Ψ as a single Poisson point process, we can decompose Ψ into multiple Poisson point processes that are associated to the closed walks in $\mathcal{G}$.

Conditionally on these Poisson processes the increment of the occupation field at a jump time τ can then be determined deterministically by observing which of these Poisson point processes contains τ.

For a closed walk $\gamma = (x_0, x_1, \ldots, x_l) \in \mathcal{P}^{\mathrm{cl}}$ write $\gamma^- := (x_0, x_1, \ldots x_{l-1})$ to denote the walk obtained by omitting the last step from γ.

Corollary 3.6.1. *For all closed walks $\gamma \in \mathcal{P}^{\mathrm{cl}}$ let Ψ_γ be independent Poisson point processes with respective intensity measures*

$$\mu_\gamma((a,b]) := w(e(\gamma)) \int_a^b \frac{1}{(1+\delta t)^2} \left(\frac{t}{1+\delta t}\right)^{l-1} \mathrm{d}t. \tag{3.29}$$

Then it holds that

$$(N_t)_{t\geq 0} \overset{d}{=} \left(\underline{1} + \sum_{\gamma \in \mathcal{P}^{\mathrm{cl}}} \ell[\gamma^-]\ \Psi_\gamma((0,t]) \right)_{t\geq 0}. \tag{3.30}$$

For fixed $t > 0$ we can define a measure ν_t on closed walks by

$$\nu_t(\gamma) := \mu_\gamma([0,t)) = w(e(\gamma)) \int_0^t \frac{1}{(1+\delta s)^2} \left(\frac{s}{1+\delta s}\right)^{l-1} \mathrm{d}s = \frac{1}{l} \frac{w(e(\gamma))}{(1/t+\delta)^l}. \tag{3.31}$$

The measure ν_t indeed corresponds to intensity measure of the Poissonian loop-ensemble, if the loops in the ensemble are obtained from the random walk with transition matrix $(I - \frac{1}{\delta}L)$ with homogeneous killing rate $\frac{1}{1+\delta t}$, see Le Jan [56, eq (2.4)].

Hence, corollary 3.6.1 shows that the 'loop measure' ν_t can be constructed by taking an integral over different random walk killing rates. This integration further explains the appearance of the factor $\frac{1}{l}$, which for continuous-time random walk ensembles causes the loop measure to explode as the length of the loop approaches 0.

3.4 Proofs

The proof of theorem 3.6 is divided into three parts, which are each subdivided into various lemmas. In the first part it is shown that the occupation field process has independent increments. This fact is then used in the second part to compute the distribution of the random set of jump times of the process. In the third part we compute the occupation field increment distribution at jump times.

3.4.1 Independent increments

The argument that is employed in [6] to demonstrate the Markovianity of the coupled forest process $(\Phi_{1/t})_{t\geq 0}$, also shows that the joint process $(\Phi_{1/t}, N_t)_{t\geq 0}$ is Markovian. More specifically, it shows that, for fixed $0 < s < t$, the distribution of $(\Phi_{1/t},\ N_t - N_s)$ depends on the joint history $(\Phi_{1/r}, N_r)_{0\leq r\leq s}$ only through $\Phi_{1/s}$.

That is, for all $k \in \mathbb{N}$, $0 \leq t_1 < t_2 < \ldots < t_k = s$, $F, F_1, F_2, \ldots, F_k \in \mathcal{F}$, $\mathbf{n} \in \mathbb{N}_0^{\mathcal{X}}$, and all $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_k \in \mathbb{N}^{\mathcal{X}}$ with $\mathbb{P}((\Phi_{1/t_i}, N_{t_i})_{i\in[k]} = (F_i, \mathbf{m}_i)_{i\in[k]}) > 0$, it holds

that

$$\begin{aligned}\mathbb{P}(\Phi_{1/t} = F,\ N_t - N_s = \mathbf{n} \mid (\Phi_{1/t_i}, N_{t_i})_{i\in[k]} = (F_i, \mathbf{m}_i)_{i\in[k]}) \\ = \mathbb{P}(\Phi_{1/t} = F,\ N_t - N_s = \mathbf{n} \mid \Phi_{1/t_k} = F_k).\end{aligned} \tag{3.32}$$

The following lemma shows that the occupation field process $(N_t)_{t\geq 0}$ is itself Markovian as well.

Lemma 3.7. *Both the Wilson occupation field process $(N_t)_{t\geq 0}$ and the running time process $(M_t)_{t\geq 0}$ have independent increments.*

Proof. We will only prove the result for the occupation field process, as the proof of for the running time is analogous. Fix $0 < s < t$. We have to show that the increment of the occupation field $N_t - N_s$ is independent of $(N_r)_{0\leq r\leq s}$.

Consider the joint process $(\Phi_{1/t}, N_t)_{t\geq 0}$, which is constructed from the DF-stacks $\{(A_i(x), U_i(x))_{i\in\mathbb{N}_0}\colon\ x \in \mathcal{X}\}$. In the first part of the proof we show that $\Phi_{1/t}$ is independent of $(N_s)_{0\leq s\leq t}$, for which it is sufficient that the events $\{\Phi_{1/t} = F\}$ and $\bigcap_{i\in[k]}\{N_{t_i} \succeq \mathbf{m}_i + \underline{1}\}$ are independent[2], for arbitrary $k \in \mathbb{N}$, $0 < t_1 < t_2 < \ldots < t_k \leq t$, $F \in \mathcal{F}$ and $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_k \in \mathbb{N}_0^{\mathcal{X}}$. Here $\succeq$ denotes the natural partial order on $\mathbb{N}_0^{\mathcal{X}}$ given by $\mathbf{n} \preceq \mathbf{m}$ if $\mathbf{n}(x) \leq \mathbf{m}(x)$ for all $x \in \mathcal{X}$.

