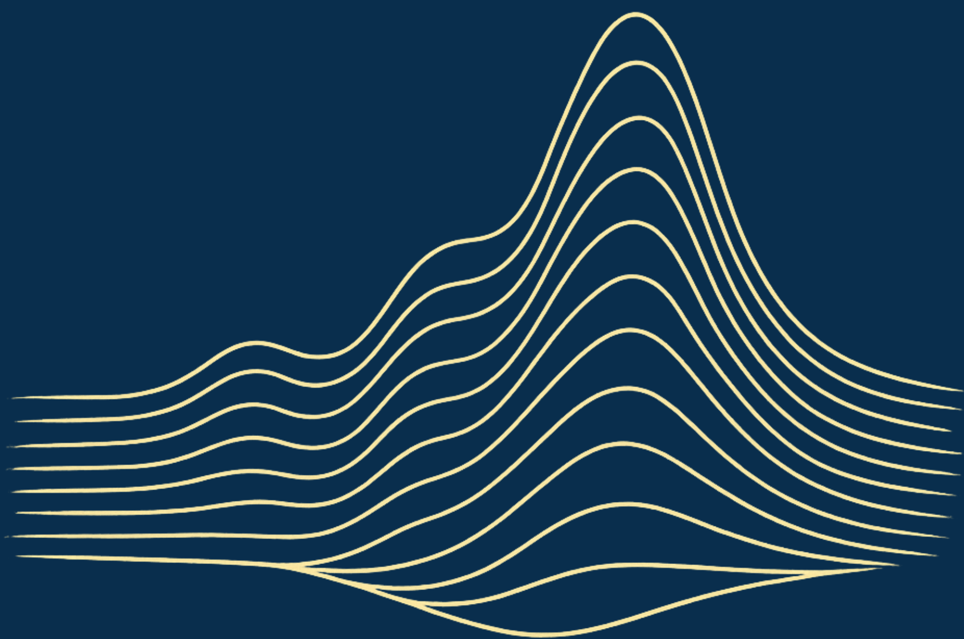
# STOCHASTIC AMPLITUDE MODULATION



# of NONLINEAR DISPERSIVE WAVES

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# Stochastic Amplitude Modulation of Nonlinear Dispersive Waves

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# CHAPTER 1
# Introduction

## 1.1 Background

This thesis studies mathematical models describing the propagation of waves affected by random disturbances. Focusing on a paradigm nonlinear wave model—the Korteweg–de Vries equation—this work explores how random fluctuations in energy modulate the amplitude of solitary waves, both rigorously and through formal analysis. While the subject matter is rooted in physics, this thesis deals with the rich *mathematical* theory underlying wave phenomena. As such, the topic sits at the intersection of *Dynamical Systems*, to describe wave dynamics, and *Probability Theory*, accounting for random effects. Chapters 2–5 form the main body of this thesis and contain its mathematical contributions. The aim of this introductory chapter is to provide both an intuitive and a mathematical foundation for the reader regarding the core concepts of this thesis. We unpack the thesis title—*Stochastic Amplitude Modulation of Nonlinear Dispersive Waves*—back to front:

1. *Waves*
2. *Dispersion*
3. *Nonlinear (Structure)*
4. *Amplitude Modulation*
5. *Stochastics*

### Waves

Wave phenomena have long been prevalent in physics, both classical and modern. While it remains an active field of research, its fundamental concepts are easily accessible. Most people are familiar with waves in nature: sound waves used for communication, surface waves on bodies of water, or even visible light. The common feature among these examples is that *something* propagates. Sound consists of pressure differences propagating through air. In bodies of water, fluid flow organizes to carry disturbances across the surface. Maxwell's laws of electromagnetism describe how electric and magnetic fields interact to transmit light through a vacuum. In each case, the underlying physics result in propagation of disturbances through the medium[1].

The wave phenomenon that inspired the topic of this thesis was first observed in 1834 by civil engineer and shipbuilder John Scott Russell:

> *"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."* [93]

Russell describes a remarkable type of water wave: a solitary elevation that continues along the channel without change of form or diminution of speed. Or rather, *almost* without change of form, as he notes that the wave height gradually diminished. Russell's encounter was, of course, not the first observation ever made of a water wave. The surprise lies in the fact that, out of a seemingly chaotic setting ("violent agitation"), a well-defined structure emerged and propagated undisturbed. Interestingly, the gradual loss of height—presumably due to imperfections in the environment—closely aligns with the topic of this thesis. Russell's observations later sparked mathematical interest: in 1871, Boussinesq derived a model for shallow water dynamics that explained the wave phenomenon [15]. His work was extended in 1895 by Korteweg and de Vries [68]. The resulting Korteweg–de Vries (KdV) equation has since become a paradigm model in nonlinear wave theory.

**Wave models** Mathematically, the physics of wave phenomena—like many topics in classical physics—are described by partial differential equations (PDEs), encoding the continuous dependence of quantities on space and time. Waves are modeled as solutions $u(t,x)$ to a PDE, where $t \in \mathbb{R}$ denotes physical time and $x \in \mathbb{R}^d$ denotes physical space in $d \geq 1$ dimensions. The function-value $u(t,x)$ models an evolving physical quantity at time $t$ and position $x$, and can be $\mathbb{R}^n$- or $\mathbb{C}^n$-valued ($n \geq 1$) depending on the physical context.

[1] Strictly speaking, light is not carried by a medium, as demonstrated by the famous Michelson-Morley experiment [83].

Models for Russell's 'Wave of Translation' usually take $x \in \mathbb{R}$ for the position along the canal, assuming that no relevant dynamics occur in the directions transverse to propagation. The scalar-valued function $u$ then models the height of the water as it evolves over time. A traveling wave, is a solution that admits the representation $u(t,x) = \phi(x - ct)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a fixed wave profile propagating at velocity $c$. A *solitary* wave as described by Russell concerns the case that $\phi$ is continuous and meets asymptotic decay conditions.

A word of caution: wave phenomena are not synonymous with the wave equation. The wave equation is a second order linear PDE whose solutions are described via d'Alemberts principle: disturbances propagate left- and rightward. Hence, in the classic wave-equation, all initial conditions give rise to a superposition of wave solutions. It is called *the* wave equation because it can be derived for simple wave phenomena such as waves on a string and electromagnetic waves. However, our interest lies in systems which *admit* a traveling wave solution. Meaning, the (unique) existence of a profile which propagates and retains its shape under the PDE dynamics. Traveling waves are among the many coherent structures that arise in nonlinear PDEs—such as stripes, spirals, breathers or spots—which form specific and nontrivial subclasses of solutions.

Chapters 2–4 of this thesis deal with (adaptations of) the KdV equation:

$$u_t(t,x) = -\partial_x^3 u(t,x) - \partial_x(u^2(t,x)), \qquad t \in \mathbb{R}^+, x \in \mathbb{R}. \tag{1.1}$$

This equation was originally derived by Boussinesq based on the physics of shallow water flow with a free boundary, which, after several approximations, simplifies to (1.1). The model admits traveling wave solutions $u(t,x) = \phi_c(x - ct)$, where the wave profiles

$$\phi_c(x) = \tfrac{3c}{2} \operatorname{sech}^2(\sqrt{c}x/2), \quad c > 0, \tag{1.2}$$

solve the traveling wave equation

$$-c\partial_x \phi_c(x) = -\partial_x^3 \phi_c(x) - \partial_x(\phi_c^2(x)), \quad x \in \mathbb{R}. \tag{1.3}$$

We remark that the solitary waves $\phi_c$ travel only to the right, i.e. towards $+\infty$, due to the fact that (1.1) breaks reflection symmetry. As seen in (1.2), the traveling waves $\phi_c$ appear as a family of waves with various amplitudes proportional to their velocity $c$. The family (1.2) satisfies the self-similarity property $\phi_c(x) = c\phi_1(\sqrt{c}x)$. This is a direct consequence of the scaling invariance

$$u(t,x) \mapsto \alpha^2 u(\alpha^3 t, \alpha x) \tag{1.4}$$

of the KdV equation, meaning that any solution $u(t,x)$ to (1.1) gives rise to rescaled solutions via the transformation (1.4), with $\alpha \in \mathbb{R}$.

In the final chapter of this thesis we turn our attention to a *discrete* wave-model related to the KdV equation: the Fermi-Pasta-Ulam-Tsingou (FPUT) lattice system. Contrary to (1.1), which models water-surface height as a function of *continu-*

*ous* space, the FPUT system describes the evolution of a *countable* set of quantities. The FPUT system was famously used to help pioneer the field of numerical mathematics in the 1950s [39]. It models an infinite collection of coupled oscillators[2], or simply, masses connected by springs. The FPUT system describes how the chain of masses oscillates when brought out of equilibrium. Like in the KdV equation, the FPUT system gives rise to solitary waves. In fact, the dynamics of the particle chain are approximated by the KdV equation in an appropriate continuum limit [109].

**Waves as an equilibrium** Since waves retain their shape in an otherwise dynamic environment, it is natural to view them as a 'balance' point, or equilibrium. From an observer's perspective, like Russell chasing the Wave of Translation on horseback, a wave is a fixed object. We might compare this to a marble placed at the bottom of a convex bowl: it rests at a spot where it can sit perfectly still. The marble can be pushed slightly out of place, after which it would settle back to its equilibrium. Such an equilibrium is stable. If the marble, on the other hand, is placed on top of an inverted bowl, the situation becomes different. Any disturbance would cause the marble to roll off. We can ask the same question about a wave: Would a disturbance of the wave profile cause it to collapse? Or would the physics of the system attract it back to equilibrium?

Mathematically, we study equilibria in systems of ordinary differential equations through the linearization around the equilibrium point. More specifically, the eigenvalues of the matrix associated with the linearized dynamics around equilibrium indicate whether disturbances in the associated eigendirection grow or decay. The same approach applies to equilibria in PDE models, like traveling waves. The linearized dynamics around the equilibrium are described by a linear operator, whose spectral properties provide criteria for stability. In the context of strongly continuous semigroups on Hilbert spaces, this relation is formalized by the Gearhart-Prüss theorem [32, Theorem V.1.11].

The linearized dynamics around the solitary waves (1.2) are governed by the linear operator

$$\mathcal{L}_c = -\partial_x^3 + c\partial_x - 2\partial_x(\phi_c\cdot), \qquad c > 0, \tag{1.5}$$

which has a double eigenvalue at zero. The presence of a zero eigenvalue is a typical feature of the linearization around traveling waves. Indeed, any spatial translate of the wave profile,

$$\phi_c(x-\xi), \quad \xi \in \mathbb{R},$$

also satisfies (1.3), so the derivative of the wave profile with respect to $x$ forms

[2]After their involvement in the Manhattan Project, the researchers from whom the FPUT system derives its name sought a problem suitable for probing the facility's state-of-the-art early computer, MANIAC. Instead of an infinite collection of oscillators, they considered only 64, with boundary conditions. Their hypothesis was that the system would thermalize after short time—i.e., that all vibrational modes would be excited evenly—a prediction that was contradicted by the numerical results.

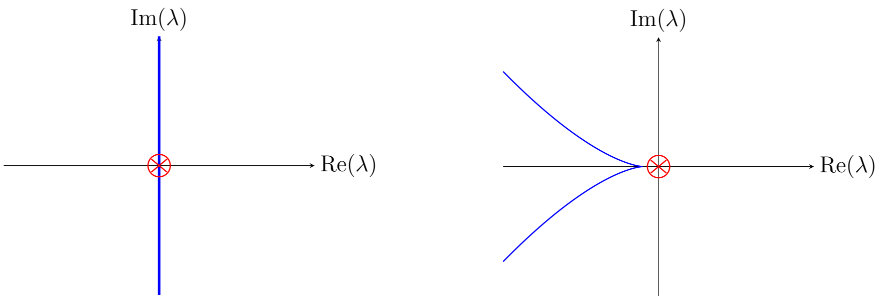



**Figure 1.1:** Spectrum of $\mathcal{L}_c$ on $L^2(\mathbb{R})$ (left) and on the exponentially weighted space $L^2(\mathbb{R}; e^{ax}\mathrm{d}x)$ with $a \in (0, \sqrt{c})$ (right). Both contain an isolated *double* eigenvalue at 0.

a neutral mode. As a result, perturbations of the wave generally lead to phase shifts, and stability must be considered modulo phase shifts. The free amplitude parameter in (1.2) introduces a second neutral mode, giving rise to the double eigenvalue at zero. Consequently, perturbations in the KdV setting additionally produce amplitude effects. On $L^2(\mathbb{R})$, the continuous spectrum of $\mathcal{L}_c$ lies on the imaginary axis, which implies that the system is not spectrally stable. For KdV solitons, this issue is often addressed by working in exponentially weighted $L^2$-spaces. This viewpoint effectively suppresses perturbations behind the solitary wave and shifts the continuous spectrum into the stable half-plane. See Figure 1.1.

## Dispersion

Aside from their relevance to the stability of waves, the linearized dynamics provide more fundamental insights into the principal physics of the system, and shed light on the formation of waves. As previously discussed, waves can be viewed as an equilibrium, arising as a balance between linear restorative forces and nonlinear destabilizing effects. In the case of water waves described by the KdV equation, the role of restorative linear mechanism is played by *dispersion*. Dispersion refers to linear dynamics that admit *plane wave* solutions

$$e^{i(kx-\omega(k)t)}$$

of different wave numbers $k \in \mathbb{R}$, which travel at a frequency-dependent velocity $\omega(k)/k$. The relation $\omega(k)$ is called the *dispersion relation*.

The linear dynamics of the KdV equation around zero are given by the Airy equation

$$w_t(t,x) = -\partial_x^3 w(t,x), \quad t \in \mathbb{R}^+, x \in \mathbb{R}, \tag{1.6}$$

which satisfies a dispersion relation $\omega(k) = -3k^2$. Solutions can be explicitly represented by the convolution

$$w(t,\cdot) = (3t)^{-1/3}\mathrm{Ai}\left((3t)^{-1/3}\cdot\right) * w(0,\cdot), \quad t \neq 0,$$

with its oscillatory fundamental solution given by the Airy function

$$\mathrm{Ai}(x) := \frac{1}{\pi}\int_0^\infty \cos\left(\frac{k^3}{3} + xk\right) \mathrm{d}k, \quad x \in \mathbb{R}. \tag{1.7}$$

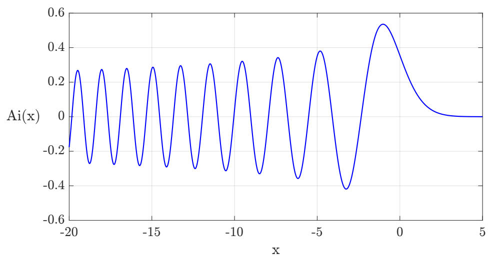



**Figure 1.2:** The Airy function $\mathrm{Ai}(x)$ describes the fundamental solution of the linear system (1.6), representing the response to an initial state concentrated at $x = 0$.

The restorative effect of the dispersive dynamics lies in the fact that the supremum (maximum) of localized initial conditions decay over time. More precisely,

$$\|w(t,\cdot)\|_{L^\infty} \leq C|t|^{-1/3}\|w(0,\cdot)\|_{L^1}, \quad t \neq 0, \tag{1.8}$$

for some $C > 0$. In particular, solitary waves cannot be supported by the linear dynamics alone! The bound (1.8) follows from the boundedness of the Airy function, and requires careful handling of the oscillatory integral (1.7). In many respects, dispersion is trickier to work with than its parabolic counterpart, diffusion. The main difference is that dispersion causes spatial modes to oscillate, whereas diffusion damps them. Consequently, the associated continuous spectrum covers all of the imaginary axis. For this reason, the stability of dispersive waves is typically studied in settings that provide additional stability. In this work, we consider weighted spaces which suppress persisting disturbances in the wake of a solitary wave.

As a consequence of its oscillatory nature, dispersion offers limited smoothing, since high-frequency components do not decay. The linear flow defined through (1.6) only provides smoothing in an averaged sense [67]. The intuition behind this dispersive smoothing, is that in view of the dispersion relation $\omega(k) = -k^2$, high-frequency components are rapidly transported to $-\infty$. This lack of straightforward smoothing complicates the local well-posedness theory for (1.1), the starting point of any rigorous analysis. Since the KdV nonlinearity $-\partial_x(u^2) = -2u\partial_x u$—see (1.1)—contains a derivative, it is hard to close estimates for a fixed-point argument. Nevertheless, this has been accomplished using delicate harmonic analysis in $L^2$-based spaces [10, 66], the most modern approach being due to Bourgain [14]. Consequently, we are somewhat restricted to $L^2$-based spaces in our study of the solitary waves and their stability.

