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## Stochastic amplitude modulation of nonlinear dispersive waves

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# Summary

This dissertation studies the behavior of waves under the influence of disturbances from their environment. Although we focus on the *mathematical* description of waves and their theoretical behavior, this description does indeed have its origins in reality. One can think of waves on the surface of shallow water: waves that are created, for example, by a passing boat on a canal, and which retain their shape for a long time even after the boat has disappeared—until they gradually merge with the surrounding waves as they meet irregularities. The power of a mathematical approach is that the insights are not limited to one specific situation. They can also clarify what is difficult to distinguish in experiments.

In this thesis, we investigate how waves in dispersive media, such as water, change in height due to realistic imperfections. We model these imperfections as random disturbances (noise) and focus on a well-known model from mathematical physics: the Korteweg–de Vries equation. This equation has been known since the 19th century for the wave phenomena it describes, and still forms the basis for modern wave analysis. A special feature of this equation is that it has solutions that predict waves of different heights (amplitudes), with higher waves traveling faster. This links amplitude and speed in a direct way—an interesting starting point for studying wave disturbance in an imperfect environment.

Chapter 2 focuses on the KdV equation with so-called multiplicative noise: a random disturbance that directly affects the wave. As a result, the waves in the KdV system undergo major changes in height and speed. The key here is the role of the energy of the system, which constantly fluctuates up and down due to the noise.

In this chapter, we introduce a technique for tracking and predicting the position and amplitude of such waves. These “effective” position and amplitude are random processes: after all, the noise causes random fluctuations. The idea is that we always look for the wave profile that best fits the disturbed wave. It never fits perfectly; the disturbance forms a “tail” behind the wave, similar to the wake of a boat. Our technique yields mathematical equations that describe the course of amplitude, position, and tail. Exact solutions are unfeasible, as is often the case with nonlinear systems, but computer simulations and systematic approaches provide insight into the predicted behavior. For example, we show that a common form of noise causes a slow average increase in amplitude. Our analysis accurately predicts this growth, and this prediction corresponds with the simulations.

In Chapter 3, we examine a more fundamental question: does a wave survive

random disturbance at all? Or does it eventually break, like a wave in the surf? And if so, after how long? To make this question more manageable, we first consider a simpler situation: instead of random noise, we study a predetermined, non-random (deterministic) energy change in the medium. This simplifies the analysis considerably. We show that as long as the supply or removal of energy occurs slowly enough, waves can withstand slow but large amplitude changes without losing their coherent form. This insight forms the basis for studying random perturbations.

In Chapter 4, we return to the random model from Chapter 2. Based on the results from Chapter 3, we show that waves can tolerate large amplitude variations while remaining coherent, even under the influence of random noise. This also demonstrates that our description of the effective amplitude evolution from Chapter 2 is mathematically valid. The techniques used even suggest that the waves remain coherent for even longer than we can strictly demonstrate. The wake extends far behind a disturbed wave, but remains low. The latter is difficult to prove mathematically and offers opportunity for further research.

In Chapter 5, we leave the KdV model and focus on a related but discrete system: the Fermi–Pasta–Ulam–Tsingou (FPUT) chain, a model of masses connected by springs. A kind of infinite chain of beads connected by elastic. The model has concrete applications in the description of crystal lattices at the molecular level, where atoms can be modeled as masses connected by springs. Wave-like solutions also arise in this discrete context: when the masses are brought out of equilibrium in a suitable way, the resulting deflection travels through the chain at a constant speed. The connection with the KdV equation is strong: the entire dynamics of the FPUT chain is approximated by the KdV equation.

We study what happens when the springs in the chain are not identical, but have small random differences in spring-force. Here too, we investigate the effect of realistic irregularities on the propagation of waves. The disturbance has a slow but clear effect: the waves decrease in height and lose energy via a tail (wake) that forms behind the wave. We explicitly calculate how strong this amplitude decrease is; the techniques from the previous chapters prove to be excellent for this purpose.

Together, these chapters show how small imperfections — both random and deterministic in nature — can have a slow but significant influence on the speed and amplitude of waves in dispersive systems. The techniques developed in this thesis offer new ways to predict such changes. They also show that these waves are surprisingly robust and can survive for a long time even in realistic imperfect environments.