Fix $\mathbf{m} \in \mathbb{N}_0^{\mathcal{X}}$ such that $\mathbf{m} \succeq \mathbf{m}_k$. There exists some subset

$$\mathcal{S}_F \subseteq \prod_{x\in\mathcal{X}} (\{y \in \mathcal{X}\colon\ (x, y) \in \mathcal{E}\} \times [0, 1])$$

such that[3]

$$\{N_t = \mathbf{m} + \underline{1}, \Phi_{1/t} = F\} = \{N_t = \mathbf{m} + \underline{1},\ \{(A_{\mathbf{m}(x)}(x), U_{\mathbf{m}(x)}(x))\colon\ x \in \mathcal{X}\} \in \mathcal{S}_F\}.$$

So, conditionally on the event $\{N_t = \mathbf{m} + \underline{1}\}$, the event $\{\Phi_{1/t} = F\}$ depends only on the variables $\{(A_{\mathbf{m}(x)}(x), U_{\mathbf{m}(x)}(x))\colon\ x \in \mathcal{X}\}$.

For each $i \in [k]$ whether the cycle popping procedure deletes the arrow above layer $\mathbf{m}_i$ does not depend on the arrows in layer $\mathbf{m}_i$ and the arrows below that layer, i.e. the event $\{N_{t_i} \succeq \mathbf{m}_i + \underline{1}\}$ is determined by the variables $\{(A_{i_x}(x), U_{i_x}(x))\colon\ x \in \mathcal{X},\ i_x = 1, 2, \ldots, \mathbf{m}_i(x) - 1\}$. Hence, the intersection $\bigcap_{i\in[k]}\{N_{t_i} \succeq \mathbf{m}_i + \underline{1}\}$ is determined by the variables $\{(A_{i_x}(x), U_{i_x}(x))\colon\ x \in \mathcal{X},\ i_x = 1, 2, \ldots, \mathbf{m}_k(x) - 1\}$. Since $\mathbf{m} \succeq \mathbf{m}_k$, we have in particular that this intersection does not depend on the variables $\{(A_{\mathbf{m}(x)}(x), U_{\mathbf{m}(x)}(x))\colon\ x \in \mathcal{X}\}$.

As this holds for all $\mathbf{m}$, it follows that the events $\bigcap_{i\in[k]}\{N_{t_i} \succeq \mathbf{m}_i + \underline{1}\}$ and $\{\Phi_{1/t} = F\}$ are conditionally independent given N_t. By lemma 3.4, the variables $\Phi_{1/t}$

[2]The addition of the all-ones vector $\underline{1}$ is an administrative trick, that is required since $N_0 = \underline{1}$, while the indices of the DF-stacks start at 0.

[3]Explicitly the subset $\mathcal{S}_F$ is given by

$$\mathcal{S}_F = \left(\prod_{x\in\mathcal{X}\setminus\rho(F)} \{y \in \mathcal{X}\colon\ (x, y) \in F\} \times [\tfrac{1}{1+\delta t}, 1]\right) \times \left(\prod_{x\in\rho(F)} \{y \in \mathcal{X}\colon\ (x, y) \in \mathcal{E}\} \times [0, \tfrac{1}{1+\delta t})\right).$$

and N_t are independent. It follows that $\bigcap_{i\in[k]}\{N_{t_i} \succeq \mathbf{m}_i + \underline{1}\}$ and $\{\Phi_{1/t} = F\}$ are also unconditionally independent, which completes the first part of the proof.

It follows from eq. (3.32) for all $\mathbf{n} \in \mathbb{N}_0^{\mathcal{X}}$ that

$$\mathbb{P}(N_t - N_s = \mathbf{n} \mid \Phi_{1/s},\ (N_r)_{0\le r\le s}) = \mathbb{P}(N_t - N_s = \mathbf{n} \mid \Phi_{1/s}). \tag{3.33}$$

Therefore, the increment $N_t - N_s$ is conditionally independent of $(N_r)_{0\le r\le s}$ given $\Phi_{1/s}$. By the first part of the proof $(N_r)_{0\le r\le s}$ and $\Phi_{1/s}$ are independent, from which it follows that $N_t - N_s$ and $(N_r)_{0\le r\le s}$ are also unconditionally independent. □

3.4.2 Characterization of jump times

Lemma 3.8 (Characterization of jump times). *The random atomic measure $\sum_{t\in\mathcal{T}}\delta_t$ supported on the set $\mathcal{T}$ of jump times of the occupation field process is a Poisson point process on $(0,\infty)$ with intensity measure*

$$\mu((a,b]) := \int_a^b \operatorname{Tr}\left[\tfrac{1}{t}(K_{1/t} - \tfrac{1}{1+\delta t}I)\right]\,\mathrm{d}t. \tag{3.34}$$

Proof. By lemma 3.7 and Kingman's representation theorem [46], the random measure $\sum_{t\in\mathcal{T}}\delta_t$ is a Poisson point process on $(0,\infty)$.

Note that the jump times $\mathcal{T}$ of the occupation field process $(N_t)_{t\ge0}$ are equal to the jump times of the running time process $(M_t)_{t\ge0}$.