This leads to a significant limitation that recurs throughout this thesis. Much of our analysis concerns the wake that forms behind a soliton when its propagation is disturbed. This wake typically extends far behind the soliton (see also Figure 1.3), and stability studies of KdV solitons therefore often disregard these persisting disturbances. The guiding idea is that the principal wave persists in front, while a long but low-magnitude wake trails behind. Consequently, the wake's $L^2$-norm may grow large over time, even though its $L^\infty$-norm remains small. Exploiting this observation is difficult, however, since much of developed theory confines us to $L^2$-based spaces. Still, linear theory supports the intuition that the maximum of the wake remains small. This stems from the oscillatory nature of the Airy function, in particular, from the fact that

$$\sup_x \Big| \int_{-\infty}^{x} \mathrm{Ai}(t)\mathrm{d}t \Big| < \infty.$$

## Nonlinear Structure

As mentioned above, this work and most research in the field concerns *nonlinear* PDEs. While this simply means that the PDE is not linear, research typically focuses on PDEs featuring some other nonlinear structure. Many PDE models arise from physical conservation laws, enforcing a natural dynamical structure. The KdV equation forms a good example. A key feature of its nonlinear structure is that it belongs to the class of Hamiltonian systems, since it has the form

$$u_t = \partial_x \mathcal{H}'(u), \tag{1.9}$$

where $\mathcal{H}$ is the Hamiltonian

$$\mathcal{H}(u) = \int_{\mathbb{R}} \tfrac{1}{2}(\partial_x u)^2 - \tfrac{1}{3}u^3 \mathrm{d}x, \tag{1.10}$$

and $\mathcal{H}'(u)$ is its functional derivative, or $L^2$-gradient, defined through the identity

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \mathcal{H}(u+\epsilon v) = \int_{\mathbb{R}} \mathcal{H}'(u) v \ \mathrm{d}x, \quad \forall v \in H^1.$$

A straightforward computation shows that for the Hamiltonian (1.10), we have $\mathcal{H}'(u) = -\partial_x^2 u - u^2$. The key structural point is that the operator $\partial_x$ in (1.9) is skew-symmetric on $L^2$. Consequently, the evolution (1.9) *conserves* the Hamiltonian $\mathcal{H}(u)$, since the dynamics are orthogonal to the $L^2$-gradient of $\mathcal{H}$:

$$\partial_t \mathcal{H}(u) = \int_{\mathbb{R}} \mathcal{H}'(u) u_t \ \mathrm{d}x = \int_{\mathbb{R}} \mathcal{H}'(u) \partial_x \mathcal{H}'(u) \ \mathrm{d}x = 0.$$

Further details can be found in [65, §5.2]. The FPUT lattice system, studied in Chapter 5, satisfies an analogous structure, with the Hamiltonian and skew-symmetric operator both acting on the sequence space $\ell^2 \times \ell^2$. In addition to the Hamiltonian, the KdV dynamics conserve an infinite number of integral expressions such as (1.10), the simplest being the $L^2$-norm. The conserved quantities of the KdV equation form a hierarchy of 'integrals of motion', featuring an increasing number of derivatives. By Noether's Theorem [90, Chapter 4], each of these stems from a symmetry of the KdV equation, such as scaling invariance (1.4). They play a key role in the global well-posedness of the KdV equation: conserved quantities featuring the first $k$ derivatives yield a priori bounds on the $H^k$-norm of solutions. The conservation of an *infinite* number of quantities forms a powerful tool. It turns out that the conserved quantities restrict the KdV dynamics to the extent that they can be decomposed into a dispersive part, and a number of right-traveling waves of the form (1.2) [48]. In the language of dynamical systems, the KdV equation is completely integrable, and can be solved using the inverse scattering transform.

The KdV nonlinearity, given by $-\partial_x(u^2) = -2u\partial_x u$, has a direct physical interpretation on its own. Upon ignoring the dispersion in (1.1), we obtain the nonlinear transport equation

$$v_t(t,x) = -2v(t,x)\partial_x v(t,x).$$

This transport-type equation describes how a field $v$ at position $x$ is transported at a speed proportional to its height. This *nonlinear steepening* effect forms the destabilizing mechanism that, counteracted by dispersion, leads to the formation of the solitary waves (1.2). It is evident that, like any solution to the KdV equation, the traveling wave $u(t,x) = \phi_c(x-ct)$ conserves the Hamiltonian (1.10) and other conserved integrals of motion. Indeed, all such conserved quantities are translation-invariant. In other words, the position of a wave does not affect its energy. However, the wave family (1.2) contains a more interesting degree of freedom: the parameter $c > 0$, which describes both the amplitude and speed of propagation. The conserved quantities do vary with *this* degree of freedom. In particular, the $L^2$-norm $\|\phi_c\|_{L^2}^2 = 6c^{3/2}$ increases with amplitude. This leads to a central question that is explored in this thesis:

**Q1**: *How is the propagation of solitary waves affected by external variation in energy?*

## Amplitude Modulation

Before giving a qualitative answer to this question, let us first clarify what is meant by 'external variation in energy'. Although the principal physics of shallow water physics is captured by (1.1), secondary effects may alter the conservative structure in realistic settings. For instance, slow energy loss may occur through viscous dissipation, or variable material properties such as bottom topography may effectively supply energy. In this thesis, we focus primarily on the mathematical structure of the forcing mechanism. We consider perturbations to (1.1) of the multiplicative

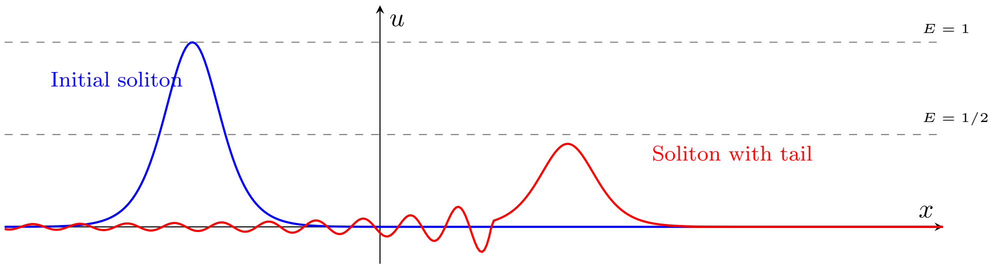



**Figure 1.3:** An initial solitary wave with energy 1 (blue) propagates through a dissipative medium, resulting in a total energy 1/2. The resulting wave (red) has formed a dispersive tail, and has slightly less than half the original energy.

form

$$u_t(t,x) = -\partial_x^3 u(t,x) - \partial_x(u^2(t,x)) + \sigma\xi(t,x)u(t,x), \qquad t \in \mathbb{R}^+, x \in \mathbb{R}, \tag{1.11}$$

where $\xi(t,x)$ models a generic forcing effect and $\sigma \geq 0$ is a small parameter that controls its strength. The multiplicative form of (1.11) ensures that changes in energy occur proportional to surface height. As a result of the $\mathcal{O}(\sigma)$ forcing, the KdV invariants such as the $L^2$-norm undergo slow changes. For the most part in this thesis, with the exception of Chapter 3, we consider forcing mechanisms $\xi(t,x)$ of a random nature. More on that later.

To understand how multiplicative forcing affects soliton propagation, let us briefly recap what we know so far. There exist solitary waves solutions to the unperturbed KdV equation, which come in a family of different amplitudes, and travel at a proportional velocity. Each amplitude corresponds to an energy level, and each associated wave profile is stable. Say we start with a wave of $L^2$-norm, or energy, $E = 1$. We wait until a dissipative forcing mechanism has reduced that to $E = 1/2$. The original wave cannot survive: there is not enough energy in the system to support it. But what then will this result in? Will a wave form with amplitude corresponding to half the energy? The answer is largely yes! But some important limitations apply.

**A1**: *Slow energy change leads to amplitude modulation: most of the energy change remains concentrated in the solitary wave, affecting its amplitude. A smaller part of the energy manifests in a dispersive tail behind the solitary wave.*

Thus, when a solitary wave is perturbed, it is no longer a pure wave given by the profile equation (1.2). One of the main goals of this work is to understand the dispersive tail that forms behind the wave, and to precisely determine the resulting amplitude.

**Q2**: *How exactly are the wave position and amplitude modulated by the energy change?*

**Q3**: *How much amplitude modulation can a solitary wave undergo before it loses*

1

*coherence?*

Question 2 is addressed in Chapter 2 of this thesis, where we present and study a method to track the amplitude and position of modulated solitons. Question 3 is more technical, and is rigorously treated in Chapter 3 and Chapter 4. One of the main technical challenges lies in the fact that we do not simply study the stability of a single object. Rather, we encounter a phenomenon of *metastability*: forced dynamics near an attracting manifold of stable wave profiles. Thus, we study dynamics near a time-varying equilibrium. Even in finite-dimensional systems, linear stability properties of equilibria in general do not persist to a time-varying setting [107]. In Chapter 3 and Chapter 4, we explore two different methods of leveraging the linear stability properties of the wave family (1.2).

Chapter 2 and Chapter 3 address this issue by analyzing the dispersive tail, or wake, in a rescaled frame. This rescaling neutralizes the amplitude changes of the soliton, thereby removing the time-dependence of the equilibrium. A key difficulty arises from the fact that the appropriate rescaling is generated by the operator $I + \frac{1}{2}x\partial_x$, as seen from the identity

$$\partial_c\phi_c(x) = c^{-1}(1 + \tfrac{1}{2}x\partial_x)\phi_c(x).$$

By the chain rule, the evolution of the dispersive tail $v(t,x)$ in the rescaled frame contains a linear contribution of the form

$$v_t(t,x) = (1 + \tfrac{1}{2}x\partial_x)v(t,x) + \dots. \tag{1.12}$$

The factor $x$ is problematic, since it obstructs closing estimates in $L^p$-spaces. In Chapter 3, we address this issue by adapting the commonly used exponentially weighted framework for studying soliton stability in the KdV setting. We introduce *time-varying* exponential weights that adjust according to the applied rescaling.

## Stochastics

As the title suggests, this thesis focuses primarily on *random* mechanisms that affect soliton propagation. In particular, two chapters of this thesis study wave propagation through stochastic partial differential equations (SPDEs). While this is a modern topic that introduces several complications, most of the probabilistic tools used in this work are by now standard, and our technical contributions are principally deterministic in nature.

Chapter 2 and Chapter 4 consider the stochastic KdV equation

$$\mathrm{d}u = -(\partial_x^3 u + 2u\partial_x u)\ \mathrm{d}t + \sigma u\ \mathrm{d}W_t^Q. \tag{1.13}$$

Here, the noise process $W_t^Q$ is a cylindrical Wiener process taking values in $L^2(\mathbb{R})$ and the small parameter $\sigma \geq 0$ controls the strength of the random forcing. The noise is white in time and colored in space with translation-invariant statistics given by the formal identity

$$\mathbb{E}\big[W^Q(t,x)W^Q(t,y)\big] = q(x-y)\min\{s,t\}, \quad x,y \in \mathbb{R}, \quad s,t > 0.$$

For those unfamiliar with SPDEs, (1.13) should be thought of as (1.11), with $\xi(t,x)$ being a random field, statistically independent at each time $t$, and spatially correlated as

$$\mathbb{E}[\xi(t,x)\xi(t,y)] = q(x-y).$$



For technical reasons, this interpretation cannot fully be justified and there truly is a need to resort to (infinite dimensional) stochastic calculus. We refer to [23] and [74] for introductions into this topic.

Some physical motivation for studying SPDE models such as (1.13) lies in the fact that many PDE models are derived as a continuum limit from random particle models. In this context, stochastic forcing serves as a tool for studying such models in circumstances where higher-order contributions play a role. Another motivation is that random forcing can be useful substitute for studying rough environmental conditions, such as an irregular bottom topography in the case of Russell's wave. While a rough bottom topography is not a random object, it can be insightful to model it as such. A derivation of (1.13) in a specific physical context can be found in [59].

It is important to distinguish SPDEs from various other settings that include randomness into PDE dynamics. Specifically, the dynamics resulting from (1.13) are not just those of the KdV equation disturbed by noise after the fact. Rather, the effect of random forcing interacts with the dynamics of the PDE, resulting in a true random dynamical system. To appreciate this point, let us briefly return to the example of a marble resting at the bottom of a convex bowl. In the stochastic setting, we consider the marble to be disturbed by random kicks. For example, by turbulent wind conditions. One needs to take into consideration that the random kicks may displace it to a point where it exits the bowl! Hence, the stochastic forcing may lead it into regions where the physics of the system act drastically different. This is a much more complicated setting than, say, observing a marble resting at the bottom of a bowl through noisy measurements.

To illustrate one of the technical challenges that arise in the SPDE setting, recall that one of our approaches for handling the time dependence of the linear operator governing tail formation is to apply a rescaling generated by $I + \frac{1}{2}x\partial_x$. In the deterministic setting, this produces linear contributions of the form (1.12). In the SPDE setting, however, such coordinate transformations are governed by the Itô formula, or stochastic chain rule, which introduces an additional term of the form $(I+\frac{1}{2}x\partial_x)^2$. This contribution cannot be readily handled using the techniques developed in Chapter 3. In Chapter 4, we analyze the dispersive tail without using a scaling transformation, which leaves the challenge of working near a time-varying equilibrium. We address this by combining local and global tracking mechanisms. The local tracking mechanisms provide stability over time intervals where the wave amplitude varies little. The global tracking mechanism monitors the accumulation of these fluctuations and integrates the stability results from the local intervals.

The answers to our guiding questions **Q1-Q3** should be reconsidered in this stochastic context: The fluctuations of the solitary wave amplitude are random in nature (see Figure 1.4), as is the formation of its dispersive tail. Whether the wave loses coherence within a given time window is a random event, whose distribution depends on the forcing strength $\sigma$. Even so, the stochastic forcing may cause 'deterministic' effects such as slow amplitude attenuation (Chapter 5). These research questions belong to the subfield of *stochastic traveling waves*, which has become a major area of interest within the field of SPDEs. Most earlier work considers traveling waves in reaction-diffusion equations, focusing on stochastic effects on the wave position. The main contribution of this thesis is that it studies stochastic traveling waves in a setting where position modulation and amplitude fluctuations are coupled. We develop both qualitative insights and rigorous results on nonlinear dispersive waves undergoing large stochastic amplitude fluctuations.

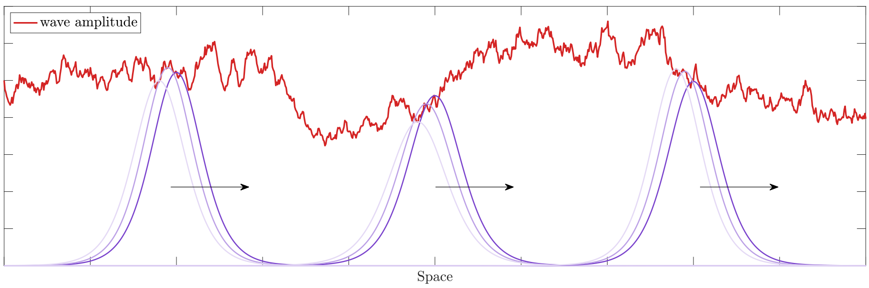



**Figure 1.4:** Schematic representation of a solitary wave undergoing random changes in amplitude. Three snapshots (purple) show the solitary wave as it moves rightward through space. The red line traces the random trajectory of the wave amplitude.

## 1.2 Contributions of this Thesis

The main body of this thesis is composed of Chapters 2–5, based on four scientific papers that resulted from this research.

**Chapter 2: Amplitude tracking**

[102] R.W.S. Westdorp, H.J. Hupkes, *Long-Timescale Soliton Dynamics in the Korteweg-de Vries Equation with Multiplicative Translation-Invariant Noise*, Physica D.

In this chapter, we present our mechanism for tracking the amplitude and position of KdV solitons affected by stochastic forcing. We introduce an effective position $\xi(t)$ and amplitude $c(t)$ of the stochastic waves, and characterize solutions

$u$ to (1.13) near the wave-family (1.2) through a decomposition

$$\underbrace{u(t,x)}_{\text{full solution}} = \underbrace{\phi_{c(t)}(x-\xi(t))}_{\text{modulated soliton}} + \text{dispersive tail.} \tag{1.14}$$

Our construction of the position $\xi(t)$ and amplitude $c(t)$ corresponds to the *variational phase*, and is based on the linear stability properties of the wave family (1.2) on exponentially weighted spaces [91]. The effective position and amplitude are stochastic processes; solutions to SDEs driven by the stochastic forcing. We characterize their evolution with help of the scaling invariance (1.4).

In order to gain insight into the stochastic soliton dynamics described by our method, we present an approximation procedure for the processes $c(t)$ and $\xi(t)$. A complicating factor is that the evolution of the amplitude is coupled to the dispersive tail. In order to account for this, we developed a non-standard nested approximation procedure, which produces SDE approximations for $c(t)$ that have random coefficients. These approximations can be computed at any desired order in the noise strength $\sigma$.

In case the noise process $W_t^Q$ in (1.13) is a scalar Brownian motion, we find that the amplitude process $c(t)$ behaves primarily according to a geometric Brownian motion. The soliton velocity follows the amplitude process $c(t)$, with additional correction due to the noise. The amplitude process $c(t)$ experiences an almost linear positive drift which develops on the time-scale $\mathcal{O}(\sigma^2 t)$. The explicit predictions of our method are verified throughout Chapter 2 via extensive numerical simulations, showcasing the predictive power of our work.

