Stochastic amplitude modulation of nonlinear dispersive waves
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CHAPTER 1
Introduction

1.1 Background

This thesis studies mathematical models describing the propagation of waves affected by random disturbances. Focusing on a paradigm nonlinear wave model—the Korteweg–de Vries equation—this work explores how random fluctuations in energy modulate the amplitude of solitary waves, both rigorously and through formal analysis. While the subject matter is rooted in physics, this thesis deals with the rich *mathematical* theory underlying wave phenomena. As such, the topic sits at the intersection of *Dynamical Systems*, to describe wave dynamics, and *Probability Theory*, accounting for random effects. Chapters 2–5 form the main body of this thesis and contain its mathematical contributions. The aim of this introductory chapter is to provide both an intuitive and a mathematical foundation for the reader regarding the core concepts of this thesis. We unpack the thesis title—*Stochastic Amplitude Modulation of Nonlinear Dispersive Waves*—back to front:

1. *Waves*
2. *Dispersion*
3. *Nonlinear (Structure)*
4. *Amplitude Modulation*
5. *Stochastics*

Waves

Wave phenomena have long been prevalent in physics, both classical and modern. While it remains an active field of research, its fundamental concepts are easily accessible. Most people are familiar with waves in nature: sound waves used for communication, surface waves on bodies of water, or even visible light. The common feature among these examples is that *something* propagates. Sound consists of pressure differences propagating through air. In bodies of water, fluid flow organizes to carry disturbances across the surface. Maxwell's laws of electromagnetism describe how electric and magnetic fields interact to transmit light through a vacuum. In each case, the underlying physics result in propagation of disturbances through the medium[1].

The wave phenomenon that inspired the topic of this thesis was first observed in 1834 by civil engineer and shipbuilder John Scott Russell:

> *"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."* [93]

Russell describes a remarkable type of water wave: a solitary elevation that continues along the channel without change of form or diminution of speed. Or rather, *almost* without change of form, as he notes that the wave height gradually diminished. Russell's encounter was, of course, not the first observation ever made of a water wave. The surprise lies in the fact that, out of a seemingly chaotic setting ("violent agitation"), a well-defined structure emerged and propagated undisturbed. Interestingly, the gradual loss of height—presumably due to imperfections in the environment—closely aligns with the topic of this thesis. Russell's observations later sparked mathematical interest: in 1871, Boussinesq derived a model for shallow water dynamics that explained the wave phenomenon [15]. His work was extended in 1895 by Korteweg and de Vries [68]. The resulting Korteweg–de Vries (KdV) equation has since become a paradigm model in nonlinear wave theory.

Wave models Mathematically, the physics of wave phenomena—like many topics in classical physics—are described by partial differential equations (PDEs), encoding the continuous dependence of quantities on space and time. Waves are modeled as solutions $u(t, x)$ to a PDE, where $t \in \mathbb{R}$ denotes physical time and $x \in \mathbb{R}^d$ denotes physical space in $d \geq 1$ dimensions. The function-value $u(t, x)$ models an evolving physical quantity at time t and position x, and can be $\mathbb{R}^n$- or $\mathbb{C}^n$-valued ($n \geq 1$) depending on the physical context.

[1]Strictly speaking, light is not carried by a medium, as demonstrated by the famous Michelson-Morley experiment [83].

Models for Russell's 'Wave of Translation' usually take $x \in \mathbb{R}$ for the position along the canal, assuming that no relevant dynamics occur in the directions transverse to propagation. The scalar-valued function u then models the height of the water as it evolves over time. A traveling wave, is a solution that admits the representation $u(t,x) = \phi(x - ct)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a fixed wave profile propagating at velocity c. A *solitary* wave as described by Russell concerns the case that ϕ is continuous and meets asymptotic decay conditions.

A word of caution: wave phenomena are not synonymous with the wave equation. The wave equation is a second order linear PDE whose solutions are described via d'Alemberts principle: disturbances propagate left- and rightward. Hence, in the classic wave-equation, all initial conditions give rise to a superposition of wave solutions. It is called *the* wave equation because it can be derived for simple wave phenomena such as waves on a string and electromagnetic waves. However, our interest lies in systems which *admit* a traveling wave solution. Meaning, the (unique) existence of a profile which propagates and retains its shape under the PDE dynamics. Traveling waves are among the many coherent structures that arise in nonlinear PDEs—such as stripes, spirals, breathers or spots—which form specific and nontrivial subclasses of solutions.