Using proposition 3.5 and lemma 3.7 we find that for fixed $0 < s < t$ the pgf of $M_t - M_s$ is given by

$$\begin{aligned}\mathbb{E}\left[z^{M_t - M_s}\right] &= \prod_{j<n}\frac{\frac{1+\lambda_j t}{1+\delta t}\left(1-\left(1-\frac{1+\lambda_j s}{1+\delta s}\right)z\right)}{\frac{1+\lambda_j s}{1+\delta s}\left(1-\left(1-\frac{1+\lambda_j t}{1+\delta t}\right)z\right)}\\ &= \prod_{j<n}\frac{(1+\lambda_j t)\,(1+\delta s-(\delta-\lambda_j)\,sz)}{(1+\lambda_j s)\,(1+\delta t-(\delta-\lambda_j)\,tz)}.\end{aligned} \tag{3.35}$$

Its intensity measure follows from eq. (3.35), since evaluating the pgf of $M_t - M_s$ at $z = 0$ gives us that

$$e^{\mu([s,t))} = \mathbb{P}(M_t = M_s) = \prod_{j<n}\frac{(1+\lambda_j t)\,(1+\delta s)}{(1+\lambda_j s)\,(1+\delta t)}. \tag{3.36}$$

By taking the logarithm we find that

$$\begin{aligned}\mu([a,b)) &= \sum_{j<n}\log\left(\frac{(1+\lambda_j a)(1+\delta b)}{(1+\delta a)(1+\lambda_j b)}\right) = \int_a^b\sum_{j<n}\frac{\delta-\lambda_j}{(1+\delta t)(1+\lambda_j t)}\,\mathrm{d}t\\ &= \int_a^b \operatorname{Tr}\left[\tfrac{1}{t}(K_{1/t} - \tfrac{1}{1+\delta t}I)\right]\,\mathrm{d}t.\end{aligned}$$

□

3.4.3 Distribution of increments at jump times

The remainder of the proof of theorem 3.6 consists of identifying the law of the increments at the jump times.

For this purpose we consider an extended graph $\mathcal{G}^\star = (\mathcal{X}^\star, \mathcal{E}^\star, w^\star)$, that depends on a parameter $\mathbf{z} \in (0,1)^{\mathcal{X}}$. The graph $\mathcal{G}^\star$ has vertices $\mathcal{X}^\star := \mathcal{X} \cup \{\bigstar, \blacklozenge\}$, directed edges $\mathcal{E}^\star := \mathcal{E} \cup \{(x, \bigstar)\colon\ x \in \mathcal{X} \cup \{\bigstar\}\} \cup \{(x, \blacklozenge)\colon\ x \in \mathcal{X} \cup \{\blacklozenge\}\}$, and edge weights

$$w^\star(x,y) := \begin{cases} w(x,y) & \text{if } x,y \in \mathcal{X}, \\ \frac{1}{t}\ \frac{1-\mathbf{z}(x)}{\mathbf{z}(x)} & \text{if } x \in \mathcal{X},\ y = \bigstar, \\ \delta\ \frac{1-\mathbf{z}(x)}{\mathbf{z}(x)} & \text{if } x \in \mathcal{X},\ y = \blacklozenge, \\ 0 & \text{otherwise.} \end{cases} \tag{3.37}$$

That is, $\mathcal{G}^\star$ is obtained from $\mathcal{G}$ by adding two absorbing vertices, and adding edges from all vertices in $\mathcal{X}$ pointing to these new vertices. The parameter $\mathbf{z}$ regulates the edge weights of the added edges. We denote the graph Laplacian of $\mathcal{G}^\star$ by $L^\star$.

In lemma 3.9 below, we will compute the probability generating function (pgf) of the Wilson occupation field at a fixed intensity.

Lemma 3.9 (PGF of Wilson occupation field). *Fix $t > 0$. For $\mathbf{z} \in (0,1)^{\mathcal{X}}$ the Wilson occupation field has multi-dimensional probability generating function*

$$\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_t(x)}\right] = \frac{\det\left[\frac{1}{t}I + L\right]}{\det\left[\left(\frac{1}{t}+\delta\right)\operatorname{diag}\left(\frac{1-\mathbf{z}}{\mathbf{z}}\right) + \frac{1}{t}I + L\right]}, \tag{3.38}$$

where $\frac{1-\mathbf{z}}{\mathbf{z}}(x) := \frac{1-\mathbf{z}(x)}{\mathbf{z}(x)}$ for all $x \in \mathcal{X}$.

Proof. The multi-dimensional pgf can be expressed as

$$\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_t(x)}\right] = \mathbb{P}\left(\bigcap_{x\in\mathcal{X}} \{N_t(x) \le T(x)\}\right), \tag{3.39}$$

where $\{T(x)\colon\ x \in \mathcal{X}\}$ is any arbitrary collection of independent random variables with distribution $T(x) \sim \mathrm{Geom}_{\mathbb{N}_0}(1 - \mathbf{z}(x))$. We will construct a specific such collection of geometrics, for which the events $\{N_t(x) \le T(x)\}$ have an interpretation in the context of Wilson's algorithm that allows us to evaluate the right hand side of eq. (3.39).

Consider the extended graph $\mathcal{G}^\star$ defined above in eq. (3.37). We construct a collection of DF-stacks $\{(A_i^\star(x), B_i^\star(x))_{i\in\mathbb{N}_0}\colon\ x \in \mathcal{X}^\star\}$ on the extended graph $\mathcal{G}^\star$, which will be coupled to the DF-stacks $\{(A_i(x), B_i(x))_{i\in\mathbb{N}_0}\colon\ x \in \mathcal{X}\}$ of the original graph $\mathcal{G}$. For for all $i \in \mathbb{N}_0$ we set the arrows by $A_i^\star(\bigstar) = \bigstar$, $A_i^\star(\blacklozenge) = \blacklozenge$ and independently set

$$A_i^\star(x) = \begin{cases} A_i(x) & \text{with probability } \frac{1/t+\delta}{1/t+\delta+w^\star(x,\bigstar)+w^\star(x,\blacklozenge)} \\ \bigstar & \text{with probability } \frac{w^\star(x,\bigstar)}{1/t+\delta+w^\star(x,\bigstar)+w^\star(x,\blacklozenge)} \\ \blacklozenge & \text{with probability } \frac{w^\star(x,\blacklozenge)}{1/t+\delta+w^\star(x,\bigstar)+w^\star(x,\blacklozenge)} \end{cases} \quad \text{for all } x \in \mathcal{X}. \tag{3.40}$$