**Chapter 3: Deterministic Stability**

[103] R.W.S. Westdorp, H.J. Hupkes, *Soliton Amplification in the Korteweg-de Vries Equation by Multiplicative Forcing*, Communications on Pure and Applied Analysis

The decomposition (1.14) describes solutions to (1.11) as near-solitons, as long as the dispersive tail remains small. In Chapter 3, we rigorously establish this fact in the context of *deterministic* forcing, using a novel stability analysis involving time-varying weights. In particular, we establish the orbital stability of the traveling-wave family (1.2) under the influence of deterministic multiplicative forcing. We show that solitons evolving via the forced KdV equation

$$u_t = -\partial_x^3 u - 2u\partial_x u + f(t)u \tag{1.15}$$

remain close to the family (1.2) in the energy norm, while undergoing a large change in amplitude. The deterministic equation (1.15) forms a stepping stone for our study of the SPDE (1.13). Chapter 3 contains no stochastic elements, and is hence more broadly accessible to readers focused on deterministic PDEs. We uncover several technical aspects regarding the coherence of solitary waves subject to amplitude modulation. In particular, we gain understanding in the dual role of weighted and unweighted spaces to characterize the formation of the dispersive tail behind the soliton.

**Chapter 4: Stochastic Stability**

[104] R.W.S. Westdorp, H.J. Hupkes, *Stability of Stochastically Forced Solitons in the Korteweg-de Vries Equation* (preprint)

Chapter 4 collects our rigorous results on the stability of KdV solitons subject to multiplicative stochastic forcing. Building on the stability results of Chapter 3, we return to the stochastic setting of Chapter 2 and assert that KdV solitons remain coherent while possibly undergoing large fluctuations in amplitude. While the methods of Chapter 3 rely on the global scaling invariance (1.4), in Chapter 4, we combine local stability methods that follow the amplitude fluctuations of the wave. This method proves to be better adapted to the stochastic setting, as it requires no coordinate transformation. As in Chapter 3, our analysis consists of two complementary parts: we control disturbances of the waves using linear stability tools on weighted spaces, and we employ energy methods to characterize the growth of the dispersive tail in unweighted spaces.

**Chapter 5: Lattice Waves**

[63] H.J. Hupkes, J.A. McGinnis, R.W.S. Westdorp, and J.D. Wright, *Radiating Solitary Waves in an FPUT Lattice with Random Coefficients* (preprint)

This last chapter of the thesis departs from the continuous setting of the KdV equation, and turns to the Fermi-Pasta-Ulam-Tsingou (FPUT) lattice system. This discrete system models an infinite chain of masses connected by springs. In this chapter, we consider wave propagation in the FPUT lattice system affected by random variation in the linear spring force. In absence of random forcing, the FPUT lattice system shares a famous connection to the KdV equation. In an appropriate continuum limit, its dynamics are described by an effective KdV equation. Moreover, the lattice system admits solitary waves which, in the long-wave limit, behave like the KdV solitons. Unlike the stochastic forcing in Chapters 2–4, the random spring coefficients respect the Hamiltonian structure of the FPUT system. As such, solitary waves propagate largely undisturbed. However, they slowly radiate energy in the form of a dispersive tail. As a consequence, their amplitude slowly decreases with time. Exploiting the connection of the FPUT lattice to the KdV equation, we explicitly characterize the amplitude attenuation caused by the random environment, showcasing the broader applicability of our methods.

# CHAPTER 2
# Amplitude tracking

2

This chapter studies the behavior of solitons in the Korteweg-de Vries equation under the influence of multiplicative noise. We introduce stochastic processes that track the amplitude and position of solitons based on a rescaled frame formulation and stability properties of the soliton family. We furthermore construct tractable approximations to the stochastic soliton amplitude and position which reveal their leading-order drift. We find that the statistical properties predicted by our method agree well with numerical evidence.

## 2.1 Introduction

In recent years, stochastic traveling waves and more general stochastic pattern dynamics have become major areas of interest in the field of SPDEs. This chapter[1] employs modern stochastic phase-tracking techniques to study traveling waves in stochastic Korteweg-de Vries (KdV) equations with multiplicative noise, such as

$$\mathrm{d}u = -(\partial_x^3 u + 2u\partial_x u)\ \mathrm{d}t + \sigma u \cdot \mathrm{d}W_t^Q. \tag{2.1}$$

Here $u$ is a real-valued process on $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, and the scalar parameter $\sigma > 0$ encodes the noise strength. The noise $W_t^Q$ is a cylindrical $Q$-Wiener process on a separable Hilbert space $\mathcal{H}$, and $\cdot$ denotes a suitable product between elements of $\mathcal{H}$ and $L^2(\mathbb{R})$. Both multiplicative space-time noise and multiplicative scalar noise can be treated in this setting.

In the deterministic case ($\sigma = 0$), it is well-known that (2.1) admits soliton solutions $u(t,x) = \phi_c(x - ct)$ of the form

$$\phi_c(x) = \tfrac{3c}{2}\,\mathrm{sech}^2(\sqrt{c}x/2), \quad c > 0. \tag{2.2}$$

[1]The contents of this chapter have been published as R.W.S. Westdorp, H.J. Hupkes, *Long-Timescale Soliton Dynamics in the Korteweg-de Vries Equation with Multiplicative Translation-Invariant Noise*, Physica D, see [102]

We describe the evolution $u(t,x)$ of such a soliton under the influence of the multiplicative stochastic forcing ($\sigma > 0$) by using the modulation Ansatz

$$u(t,x) = \phi_{c(t)}(x - \xi(t)) + r(t,x). \tag{2.3}$$

Here, $c(t)$ and $\xi(t)$ are stochastic processes that track the amplitude and position of the modulated soliton, respectively, while the perturbation $r$ remains small in a suitable sense. Numerical evidence based on these phase definitions strongly suggests that such solutions remain close to the soliton family for exponentially long times. We furthermore construct tractable approximations to the modulation parameters which reveal their leading-order drift.

**Solitons in the Korteweg-de Vries equation** The family of solitons (2.2) has been central to the analysis of the deterministic KdV equation ($\sigma = 0$). At the time of its introduction by Boussinesq [15] and rediscovery by Korteweg and de Vries [68] in the late nineteenth century, the KdV equation was primarily used as a model for shallow water waves along a canal. The equation has since appeared as an amplitude equation to describe a wide variety of physical wave phenomena, such as internal waves in stratified oceans [21] and acoustic waves in plasmas [64]. Let us specifically mention the Fermi-Pasta-Ulam-Tsingou (FPUT) chain, whose dynamics in the continuum regime can be described by the KdV equation [41, 42, 62].

As a dispersive system, the dynamics generated by the KdV equation spread out localized initial conditions. Yet, due to nonlinear effects, (2.1) with $\sigma = 0$ admits the family of traveling wave solutions (2.2) that can have arbitrary amplitude, propagating at the proportional velocity. The relation $\phi_c(x) = c\phi_1(\sqrt{c}x)$ apparent in (2.2) is a direct consequence of the fact that the KdV equation enjoys the scaling invariance

$$u(t,x) \mapsto \alpha^2 u(\alpha^3 t, \alpha x), \tag{2.4}$$

in addition to its translational symmetry.

Zabusky and Kruskal observed in numerical experiments [109] that, asymptotically, solutions to the KdV evolution decompose into several solitons of the form (2.2) followed by a radiation component, which undergoes a dispersive evolution. See the work of Schuur [95] for rigorous results in this direction. Another key property of the deterministic KdV equation is that it is a completely integrable system, and consequently has an infinite number of conserved quantities [97]. For example, the KdV evolution conserves the energy $\int_{\mathbb{R}} u^2(t,x)\mathrm{d}x$.

**Deterministic stability** In the deterministic setting, the family of solitons in (2.2) was shown to be orbitally stable by Bona, Souganidis and Strauss [11]. Their work asserts that a slightly perturbed soliton remains, upto translations, within an $H^1$-neighborhood of the initial soliton. This result was improved upon by Pego and Weinstein, who established *asymptotic* stability of the soliton family [91]. More precisely, the authors show that if $u(t,x)$ is initially a small perturbation of the soliton $\phi_{c_*}(\cdot - \xi_*)$, then there exists a final velocity $c > 0$ and a final phase-shift $\xi \in \mathbb{R}$ such that

$$u(t, \cdot + \xi + ct) - \phi_c \to 0 \quad \text{as} \quad t \uparrow \infty,$$

in the weighted spaces

$$L^2_a := L^2(\mathbb{R}, e^{2ax}\mathrm{d}x), \quad 0 < a < \sqrt{c_*}. \tag{2.5}$$

Here, the final velocity $c$ and phase-shift $\xi$ contain small corrections to their starting values $c_*$ and $\xi_*$. The exponential weight ensures that disturbances in the wake of the soliton decay at an exponential rate. The result holds under the assumption that the initial perturbation and its derivative lie in $L^2 \cap L^2_a$. This assumption was later relaxed by Merle and Vega [82] to accommodate general $L^2$-perturbations, with convergence in $L^2_{loc}$.

The proof of Pego and Weinstein relies on the spectral stability of the linearization of the KdV evolution around a soliton $\phi_c$, encoded by the operator

$$\mathcal{L}_c = -\partial_x^3 + (c - 2\phi_c)\partial_x - 2\partial_x\phi_c = -\partial_x^3 + c\partial_x - 2\partial_x[\phi_c \cdot]. \tag{2.6}$$

As a linear operator on the space $L^2(\mathbb{R})$, the operator $\mathcal{L}_c$ has spectrum $i\mathbb{R}$. Embedded in the continuous spectrum is a double eigenvalue at 0 with an associated two-dimensional generalized kernel spanned by $\partial_x\phi_c$ and $\partial_c\phi_c$. Considering the operator $\mathcal{L}_c$ on a weighted space $L^2_a$ with $0 < a < \sqrt{c}$ moves the essential spectrum to the left of the imaginary axis.[2] Its (formal) adjoint $\mathcal{L}_c^*$ on the space $L^2_{-a}$ has a generalized kernel spanned by the soliton $\phi_c$ and the primitive

$$\zeta_c(x) = \int_{-\infty}^x \partial_c\phi_c(y)\mathrm{d}y \in L^2_{-a}.$$

This primitive $\zeta_c$ is not a localized function, which is evident from the fact that $\zeta_c(x)$ tends to $\int_{\mathbb{R}} \partial_c\phi_c \mathrm{d}x = 3c^{-1/2}$ as $x \to \infty$. Pego and Weinstein show that $\mathcal{L}_c$ generates an exponentially stable $C_0$-semigroup $\{e^{\mathcal{L}_c t}\}_{t \geq 0}$ on the subspace of $L^2_a$ consisting of functions $v \in L^2_a$ which satisfy the orthogonality conditions[3]

$$\langle v, \zeta_c \rangle_{L^2} = \langle v, \phi_c \rangle_{L^2} = 0. \tag{2.7}$$

**Stochastic KdV equations** Several stochastic versions of the KdV equation have been introduced in the literature, which incorporate random perturbations that affect the propagation of solitons. In [59], Herman derives a KdV equation perturbed by a single Brownian motion to model the propagation of an ion-acoustic soliton in the presence of noise. Since the KdV equation arises as an approximation

[2] The width of the resulting spectral gap is $a(c - a^2)$, which is maximal at $a = \sqrt{c/3}$. In the literature, one often encounters the restriction $0 < a < \sqrt{c/3}$. This avoids the use of spaces which are more restrictive without gaining a faster decay rate.

[3] In expression (2.7) we slightly abuse notation, as $\zeta_c \notin L^2$. The product $\int_{\mathbb{R}} v\zeta_c \, \mathrm{d}x$ is, however, a well-defined real number, since $e^{-ax}\zeta_c \in L^2$ and $e^{ax}v \in L^2$.

for more involved models, such as for fluid dynamics or wave dynamics in the FPUT lattice, stochastic KdV equations also serve as starting point for studying the effects of random perturbations in such systems [81, 4].

In [26], de Bouard and Debussche establish the well-posedness of (2.1) in the space $H^1(\mathbb{R})$, in the case that the covariance operator $Q$ is a translation-invariant and non-negative convolution operator on $L^2(\mathbb{R})$ given by

$$Qf(x) = \int_{\mathbb{R}} q(x-y)f(y)\,\mathrm{d}y, \tag{2.8}$$

with a convolution kernel $q$ in $H^1(\mathbb{R}) \cap L^1(\mathbb{R})$. The same authors also prove the existence of solutions to (2.1) in two cases that approximate a space-time white noise on $L^2(\mathbb{R})$ [28, 27], corresponding to $Q = I_{L^2}$.

The multiplicative forcing in (2.1) breaks the conservative nature of the KdV equation, which can readily be seen in the scalar case where $W_t^Q$ is a real-valued Brownian motion $\beta_t$. Indeed, by formally applying Itô's lemma to the SPDE

$$\mathrm{d}u = -(\partial_x^3 u + 2u\partial_x u)\,\mathrm{d}t + \sigma u\,\mathrm{d}\beta_t,$$

one finds

$$\begin{aligned}\mathrm{d}\langle u, u\rangle_{L^2} &= \left[-2\langle u, \partial_x^3 u + 2u\partial_x u\rangle_{L^2} + \sigma^2\langle u, u\rangle_{L^2}\right]\,\mathrm{d}t \\ &\quad + 2\sigma\langle u, u\rangle_{L^2}\,\mathrm{d}\beta_t \\ &= \sigma^2\langle u, u\rangle_{L^2}\,\mathrm{d}t + 2\sigma\langle u, u\rangle_{L^2}\,\mathrm{d}\beta_t, \end{aligned}\tag{2.9}$$

which shows that the energy $\int_{\mathbb{R}} u^2(t,x)\mathrm{d}x$ undergoes a geometric Brownian motion with positive drift. See [26, Proposition 3.1] for a similar result in the case of space-time noise. As the stochastic forcing slightly perturbs the soliton continually, its aggregated effect produces a stochastic phase-shift and a substantially varying soliton parameter $c(t)$.

The conservative nature of the equation is, however, not entirely lost. One easily verifies that the *average* $L^2$-norm of solutions to the stochastic KdV equation is conserved:

$$\mathbb{E}\left[\|u(t,\cdot)\|_{L^2}\right] = \|u_0\|_{L^2}.$$

Due to its diffusion, the same does not hold for nonzero powers $p \in \mathbb{R}$ of the $L^2$-norm. In this context it is worthwhile to point out that the soliton parameter $c$ is related to the $L^2$-energy as $\|\phi_c\|_{L^2}^2 = 6c^{3/2}$. This hints at a relation between the stochastic soliton parameter $c(t)$ and the process

$$\|u(t,\cdot)\|_{L^2}^{4/3} = \|u_0\|_{L^2}^{4/3} e^{-\frac{2}{3}\sigma^2 t + \frac{4}{3}\sigma\beta_t}, \tag{2.10}$$

a basic prediction that the results in this chapter will reproduce and refine.

**Stochastic traveling waves** The effect of noise on traveling waves has been previously been analyzed in various settings, primarily for equations of reaction-diffusion type. The subject has seen considerable activity in the physics literature, see for instance the works by García-Ojalvo and Schimansky-Geier[47, 73] which introduce stochastic corrections to traveling waves in bistable RDEs. One well-known example of a noise-induced velocity correction is the Brunet-Derrida conjecture [17, 16], which describes a speed correction for traveling fronts in randomly perturbed Fischer-KPP equations and was proved by Mueller [87]. We also refer to the works [76, 75], which analyze numerical methods for the simulation of stochastic traveling waves and provide numerous intriguing observations.

Various contributions have appeared in the mathematics literature to give (further) rigorous meaning to such results. Krüger and Stannat introduced a multiscale expansion of a stochastic phase for traveling waves in bistable RDEs [70, 96]. This method was later applied to the FitzHugh-Nagumo equation in [31] and extended upon in [77]. Hamster and Hupkes have developed a phase-tracking method in the setting of reaction-diffusion equations that tracks stochastic traveling waves over exponentially long timescales [56, 58, 57]. We refer to the review by Kuehn [71] for a more detailed overview of results on stochastic traveling waves in reaction-diffusion equations.



**Stochastic KdV waves** The stochastic dynamics of solitons in a randomly perturbed KdV equation was first considered by Wadati [99], who derived statistical properties of solitons in the KdV equation with additive scalar noise using a Galilean coordinate transformation. See also the works [100, 49], which expand on this method.

De Bouard and Debussche analyzed the stochastic soliton dynamics produced by the KdV equation (2.1) with multiplicative space-time noise in [25]. Similar to the approach taken in this work, the authors consider a decomposition of the form (2.3) and formulate modulation equations for the soliton parameters. The authors, moreover, estimate the exit-time of the solution from a neighborhood of the modulated soliton. See also the works [24, 29], where the same authors study solitons in the KdV equation with additive noise and a stochastic Gross-Pitaevskii equation.