Chapters 2–4 of this thesis deal with (adaptations of) the KdV equation:

$$u_t(t,x) = -\partial_x^3 u(t,x) - \partial_x(u^2(t,x)), \qquad t \in \mathbb{R}^+, x \in \mathbb{R}. \tag{1.1}$$

This equation was originally derived by Boussinesq based on the physics of shallow water flow with a free boundary, which, after several approximations, simplifies to (1.1). The model admits traveling wave solutions $u(t,x) = \phi_c(x - ct)$, where the wave profiles

$$\phi_c(x) = \tfrac{3c}{2} \operatorname{sech}^2(\sqrt{c}x/2), \quad c > 0, \tag{1.2}$$

solve the traveling wave equation

$$-c\partial_x \phi_c(x) = -\partial_x^3 \phi_c(x) - \partial_x(\phi_c^2(x)), \quad x \in \mathbb{R}. \tag{1.3}$$

We remark that the solitary waves ϕ_c travel only to the right, i.e. towards $+\infty$, due to the fact that (1.1) breaks reflection symmetry. As seen in (1.2), the traveling waves ϕ_c appear as a family of waves with various amplitudes proportional to their velocity c. The family (1.2) satisfies the self-similarity property $\phi_c(x) = c\phi_1(\sqrt{c}x)$. This is a direct consequence of the scaling invariance

$$u(t,x) \mapsto \alpha^2 u(\alpha^3 t, \alpha x) \tag{1.4}$$

of the KdV equation, meaning that any solution $u(t,x)$ to (1.1) gives rise to rescaled solutions via the transformation (1.4), with $\alpha \in \mathbb{R}$.

In the final chapter of this thesis we turn our attention to a *discrete* wave-model related to the KdV equation: the Fermi-Pasta-Ulam-Tsingou (FPUT) lattice system. Contrary to (1.1), which models water-surface height as a function of *continu-*

ous space, the FPUT system describes the evolution of a *countable* set of quantities. The FPUT system was famously used to help pioneer the field of numerical mathematics in the 1950s [39]. It models an infinite collection of coupled oscillators[2], or simply, masses connected by springs. The FPUT system describes how the chain of masses oscillates when brought out of equilibrium. Like in the KdV equation, the FPUT system gives rise to solitary waves. In fact, the dynamics of the particle chain are approximated by the KdV equation in an appropriate continuum limit [109].

Waves as an equilibrium Since waves retain their shape in an otherwise dynamic environment, it is natural to view them as a 'balance' point, or equilibrium. From an observer's perspective, like Russell chasing the Wave of Translation on horseback, a wave is a fixed object. We might compare this to a marble placed at the bottom of a convex bowl: it rests at a spot where it can sit perfectly still. The marble can be pushed slightly out of place, after which it would settle back to its equilibrium. Such an equilibrium is stable. If the marble, on the other hand, is placed on top of an inverted bowl, the situation becomes different. Any disturbance would cause the marble to roll off. We can ask the same question about a wave: Would a disturbance of the wave profile cause it to collapse? Or would the physics of the system attract it back to equilibrium?

Mathematically, we study equilibria in systems of ordinary differential equations through the linearization around the equilibrium point. More specifically, the eigenvalues of the matrix associated with the linearized dynamics around equilibrium indicate whether disturbances in the associated eigendirection grow or decay. The same approach applies to equilibria in PDE models, like traveling waves. The linearized dynamics around the equilibrium are described by a linear operator, whose spectral properties provide criteria for stability. In the context of strongly continuous semigroups on Hilbert spaces, this relation is formalized by the Gearhart-Prüss theorem [32, Theorem V.1.11].

The linearized dynamics around the solitary waves (1.2) are governed by the linear operator

$$\mathcal{L}_c = -\partial_x^3 + c\partial_x - 2\partial_x(\phi_c \cdot), \qquad c > 0, \tag{1.5}$$

which has a double eigenvalue at zero. The presence of a zero eigenvalue is a typical feature of the linearization around traveling waves. Indeed, any spatial translate of the wave profile,

$$\phi_c(x - \xi), \quad \xi \in \mathbb{R},$$

also satisfies (1.3), so the derivative of the wave profile with respect to x forms

[2] After their involvement in the Manhattan Project, the researchers from whom the FPUT system derives its name sought a problem suitable for probing the facility's state-of-the-art early computer, MANIAC. Instead of an infinite collection of oscillators, they considered only 64, with boundary conditions. Their hypothesis was that the system would thermalize after short time—i.e., that all vibrational modes would be excited evenly—a prediction that was contradicted by the numerical results.