The killing marks are defined by $B_i^\star(\bigstar) = 1$, $B_i^\star(\blacklozenge) = 1$ and

$$B_i^\star(x) = \begin{cases} B_i(x) & \text{if } A_i^\star(x) = A_i(x) \\ 0 & \text{if } A_i^\star(x) \in \{\bigstar, \blacklozenge\} \end{cases} \quad \text{for all } x \in \mathcal{X}. \tag{3.41}$$

These DF-stacks do not have the distribution given in eq. (3.13), and as a consequence the killed Markov chain on $\mathcal{G}^\star$ constructed from these stacks, as defined in section 3.1.2.3, does not have transition matrix $I - \frac{1}{\delta^\star}L^\star$ for some parameter $\delta^\star$. However, the law of the loop-erased trajectory of the killed Markov chain constructed from these stacks, is identical to the law of the loop-erased trajectory of the killed random walk with transition matrix $I - \frac{1}{\delta^\star}L^\star$. Hence, applying the cycle popping procedure to the stacks $\{(A_i^\star(x), B_i^\star(x))_{i\in\mathbb{N}_0}\colon\ x \in \mathcal{X}^\star\}$ does produce a Kirchhoff forest of the graph $\mathcal{G}^\star$ of intensity $1/t$, which we denote by $\Phi^\star_{1/t}$.

Cycle popping the DF-stacks on $\mathcal{G}^\star$ also produces a field of stack depths $\ell^\star[\Phi^\star_{1/t}]$, as defined in eq. (3.18), where $\ell[\Phi^\star_{1/t}](x)-1$ denotes the index of the arrow at x that is contained in $\Phi^\star_{1/t}$. We remark that since the stacks on $\mathcal{G}^\star$ define a random walk with a different transition matrix, the field $\ell^\star[\Phi^\star_{1/t}]$ is not distributed as a Wilson occupation field on $\mathcal{G}^\star$, as expained in section 3.2.1.1.

Define the collection of geometrics $\{T(x)\colon\ x \in \mathcal{X}\}$ by

$$T(x) := \min\{i \in \mathbb{N}_0\colon\ A_i^\star(x) \in \{\bigstar, \blacklozenge\}\}, \qquad \text{for all } x \in \mathcal{X}, \tag{3.42}$$

which indeed has success parameter $\frac{w^\star(x,\bigstar)+w^\star(x,\blacklozenge)}{1/t+\delta+w^\star(x,\bigstar)+w^\star(x,\blacklozenge)} = 1 - \mathbf{z}(x)$, and is independent of $\{T(y)\colon\ y \in \mathcal{X}\setminus\{x\}\}$ and $\ell[\Phi_{1/t}]$.

Since $T(x)$ denotes the index of the topmost arrow that points to one of the two added vertices, it holds that

$$\{\ell[\Phi^\star_{1/t}](x) \le T(x)\} = \{(x,\bigstar) \notin \Phi^\star_{1/t} \text{ and } (x,\blacklozenge) \notin \Phi^\star_{1/t}\}, \tag{3.43}$$

where we use that arrows of the form $(x,\bigstar)$ or $(x,\blacklozenge)$ cannot be part of a cycle in $\mathcal{G}^\star$, so they can never be popped by the cycle popping procedure. It follows that

$$\bigcap_{x\in\mathcal{X}} \{\ell[\Phi^\star_{1/t}](x) \le T(x)\} = \{\bigstar \text{ and } \blacklozenge \text{ are isolated vertices of } \Phi^\star_{1/t}\}. \tag{3.44}$$

By the coupling of $\{(A_i^\star(x), B_i^\star(x))_{i\in\mathbb{N}_0}\colon\ x \in \mathcal{X}^\star\}$ and $\{(A_i(x), B_i(x))_{i\in\mathbb{N}_0}\colon\ x \in \mathcal{X}\}$ it holds that

$$\bigcap_{x\in\mathcal{X}} \{\ell[\Phi^\star_{1/t}](x) \le T(x)\} = \bigcap_{x\in\mathcal{X}} \{\ell[\Phi_{1/t}](x) \le T(x)\}. \tag{3.45}$$

Hence, recalling that $\mathcal{F}$ denotes the set of rooted forests of $\mathcal{G}$, we have by the matrix-tree theorem that

$$\begin{aligned}
\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_t(x)}\right] &= \mathbb{P}(\bigstar \text{ and } \blacklozenge \text{ are isolated vertices of } \Phi^\star_{1/t}) \\
&= \mathbb{P}(\Phi^\star_{1/t} \in \mathcal{F}) = \frac{\frac{1}{t^2}\det[\frac{1}{t}I + L]}{\det[\frac{1}{t}I + L^\star]} \\
&= \frac{\det\left[\frac{1}{t}I + L\right]}{\det\left[\left(\frac{1}{t}+\delta\right)\operatorname{diag}\left(\frac{1-\mathbf{z}}{\mathbf{z}}\right) + \frac{1}{t}I + L\right]}.
\end{aligned}$$