While the method in [25] yields rigorous stability results, it only allows for a small variation of the soliton parameter $c(t)$. The resulting modulation equations, therefore, do not incorporate the full dynamics of the soliton amplitude and are valid only on timescales of order $\mathcal{O}(1/\sigma^2)$, where the parameter $c(t)$ remains close to its starting value $c_*$. Our method allows for large variations in the parameter $c(t)$ by not only translating the solution, but also rescaling the solution in accordance with the natural scaling $u(t,\cdot) \mapsto \alpha^2 u(t, \alpha\cdot)$ of the soliton family (2.2).

In a more recent work by Cartwright and Gottwald [19], the authors apply a collective coordinate framework developed in [18] to (2.1) in the scalar case, where $W_t^Q$ in (2.1) is a scalar Brownian motion $\beta_t$. One version of their approach modulates the amplitude and width of the soliton independently, whereas another version requires that the modulated soliton remains in the family (2.2). The former method is found to be advantageous in case the perturbations are large. The authors develop an $\mathcal{O}(\sigma)$ modulation system based on this method. Our work here enables the inclusion of higher order effects, which are essential to capture the dominant behaviour of fundamental soliton features such as the width.

**Soliton tracking** In this chapter, we introduce a soliton-tracking method that adapts a phase-tracking method developed in [58] by Hamster and Hupkes for traveling waves in reaction-diffusion equations. The method relies on a transformation of (2.1) that translates and rescales a solution $u(t,x)$ to closely match a fixed soliton $\phi_{c_*}$ at the origin. More precisely, we introduce the process

$$v(t,x) = \alpha^2(t)u\big(t, \alpha(t)x + \xi(t)\big) - \phi_{c_*}(x), \tag{2.11}$$

which is the remaining difference between the translated and rescaled solution, and the fixed soliton $\phi_{c_*}$. From the scaling process $\alpha(t)$, we can recover the effective soliton parameter $c(t)$ of the solution $u(t,x)$ as $c(t) = c_*\alpha^{-2}(t)$.

The translation process $\xi(t)$ and scaling process $\alpha(t)$ are a-priori not specified. Their drift and martingale components bring about four degrees of freedom. We formulate an SPDE that governs the dynamics of the remainder $v$ produced by translation and rescaling as in (2.11) by noise-driven processes $\xi(t)$ and $\alpha(t)$. We use the four degrees of freedom brought about by $\xi(t)$ and $\alpha(t)$ to ensure that $v$ satisfies the orthogonality conditions of (2.7) at all times. Intuitively, this should mean that the remainder $v$ continuously experiences exponential damping, and will remain small. This argument has been made rigorous by Hamster and Hupkes for traveling waves in reaction-diffusion equations in [57], where the authors establish exit-times on the remainder which are exponentially long with respect to the parameter $1/\sigma$.

We restrict ourselves in this work to formal arguments to support the construction of the soliton-tracking method. Our method produces a coupled SPDE system that describes the evolution of $v(t), c(t)$ and $\xi(t)$, which we characterize in considerable detail. On the one hand, this is intended to facilitate the re-use of our ideas in other contexts. On the other hand, we view this chapter as the basis for ongoing efforts to rigorously establish long-time stability of the KdV soliton under stochastic perturbations. We complement the presentation of our general method with three worked-out examples, which allow us to showcase the strong predictive power of our method, as validated by extensive numerical simulations.

Although we anticipate that a meta-stability result based on the methods presented in this work is possible, one needs to control transformations of the perturbation due to the rescaling. It seems that this requires spatial information on the linearized frozen-frame evolution beyond the semigroup bounds obtained in [91].

**Stochastic soliton dynamics** In order to gain insight into the stochastic soliton dynamics described by our method, we design an approximation procedure for the processes $\alpha(t)$ and $\xi(t)$. In contrast to the stochastic wave position studied in [56] and [58], the evolution of the perturbation $v(t)$ does not decouple from the rescaling process $\alpha(t)$. This significantly complicates our analysis, since $\alpha(t)$ develops significant fluctuations on short timescales. In order to account for this, we first expand the perturbation $v(t)$ in terms of the small parameter $\sigma$, using $\alpha$ as an external input.

Expanding the modulation equation for $\alpha(t)$ in terms of $v$ subsequently produces SDE approximations for $\alpha(t)$ that have random coefficients. Solving these SDEs then provides the desired final approximations, which can in principle be computed at any desired order in $\sigma$. This non-standard nested approximation procedure is discussed in detail in §2.3.

We find that the soliton velocity primarily follows the amplitude process $c(t)$, with additional correction due to the noise. The amplitude process $c(t)$ experiences an almost linear positive drift which develops on the time-scale $\mathcal{O}(\sigma^2 t)$. Numerical simulations of (2.1) show that this drift is captured quite well by an approximation of $c(t)$ that takes into account coupling terms that are quadratic in $\sigma$.

The amplitude growth rate and velocity correction are determined by statistical properties of the perturbation with respect to the modulated soliton. This further motivates the use of the frozen-frame formulation (2.11), which keeps the stochastic traveling wave in a fixed reference frame and facilitates analysis of the perturbation shape. We emphasize that the general approach is not restricted to the KdV equation but can also be used in other settings where (approximate) self-similarity properties hold.



**Outlook** This work extends the phase-tracking results of [56] and [58] to a system where stochastic perturbations not only induce a translation, but also a rescaling of the traveling wave. Numerical simulations indicate that our method tracks the KdV soliton over exponentially long times. However, this work does not provide a rigorous stability result to support this claim. We hope that the decompositions and approximations obtained here will provide a pathway towards such a result.

**Outline** This chapter is organized as follows. In §2.2, we provide a detailed derivation of our phase-tracking approach and the resulting modulation system for the soliton parameters. We also introduce the three example systems that we use throughout the work to illustrate our techniques and results. In §2.3, we formulate an expansion of the SPDE system that governs the dynamics of $\alpha(t)$ and $v(t)$, and from there derive the leading-order behavior of the mean and variance of the effective soliton parameter $c(t)$ and position $\xi(t)$. We verify these findings via numerical simulations throughout §2.3.

## 2.2 Stochastic soliton tracking

In this section, we introduce a system of modulation equations to describe the evolution of the solution $u(t, x)$ to the stochastic KdV equation in a stochastic co-moving frame. In particular, we consider the SPDE

$$\mathrm{d}u = -(\partial_x^3 u + 2u\partial_x u)\mathrm{d}t + \sigma M(u)[\mathrm{d}W_t^Q], \tag{2.12}$$

with initial condition $u(0, x) = \phi_{c_*}(x)$, where $\phi_{c_*}$ is the soliton defined in (2.2). The noise term $M(u)$ is of general multiplicative form and is driven by a translation-invariant noise process $W_t^Q$, which we both describe in more detail in §2.2.1.

We introduce a position process $\xi$, which (roughly) keeps the stochastically evolving soliton centered at the origin, and a rescaling process $\alpha$, which neutralizes its amplitude fluctuations. More precisely, we introduce the remainder

$$v(t,x) = \alpha^2(t)u\big(t, \alpha(t)x + \xi(t)\big) - \phi_{c_*}(x), \tag{2.13}$$

which describes the deviation from the soliton $\phi_{c_*}$ in a frame where the solution $u(t,x)$ has been translated and rescaled according to the natural scaling $\phi_c \mapsto \alpha^2\phi_c(\alpha\cdot)$ of the soliton family. The aim is to choose the processes $\alpha$ and $\xi$ in a fashion that keeps the perturbation $v$ in the space characterized by (2.7), where the linearized evolution is stable.

First, in §2.2.1, we describe the forms of multiplicative noise that can be treated by our method. We introduce three concrete settings that will appear throughout the chapter: scalar noise, translation-invariant colored noise and space-time white noise. In §2.2.2, we derive an SPDE that describes the evolution of the perturbation $v$ in the co-moving frame, which we use in §2.2.3 to prescribe the dynamics of our processes $\xi$ and $\alpha$. This results in a modulation system of the form

$$\begin{aligned}
\mathrm{d}v &= \alpha^{-3}\big[\mathcal{L}_{c_*}v + \mathcal{O}(v^2)\big]\ \mathrm{d}t + \sigma^2\mathcal{O}(1)\ \mathrm{d}t + \sigma\mathcal{O}(1)\ \hat{T}_\alpha \mathrm{d}W_t^Q,\\
\mathrm{d}\alpha &= \big[\alpha^{-2}\mathcal{O}(v^2) + \sigma^2\overline{\gamma}_d(v,\alpha)\big]\ \mathrm{d}t + \sigma\alpha\mathcal{O}(1)\ \hat{T}_\alpha \mathrm{d}W_t^Q,\\
\mathrm{d}\xi &= \big[\alpha^{-2}(c_* + \mathcal{O}(v^2)) + \sigma^2\overline{\mu}_d(v,\alpha)\big]\ \mathrm{d}t + \sigma\alpha\mathcal{O}(1)\ \hat{T}_\alpha \mathrm{d}W_t^Q.
\end{aligned}$$

Here, $\mathcal{L}_{c_*}$ is the linear operator introduced in (2.6) and the operator $\hat{T}_\alpha$ corrects for the spatial rescaling by $\alpha$. The drift terms $\overline{\gamma}_d(v,\alpha)$ and $\overline{\mu}_d(v,\alpha)$ are both $\mathcal{O}(1)$ in $v$, but their dependence on $\alpha$ varies per noise type. We explore these differences by returning to the three example settings.

Finally, we illustrate the effectiveness of our decomposition in §2.2.4 via numerical simulations. Since our approach keeps $v$ in the stable subspace of $\mathcal{L}_{c_*}$, we expect that the perturbation $v$ only grows logarithmically in time. Our numerical simulations strongly suggest that this is indeed the case.

### 2.2.1 Stochastic set-up

Let us outline the setting of (2.12) in more detail. We follow the approach of [23] and [74], and consider noise from a separable Hilbert space $\mathcal{H}$ with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and an orthonormal basis $\{e_k\}_{k=0}^{\infty}$. We then pick a covariance operator $Q$ that satisfies the following properties:

**Assumption 2.2.1.** The operator $Q : \mathcal{H} \to \mathcal{H}$ is linear and bounded, and for each $h, g \in \mathcal{H}$ we have

- $\langle Qh, h\rangle_{\mathcal{H}} \geq 0$;  (non-negativity)
- $\langle Qh, g\rangle_{\mathcal{H}} = \langle h, Qg\rangle_{\mathcal{H}}$.  (symmetry)

With this assumption in place, we let $W_t^Q$ be a $Q$-cylindrical Wiener process on $\mathcal{H}$; cf. [23, §4.1.2] and [74, §2.5.1]. This $\mathcal{H}$-valued Wiener process has the property that for each $h \in \mathcal{H}$, the process $\langle W_t^Q, h\rangle$ defines a real-valued Brownian motion, which satisfies the correlation identity

$$\mathbb{E}\big[\langle W_t^Q, h\rangle\langle W_s^Q, g\rangle\big] = (t \wedge s)\langle Qh, g\rangle,$$

for $t, s \geq 0$ and $h, g \in \mathcal{H}$.

In order to define a stochastic integral with respect to $W_t^Q$, we follow [23, 74] and introduce the space $\mathcal{H}_Q := Q^{1/2}(\mathcal{H})$. Equipped with the inner product

$$\langle v, w\rangle_{\mathcal{H}_Q} = \langle Q^{-1/2}v, Q^{-1/2}w\rangle_{\mathcal{H}},$$

$\mathcal{H}_Q$ is a separable Hilbert space for which $\{Q^{1/2}e_k\}_{k=0}^{\infty}$ is an orthonormal basis. Let us furthermore introduce the notation $\mathrm{HS}(\mathcal{H}_Q, \mathcal{H})$ for the space of Hilbert-Schmidt operators between $\mathcal{H}_Q$ and $\mathcal{H}$. We recall that $\mathrm{HS}(\mathcal{H}_Q, \mathcal{H})$ is a Hilbert space with the inner-product

$$\langle A, B\rangle_{\mathrm{HS}(\mathcal{H}_Q,\mathcal{H})} = \sum_{k=0}^{\infty}\big\langle A[Q^{1/2}e_k], B[Q^{1/2}e_k]\big\rangle_{\mathcal{H}}.$$

We refer to [23, §4.2.1] and [74, §2.5.2] for the construction of the stochastic integral

$$\int_0^t \Phi(s)\,\mathrm{d}W_s^Q \tag{2.14}$$

with respect to $W_t^Q$, which defines an $\mathcal{H}$-valued stochastic process for $\mathrm{HS}(\mathcal{H}_Q, \mathcal{H})$-valued integrands $\Phi$ that satisfy the integrability condition

$$\mathbb{E}\int_0^t \|\Phi(s)\|^2_{\mathrm{HS}(\mathcal{H}_Q,\mathcal{H})}\,\mathrm{d}s < \infty.$$

For convenience, we also introduce a rescaling and translation transformation $T_{\alpha,\xi}$, which acts on functions $u \in L^2(\mathbb{R})$ as

$$T_{\alpha,\xi}u = u(\alpha \cdot + \xi). \tag{2.15}$$

This allows us to write (2.13) as

$$v(t,x) = \alpha^2(t)T_{\alpha(t),\xi(t)}u(t,x) - \phi_{c_*}(x). \tag{2.16}$$

With these preliminaries in place, we impose the following conditions on the noise-term $M(u)$ and its relation to the scaling operators $T_{\alpha,\xi}$.

**Assumption 2.2.2.** For each $u \in L^2(\mathbb{R})$:

1. $M(u)$ defines a Hilbert-Schmidt operator from $\mathcal{H}_Q$ to $L^2(\mathbb{R})$. If furthermore

$u \in H^1(\mathbb{R})$, then $M(u)$ defines a bounded linear operator from $\mathcal{H}$ to $L^2(\mathbb{R})$.

2. For each $\beta \in \mathbb{R}$ and $h \in \mathcal{H}$ we have the identity[4]

$$\beta M(u)[h] = M(\beta u)[h].$$

3. There exists a bounded linear operator $\hat{T}_{\alpha,\xi}$ on $\mathcal{H}$, such that for each $\alpha > 0$ and $\xi \in \mathbb{R}$ we have

$$T_{\alpha,\xi}\big[M(u)[h]\big] = M(T_{\alpha,\xi}u)[\hat{T}_{\alpha,\xi}h].$$

4. There exists a linear operator $\hat{\partial}_x$ on $\mathcal{H}_Q$ such that we have[4]

$$\partial_x\big[M(u)[h]\big] = M(u_x)[h] + M(u)[\hat{\partial}_x h],$$

for every $h \in \mathcal{H}_Q$ and $u \in H^1(\mathbb{R})$.

5. The translation invariance identities

$$\hat{T}_{\alpha,\xi} Q \hat{T}^*_{\alpha,\xi} = \hat{T}_\alpha Q \hat{T}^*_\alpha \quad \text{and} \quad \hat{T}_{\alpha,\xi}\hat{\partial}_x Q \hat{T}^*_{\alpha,\xi} = \hat{T}_\alpha \hat{\partial}_x Q \hat{T}^*_\alpha$$

hold for each $\alpha > 0$ and $\xi \in \mathbb{R}$, where $\hat{T}_\alpha$ is shorthand for $\hat{T}_{\alpha,0}$ and $\hat{T}^*_{\alpha,\xi}$ denotes the $\mathcal{H}$-adjoint of $\hat{T}_{\alpha,\xi}$.

The operator $M$ takes on the role of multiplication between $u$ and an element of $\mathcal{H}$. We remark that for each $u \in H^1(\mathbb{R})$ the operator $M(u)$ has an adjoint $M^*(u) : L^2(\mathbb{R}) \to \mathcal{H}$, which by definition satisfies

$$\big\langle M(u)[h], f\big\rangle_{L^2(\mathbb{R})} = \big\langle h, M^*(u)[f]\big\rangle_{\mathcal{H}} \tag{2.17}$$

for each $h \in \mathcal{H}$ and $f \in L^2(\mathbb{R})$.

**Solution types** With this assumption in place, we can assign a rigorous meaning to the SPDE (2.12). Based on the flow $S(t) = e^{-t\partial_x^3}$ generated by the linear equation $u_t = -\partial_x^3 u$, we call an $H^1$-valued process $u$ a *mild solution* to (2.12) with initial condition $u(0,\cdot) = u_0$ if the mild formula

$$u(t) = S(t)u_0 - \int_0^t S(t-s)\partial_x\big(u^2(s)\big)\,\mathrm{d}s + \sigma\int_0^t S(t-s)M\big(u(s)\big)[\mathrm{d}W_s^Q] \tag{2.18}$$

holds for all $t > 0$. The integral equation (2.18) generalizes the variation of constants formula, or Duhamel formula, to the SPDE setting and allows for solutions of lower

[4]Items 2 and 4 of Assumption 2.2.2 restrict the setting to linear noise terms, in order to keep our computations tractable. More general noise terms can be treated by applying a function $g : \mathbb{R} \to \mathbb{R}$ point-wise and considering the noise term $M\big(g(u)\big)$. Items 2 and 4 then generalise to $\beta M\big(g(u)\big)[h] = M\big(\beta g(u)\big)[h]$ and $\partial_x\big[M\big(g(u)\big)[h]\big] = M\big(g'(u)u_x\big)[h] + M\big(g(u)\big)[\hat{\partial}_x h]$.

regularity. As is the case with PDEs, a mild solution is also classical if it takes values in the domain of the linear operator.