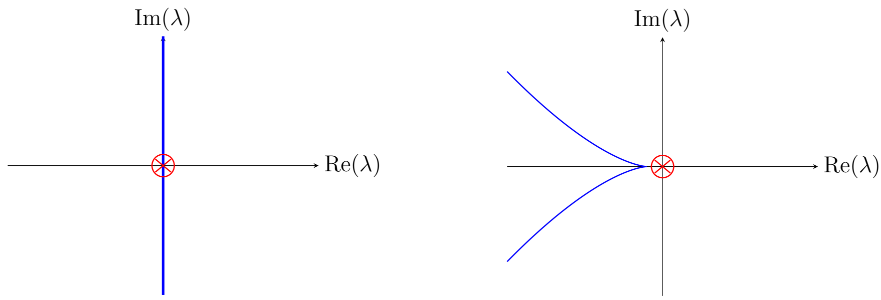

Figure 1.1: Spectrum of $\mathcal{L}_c$ on $L^2(\mathbb{R})$ (left) and on the exponentially weighted space $L^2(\mathbb{R}; e^{ax}\mathrm{d}x)$ with $a \in (0, \sqrt{c})$ (right). Both contain an isolated *double* eigenvalue at 0.

a neutral mode. As a result, perturbations of the wave generally lead to phase shifts, and stability must be considered modulo phase shifts. The free amplitude parameter in (1.2) introduces a second neutral mode, giving rise to the double eigenvalue at zero. Consequently, perturbations in the KdV setting additionally produce amplitude effects. On $L^2(\mathbb{R})$, the continuous spectrum of $\mathcal{L}_c$ lies on the imaginary axis, which implies that the system is not spectrally stable. For KdV solitons, this issue is often addressed by working in exponentially weighted L^2-spaces. This viewpoint effectively suppresses perturbations behind the solitary wave and shifts the continuous spectrum into the stable half-plane. See Figure 1.1.

Dispersion

Aside from their relevance to the stability of waves, the linearized dynamics provide more fundamental insights into the principal physics of the system, and shed light on the formation of waves. As previously discussed, waves can be viewed as an equilibrium, arising as a balance between linear restorative forces and nonlinear destabilizing effects. In the case of water waves described by the KdV equation, the role of restorative linear mechanism is played by *dispersion*. Dispersion refers to linear dynamics that admit *plane wave* solutions

$$e^{i(kx-\omega(k)t)}$$

of different wave numbers $k \in \mathbb{R}$, which travel at a frequency-dependent velocity $\omega(k)/k$. The relation $\omega(k)$ is called the *dispersion relation*.

The linear dynamics of the KdV equation around zero are given by the Airy equation

$$w_t(t,x) = -\partial_x^3 w(t,x), \quad t \in \mathbb{R}^+, x \in \mathbb{R}, \tag{1.6}$$

which satisfies a dispersion relation $\omega(k) = -3k^2$. Solutions can be explicitly represented by the convolution

$$w(t,\cdot) = (3t)^{-1/3}\mathrm{Ai}\left((3t)^{-1/3}\cdot\right) * w(0,\cdot), \quad t \neq 0,$$

with its oscillatory fundamental solution given by the Airy function

$$\mathrm{Ai}(x) := \frac{1}{\pi}\int_0^\infty \cos\left(\frac{k^3}{3} + xk\right) \mathrm{d}k, \quad x \in \mathbb{R}. \tag{1.7}$$

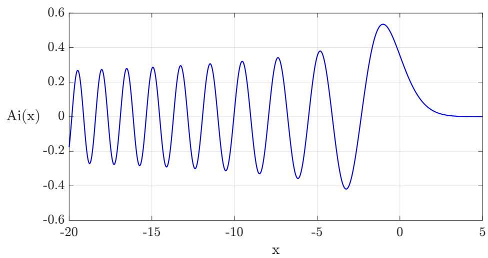

Figure 1.2: The Airy function $\mathrm{Ai}(x)$ describes the fundamental solution of the linear system (1.6), representing the response to an initial state concentrated at $x = 0$.