□

Lemma 3.10 (PGF of increment of Wilson occupation field). *Fix $\mathbf{z} \in (0,1)^{\mathcal{X}}$. For the pgf of the increment $N_t - N_s$ of the Wilson occupation field conditioned on being non-zero, it holds in the limit $s \uparrow t$ that*

$$\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_t(x)-N_s(x)} \;\middle|\; N_t - N_s \neq \underline{0}\right] = \frac{\mathrm{Tr}\left[K^{\delta\mathbf{z}}_{t\mathbf{z},t}\right] - \frac{n}{1+\delta t}}{\mathrm{Tr}\left[K_{1/t}\right] - \frac{n}{1+\delta t}} + \mathcal{O}(t-s), \tag{3.46}$$

where $K_{1/t} := \frac{1}{t}(\frac{1}{t}I + L)^{-1}$ and

$$K^{\delta\mathbf{z}}_{t\mathbf{z},t} := \left(\left(\tfrac{1}{t}+\delta\right)\mathrm{diag}\left(\tfrac{\underline{1}-\mathbf{z}}{\mathbf{z}}\right) + \tfrac{1}{t}I + L\right)^{-1}\left(\tfrac{1}{t}\mathrm{diag}\left(\tfrac{\underline{1}-\mathbf{z}}{\mathbf{z}}\right) + \tfrac{1}{t}I\right). \tag{3.47}$$

Proof. Write

$$\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_t(x)-N_s(x)} \;\middle|\; N_t - N_s \neq \underline{0}\right] = \frac{\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_t-N_s}\right] - \mathbb{P}(N_t - N_s = \underline{0})}{\mathbb{P}(N_t - N_s \neq \underline{0})}. \tag{3.48}$$

It holds by lemmas 3.7 and 3.9 that

$$\begin{aligned}\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_t-N_s}\right] &= \frac{\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_t}\right]}{\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_s}\right]} \\ &= \frac{\det\left[\frac{1}{t}I + L\right]\det\left[(1/s+\delta)\mathrm{diag}\left(\frac{\underline{1}-\mathbf{z}}{\mathbf{z}}\right) + \frac{1}{s}I + L\right]}{\det\left[\frac{1}{s}I + L\right]\det\left[(1/t+\delta)\mathrm{diag}\left(\frac{\underline{1}-\mathbf{z}}{\mathbf{z}}\right) + \frac{1}{t}I + L\right]}.\end{aligned}$$

As $s \uparrow t$, Jacobi's formula gives us that

$$\begin{aligned}\det\left[\tfrac{1}{s}I + L\right] &= \det\left[\tfrac{1}{t}I + L\right] - (t-s)\det\left[\tfrac{1}{t}I + L\right]\mathrm{Tr}\left[(\tfrac{1}{t}I + L)^{-1}(-\tfrac{1}{t^2}I)\right] + \mathcal{O}\left((t-s)^2\right) \\ &= \det\left[\tfrac{1}{t}I + L\right]\left(1 + \tfrac{(t-s)}{t}\mathrm{Tr}\left[K_{1/t}\right]\right) + \mathcal{O}\left((t-s)^2\right),\end{aligned}$$

and similarly that

$$\begin{aligned}&\det\left[(1/s+\delta)\mathrm{diag}\left(\tfrac{\underline{1}-\mathbf{z}}{\mathbf{z}}\right) + \tfrac{1}{s}I + L\right] \\ &= \det\left[(1/t+\delta)\mathrm{diag}\left(\tfrac{\underline{1}-\mathbf{z}}{\mathbf{z}}\right) + \tfrac{1}{t}I + L\right]\left(1 + \tfrac{(t-s)}{t}\mathrm{Tr}\left[K^{\delta\mathbf{z}}_{t\mathbf{z},t}\right]\right) + \mathcal{O}\left((t-s)^2\right).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\left[\prod_{x\in\mathcal{X}} \mathbf{z}(x)^{N_t(x)-N_s(x)}\right] &= \frac{1 + \frac{(t-s)}{t}\mathrm{Tr}\left[K^{\delta\mathbf{z}}_{t\mathbf{z},t}\right]}{1 + \frac{(t-s)}{t}\mathrm{Tr}\left[K_{1/t}\right]} + \mathcal{O}\left((t-s)^2\right) \\ &= 1 + \tfrac{(t-s)}{t}\mathrm{Tr}\left[K^{\delta\mathbf{z}}_{t\mathbf{z},t} - K_{1/t}\right] + \mathcal{O}\left((t-s)^2\right).\end{aligned}$$

From lemma 3.8 we know that

$$\begin{aligned}\mathbb{P}(N_t - N_s \neq \underline{0}) &= 1 - \prod_{j<n}\frac{(1+\delta s)(1+\lambda_j t)}{(1+\lambda_j s)(1+\delta t)} \\ &= (t-s)\sum_{j<n}\frac{\delta - \lambda_j}{(1+\lambda_j t)(1+\delta t)} + \mathcal{O}((t-s)^2).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
&\mathbb{E}\left[\prod_{x\in\mathcal{X}}\mathbf{z}(x)^{N_t-N_s}\;\middle|\; N_t-N_s\neq\underline{0}\right]\\
&=\frac{\frac{1}{t}\mathrm{Tr}\left[K^{\delta\mathbf{z}}_{t\mathbf{z},t}-K_{1/t}\right]+\sum_{j<n}\frac{\delta-\lambda_j}{(1+\lambda_j t)(1+\delta t)}}{\sum_{j<n}\frac{\delta-\lambda_j}{(1+\lambda_j t)(1+\delta t)}}+\mathcal{O}(t-s)\\
&=\frac{\mathrm{Tr}\left[K^{\delta\mathbf{z}}_{t\mathbf{z},t}-K_{1/t}\right]+\sum_{j<n}\left(\frac{1}{1+\lambda_j t}-\frac{1}{1+\delta t}\right)}{\sum_{j<n}\left(\frac{1}{1+\lambda_j t}-\frac{1}{1+\delta t}\right)}+\mathcal{O}(t-s)\\
&=\frac{\mathrm{Tr}\left[K^{\delta\mathbf{z}}_{t\mathbf{z},t}\right]-\frac{n}{1+\delta t}}{\mathrm{Tr}\left[K_{1/t}\right]-\frac{n}{1+\delta t}}+\mathcal{O}(t-s).
\end{aligned}$$