Alternatively, one can also consider weak solutions: an $H^1$-valued process $u$ is a *weak solution* to (2.12) with initial condition $u(0,\cdot) = u_0$ if the identity

$$\begin{aligned}\langle u(t),\zeta\rangle_{L^2} = \langle u_0,\zeta\rangle_{L^2} - \int_0^t \langle u_x(s),\partial_x^2\zeta\rangle_{L^2} - 2\langle u(s)u_x(s),\zeta\rangle_{L^2}\ \mathrm{d}s \\ + \sigma\int_0^t \langle M(u(s))[\mathrm{d}W_s^Q],\zeta\rangle_{L^2} \qquad (2.19)\end{aligned}$$

holds for all $\zeta \in H^2(\mathbb{R})$ and $t > 0$. See for instance [46], which asserts the existence of weak solutions for (2.12) posed on a bounded domain with scalar noise. This solution type is, however, less common in the stochastic KdV literature. We refer to [74, Appendix G] for an overview of various solution concepts for SPDEs and the relationships between them.

Let us illustrate the broad applicability of this abstract setting with three examples, which we use throughout the chapter to showcase our results.

**Example I: Scalar multiplicative noise**

As a first example, consider the KdV equation perturbed by a single Brownian motion $\beta_t$, which we write as

$$\mathrm{d}u = -(\partial_x^3 u + 2u\partial_x u)\ \mathrm{d}t + \sigma u\ \mathrm{d}\beta_t. \qquad (2.20)$$

Here, the Brownian motion takes values in the Hilbert space $\mathcal{H} = \mathbb{R}$. Since the perturbation is uniform in space, the covariance operator acts trivially as $Q = I_{\mathbb{R}}$. In this case, we have $\mathcal{H}_Q = \mathbb{R}$ and the multiplication between the noise and the function $u$ is simply given by $M_I(u) : \mathbb{R} \to L^2(\mathbb{R})$, which acts on $h \in \mathbb{R}$ as

$$M_I(u)[h] = hu.$$

By the defining identity (2.17), the formal adjoint $M_I^*(u)$ must satisfy

$$\langle hu, f\rangle_{L^2} = hM_I^*(u)[f],$$

which implies that $M_I^*(u) : L^2(\mathbb{R}) \to \mathbb{R}$ acts on $f \in L^2(\mathbb{R})$ as

$$M_I^*(u)[f] = \langle u, f\rangle_{L^2(\mathbb{R})}.$$

We furthermore compute

$$\begin{aligned}T_{\alpha,\xi}\big[M_I(u)[h]\big] &= hT_{\alpha,\xi}u = M_I(T_{\alpha,\xi}u)[h], \\ \partial_x\big[M_I(u)[h]\big] &= hu_x = M_I(u_x)[h],\end{aligned}$$

which shows that $M_I$ fits items 3 and 4 of Assumption 2.2.2 with $\hat{T}_{\alpha,\xi} = I_{\mathbb{R}}$ and $\hat{\partial}_x$ acting as $\hat{\partial}_x h = 0$ for all $h \in \mathbb{R}$.

As pointed out in [19], we remark that (2.20) can be transformed into a KdV equation with a random coefficient. Indeed, by factoring out the geometric Brownian motion

$$g(t) = e^{-\frac{\sigma^2}{2}t+\sigma\beta_t}$$

as $v = g^{-1}u$, we find

$$\mathrm{d}v = -(\partial_x^3 v + 2g(t)v\partial_x v)\ \mathrm{d}t.$$

This reduces matters of well-posedness to the well-posedness of a KdV equation where the nonlinearity is multiplied with a varying coefficient. We are, however, unaware of results that establish the well-posedness of such an equation with a non-smooth coefficient.

### Example II: Translation-invariant colored noise

Our setting also allows for space-time noise from the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, with the usual inner product. We consider translation-invariant noise, with a spatial correlation structure given by an even function $q \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ that has a non-negative Fourier transform $\hat{q}$. We then introduce the covariance operator $Q$ that acts on $f \in L^2(\mathbb{R})$ as the convolution

$$Qf(x) = \int_{\mathbb{R}} q(x-y)f(y)\ \mathrm{d}y \tag{2.21}$$

with respect to the kernel $q$. The integrability of $q$ ensures that $Q$ is a bounded operator on $L^2(\mathbb{R})$, and the non-negativity of the Fourier transform $\hat{q}$ provides the remaining properties of $Q$ in Assumption 2.2.1.

Well-posedness in this setting is asserted in [26], where the authors construct mild solutions as per (2.18). We have the formal covariance identity

$$\mathbb{E}\big[\mathrm{d}W^Q(x,t)\mathrm{d}W^Q(y,s)\big] = \delta(t-s)q(x-y),$$

which quantifies how $q$ determines the spatial correlation of the noise, depending only on the distance between two points.

Using the kernel $q$, it is possible to provide an explicit formulation of the operator $Q^{1/2}$. To this end, note that $Q$ acts as a Fourier multiplier with symbol $\hat{q}$. It follows from the assumption $q \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and elementary properties of the Fourier transform that $\hat{q}$ lies in $L^1(\mathbb{R})$ and is bounded. As a consequence, the function $\xi \mapsto \sqrt{\hat{q}(\xi)}$ is also bounded, and defines a Fourier multiplier which is bounded on $L^2(\mathbb{R})$. Denoting the inverse Fourier transform of $\sqrt{\hat{q}}$ by $q_{1/2}$ allows us to characterize $Q^{1/2}$ as

$$Q^{1/2}f(x) = \int_{\mathbb{R}} q_{1/2}(x-y)f(y)\ \mathrm{d}y.$$

For future reference, it is convenient to introduce here a rescaled family $\{Q_\alpha\}_{\alpha>0}$ of the convolution operator $Q$, which rescales correlation lengths of the kernel $q$ as

$$Q_\alpha f = \alpha q(\alpha\cdot) * f. \tag{2.22}$$

Similarly, we set

$$(Q^{1/2})_\alpha f = \alpha q_{1/2}(\alpha\cdot) * f.$$

In applications, one often encounters the Gaussian kernel

$$q(x) = \frac{1}{2\zeta} e^{\frac{-\pi x^2}{4\zeta^2}}, \tag{2.23}$$

where the parameter $\zeta > 0$ is a measure for the correlation length. In this case, $\hat{q}$ is given by

$$\hat{q}(\xi) = \frac{1}{\sqrt{2\pi\zeta^2}} e^{\frac{-\zeta^2\xi^2}{\pi}}.$$

We note that, with the kernel (2.23), $Q_\alpha$ acts as

$$Q_\alpha f = \frac{\alpha}{2\zeta} e^{\frac{-\pi\alpha^2 x^2}{4\zeta^2}} * f,$$

so that the correlation length $\zeta$ of the Gaussian kernel is effectively rescaled to $\zeta/\alpha$.

In the setting of this example, the space-time noise enters the KdV equation via the operator $M_{I\!I}(u) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ that acts on $h \in L^2(\mathbb{R})$ as the point-wise multiplication

$$\big(M_{I\!I}(u)[h]\big)(x) = h(x)u(x).$$

In order to verify that $M_{I\!I}(u)$ is in the class $\mathrm{HS}(L^2_Q, L^2)$ for each $u \in L^2(\mathbb{R})$, we compute

$$\begin{aligned} \|M_{I\!I}(u)\|^2_{\mathrm{HS}(L^2_Q,L^2)} &= \sum_{k=0}^{\infty} \langle uQ^{1/2}e_k, uQ^{1/2}e_k\rangle_{L^2} \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{R}} u(x)^2 \big\langle q_{1/2}(x-\cdot), e_k\big\rangle^2_{L^2} \mathrm{d}x \\ &= \int_{\mathbb{R}} u(x)^2 \big\langle q_{1/2}(x-\cdot), q_{1/2}(x-\cdot)\big\rangle_{L^2} \mathrm{d}x = \|q_{1/2}\|^2_{L^2}\|u\|^2_{L^2}. \end{aligned}$$

We then note that

$$\|q_{1/2}\|^2_{L^2} = \|\sqrt{\hat{q}}\|^2_{L^2} = \|\hat{q}\|_{L^1},$$

via Parseval's theorem, and we conclude that the operator $M_{I\!I}(u)$ is indeed Hilbert-Schmidt.

Applying the adjoint identity (2.17) to the multiplication operator $M_{I\!I}$ yields

$$\langle hu, f\rangle_{L^2} = \big\langle h, M^*_{I\!I}(u)[f]\big\rangle_{L^2},$$

which reveals that $M_{II}(u)$ is self-adjoint:

$$M_{II}^*(u)[f] = uf = M_{II}(u)[f].$$

We can furthermore compute

$$\begin{aligned} T_{\alpha,\xi}\big[M_{II}(u)[h]\big] &= T_{\alpha,\xi}uT_{\alpha,\xi}h = M_{II}(T_{\alpha,\xi}u)[T_{\alpha,\xi}h], \\ \partial_x\big[M_{II}(u)[h]\big] &= u_xh + uh_x = M_{II}(u_x)[h] + M_{II}(u)[h_x], \end{aligned}$$

which shows that $M_{II}$ satisfies items 3 and 4 of Assumption 2.2.2 with $\hat{T}_{\alpha,\xi} = T_{\alpha,\xi}$ and $\hat{\partial}_x = \partial_x$. Lemma 2.5.1 then shows that we have the identities

$$\hat{T}_{\alpha,\xi}Q\hat{T}^*_{\alpha,\xi} = \alpha^{-1}Q_\alpha \quad \text{and} \quad \hat{T}_{\alpha,\xi}\hat{\partial}_xQ\hat{T}^*_{\alpha,\xi} = \alpha^{-2}Q_\alpha\partial_x, \tag{2.24}$$

where we recall that the operator $Q_\alpha$ is defined in (2.22). Consequently, item 5 of Assumption 2.2.2 is met.

**Example III: Space-time white noise**

The setting described above formalizes translation-invariant space-time noise with arbitrary correlation length. It is inviting to consider the limiting case where the correlation length $\zeta$ in (2.23) tends to zero. This leads to a space-time white noise $W_t$, which is completely uncorrelated in space and time as specified by the formal identity

$$\mathbb{E}[\mathrm{d}W(x,t)\mathrm{d}W(y,s)] = \delta(t-s)\delta(x-y).$$

Upon doing so, the regularising effect of the convolution with respect to the kernel $q$ is lost. As a consequence, it is unclear how to interpret the noise term $M_{II}(u)\mathrm{d}W_t$, since $M_{II}(u)$ is not in the class $\mathrm{HS}(L^2, L^2)$ and violates item 1 of Assumption 2.2.2. Indeed, to our knowledge, no well-posedness theory is currently available for the KdV equation with multiplicative space-time white noise.

Despite these limitations, we can proceed formally by considering the covariance operator $Q = I_{L^2}$, since convolution with respect to the Dirac distribution acts as the identity operator $I_{L^2}$. We stress that this procedure only amounts to a formal computation, but as we shall see in the sequel the results are very insightful. In this case, the translational invariance identities simplify to

$$\hat{T}_\alpha Q\hat{T}^*_\alpha = \alpha^{-1} \quad \text{and} \quad \hat{T}_\alpha\hat{\partial}_xQ\hat{T}^*_\alpha = \alpha^{-2}\partial_x.$$

### 2.2.2 Frozen-frame transformation

In this section, we formulate an SPDE that governs the evolution of the remainder $v$ in the co-moving frame introduced in (2.16). We postulate that the rescaling and translation processes $\alpha$ and $\xi$ satisfy the system

$$d\alpha = \gamma_d^\sigma(v,\alpha)\ dt + \gamma_s^\sigma(v,\alpha,\xi)\ dW_t^Q, \tag{2.25}$$

$$d\xi = \mu_d^\sigma(v,\alpha)\ dt + \mu_s^\sigma(v,\alpha,\xi)\ dW_t^Q, \tag{2.26}$$

where $\gamma_d^\sigma, \mu_d^\sigma$ are real-valued and $\gamma_s^\sigma, \mu_s^\sigma$ are linear operators from $\mathcal{H}$ to $\mathbb{R}$. The drift components $\gamma_d^\sigma, \mu_d^\sigma$ and martingale components $\gamma_s^\sigma, \mu_s^\sigma$ will be explicitly provided in §2.2.3. For now we note that $\gamma_s^\sigma$ and $\mu_s^\sigma$ can be written as

$$\gamma_s^\sigma(v,\alpha,\xi)[h] = -\sigma\alpha\langle \hat{T}_{\alpha,\xi}h, \overline{\gamma}_s(v)\rangle_{\mathcal{H}}, \tag{2.27}$$

$$\mu_s^\sigma(v,\alpha,\xi)[h] = -\sigma\alpha\langle \hat{T}_{\alpha,\xi}h, \overline{\mu}_s(v)\rangle_{\mathcal{H}}, \tag{2.28}$$

where $\overline{\gamma}_s$ and $\overline{\mu}_s$ are $\mathcal{H}$-valued, depending only on $v$. In addition, the drift components $\gamma_d^\sigma$ and $\mu_d^\sigma$ are of the form

$$\gamma_d^\sigma(v,\alpha) = -\alpha^{-2}\overline{\gamma}_d^0(v) + \sigma^2\overline{\gamma}_d(v,\alpha), \tag{2.29}$$

$$\mu_d^\sigma(v,\alpha) = c_*\alpha^{-2} - \alpha^{-2}\overline{\mu}_d^0(v) + \sigma^2\overline{\mu}_d(v,\alpha) \tag{2.30}$$

where $\overline{\gamma}_d^0, \overline{\mu}_d^0, \overline{\gamma}_d$ and $\overline{\mu}_d$ are real-valued.

In 2.4 we apply Itô's lemma[5] to show that the remainder $v$ defined in (2.13) with $\xi$ and $\alpha$ as in (2.25)–(2.26) satisfies the SPDE

$$dv = \alpha^{-3}\mathcal{L}_{c_*}v\ dt + R^\sigma(v,\alpha)\ dt + \sigma S(v)[\hat{T}_{\alpha,\xi}dW_t^Q], \tag{2.31}$$

where the drift term $R^\sigma(v,\alpha)$ is of the form

$$R^\sigma(v,\alpha) = \alpha^{-3}[N(v) + R_0(v)] + \sigma^2\sum_{i=1}^{6} R_i(v,\alpha). \tag{2.32}$$

Here $N(v)$ is the KdV nonlinearity

$$N(v) = -\partial_x(v^2) = -2v\partial_x v,$$

while $R_0$ through $R_6$ are given by

$$\begin{aligned}
R_0(v) &= -\overline{\gamma}_d^0(v)(2 + x\partial_x)[\phi_{c_*} + v] - \overline{\mu}_d^0(v)\partial_x[\phi_{c_*} + v],\\
R_1(v,\alpha) &= \tfrac{1}{2}\left\|Q^{1/2}\hat{T}_\alpha^*\overline{\mu}_s(v)\right\|_{\mathcal{H}}^2\partial_x^2[\phi_{c_*} + v],\\
R_2(v,\alpha) &= \left\|Q^{1/2}\hat{T}_\alpha^*\overline{\gamma}_s(v)\right\|_{\mathcal{H}}^2(\tfrac{1}{2}x^2\partial_x^2 + 2x\partial_x + 1)[\phi_{c_*} + v],\\
R_3(v,\alpha) &= \left\langle Q^{1/2}\hat{T}_\alpha^*\overline{\gamma}_s(v), Q^{1/2}\hat{T}_\alpha^*\overline{\mu}_s(v)\right\rangle_{\mathcal{H}}(x\partial_x^2 + 2\partial_x)[\phi_{c_*} + v],
\end{aligned}$$

[5] There are several Itô-type formulas available in the literature, tailored to different solution types. In 2.4, we apply the regular Itô formula [23, Theorem 4.32] to the weak formulation (2.19). Applying the mild Itô formula [22, Theorem 1] to the mild formulation (2.18) gives, after tedious computations, an equivalent result.

$$
\begin{aligned}
R_4(v,\alpha) &= -2M(\phi_{c_*}+v)\big[\hat{T}_\alpha Q\hat{T}^*_\alpha\overline{\gamma}_s(v)\big] - xM(\partial_x\phi_{c_*}+v_x)\big[\hat{T}_\alpha Q\hat{T}^*_\alpha\overline{\gamma}_s(v)\big] \\
&\quad - \alpha x M(\phi_{c_*}+v)\big[\hat{T}_\alpha\hat{\partial}_x Q\hat{T}^*_\alpha\overline{\gamma}_s(v)\big], \\
R_5(v,\alpha) &= -M(\partial_x\phi_{c_*}+v_x)\big[\hat{T}_\alpha Q\hat{T}^*_\alpha\overline{\mu}_s(v)\big] - \alpha M(\phi_{c_*}+v)\big[\hat{T}_\alpha\hat{\partial}_x Q\hat{T}^*_\alpha\overline{\mu}_s(v)\big], \\
R_6(v,\alpha) &= \alpha^{-1}\big(\overline{\gamma}_d(v,\alpha)(2+x\partial_x)[\phi_{c_*}+v] + \overline{\mu}_d(v,\alpha)\partial_x[\phi_{c_*}+v]\big).
\end{aligned} \tag{2.33}
$$

In addition, the operator $S$ in (2.31) acts on $h\in\mathcal{H}$ as

$$
\begin{aligned}
S(v)[h] &= M(\phi_{c_*}+v)[h] - (x\partial_x+2)[\phi_{c_*}+v]\langle h,\overline{\gamma_s}(v)\rangle_{\mathcal{H}} \\
&\quad - \partial_x[\phi_{c_*}+v]\langle h,\overline{\mu}_s(v)\rangle_{\mathcal{H}}.
\end{aligned} \tag{2.34}
$$

Note that the noise term in (2.31) has been transformed via $\hat{T}_{\alpha,\xi}$. The translation invariance of $\hat{T}_{\alpha,\xi}Q\hat{T}^*_{\alpha,\xi}$ (see item 5 of Assumption 2.2.2) implies that the transformed noise process $\hat{T}_{\alpha,\xi}W^Q_t$ does, in distribution, not depend on $\xi$. In what follows, we therefore omit the dependence of the noise transformation on $\xi$. We do stress that it should be taken into account during numerical simulations if a pathwise correspondence is desired.