The restorative effect of the dispersive dynamics lies in the fact that the supremum (maximum) of localized initial conditions decay over time. More precisely,

$$\|w(t,\cdot)\|_{L^\infty} \leq C|t|^{-1/3}\|w(0,\cdot)\|_{L^1}, \quad t \neq 0, \tag{1.8}$$

for some $C > 0$. In particular, solitary waves cannot be supported by the linear dynamics alone! The bound (1.8) follows from the boundedness of the Airy function, and requires careful handling of the oscillatory integral (1.7). In many respects, dispersion is trickier to work with than its parabolic counterpart, diffusion. The main difference is that dispersion causes spatial modes to oscillate, whereas diffusion damps them. Consequently, the associated continuous spectrum covers all of the imaginary axis. For this reason, the stability of dispersive waves is typically studied in settings that provide additional stability. In this work, we consider weighted spaces which suppress persisting disturbances in the wake of a solitary wave.

As a consequence of its oscillatory nature, dispersion offers limited smoothing, since high-frequency components do not decay. The linear flow defined through (1.6) only provides smoothing in an averaged sense [67]. The intuition behind this dispersive smoothing, is that in view of the dispersion relation $\omega(k) = -k^2$, high-frequency components are rapidly transported to $-\infty$. This lack of straightforward smoothing complicates the local well-posedness theory for (1.1), the starting point of any rigorous analysis. Since the KdV nonlinearity $-\partial_x(u^2) = -2u\partial_x u$—see (1.1)—contains a derivative, it is hard to close estimates for a fixed-point argument. Nevertheless, this has been accomplished using delicate harmonic analysis in L^2-based spaces [10, 66], the most modern approach being due to Bourgain [14]. Consequently, we are somewhat restricted to L^2-based spaces in our study of the solitary waves and their stability.

This leads to a significant limitation that recurs throughout this thesis. Much of our analysis concerns the wake that forms behind a soliton when its propagation is disturbed. This wake typically extends far behind the soliton (see also Figure 1.3), and stability studies of KdV solitons therefore often disregard these persisting disturbances. The guiding idea is that the principal wave persists in front, while a long but low-magnitude wake trails behind. Consequently, the wake's L^2-norm may grow large over time, even though its L^∞-norm remains small. Exploiting this observation is difficult, however, since much of developed theory confines us to L^2-based spaces. Still, linear theory supports the intuition that the maximum of the wake remains small. This stems from the oscillatory nature of the Airy function, in particular, from the fact that

$$\sup_x \left| \int_{-\infty}^{x} \mathrm{Ai}(t)\mathrm{d}t \right| < \infty.$$

Nonlinear Structure

As mentioned above, this work and most research in the field concerns *nonlinear* PDEs. While this simply means that the PDE is not linear, research typically focuses on PDEs featuring some other nonlinear structure. Many PDE models arise from physical conservation laws, enforcing a natural dynamical structure. The KdV equation forms a good example. A key feature of its nonlinear structure is that it belongs to the class of Hamiltonian systems, since it has the form

$$u_t = \partial_x \mathcal{H}'(u), \tag{1.9}$$

where $\mathcal{H}$ is the Hamiltonian

$$\mathcal{H}(u) = \int_{\mathbb{R}} \tfrac{1}{2}(\partial_x u)^2 - \tfrac{1}{3}u^3 \mathrm{d}x, \tag{1.10}$$

and $\mathcal{H}'(u)$ is its functional derivative, or L^2-gradient, defined through the identity

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \mathcal{H}(u+\epsilon v) = \int_{\mathbb{R}} \mathcal{H}'(u)v \,\mathrm{d}x, \quad \forall v \in H^1.$$

A straightforward computation shows that for the Hamiltonian (1.10), we have $\mathcal{H}'(u) = -\partial_x^2 u - u^2$. The key structural point is that the operator ∂_x in (1.9) is skew-symmetric on L^2. Consequently, the evolution (1.9) *conserves* the Hamiltonian $\mathcal{H}(u)$, since the dynamics are orthogonal to the L^2-gradient of $\mathcal{H}$:

$$\partial_t \mathcal{H}(u) = \int_{\mathbb{R}} \mathcal{H}'(u)u_t \,\mathrm{d}x = \int_{\mathbb{R}} \mathcal{H}'(u)\partial_x \mathcal{H}'(u) \,\mathrm{d}x = 0.$$

Further details can be found in [65, §5.2]. The FPUT lattice system, studied in Chapter 5, satisfies an analogous structure, with the Hamiltonian and skew-symmetric operator both acting on the sequence space $\ell^2 \times \ell^2$. In addition to the Hamiltonian, the KdV dynamics conserve an infinite number of integral expressions such as (1.10), the simplest being the L^2-norm. The conserved quantities of the KdV equation form a hierarchy of 'integrals of motion', featuring an increasing number of derivatives. By Noether's Theorem [90, Chapter 4], each of these stems from a symmetry of the KdV equation, such as scaling invariance (1.4). They play a key role in the global well-posedness of the KdV equation: conserved quantities featuring the first k derivatives yield a priori bounds on the H^k-norm of solutions. The conservation of an *infinite* number of quantities forms a powerful tool. It turns out that the conserved quantities restrict the KdV dynamics to the extent that they can be decomposed into a dispersive part, and a number of right-traveling waves of the form (1.2) [48]. In the language of dynamical systems, the KdV equation is completely integrable, and can be solved using the inverse scattering transform.