□

The matrix $K_{1/t}$, that appears in lemma 3.10, has an interpretation in terms of Kirchhoff forests. The entry $K_{1/t}(x,y)$ equals the probability of the event that vertex y is a root of $\Phi_{1/t}$ and that vertex x belongs to the same component as y [19]. In particular, the diagonal entry $K_{1/t}(x,x)$ gives the probability that x is a root. The entries of $K_{1/t}$ also have an interpretation in terms of the killed random walk

$$K_{1/t}(x,y)=\mathbf{P}_x(X_{T_t}=y). \tag{3.49}$$

The matrix $K^{\delta\mathbf{z}}_{t\mathbf{z},t}$ can be interpretated using the Kirchhoff forest $\Phi^\star_{1/t}$ on the extended graph $\mathcal{G}^\star$, which was defined in eq. (3.37), as it holds that

$$K^{\delta\mathbf{z}}_{t\mathbf{z},t}(x,y)=\mathbb{P}(x\leftrightarrow_{\Phi^\star_{1/t}}y,\ y\in\rho^\star_{1/t})+\mathbb{P}(x\leftrightarrow_{\Phi^\star_{1/t}}y,\ (y,\bigstar)\in\Phi^\star_{1/t}). \tag{3.50}$$

That is, $K^{\delta\mathbf{z}}_{t\mathbf{z},t}(x,y)$ denotes the probability of the event that x and y belong to the same component of $\Phi^\star_{1/t}$ and that either y is a root or a neighbor of $\bigstar$.

Lemma 3.11 (PGF of occupation field of closed random walk). *Fix $t>0$. Let Λ_t be an $\mathbb{N}_0^{\mathcal{X}}$-valued random variable with law*

$$P(\Lambda_t=\mathbf{m}):=\mathbf{P}_{\mathrm{unif}}(\ell[(X_k)_{0\le k\le T_t}]=\mathbf{m}\mid X_{T_t+1}=X_0),\qquad \textit{for all } \mathbf{m}\in\mathbb{N}_0^{\mathcal{X}}. \tag{3.51}$$

Then for $\mathbf{z}\in(0,1)^{\mathcal{X}}$ the pgf of Λ_t is given by

$$E\left[\prod_{x\in\mathcal{X}}\mathbf{z}(x)^{\Lambda_t(x)}\right]=\frac{\mathrm{Tr}\left[K^{\delta\mathbf{z}}_{t\mathbf{z},t}\right]-\frac{n}{1+\delta t}}{\mathrm{Tr}\left[K_{1/t}\right]-\frac{n}{1+\delta t}}, \tag{3.52}$$

where $K_{1/t}:=\frac{1}{t}(\frac{1}{t}I+L)^{-1}$ and

$$K^{\delta\mathbf{z}}_{t\mathbf{z},t}:=\left(\left(\tfrac{1}{t}+\delta\right)\mathrm{diag}\left(\tfrac{1-\mathbf{z}}{\mathbf{z}}\right)+\tfrac{1}{t}I+L\right)^{-1}\left(\tfrac{1}{t}\mathrm{diag}\left(\tfrac{1-\mathbf{z}}{\mathbf{z}}\right)+\tfrac{1}{t}I\right). \tag{3.53}$$

Proof. First note that by eq. (3.49), we have that

$$\begin{aligned}\operatorname{Tr}\left[K_{1/t}\right] - \tfrac{n}{1+\delta t} &= \sum_{x\in\mathcal{X}}\left(K_{1/t}(x,x) - \tfrac{1}{1+\delta t}\right) = \sum_{x\in\mathcal{X}}\left(\mathbf{P}_x(X_{T_t}=X_0) - \mathbf{P}_{\mathrm{unif}}(T_t=0)\right)\\ &= \sum_{x\in\mathcal{X}}\mathbf{P}_x(X_{T_t}=X_0,\ T_t\geq 1) = n\mathbf{P}_{\mathrm{unif}}(X_{T_t+1}=X_0)\mathbf{P}_{\mathrm{unif}}(T_t\geq 1).\end{aligned} \tag{3.54}$$

Below we assume that the killed random walk $X=(X_k)_{0\leq k\leq T_t}$ is constructed from the DF-stacks $\{(A_i(x),B_i(x))_{i\in\mathbb{N}_0}\colon\ x\in\mathcal{X}\}$, as described in section 3.1.2.3, with X_0 uniformly distributed on $\mathcal{X}$ and independent of the DF-stacks.

Consider the DF-stacks $\{(A_i^\star(x),B_i^\star(x))_{i\in\mathbb{N}_0}\colon\ x\in\mathcal{X}^\star\}$ on the extended graph $\mathcal{G}^\star$, defined in the proof of lemma 3.9, and recall that they are coupled to the DF-stacks $\{(A_i(x),B_i(x))_{i\in\mathbb{N}_0}\colon\ x\in\mathcal{X}\}$.

For each $x\in\mathcal{X}$ we define three geometric random variables

$$\begin{aligned}T^\bigstar(x) &:= \min\{i\in\mathbb{N}_0\colon\ A_i^\star(x)=\bigstar\},\\ T^\blacklozenge(x) &:= \min\{i\in\mathbb{N}_0\colon\ A_i^\star(x)=\blacklozenge\},\\ T^\dagger(x) &:= \min\{i\in\mathbb{N}_0\colon\ B_i^\star(x)=1\},\end{aligned}$$

and we note that $T^\bigstar(x)\wedge T^\blacklozenge(x)\sim\mathrm{Geom}_{\mathbb{N}_0}(1-\mathbf{z}(x))$.