### 2.2.3 Modulation equations

The SPDE (2.31) describes the evolution of the remainder (2.16) where the shift $\xi(t)$ and rescaling by $\alpha(t)$ are of the form (2.27)–(2.30). This leaves the freedom to make an informed choice for the drift components $\gamma^\sigma_d,\mu^\sigma_d$ and the martingale components $\overline{\gamma}_s,\overline{\mu}_s$ of $\alpha(t)$ and $\xi(t)$. The objective underlying this choice is to ensure that the remainder $v$ remains small in the weighted spaces $L^2_a$ defined in (2.5), with $0<a<\sqrt{c_*}$.

We note that (2.31) is a non-autonomous semi-linear equation on account of the fact that the linear operator $\mathcal{L}_{c_*}$ carries a $(t,\omega)$-dependent factor $\alpha^{-3}(t,\omega)$. This comes at no surprise in view of the scaling symmetry (2.4), where the time-variable receives a factor $\alpha^3$. By transforming the time-variable $t$ path-wise as $(t,\omega)\mapsto\int_0^t\alpha^{-3}(s,\omega)\,\mathrm{d}s$, we can scale out the $(t,\omega)$-dependent factor in (2.31) as

$$
\mathrm{d}\tilde{v} = \mathcal{L}_{c_*}\tilde{v}\;\mathrm{d}\tau + \tilde{\alpha}^3R^\sigma(\tilde{v},\tilde{\alpha})\;\mathrm{d}\tau + \tilde{\alpha}^{3/2}\sigma S(\tilde{v})[\hat{T}_{\tilde{\alpha}}\mathrm{d}W^Q_\tau], \tag{2.35}
$$

where $\tilde{v},\tilde{\alpha}$ and $\tilde{\xi}$ are time-transformed versions of $v,\alpha$ and $\xi$. For details, we refer to [56, Lemma 6.2], where the same argument is carried out in the setting of reaction-diffusion equations. After this transformation, (2.35) is semi-linear, and we can recast it into the mild form

$$
\tilde{v}(\tau) = \int_0^\tau \tilde{\alpha}^3e^{\mathcal{L}_{c_*}(\tau-\tau')}R^\sigma(\tilde{v},\tilde{\alpha})\;\mathrm{d}\tau' + \int_0^\tau \tilde{\alpha}^{3/2}e^{\mathcal{L}_{c_*}(\tau-\tau')}\sigma S(\tilde{v})[\hat{T}_{\tilde{\alpha}}\mathrm{d}W^Q_{\tau'}]. \tag{2.36}
$$

In order to control the drift and martingale components of $\tilde{v}$, and equivalently $v$, we demand that the drift component $R^\sigma(v,\alpha)$ only takes values in the subspace of $L^2_a$ characterized by (2.7), where the semigroup generated by $\mathcal{L}_{c_*}$ is contractive.

Similarly, we demand that the stochastic integrand $S(v)$ defined in (2.34) maps $\mathcal{H}$ into this stable subspace.

**Martingale components** Let us determine what form the martingale components of $\alpha$ and $\xi$ must have to ensure that $S$, as defined in (2.34), maps into the stable subspace. More precisely, we require that

$$\big\langle S(v)[h], \zeta_{c_*} \big\rangle_{L^2} = \big\langle S(v)[h], \phi_{c_*} \big\rangle_{L^2} = 0,$$

for each $v \in L^2(\mathbb{R})$ and $h \in \mathcal{H}$. This is achieved if $\overline{\gamma}_s, \overline{\mu}_s$ are chosen in such a way that

$$K(v) \begin{bmatrix} \big\langle h, \overline{\gamma}_s(v) \big\rangle_{\mathcal{H}} \\ \big\langle h, \overline{\mu}_s(v) \big\rangle_{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} \big\langle M(v+\phi_{c_*})[h], \phi_{c_*} \big\rangle_{L^2} \\ \big\langle M(v+\phi_{c_*})[h], \zeta_{c_*} \big\rangle_{L^2} \end{bmatrix}$$



holds for all $v \in L^2(\mathbb{R})$ and $h \in \mathcal{H}$, where $K(v)$ is the matrix

$$K(v) = \begin{bmatrix} \big\langle (x\partial_x + 2)[\phi_{c_*} + v], \phi_{c_*} \big\rangle_{L^2} & \big\langle \partial_x v, \phi_{c_*} \big\rangle_{L^2} \\ \big\langle (x\partial_x + 2)[\phi_{c_*} + v], \zeta_{c_*} \big\rangle_{L^2} & \big\langle \partial_x [\phi_{c_*} + v], \zeta_{c_*} \big\rangle_{L^2} \end{bmatrix}. \tag{2.37}$$

The matrix $K(v)$ is invertible in case $\|v\|_{L^2_a} < \delta$ for some $\delta > 0$, since $K(v)$ is invertible at $v = 0$ and the mapping $v \mapsto \det K(v)$ is continuous from $L^2_a$ to $\mathbb{R}$. We then set

$$\begin{bmatrix} \overline{\gamma}_s(v) \\ \overline{\mu}_s(v) \end{bmatrix} = K^{-1}(v) \begin{bmatrix} M^*(v+\phi_{c_*})[\phi_{c_*}] \\ M^*(v+\phi_{c_*})[\zeta_{c_*}] \end{bmatrix}. \tag{2.38}$$

**Drift components** Applying the orthogonality conditions (2.7) to the drift component $R^\sigma(v,\alpha)$ also gives rises to a system of two linear equations

$$\langle R^\sigma(v,\alpha), \zeta_{c_*} \rangle_{L^2} = \langle R^\sigma(v,\alpha), \phi_{c_*} \rangle_{L^2} = 0,$$

which is solved by setting

$$\begin{bmatrix} \overline{\gamma}^0_d(v) \\ \overline{\mu}^0_d(v) \end{bmatrix} = K(v)^{-1} \begin{bmatrix} \langle N(v), \phi_{c_*} \rangle_{L^2} \\ \langle N(v), \zeta_{c_*} \rangle_{L^2} \end{bmatrix}, \tag{2.39}$$

together with

$$\begin{bmatrix} \overline{\gamma}_d(v,\alpha) \\ \overline{\mu}_d(v,\alpha) \end{bmatrix} = -\alpha K^{-1}(v) \sum_{i=1}^{5} \begin{bmatrix} \langle R_i(v,\alpha), \phi_{c_*} \rangle_{L^2} \\ \langle R_i(v,\alpha), \zeta_{c_*} \rangle_{L^2} \end{bmatrix}. \tag{2.40}$$

We collect that the position and scaling processes $\xi$ and $\alpha$ are governed by the modulation system

$$\mathrm{d}v = \alpha^{-3} \mathcal{L}_{c_*} v \; \mathrm{d}t + R^\sigma(v,\alpha) \; \mathrm{d}t + \sigma S(v)[\hat{T}_\alpha \mathrm{d}W^Q_t], \tag{2.41}$$

$$\mathrm{d}\alpha = \left[-\alpha^{-2}\overline{\gamma}_d^0(v) + \sigma^2\overline{\gamma}_d(v,\alpha)\right]\,\mathrm{d}t - \sigma\alpha\langle \hat{T}_\alpha \mathrm{d}W_t^Q, \overline{\gamma}_s(v)\rangle_{\mathcal{H}}, \tag{2.42}$$
$$\mathrm{d}\xi = \left[c_*\alpha^{-2} - \alpha^{-2}\overline{\mu}_d^0(v) + \sigma^2\overline{\mu}_d(v,\alpha)\right]\,\mathrm{d}t - \sigma\alpha\langle \hat{T}_\alpha \mathrm{d}W_t^Q, \overline{\mu}_s(v)\rangle_{\mathcal{H}}, \tag{2.43}$$

and remark that the $v$-dependence on the right-hand side of (2.41) can be summarised as

$$\mathrm{d}v = \alpha^{-3}\left[\mathcal{L}_{c_*}v + \mathcal{O}(v^2)\right]\,\mathrm{d}t + \sigma^2\mathcal{O}(1)\,\mathrm{d}t + \sigma\mathcal{O}(1)\,\hat{T}_\alpha \mathrm{d}W_t^Q.$$

We now return to the examples presented in Sections 2.2.1-2.2.1, which allows various terms in the modulation system (2.41)–(2.43) to be simplified.

**Example I: Modulation equations for scalar noise**

2

In the setting of §2.2.1, the modulation system takes the form

$$\mathrm{d}v = \alpha^{-3}\mathcal{L}_{c_*}v\,\mathrm{d}t + R_I^\sigma(v,\alpha)\,\mathrm{d}t + \sigma S_I(v)\,\mathrm{d}\beta_t, \tag{2.44}$$
$$\mathrm{d}\alpha = \left[-\alpha^{-2}\overline{\gamma}_d^0(v) + \sigma^2\alpha\overline{\gamma}_{d;I}(v)\right]\,\mathrm{d}t - \sigma\alpha\overline{\gamma}_{s;I}(v)\,\mathrm{d}\beta_t, \tag{2.45}$$
$$\mathrm{d}\xi = \left[c_*\alpha^{-2} - \alpha^{-2}\overline{\mu}_d^0(v) + \sigma^2\alpha\overline{\mu}_{d;I}(v)\right]\,\mathrm{d}t - \sigma\alpha\overline{\mu}_{s;I}(v)\,\mathrm{d}\beta_t. \tag{2.46}$$

Here the drift component is given by

$$R_I^\sigma(v,\alpha) = \alpha^{-3}\left[N(v) + R_0(v)\right] + \sigma^2\sum_{i=1}^{6} R_{i;I}(v),$$

where

$$R_{1;I}(v) = \tfrac{1}{2}\overline{\mu}_{s;I}(v)^2\partial_x^2[\phi_{c_*} + v], \tag{2.47}$$
$$R_{2;I}(v) = \overline{\gamma}_{s;I}(v)^2(\tfrac{1}{2}x^2\partial_x^2 + 2x\partial_x + 1)[\phi_{c_*} + v], \tag{2.48}$$
$$R_{3;I}(v) = \overline{\gamma}_{s;I}(v)\overline{\mu}_{s;I}(v)(x\partial_x^2 + 2\partial_x)[\phi_{c_*} + v], \tag{2.49}$$
$$R_{4;I}(v) = -\big(2\overline{\gamma}_{s;I}(v)(\phi_{c_*} + v) + x\overline{\gamma}_{s;I}(v)(\partial_x\phi_{c_*} + v_x)\big), \tag{2.50}$$
$$R_{5;I}(v) = -\overline{\mu}_{s;I}(v)(\partial_x\phi_{c_*} + v_x), \tag{2.51}$$
$$R_{6;I}(v) = \overline{\gamma}_{d;I}(v)(2 + x\partial_x)[\phi_{c_*} + v] + \overline{\mu}_{d;I}(v)\partial_x[\phi_{c_*} + v], \tag{2.52}$$

while the martingale component is given by

$$S_I(v) = \phi_{c_*} + v - 2\overline{\gamma}_{s;I}(v)[\phi_{c_*} + v] - \overline{\gamma}_{s;I}(v)x\partial_x[\phi_{c_*} + v] - \overline{\mu}_{s;I}(v)\partial_x[\phi_{c_*} + v].$$

The martingale components $\overline{\gamma}_{s;I}$ and $\overline{\mu}_{s;I}$ are real-valued and take the form

$$\begin{bmatrix}\overline{\gamma}_{s;I}(v)\\ \overline{\mu}_{s;I}(v)\end{bmatrix} = K^{-1}(v)\begin{bmatrix}\langle v + \phi_{c_*}, \phi_{c_*}\rangle_{L^2}\\ \langle v + \phi_{c_*}, \zeta_{c_*}\rangle_{L^2}\end{bmatrix}, \tag{2.53}$$

where we recall that the matrix $K(v)$ is defined in (2.37). The drift components $\overline{\gamma}_{d;I}$ and $\overline{\mu}_{d;I}$ are given by

$$\begin{aligned}\begin{bmatrix}\overline{\gamma}_{d;I}(v)\\ \overline{\mu}_{d;I}(v)\end{bmatrix} &= -\overline{\mu}_{s;I}(v)^2\begin{bmatrix}\overline{\gamma}_d^1(v)\\ \overline{\mu}_d^1(v)\end{bmatrix} - \overline{\gamma}_{s;I}(v)^2\begin{bmatrix}\overline{\gamma}_d^2(v)\\ \overline{\mu}_d^2(v)\end{bmatrix} - \overline{\gamma}_{s;I}(v)\overline{\mu}_{s;I}(v)\begin{bmatrix}\overline{\gamma}_d^3(v)\\ \overline{\mu}_d^3(v)\end{bmatrix}\\ &\quad + \overline{\gamma}_{s;I}(v)K(v)^{-1}\begin{bmatrix}\langle (x\partial_x+2)[\phi_{c_*}+v],\phi_{c_*}\rangle_{L^2}\\ \langle (x\partial_x+2)[\phi_{c_*}+v],\zeta_{c_*}\rangle_{L^2}\end{bmatrix}\\ &\quad + \overline{\mu}_{s;I}(v)K(v)^{-1}\begin{bmatrix}\langle \partial_x[\phi_{c_*}+v],\phi_{c_*}\rangle_{L^2}\\ \langle \partial_x[\phi_{c_*}+v],\zeta_{c_*}\rangle_{L^2}\end{bmatrix},\end{aligned} \tag{2.54}$$

where the terms $\overline{\gamma}_d^1,\dots,\overline{\gamma}_d^3$ and $\overline{\mu}_d^1,\dots,\overline{\mu}_d^3$ are defined by the expressions

$$\begin{bmatrix}\overline{\gamma}_d^1(v)\\ \overline{\mu}_d^1(v)\end{bmatrix} = K(v)^{-1}\begin{bmatrix}\frac{1}{2}\langle \partial_x^2[\phi_{c_*}+v],\phi_{c_*}\rangle_{L^2}\\ \frac{1}{2}\langle \partial_x^2[\phi_{c_*}+v],\zeta_{c_*}\rangle_{L^2}\end{bmatrix}, \tag{2.55}$$

$$\begin{bmatrix}\overline{\gamma}_d^2(v)\\ \overline{\mu}_d^2(v)\end{bmatrix} = K(v)^{-1}\begin{bmatrix}\langle (\frac{1}{2}x^2\partial_x^2+2x\partial_x+1)[\phi_{c_*}+v],\phi_{c_*}\rangle_{L^2}\\ \langle (\frac{1}{2}x^2\partial_x^2+2x\partial_x+1)[\phi_{c_*}+v],\zeta_{c_*}\rangle_{L^2}\end{bmatrix}, \tag{2.56}$$

$$\begin{bmatrix}\overline{\gamma}_d^3(v)\\ \overline{\mu}_d^3(v)\end{bmatrix} = K(v)^{-1}\begin{bmatrix}\langle (x\partial_x^2+2\partial_x)[\phi_{c_*}+v],\phi_{c_*}\rangle_{L^2}\\ \langle (x\partial_x^2+2\partial_x)[\phi_{c_*}+v],\zeta_{c_*}\rangle_{L^2}\end{bmatrix}. \tag{2.57}$$

We remark that $\overline{\gamma}_{d;I}$ and $\overline{\mu}_{d;I}$ induce an $\mathcal{O}(\sigma^2)$ drift on $\alpha$ and $\xi$.