The KdV nonlinearity, given by $-\partial_x(u^2) = -2u\partial_x u$, has a direct physical interpretation on its own. Upon ignoring the dispersion in (1.1), we obtain the nonlinear transport equation

$$v_t(t,x) = -2v(t,x)\partial_x v(t,x).$$

This transport-type equation describes how a field v at position x is transported at a speed proportional to its height. This *nonlinear steepening* effect forms the destabilizing mechanism that, counteracted by dispersion, leads to the formation of the solitary waves (1.2). It is evident that, like any solution to the KdV equation, the traveling wave $u(t,x) = \phi_c(x-ct)$ conserves the Hamiltonian (1.10) and other conserved integrals of motion. Indeed, all such conserved quantities are translation-invariant. In other words, the position of a wave does not affect its energy. However, the wave family (1.2) contains a more interesting degree of freedom: the parameter $c > 0$, which describes both the amplitude and speed of propagation. The conserved quantities do vary with *this* degree of freedom. In particular, the L^2-norm $\|\phi_c\|_{L^2}^2 = 6c^{3/2}$ increases with amplitude. This leads to a central question that is explored in this thesis:

Q1: *How is the propagation of solitary waves affected by external variation in energy?*

Amplitude Modulation

Before giving a qualitative answer to this question, let us first clarify what is meant by 'external variation in energy'. Although the principal physics of shallow water physics is captured by (1.1), secondary effects may alter the conservative structure in realistic settings. For instance, slow energy loss may occur through viscous dissipation, or variable material properties such as bottom topography may effectively supply energy. In this thesis, we focus primarily on the mathematical structure of the forcing mechanism. We consider perturbations to (1.1) of the multiplicative

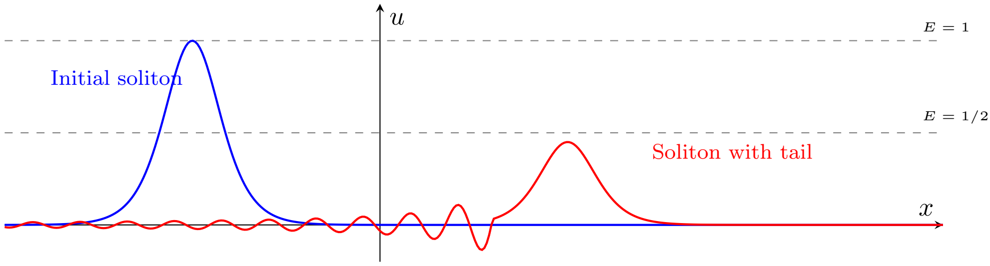

Figure 1.3: An initial solitary wave with energy 1 (blue) propagates through a dissipative medium, resulting in a total energy 1/2. The resulting wave (red) has formed a dispersive tail, and has slightly less than half the original energy.

form

$$u_t(t,x) = -\partial_x^3 u(t,x) - \partial_x(u^2(t,x)) + \sigma\xi(t,x)u(t,x), \qquad t \in \mathbb{R}^+, x \in \mathbb{R}, \tag{1.11}$$

where $\xi(t,x)$ models a generic forcing effect and $\sigma \geq 0$ is a small parameter that controls its strength. The multiplicative form of (1.11) ensures that changes in energy occur proportional to surface height. As a result of the $\mathcal{O}(\sigma)$ forcing, the KdV invariants such as the L^2-norm undergo slow changes. For the most part in this thesis, with the exception of Chapter 3, we consider forcing mechanisms $\xi(t,x)$ of a random nature. More on that later.

To understand how multiplicative forcing affects soliton propagation, let us briefly recap what we know so far. There exist solitary waves solutions to the unperturbed KdV equation, which come in a family of different amplitudes, and travel at a proportional velocity. Each amplitude corresponds to an energy level, and each associated wave profile is stable. Say we start with a wave of L^2-norm, or energy, $E = 1$. We wait until a dissipative forcing mechanism has reduced that to $E = 1/2$. The original wave cannot survive: there is not enough energy in the system to support it. But what then will this result in? Will a wave form with amplitude corresponding to half the energy? The answer is largely yes! But some important limitations apply.