Using the observation in eq. (3.39) gives us for the pgf of Λ_t that

$$\begin{aligned}E\left[\prod_{x\in\mathcal{X}}\mathbf{z}(x)^{\Lambda_t(x)}\right] &= \mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\{\ell[(X_k)_{0\leq k\leq T_t}](x)\leq T^\bigstar(x)\wedge T^\blacklozenge(x)\}\ \middle|\ X_{T_t+1}=X_0\right)\\ &= \frac{\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\{\ell[(X_k)_{0\leq k\leq T_t}](x)\leq T^\bigstar(x)\wedge T^\blacklozenge(x)\},\ X_{T_t+1}=X_0\right)}{\mathbb{P}(X_{T_t+1}=X_0)}\\ &= \frac{n\mathbb{P}(T_t\geq 1)\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\{\ell[(X_k)_{0\leq k\leq T_t}](x)\leq T^\bigstar(x)\wedge T^\blacklozenge(x)\},\ X_{T_t+1}=X_0\right)}{\operatorname{Tr}\left[K_{1/t}\right]-\frac{n}{1+\delta t}}.\end{aligned}$$

Since the denominator is as required, it remains to consider the numerator

$$\begin{aligned}&n\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\{\ell[(X_k)_{0\leq k\leq T_t}](x)\leq T^\bigstar(x)\wedge T^\blacklozenge(x)\},\ X_{T_t+1}=X_0\right)\mathbb{P}(T_t\geq 1)\\ &= n\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\{\ell[(X_k)_{0\leq k\leq T_t-1}](x)\leq T^\bigstar(x)\wedge T^\blacklozenge(x)\},\ X_{T_t}=X_0\ \middle|\ T_t\geq 1\right)\mathbb{P}(T_t\geq 1)\\ &= n\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\{\ell[(X_k)_{0\leq k\leq T_t-1}](x)\leq T^\bigstar(x)\wedge T^\blacklozenge(x)\},\ X_{T_t}=X_0,\ T_t\geq 1\right)\\ &= n\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\{\ell[(X_k)_{0\leq k\leq T_t}](x)-\mathbf{1}_{\{X_0\}}(x)\leq T^\bigstar(x)\wedge T^\blacklozenge(x)\},\ X_{T_t}=X_0,\ T_t\geq 1\right),\end{aligned}$$

where $\mathbf{1}_{\{X_0\}}\in\{0,1\}^{\mathcal{X}}$ denotes the indicator of X_0.

On the event $\{X_0 = y\}$ the random variable $\mathbf{1}_{\{X_0\}}$ is deterministic, so that $\ell[(X_k)_{0\le k\le T_t-1}]$ and $\mathbf{1}_{\{X_0\}}$ are conditionally independent. Since the pgf of two independent random variables equals the product of their respective pgfs, this gives us that

$$
\begin{aligned}
&n\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\left\{\ell[(X_k)_{0\le k\le T_t}](x)-\mathbf{1}_{\{X_0\}}(x)\le T^\bigstar(x)\wedge T^\blacklozenge(x)\right\},\ X_{T_t}=X_0,\ T_t\ge 1\right)\\
&=\sum_{y\in\mathcal{X}}\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\left\{\ell[(X_k)_{0\le k\le T_t}](x)-\mathbf{1}_{\{y\}}(x)\le T^\bigstar(x)\wedge T^\blacklozenge(x)\right\},\ X_{T_t}=y,\ T_t\ge 1\ \middle|\ X_0=y\right)\\
&=\sum_{y\in\mathcal{X}}\frac{\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\left\{\ell[(X_k)_{0\le k\le T_t}](x)\le T^\bigstar(x)\wedge T^\blacklozenge(x)\right\},\ X_{T_t}=y,\ T_t\ge 1\ \middle|\ X_0=y\right)}{\mathbb{P}\left(1\le T^\bigstar(y)\wedge T^\blacklozenge(y)\right)}\\
&=\sum_{y\in\mathcal{X}}\mathbf{z}(y)\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\left\{\ell[(X_k)_{0\le k\le T_t}](x)\le T^\bigstar(x)\wedge T^\blacklozenge(x)\right\},\ X_{T_t}=y,\ T_t\ge 1\ \middle|\ X_0=y\right).
\end{aligned}
$$

Defining

$$
\begin{aligned}
\tau^\bigstar &:= \min\{m\in\mathbb{N}_0\colon\ \exists x\in\mathcal{X}\text{ s.t. }\ell[(X_k)_{0\le k\le m}](x)>T^\bigstar(x)\},\\
\tau^\blacklozenge &:= \min\{m\in\mathbb{N}_0\colon\ \exists x\in\mathcal{X}\text{ s.t. }\ell[(X_k)_{0\le k\le m}](x)>T^\blacklozenge(x)\},
\end{aligned}
$$

gives us that

$$
\bigcap_{x\in\mathcal{X}}\{\ell[(X_k)_{0\le k\le T_t}](x)\le T^\bigstar(x)\wedge T^\blacklozenge(x)\}=\{T_t<\tau^\bigstar\wedge\tau^\blacklozenge\}. \tag{3.55}
$$