**Example II: Modulation equations for translation-invariant colored noise**

In the setting of translation-invariant colored noise of §2.2.1, the modulation equations for $v,\alpha$ and $\xi$ take the form

$$\begin{aligned}\mathrm{d}v &= \alpha^{-3}\mathcal{L}_{c_*}v\ \mathrm{d}t + R^\sigma_{II}(v,\alpha)\ \mathrm{d}t + \sigma S_\diamond(v)[T_\alpha \mathrm{d}W_t^Q],\\ \mathrm{d}\alpha &= \left[-\alpha^{-2}\overline{\gamma}_d^0(v)+\sigma^2\overline{\gamma}_{d;II}(v,\alpha)\right]\ \mathrm{d}t - \sigma\alpha\langle T_\alpha \mathrm{d}W_t^Q,\overline{\gamma}_\diamond(v)\rangle_{L^2},\\ \mathrm{d}\xi &= \left[c_*\alpha^{-2}-\alpha^{-2}\overline{\mu}_d^0(v)+\sigma^2\overline{\mu}_{d;II}(v,\alpha)\right]\ \mathrm{d}t - \sigma\alpha\langle T_\alpha \mathrm{d}W_t^Q,\overline{\mu}_\diamond(v)\rangle_{L^2}.\end{aligned}$$

Here, the martingale component $S_\diamond(v)$ acts on $h\in L^2(\mathbb{R})$ as

$$\begin{aligned}S_\diamond(v)[h] &= (\phi_{c_*}+v)h - (x\partial_x+2)[\phi_{c_*}+v]\langle h,\overline{\gamma}_\diamond(v)\rangle_{L^2}\\ &\quad - \partial_x[\phi_{c_*}+v]\langle h,\overline{\mu}_\diamond(v)\rangle_{L^2}.\end{aligned} \tag{2.58}$$

The functions $\overline{\gamma}_\diamond$ and $\overline{\mu}_\diamond$ are $L^2$-valued and given by

$$\begin{bmatrix}\overline{\gamma}_\diamond(v)\\ \overline{\mu}_\diamond(v)\end{bmatrix} = K^{-1}(v)\begin{bmatrix}(v+\phi_{c_*})\phi_{c_*}\\ (v+\phi_{c_*})\zeta_{c_*}\end{bmatrix}. \tag{2.59}$$

The drift component $R^\sigma_{II}(v,\alpha)$ follows from the general formulation (2.32), where one evaluates the terms $R_4$ and $R_5$ using (2.24). These identities also show that the inner products in $R_1, R_2$ and $R_3$ can be computed for $f, g \in L^2(\mathbb{R})$ as

$$\langle Q^{1/2}\hat{T}_\alpha^* f, Q^{1/2}\hat{T}_\alpha^* g\rangle_{L^2} = \langle \hat{T}_\alpha Q \hat{T}_\alpha^* f, g\rangle_{L^2} = \alpha^{-1}\langle Q_\alpha^{1/2} f, Q_\alpha^{1/2} g\rangle_{L^2},$$

where $Q_\alpha$ is a rescaled version of the covariance operator $Q$ as introduced in (2.22). The drift components $\overline{\gamma}_{d;\mathit{II}}$ and $\overline{\mu}_{d;\mathit{II}}$ are given by the expression

$$\begin{aligned}
\begin{bmatrix} \overline{\gamma}_{d;\mathit{II}}(v,\alpha) \\ \overline{\mu}_{d;\mathit{II}}(v,\alpha) \end{bmatrix} &= -\big\|Q_\alpha^{1/2}\overline{\mu}_\diamond(v)\big\|_{L^2}^2 \begin{bmatrix} \overline{\gamma}_d^1(v) \\ \overline{\mu}_d^1(v) \end{bmatrix} - \big\|Q_\alpha^{1/2}\overline{\gamma}_\diamond(v)\big\|_{L^2}^2 \begin{bmatrix} \overline{\gamma}_d^2(v) \\ \overline{\mu}_d^2(v) \end{bmatrix} \\
&\quad - \big\langle Q_\alpha^{1/2}\overline{\gamma}_\diamond(v), Q_\alpha^{1/2}\overline{\mu}_\diamond(v)\big\rangle_{L^2} \begin{bmatrix} \overline{\gamma}_d^3(v) \\ \overline{\mu}_d^3(v) \end{bmatrix} \\
&\quad + K(v)^{-1} \begin{bmatrix} \big\langle (x\partial_x + 2)[\phi_{c_*} + v] Q_\alpha \overline{\gamma}_\diamond(v), \phi_{c_*}\big\rangle_{L^2} \\ \big\langle (x\partial_x + 2)[\phi_{c_*} + v] Q_\alpha \overline{\gamma}_\diamond(v), \zeta_{c_*}\big\rangle_{L^2} \end{bmatrix} \\
&\quad + K(v)^{-1} \begin{bmatrix} \big\langle \partial_x[\phi_{c_*} + v] Q_\alpha \overline{\mu}_\diamond(v), \phi_{c_*}\big\rangle_{L^2} \\ \big\langle \partial_x[\phi_{c_*} + v] Q_\alpha \overline{\mu}_\diamond(v), \zeta_{c_*}\big\rangle_{L^2} \end{bmatrix} \\
&\quad + K(v)^{-1} \begin{bmatrix} \big\langle (x Q_\alpha \partial_x \overline{\gamma}_\diamond(v) + Q_\alpha \partial_x \overline{\mu}_\diamond(v))(\phi_{c_*} + v), \phi_{c_*}\big\rangle_{L^2} \\ \big\langle (x Q_\alpha \partial_x \overline{\gamma}_\diamond(v) + Q_\alpha \partial_x \overline{\mu}_\diamond(v))(\phi_{c_*} + v), \zeta_{c_*}\big\rangle_{L^2} \end{bmatrix}, \qquad (2.60)
\end{aligned}$$

where we recall that the terms $\overline{\gamma}_d^0, \ldots, \overline{\gamma}_d^3$ and $\overline{\mu}_d^0, \ldots, \overline{\mu}_d^3$ are defined in (2.39) and (2.55)–(2.57).

### Example III: Modulation equations for space-time white noise

In the setting of space-time noise of §2.2.1, the modulation system takes a slightly simpler form, in the sense that the dependence on the rescaling process $\alpha$ is more straightforward:

$$\begin{aligned}
\mathrm{d}v &= \alpha^{-3}\mathcal{L}_{c_*} v\ \mathrm{d}t + R_{\mathit{III}}^\sigma(v,\alpha)\ \mathrm{d}t + \sigma S_\diamond(v)[T_\alpha \mathrm{d}W_t], && (2.61)\\
\mathrm{d}\alpha &= \big[-\alpha^{-2}\overline{\gamma}_d^0(v) + \sigma^2 \overline{\gamma}_{d;\mathit{III}}(v)\big]\ \mathrm{d}t - \sigma\alpha\big\langle T_\alpha \mathrm{d}W_t, \overline{\gamma}_\diamond(v)\big\rangle_{L^2}, && (2.62)\\
\mathrm{d}\xi &= \big[c_*\alpha^{-2} - \alpha^{-2}\overline{\mu}_d^0(v) + \sigma^2\overline{\mu}_{d;\mathit{III}}(v)\big]\ \mathrm{d}t - \sigma\alpha\big\langle T_\alpha \mathrm{d}W_t, \overline{\mu}_\diamond(v)\big\rangle_{L^2}, && (2.63)
\end{aligned}$$

where

$$R_{\mathit{III}}^\sigma(v,\alpha) = \alpha^{-3}\big[N(v) + R_0(v)\big] + \sigma^2\alpha^{-1}\sum_{i=1}^{6} R_{i;\mathit{III}}(v),$$

and

$$\begin{aligned}
R_{1;\mathit{III}}(v) &= \tfrac{1}{2}\big\|\overline{\mu}_\diamond(v)\big\|_{\mathcal{H}}^2 \partial_x^2[\phi_{c_*} + v] \\
R_{2;\mathit{III}}(v) &= \big\|\overline{\gamma}_\diamond(v)\big\|_{\mathcal{H}}^2 (\tfrac{1}{2}x^2\partial_x^2 + 2x\partial_x + 1)[\phi_{c_*} + v] \\
R_{3;\mathit{III}}(v) &= \big\langle \overline{\gamma}_\diamond(v), \overline{\mu}_\diamond(v)\big\rangle_{\mathcal{H}} (x\partial_x^2 + 2\partial_x)[\phi_{c_*} + v] \\
R_{4;\mathit{III}}(v) &= -2(\phi_{c_*} + v)\overline{\gamma}_\diamond(v) - x(\partial_x\phi_{c_*} + v_x)\overline{\gamma}_\diamond(v) \\
&\quad - x(\phi_{c_*} + v)\partial_x\overline{\gamma}_\diamond(v)
\end{aligned}$$

$$R_{5;III}(v) = -(\partial_x \phi_{c_*} + v_x)\overline{\mu}_\diamond(v) - (\phi_{c_*} + v)\partial_x \overline{\mu}_\diamond(v)$$
$$R_{6;III}(v) = \overline{\gamma}_{d;III}(v)(2 + x\partial_x)[\phi_{c_*} + v] + \overline{\mu}_{d;III}(v)\partial_x[\phi_{c_*} + v].$$

The martingale components $S_\diamond, \overline{\gamma}_\diamond$ and $\overline{\mu}_\diamond$ are as in (2.59) and the drift components of $\alpha$ and $\xi$ take the form

$$\begin{aligned}
\begin{bmatrix} \overline{\gamma}_{d;III}(v) \\ \overline{\mu}_{d;III}(v) \end{bmatrix} = & -\left\|\overline{\mu}_\diamond(v)\right\|_{L^2}^2 \begin{bmatrix} \overline{\gamma}_d^1(v) \\ \overline{\mu}_d^1(v) \end{bmatrix} - \left\|\overline{\gamma}_\diamond(v)\right\|_{L^2}^2 \begin{bmatrix} \overline{\gamma}_d^2(v) \\ \overline{\mu}_d^2(v) \end{bmatrix} \\
& - \left\langle \overline{\gamma}_\diamond(v), \overline{\mu}_\diamond(v) \right\rangle_{L^2} \begin{bmatrix} \overline{\gamma}_d^3(v) \\ \overline{\mu}_d^3(v) \end{bmatrix} \\
& + K(v)^{-1} \begin{bmatrix} \left\langle (x\partial_x + 2)[\phi_{c_*} + v]\overline{\gamma}_\diamond(v), \phi_{c_*} \right\rangle_{L^2} \\ \left\langle (x\partial_x + 2)[\phi_{c_*} + v]\overline{\gamma}_\diamond(v), \zeta_{c_*} \right\rangle_{L^2} \end{bmatrix} \\
& + K(v)^{-1} \begin{bmatrix} \left\langle \partial_x[\phi_{c_*} + v]\overline{\mu}_\diamond(v), \phi_{c_*} \right\rangle_{L^2} \\ \left\langle \partial_x[\phi_{c_*} + v]\overline{\mu}_\diamond(v), \zeta_{c_*} \right\rangle_{L^2} \end{bmatrix} \\
& + K(v)^{-1} \begin{bmatrix} \left\langle (x\partial_x \overline{\gamma}_\diamond(v) + \partial_x \overline{\mu}_\diamond(v))(\phi_{c_*} + v), \phi_{c_*} \right\rangle_{L^2} \\ \left\langle (x\partial_x \overline{\gamma}_\diamond(v) + \partial_x \overline{\mu}_\diamond(v))(\phi_{c_*} + v), \zeta_{c_*} \right\rangle_{L^2} \end{bmatrix}.
\end{aligned} \tag{2.64}$$

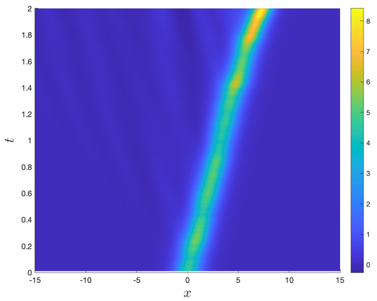
We recall again that the terms $\overline{\gamma}_d^0, \ldots, \overline{\gamma}_d^3$ and $\overline{\mu}_d^0, \ldots, \overline{\mu}_d^3$ are defined in (2.39) and (2.55)–(2.57).

### 2.2.4 Numerical simulations

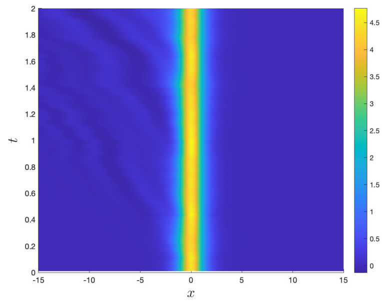
Having fully laid out the modulation systems in the stochastic co-moving frame for our three example setups, we now explore the dynamics that the systems produce via numerical simulations. We restrict ourselves to simulations of scalar noise (Example I) and space-time white noise (Example III), as the modulation systems for these examples are the most tractable. See 2.7 for the numerical schemes that were employed to simulate the stochastic KdV equation (2.1) and the modulation systems (2.44)–(2.46) and (2.62)–(2.63).

**Pathwise simulation** It is widely acknowledged that accurate numerical simulations of SPDEs are challenging to obtain [75]. In order to validate our results, we present a numerical comparison between our constructed modulation parameters and soliton parameters derived from a direct simulation. Figure 2.1 shows one realization obtained from a simulation of the KdV equation (2.20) with multiplicative scalar noise in both the original frame and co-moving frame.

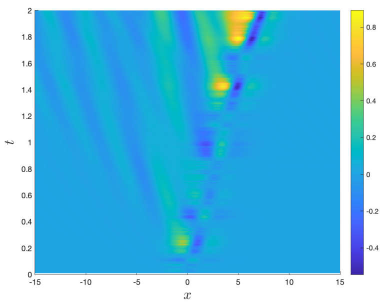
In Figure 2.1a, we observe that the soliton propagates approximately at a constant velocity, and at times slightly speeds up or slows down when it increases or decreases in amplitude, respectively. The transformation from the stochastic KdV equation (2.20) to the modulation system (2.44)–(2.46) allows us to 'freeze' the stochastic soliton. In Figure 2.1b, the soliton remains centered and roughly has constant amplitude. To the left of the soliton we observe slight perturbations due to the noise. These can be observed more clearly upon removing the soliton



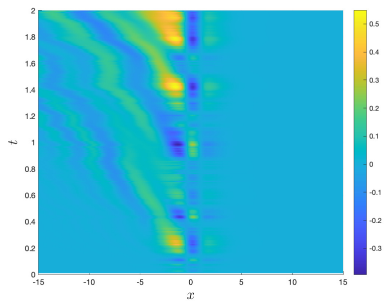
**(a)** Original frame.



**(b)** Stochastic co-moving frame.



**(c)** Original frame, soliton removed.



**(d)** Stochastic co-moving frame, soliton removed.

**Figure 2.1:** Simulation of the KdV equation with scalar noise of strength $\sigma = 0.25$. Panel (a) shows the original frame realization $u(t, x)$, from a simulation of (2.20). Panel (b) shows $\phi_{c_*}(x) + v(t, x)$, from simulation in the frozen frame of (2.44)–(2.46) with the same realization of the noise. Panels (c) and (d) show the perturbation with respect to the soliton, that is $u(t, x) - \phi_{c(t)}(x - \xi(t))$ with the phase-definitions (2.65) in panel (c), and $v(t, x)$ in panel (d).

in Figure 2.1c, which reveals the 'wake' of the stochastic soliton. The stochastic perturbations encountered by the soliton result in a radiation field to the left of the soliton. In Figure 2.1d, we furthermore observe that the radiation field has undergone a rescaling in the $x$-direction, as is evident from the distortion of the radiation waves. The effect of the stochastic frozen-frame transformation can also be visualised in the case of space-time white noise, see Figure 2.11 in 2.8.

For comparison purposes, we define 'fitted' versions of the position $\xi_{\rm fit}(t)$ and amplitude $c_{\rm fit}(t)$ of a solution $u(t, x)$ to (2.12) implicitly via the identities

$$\begin{aligned} \big\langle u\big(t, \cdot + \xi_{\rm fit}(t)\big) - \phi_{c_{\rm fit}(t)}, \zeta_{c_{\rm fit}(t)} \big\rangle_{L^2} &= 0, \\ \big\langle u\big(t, \cdot + \xi_{\rm fit}(t)\big) - \phi_{c_{\rm fit}(t)}, \phi_{c_{\rm fit}(t)} \big\rangle_{L^2} &= 0, \end{aligned} \tag{2.65}$$

which we solve numerically. This allows us to compare the evolution of soliton parameters obtained by direct simulation of (2.12) and the modulation system (2.41)–(2.43). Recall therefore that the amplitude process $c(t)$ can be recovered from the rescaling process $\alpha(t)$ as $c(t) = c_*\alpha^{-2}(t)$. We also introduce the phase shift processes

$$\Omega(t) = \xi(t) - \int_0^t c(s)\ \mathrm{d}s \quad \text{and} \quad \Omega_{\text{fit}}(t) = \xi_{\text{fit}}(t) - \int_0^t c_{\text{fit}}(s)\ \mathrm{d}s, \tag{2.66}$$

which track the deviation of the soliton position from the integrated stochastic velocities $c(t)$ and $c_{\text{fit}}(t)$ and isolate the noise-induced effects.