A1: *Slow energy change leads to amplitude modulation: most of the energy change remains concentrated in the solitary wave, affecting its amplitude. A smaller part of the energy manifests in a dispersive tail behind the solitary wave.*

Thus, when a solitary wave is perturbed, it is no longer a pure wave given by the profile equation (1.2). One of the main goals of this work is to understand the dispersive tail that forms behind the wave, and to precisely determine the resulting amplitude.

Q2: *How exactly are the wave position and amplitude modulated by the energy change?*

Q3: *How much amplitude modulation can a solitary wave undergo before it loses* *coherence?*

Question 2 is addressed in Chapter 2 of this thesis, where we present and study a method to track the amplitude and position of modulated solitons. Question 3 is more technical, and is rigorously treated in Chapter 3 and Chapter 4. One of the main technical challenges lies in the fact that we do not simply study the stability of a single object. Rather, we encounter a phenomenon of *metastability*: forced dynamics near an attracting manifold of stable wave profiles. Thus, we study dynamics near a time-varying equilibrium. Even in finite-dimensional systems, linear stability properties of equilibria in general do not persist to a time-varying setting [107]. In Chapter 3 and Chapter 4, we explore two different methods of leveraging the linear stability properties of the wave family (1.2).

Chapter 2 and Chapter 3 address this issue by analyzing the dispersive tail, or wake, in a rescaled frame. This rescaling neutralizes the amplitude changes of the soliton, thereby removing the time-dependence of the equilibrium. A key difficulty arises from the fact that the appropriate rescaling is generated by the operator $I + \frac{1}{2}x\partial_x$, as seen from the identity

$$\partial_c \phi_c(x) = c^{-1}(1 + \tfrac{1}{2}x\partial_x)\phi_c(x).$$

By the chain rule, the evolution of the dispersive tail $v(t,x)$ in the rescaled frame contains a linear contribution of the form

$$v_t(t,x) = (1 + \tfrac{1}{2}x\partial_x)v(t,x) + \dots. \tag{1.12}$$

The factor x is problematic, since it obstructs closing estimates in L^p-spaces. In Chapter 3, we address this issue by adapting the commonly used exponentially weighted framework for studying soliton stability in the KdV setting. We introduce *time-varying* exponential weights that adjust according to the applied rescaling.

Stochastics

As the title suggests, this thesis focuses primarily on *random* mechanisms that affect soliton propagation. In particular, two chapters of this thesis study wave propagation through stochastic partial differential equations (SPDEs). While this is a modern topic that introduces several complications, most of the probabilistic tools used in this work are by now standard, and our technical contributions are principally deterministic in nature.

Chapter 2 and Chapter 4 consider the stochastic KdV equation

$$\mathrm{d}u = -(\partial_x^3 u + 2u\partial_x u)\ \mathrm{d}t + \sigma u\ \mathrm{d}W_t^Q. \tag{1.13}$$

Here, the noise process W_t^Q is a cylindrical Wiener process taking values in $L^2(\mathbb{R})$ and the small parameter $\sigma \geq 0$ controls the strength of the random forcing. The noise is white in time and colored in space with translation-invariant statistics given by the formal identity

$$\mathbb{E}\big[W^Q(t,x)W^Q(t,y)\big] = q(x-y)\min\{s,t\}, \quad x,y \in \mathbb{R}, \quad s,t > 0.$$

For those unfamiliar with SPDEs, (1.13) should be thought of as (1.11), with $\xi(t,x)$ being a random field, statistically independent at each time t, and spatially correlated as

$$\mathbb{E}[\xi(t,x)\xi(t,y)] = q(x-y).$$

For technical reasons, this interpretation cannot fully be justified and there truly is a need to resort to (infinite dimensional) stochastic calculus. We refer to [23] and [74] for introductions into this topic.

Some physical motivation for studying SPDE models such as (1.13) lies in the fact that many PDE models are derived as a continuum limit from random particle models. In this context, stochastic forcing serves as a tool for studying such models in circumstances where higher-order contributions play a role. Another motivation is that random forcing can be useful substitute for studying rough environmental conditions, such as an irregular bottom topography in the case of Russell's wave. While a rough bottom topography is not a random object, it can be insightful to model it as such. A derivation of (1.13) in a specific physical context can be found in [59].