On the event $\{T_t<\tau^\bigstar\wedge\tau^\blacklozenge\}$ we have that

$$
T_t=\min\{m\in\mathbb{N}_0\colon\ \exists x\in\mathcal{X}\text{ s.t. }\ell[(X_k)_{0\le k\le m}](x)>T^\dagger(x)\}, \tag{3.56}
$$

from which follows that

$$
\begin{aligned}
&\sum_{y\in\mathcal{X}}\mathbf{z}(y)\mathbb{P}\left(\bigcap_{x\in\mathcal{X}}\{\ell[(X_k)_{0\le k\le T_t}](x)\le T^\bigstar(x)\wedge T^\blacklozenge(x)\},\ X_{T_t}=y,\ T_t\ge 1\ \middle|\ X_0=y\right)\\
&\qquad=\sum_{y\in\mathcal{X}}\mathbf{z}(y)\mathbb{P}\left(1\le T_t<\tau^\bigstar\wedge\tau^\blacklozenge,\ X_{T_t}=y\mid X_0=y\right)\\
&\qquad=\sum_{y\in\mathcal{X}}\mathbf{z}(y)\mathbb{P}\left(1\le T_t<\tau^\bigstar\wedge\tau^\blacklozenge,\ X_{T_t}=y,\ T^\dagger(y)<T^\bigstar(y)\ \middle|\ X_0=y\right)\\
&\qquad=\sum_{y\in\mathcal{X}}\mathbf{z}(y)\mathbb{P}\left(1\le T_t\wedge\tau^\bigstar<\tau^\blacklozenge,\ X_{T_t\wedge\tau^\bigstar}=y,\ T^\dagger(y)<T^\bigstar(y)\ \middle|\ X_0=y\right).
\end{aligned}
$$

By independence we find that

$$\begin{aligned}
&\sum_{y\in\mathcal{X}} \mathbf{z}(y)\mathbb{P}\left(1 \le T_t \wedge \tau^\star < \tau^\blacklozenge,\ X_{T_t\wedge\tau^\star} = y,\ T^\dagger(y) < T^\star(y) \;\middle|\; X_0 = y\right) \\
&= \sum_{y\in\mathcal{X}} \mathbf{z}(y)\mathbb{P}\left(1 \le T_t \wedge \tau^\star < \tau^\blacklozenge,\ X_{T_t\wedge\tau^\star} = y \mid X_0 = y\right)\mathbb{P}\left(T^\dagger(y) < T^\star(y)\right) \\
&= \sum_{y\in\mathcal{X}} \mathbb{P}\left(1 \le T_t \wedge \tau^\star < \tau^\blacklozenge,\ X_{T_t\wedge\tau^\star} = y \mid X_0 = y\right) \\
&= \sum_{y\in\mathcal{X}} \left(\mathbb{P}\left(T_t \wedge \tau^\star < \tau^\blacklozenge,\ X_{T_t\wedge\tau^\star} = y \mid X_0 = y\right) - \tfrac{1}{1+\delta t}\right).
\end{aligned}$$

Using eq. (3.50) gives us that

$$\begin{aligned}
&\sum_{y\in\mathcal{X}} \left(\mathbb{P}\left(T_t \wedge \tau^\star < \tau^\blacklozenge,\ X_{T_t\wedge\tau^\star} = y \mid X_0 = y\right) - \tfrac{1}{1+\delta t}\right) \\
&= \sum_{y\in\mathcal{X}} \left(\mathbb{P}\left((y,\star) \in \Phi^\star_{1/t} \text{ or } y \in \rho^\star_{1/t}\right) - \tfrac{1}{1+\delta t}\right) \\
&= \sum_{y\in\mathcal{X}} \left(K^{\delta\mathbf{z}}_{t\mathbf{z},t}(y,y) - \tfrac{1}{1+\delta t}\right) = \operatorname{Tr}\left[K^{\delta\mathbf{z}}_{t\mathbf{z},t}\right] - \tfrac{n}{1+\delta t},
\end{aligned}$$

which completes the proof. □

Proof of theorem 3.6. The result of theorem 3.6 follows from lemmas 3.7, 3.8, 3.10 and 3.11. □

3.4.4 Closed walk decomposition

Corollary 3.6.1 is a simple consequence of theorem 3.6.

Proof of corollary 3.6.1. Fix $\gamma = (x_0, x_1, \dots, x_l) \in \mathcal{P}^{\mathrm{cl}}$. By theorem 3.6 the intensity measure μ_γ is given by

$$\mu_\gamma((a,b]) = \int_a^b \kappa_t \mathbf{P}_{\mathrm{unif}}((X_k)_{0\le k\le T_t+1} = \gamma \mid X_{T_t+1} = X_0)\ \mathrm{d}t.$$

Write

$$\mathbf{P}_{\mathrm{unif}}((X_k)_{0\le k\le T_t+1} = \gamma \mid X_{T_t+1} = X_0) = \frac{\mathbf{P}_{\mathrm{unif}}((X_k)_{0\le k\le T_t+1} = \gamma)}{\mathbf{P}_{\mathrm{unif}}(X_{T_t+1} = X_0)}.$$

For the denominator we have by eq. (3.54) that

$$\mathbf{P}_{\mathrm{unif}}(X_{T_t+1} = X_0) = \frac{1+\delta t}{n\delta t}\left(\operatorname{Tr}\left[K_{1/t}\right] - \frac{n}{1+\delta t}\right) = \frac{(1+\delta t)\kappa_t}{n\delta}. \tag{3.57}$$

We therefore have that

$$\begin{aligned}
&\mathbf{P}_{\mathrm{unif}}((X_k)_{0\le k\le T_t+1} = \gamma \mid X_{T_t+1} = X_0) = \frac{n\delta}{(1+\delta t)\kappa_t}\mathbf{P}_{\mathrm{unif}}((X_k)_{0\le k\le T_t+1} = \gamma) \\
&= \frac{\delta}{(1+\delta t)^2\kappa_t}\left(\frac{\delta t}{1+\delta t}\right)^{l-1}\mathbf{P}_{x_0}((X_k)_{0\le k\le l} = \gamma) \\
&= \frac{\delta}{(1+\delta t)^2\kappa_t}\left(\frac{\delta t}{1+\delta t}\right)^{l-1}\prod_{i=1}^{l}\frac{w(x_{i-1},x_i)}{\delta} = \frac{1}{(1+\delta t)^2\kappa_t}\left(\frac{t}{1+\delta t}\right)^{l-1} w(e(\gamma)),
\end{aligned}$$

which completes the proof. □