It is important to distinguish SPDEs from various other settings that include randomness into PDE dynamics. Specifically, the dynamics resulting from (1.13) are not just those of the KdV equation disturbed by noise after the fact. Rather, the effect of random forcing interacts with the dynamics of the PDE, resulting in a true random dynamical system. To appreciate this point, let us briefly return to the example of a marble resting at the bottom of a convex bowl. In the stochastic setting, we consider the marble to be disturbed by random kicks. For example, by turbulent wind conditions. One needs to take into consideration that the random kicks may displace it to a point where it exits the bowl! Hence, the stochastic forcing may lead it into regions where the physics of the system act drastically different. This is a much more complicated setting than, say, observing a marble resting at the bottom of a bowl through noisy measurements.

To illustrate one of the technical challenges that arise in the SPDE setting, recall that one of our approaches for handling the time dependence of the linear operator governing tail formation is to apply a rescaling generated by $I + \frac{1}{2}x\partial_x$. In the deterministic setting, this produces linear contributions of the form (1.12). In the SPDE setting, however, such coordinate transformations are governed by the Itô formula, or stochastic chain rule, which introduces an additional term of the form $(I + \frac{1}{2}x\partial_x)^2$. This contribution cannot be readily handled using the techniques developed in Chapter 3. In Chapter 4, we analyze the dispersive tail without using a scaling transformation, which leaves the challenge of working near a time-varying equilibrium. We address this by combining local and global tracking mechanisms. The local tracking mechanisms provide stability over time intervals where the wave amplitude varies little. The global tracking mechanism monitors the accumulation of these fluctuations and integrates the stability results from the local intervals.

The answers to our guiding questions **Q1-Q3** should be reconsidered in this stochastic context: The fluctuations of the solitary wave amplitude are random in nature (see Figure 1.4), as is the formation of its dispersive tail. Whether the wave loses coherence within a given time window is a random event, whose distribution depends on the forcing strength σ. Even so, the stochastic forcing may cause 'deterministic' effects such as slow amplitude attenuation (Chapter 5). These research questions belong to the subfield of *stochastic traveling waves*, which has become a major area of interest within the field of SPDEs. Most earlier work considers traveling waves in reaction-diffusion equations, focusing on stochastic effects on the wave position. The main contribution of this thesis is that it studies stochastic traveling waves in a setting where position modulation and amplitude fluctuations are coupled. We develop both qualitative insights and rigorous results on nonlinear dispersive waves undergoing large stochastic amplitude fluctuations.

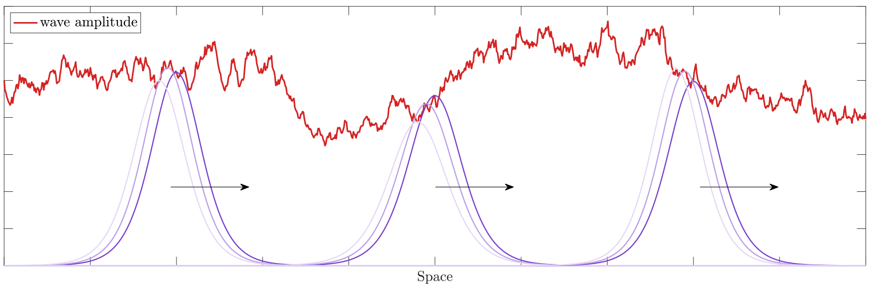

Figure 1.4: Schematic representation of a solitary wave undergoing random changes in amplitude. Three snapshots (purple) show the solitary wave as it moves rightward through space. The red line traces the random trajectory of the wave amplitude.

1.2 Contributions of this Thesis

The main body of this thesis is composed of Chapters 2–5, based on four scientific papers that resulted from this research.

Chapter 2: Amplitude tracking

[102] R.W.S. Westdorp, H.J. Hupkes, *Long-Timescale Soliton Dynamics in the Korteweg-de Vries Equation with Multiplicative Translation-Invariant Noise*, Physica D.

In this chapter, we present our mechanism for tracking the amplitude and position of KdV solitons affected by stochastic forcing. We introduce an effective position $\xi(t)$ and amplitude $c(t)$ of the stochastic waves, and characterize solutions

u to (1.13) near the wave-family (1.2) through a decomposition

$$\underbrace{u(t,x)}_{\text{full solution}} = \underbrace{\phi_{c(t)}(x-\xi(t))}_{\text{modulated soliton}} + \text{dispersive tail}. \tag{1.14}$$

Our construction of the position $\xi(t)$ and amplitude $c(t)$ corresponds to the *variational phase*, and is based on the linear stability properties of the wave family (1.2) on exponentially weighted spaces [91]. The effective position and amplitude are stochastic processes; solutions to SDEs driven by the stochastic forcing. We characterize their evolution with help of the scaling invariance (1.4).

In order to gain insight into the stochastic soliton dynamics described by our method, we present an approximation procedure for the processes $c(t)$ and $\xi(t)$. A complicating factor is that the evolution of the amplitude is coupled to the dispersive tail. In order to account for this, we developed a non-standard nested approximation procedure, which produces SDE approximations for $c(t)$ that have random coefficients. These approximations can be computed at any desired order in the noise strength σ.

In case the noise process W_t^Q in (1.13) is a scalar Brownian motion, we find that the amplitude process $c(t)$ behaves primarily according to a geometric Brownian motion. The soliton velocity follows the amplitude process $c(t)$, with additional correction due to the noise. The amplitude process $c(t)$ experiences an almost linear positive drift which develops on the time-scale $\mathcal{O}(\sigma^2 t)$. The explicit predictions of our method are verified throughout Chapter 2 via extensive numerical simulations, showcasing the predictive power of our work.

Chapter 3: Deterministic Stability

[103] R.W.S. Westdorp, H.J. Hupkes, *Soliton Amplification in the Korteweg-de Vries Equation by Multiplicative Forcing*, Communications on Pure and Applied Analysis

The decomposition (1.14) describes solutions to (1.11) as near-solitons, as long as the dispersive tail remains small. In Chapter 3, we rigorously establish this fact in the context of *deterministic* forcing, using a novel stability analysis involving time-varying weights. In particular, we establish the orbital stability of the traveling-wave family (1.2) under the influence of deterministic multiplicative forcing. We show that solitons evolving via the forced KdV equation

$$u_t = -\partial_x^3 u - 2u\partial_x u + f(t)u \tag{1.15}$$

remain close to the family (1.2) in the energy norm, while undergoing a large change in amplitude. The deterministic equation (1.15) forms a stepping stone for our study of the SPDE (1.13). Chapter 3 contains no stochastic elements, and is hence more broadly accessible to readers focused on deterministic PDEs. We uncover several technical aspects regarding the coherence of solitary waves subject to amplitude modulation. In particular, we gain understanding in the dual role of weighted and unweighted spaces to characterize the formation of the dispersive tail behind the soliton.

Chapter 4: Stochastic Stability

[104] R.W.S. Westdorp, H.J. Hupkes, *Stability of Stochastically Forced Solitons in the Korteweg-de Vries Equation* (preprint)

Chapter 4 collects our rigorous results on the stability of KdV solitons subject to multiplicative stochastic forcing. Building on the stability results of Chapter 3, we return to the stochastic setting of Chapter 2 and assert that KdV solitons remain coherent while possibly undergoing large fluctuations in amplitude. While the methods of Chapter 3 rely on the global scaling invariance (1.4), in Chapter 4, we combine local stability methods that follow the amplitude fluctuations of the wave. This method proves to be better adapted to the stochastic setting, as it requires no coordinate transformation. As in Chapter 3, our analysis consists of two complementary parts: we control disturbances of the waves using linear stability tools on weighted spaces, and we employ energy methods to characterize the growth of the dispersive tail in unweighted spaces.

Chapter 5: Lattice Waves

[63] H.J. Hupkes, J.A. McGinnis, R.W.S. Westdorp, and J.D. Wright, *Radiating Solitary Waves in an FPUT Lattice with Random Coefficients* (preprint)

This last chapter of the thesis departs from the continuous setting of the KdV equation, and turns to the Fermi-Pasta-Ulam-Tsingou (FPUT) lattice system. This discrete system models an infinite chain of masses connected by springs. In this chapter, we consider wave propagation in the FPUT lattice system affected by random variation in the linear spring force. In absence of random forcing, the FPUT lattice system shares a famous connection to the KdV equation. In an appropriate continuum limit, its dynamics are described by an effective KdV equation. Moreover, the lattice system admits solitary waves which, in the long-wave limit, behave like the KdV solitons. Unlike the stochastic forcing in Chapters 2–4, the random spring coefficients respect the Hamiltonian structure of the FPUT system. As such, solitary waves propagate largely undisturbed. However, they slowly radiate energy in the form of a dispersive tail. As a consequence, their amplitude slowly decreases with time. Exploiting the connection of the FPUT lattice to the KdV equation, we explicitly characterize the amplitude attenuation caused by the random environment, showcasing the broader applicability of our methods.