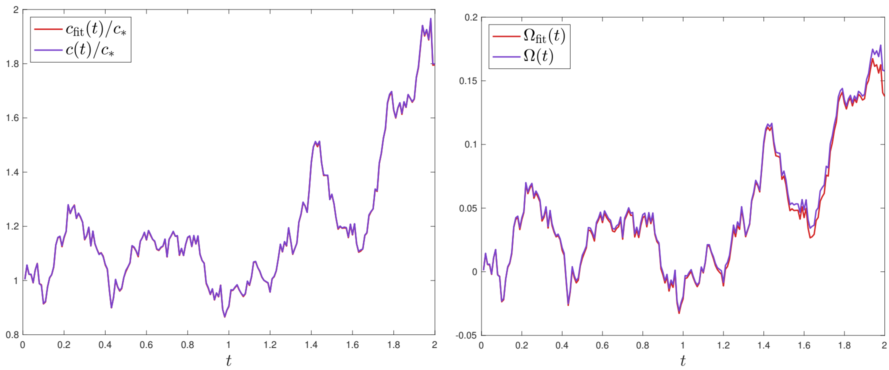
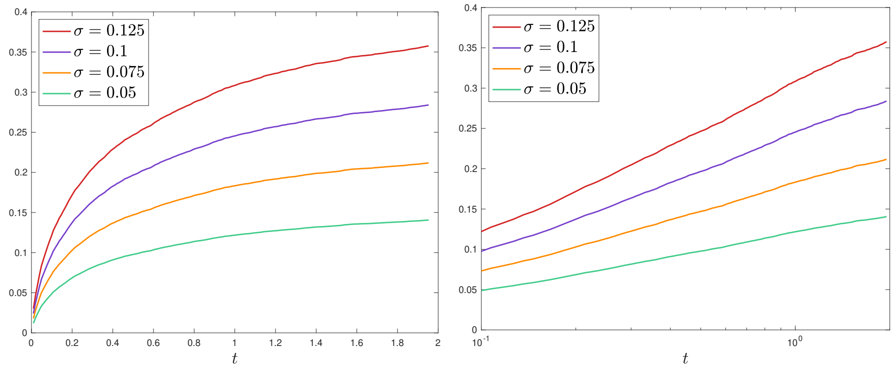



**Figure 2.2:** Path-wise comparison of soliton amplitudes $c(t)$ to $c_{\text{fit}}(t)$ (left) and phase shifts $\Omega(t)$ to $\Omega_{\text{fit}}(t)$ (right) at noise strength $\sigma = 0.25$ and initial amplitude $c_* = 3$. The parameters $c_{\text{fit}}(t)$ and $\Omega_{\text{fit}}(t)$, defined in (2.65) and (2.66), are obtained from direct simulation in the original frame of (2.20). The soliton amplitude $c(t)$ and phase shift $\Omega(t)$ are obtained from simulation of the frozen frame system (2.44)–(2.46).

Figure 2.2 shows the correspondence between the evolution of the soliton amplitude and phase shift in both frames. Note that the soliton amplitude in this realization attains almost twice its original value at $t = 2$. The phase shifts $\Omega$ and $\Omega_{\text{fit}}$ develop a small discrepancy over time, which we attribute mainly to truncation effects and the fact that errors in $c_{\text{fit}}$ are compounded through the integral in (2.66).

**Stability** The construction of the modulation system in §2.2.3 should ensure that the perturbation $v$ remains small in the exponentially weighted spaces $L^2_a$ defined in (2.5). Figure 2.3 shows the average growth of the in $L^2_a$-norm of the perturbation $v$ with respect to the soliton. The spatial norm of the perturbation appears to grows logarithmically, as indicated by Figure 2.3b, where we observe a linear growth of the perturbation size on logarithmic scale. For the case of space-time white noise we refer to Figure 2.12.

The logarithmic growth strongly suggests that our soliton-tracking method is valid over exponentially long timescales. A logarithmic growth of the remaining perturbation is also observed in [58, Figure 3.8], where the spirit of our approach is applied to traveling waves in reaction-diffusion equations. Using this fact, Hamster




**(a)** Perturbation size over time. **(b)** Perturbation size over time, log-scale.

**Figure 2.3:** Sample mean of the process $\sup_{s\leq t} \|v(s)\|_{L^2_a([-50,20])}$ for scalar noise, see (2.44), computed over 500 realisations for $\sigma \in \{0.05, 0.075, 0.1, 0.125\}$ and $c_* = 3$. The exponential weight $e^{ax}$ in the $L^2_a$-norm strongly amplifies numerical effects entering from the right boundary of the computational domain $[-50, 50]$. We take care to avoid these by computing the $L^2_a$-norm on $[-50, 20]$, with $a = 0.5$. For the initial soliton-parameter used in this simulation, the relevant dynamics occur well within $[-50, 20]$ (see Figure 2.1).

and Hupkes rigorously prove in [57] that the exit-time from the soliton family is exponentially long with respect to the parameter $1/\sigma$.

## 2.3 Soliton dynamics

In this section, we set out to derive explicit, tractable expansions to uncover the effects of the multiplicative noise on the soliton amplitude $c$ and position $\xi$. In §2.2, we have seen that the dynamics of the soliton parameters $c$ and $\xi$ are governed by the rescaling process $\alpha$ and the infinite dimensional perturbation $v$, which follow the coupled equations

$$\begin{aligned} v(t) &= I^\sigma_v(v, \alpha, t),\\ \alpha(t) &= I^\sigma_\alpha(v, \alpha, t). \end{aligned} \tag{2.67}$$

For $v$, we choose to work with the mild formulation

$$\begin{aligned} I^\sigma_v(v, \alpha, t) = &\int_0^t e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} R^\sigma(v, \alpha)\, \mathrm{d}s \\ &+ \sigma \int_0^t e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} S(v)[\hat{T}_\alpha \mathrm{d}W^Q_s], \end{aligned} \tag{2.68}$$

which follows from (2.36) by undoing the time transformation. This has the advantage of being suitable for constructing explicit approximations. For $I^\sigma_\alpha$ we use the strong form

$$I^\sigma_\alpha(v,\alpha,t) = 1 + \int_0^t \left[-\alpha^{-2}\overline{\gamma}^0_d(v) + \sigma^2\overline{\gamma}_d(v,\alpha)\right] \mathrm{d}s \\ - \sigma\int_0^t \alpha\langle \hat{T}_\alpha \mathrm{d}W^Q_s, \overline{\gamma}_s(v)\rangle_{\mathcal{H}}, \tag{2.69}$$

which corresponds to (2.45).

To develop our approximation procedure, it is relevant to note that the process $\alpha$ exhibits significant fluctuations, while the perturbation $v$ remains relatively small due to the damping of the semigroup. Indeed, we observe that $\alpha$ grows as $\mathcal{O}(\sigma\sqrt{t})$, which can be anticipated by noting that $I^\sigma_\alpha$ contains no damping terms. On the other hand, Figure 2.3 indicates that $v$ grows at a slower rate, namely $\mathcal{O}(\sigma \ln t)$. As such, a carefully tailored approximation procedure is required to accurately capture the interplay between $\alpha$ and $v$. To construct approximations of $\alpha$ that account for the influence of the perturbation $v$, we introduce SDEs based on an expansion of the $v$-dependent coupling terms. In broad terms, we will expand the $\alpha$ dynamics in terms of $v$, while expanding the $v$ dynamics in terms of $\sigma$, treating $\alpha$ as an external input.

Below, in §2.3.1–§2.3.3, we describe the expansions of the coupled system (2.67) in more detail. Combining these, we obtain a sequence of increasingly refined approximations for the modulation parameters, which we present in §2.3.4. We proceed by evaluating the first few approximations for the cases of Example I (scalar noise) and Example Ⅲ (space-time white noise) in §2.3.5 and §2.3.6, respectively. In the first setting, the soliton amplitude roughly behaves as a geometric Brownian motion. The explicit approximations are used to compute leading-order statistical properties, which we compare with sample statistics of the numerical observations $c_{\mathrm{fit}}$ and $\Omega_{\mathrm{fit}}$ defined in (2.65) and (2.66).

### 2.3.1 Expansion of the rescaling process

In order to unravel how the rescaling process $\alpha$ is influenced by the perturbation $v$, we introduce an expansion of $\alpha$ in terms of $v$. As a first step, consider the situation where the perturbation $v$ is set to zero in (2.67). That is, we introduce a process $A_0$ which satisfies

$$A_0(t) = I^\sigma_\alpha(0, A_0, t) \tag{2.70}$$

and we supply this SDE with the initial condition $A_0(0) = 1$. The process $A_0$ then constitutes a relatively crude first approximation to $\alpha$, which we subsequently refine by increasing the order of the perturbation $v$ that we take into account. In particular, let us now include terms in the Itô form $I^\sigma_\alpha(v,\alpha,t)$ that depend *linearly* on $v$. We assume that there is an approximation of $v$ available, for which we introduce the variable $\tilde{v}_1$. This variable should be thought of as approximating $v$ with an error of $\mathcal{O}(\sigma^2)$. The next approximation $A_1$, given the process $\tilde{v}_1$, is defined through the SDE

$$A_1(\tilde{v}_1, t) = I_\alpha^\sigma(0, A_1, t) + [I_\alpha^\sigma]^{(1)}(\tilde{v}_1; A_1, t).$$

Here, $[I_\alpha^\sigma]^{(1)}$ is defined as

$$[I_\alpha^\sigma]^{(1)}(v; \alpha, t) = \sigma^2 \int_0^t [\overline{\gamma}_d]^{(1)}(v; \alpha)\, \mathrm{d}s - \sigma \int_0^t \alpha \langle \hat{T}_\alpha \mathrm{d}W_s^Q, [\overline{\gamma}_s]^{(1)}(v) \rangle_{\mathcal{H}},$$

with $[\overline{\gamma}_d]^{(1)}$ and $[\overline{\gamma}_s]^{(1)}$ denoting the linear parts of the mappings $v \mapsto \overline{\gamma}_d(v, \alpha)$ and $v \mapsto \overline{\gamma}_s(v)$. We remark that the functional $\overline{\gamma}_d^0(v)$ in (2.69) contains no linear part, and is therefore not included in the definition of $[I_\alpha^\sigma]^{(1)}$.

2

In general, if $f$ is a map from a Banach space $X$ into a Banach space $Y$ that is $N+1$ times differentiable at $v = 0$, we write

$$f(v) = \sum_{k=0}^{N} [f]^{(k)}(v) + O\big(\|v\|_X^{N+1}\big)$$

where $[f]^{(k)}(v) \sim v^k$ is the symmetric $k$-linear map that collects the order $k$ powers of $v$ in $f(v)$. Alternatively, one can say that $[f]^{(k)}(v)$ denotes the order $k$ term in the Taylor expansion of $v \mapsto f(v)$ around zero.

This expansion procedure extends naturally to higher orders. For the next approximation, we also include quadratic terms. Furthermore, we base this approximation on an additional variable $\tilde{v}_2$, which should be thought of as an approximation to $v$ with error $\mathcal{O}(\sigma^3)$. Given two processes $\tilde{v}_1, \tilde{v}_2$, we define $A_2$ as the solution to

$$A_2(\tilde{v}_1, \tilde{v}_2, t) = I_\alpha^\sigma(0, A_2, t) + [I_\alpha^\sigma]^{(1)}(\tilde{v}_2; A_2, t) + [I_\alpha^\sigma]^{(2)}(\tilde{v}_1; A_2, t).$$

In general, $A_k$ is defined implicitly in terms of the processes $\tilde{v}_1, \ldots, \tilde{v}_k$ via the SDE

$$A_k(\tilde{v}_1, \ldots, \tilde{v}_k, t) = I_\alpha^\sigma(0, A_k, t) + \sum_{i=1}^{k} [I_\alpha^\sigma]^{(i)}(\tilde{v}_{\lfloor \frac{k}{i} \rfloor}; A_k, t) \tag{2.71}$$

with

$$\begin{aligned}[I_\alpha^\sigma]^{(k)}(v; \alpha, t) = &\int_0^t \big[-\alpha^{-2}[\overline{\gamma}_d^0]^{(k)}(v) + \sigma^2 [\overline{\gamma}]_d^{(k)}(v; \alpha)\big]\, \mathrm{d}s \\ &- \sigma \int_0^t \alpha \langle \hat{T}_\alpha \mathrm{d}W_s^Q, [\overline{\gamma}_s]^{(k)}(v) \rangle_{\mathcal{H}}.\end{aligned}$$

### 2.3.2 Expansion of the phase shift

Following the same procedure, we expand the modulation equation for the position $\xi$ in terms of $v$. In §2.2 we have seen that $\xi$ can be recovered from the identity

$$\xi(t) = \int_0^t c(s)\ \mathrm{d}s + I_\Omega^\sigma(v, \alpha, t),$$

where

$$I_\Omega^\sigma(v, \alpha, t) = -\int_0^t \alpha^{-2}\overline{\mu}_d^0(v)\ \mathrm{d}s + \sigma^2 \int_0^t \overline{\mu}_d(v, \alpha)\ \mathrm{d}s - \sigma \int_0^t \alpha \langle \hat{T}_\alpha \mathrm{d}W_s^Q, \overline{\mu}_s(v) \rangle_{\mathcal{H}}.$$

Note that the position primarily follows the velocity $c(t)$, with additional noise-induced corrections resulting in the phase shift

$$\Omega(t) = \xi(t) - \int_0^t c(s)\ \mathrm{d}s = I_\Omega^\sigma(v, \alpha, t). \tag{2.72}$$

Analogously to (2.70) and (2.71), we define approximations to the phase shift $\Omega(t)$ as

$$\bar{\Omega}_0(\alpha, t) = I_\Omega^\sigma(0, \alpha, t),$$

and for $k \geq 1$

$$\bar{\Omega}_k(\tilde{v}_1, \ldots, \tilde{v}_k, \alpha, t) = I_\Omega^\sigma(0, \alpha, t) + \sum_{i=1}^{k} [I_\Omega^\sigma]^{(i)}\big(\tilde{v}_{\lfloor \frac{k}{i} \rfloor}; \alpha, t\big).$$

### 2.3.3 Expansion of the perturbation

We now turn to the complementary problem and examine how the perturbation $v$ depends on the rescaling process $\alpha$. Treating $\alpha$ as an input, we expand the perturbation $v$ in terms of the small parameter $\sigma$ as

$$V_k(\alpha, t) = \sigma V^{(1)}(\alpha, t) + \ldots + \sigma^k V^{(k)}(\alpha, t) \tag{2.73}$$

based on the integral form (2.67). Here, $V^{(k)}$ collects all terms of $\mathcal{O}(\sigma^k)$ in the Itô form (2.67). Collecting all $\mathcal{O}(\sigma)$ terms in $I_v^\sigma$, gives

$$V^{(1)}(\alpha, t) = \sigma^{-1} I_v^\sigma(0, \alpha, t) = \int_0^t e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c*}} S(0)[\hat{T}_\alpha \mathrm{d}W_s^Q].$$

In order to find the subsequent term $V^{(2)}$ in the expansion (2.73), we note first that the drift component in the Itô form (2.68) satisfies $R^\sigma(v, \alpha) = \mathcal{O}(v^2 + \sigma^2)$. Consequently, we can explicitly define $V^{(2)}$ in (2.73) using $V^{(1)}$. Indeed, collecting the $\mathcal{O}(\sigma^2)$ terms in $I_v^\sigma$, we arrive at

$$\begin{aligned}
V^{(2)}(\alpha,t) = &\int_0^t \alpha^{-3} e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} \Big[N\big(V^{(1)}(\alpha,s)\big) + [R_0]^{(2)}\big(V^{(1)}(\alpha,s)\big)\Big] \,\mathrm{d}s \\
&+ \int_0^t e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} \sum_{i=1}^{6} R_i(0,\alpha) \,\mathrm{d}s \\
&+ \int_0^t e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} [S]^{(1)}\big(V^{(1)}(\alpha,s)\big)[\hat{T}_\alpha \mathrm{d}W_s^Q].
\end{aligned}$$

Any subsequent term $V^{(k)}$ in (2.73) can now be found by continuing systematically. In general, we have

$$\begin{aligned}
V^{(k)}(\alpha,t) = &\int_0^t \alpha^{-3} e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} \Big[\chi(k) N\big(V^{(\frac{k}{2})}(\alpha,s)\big) + \sum_{i|k} [R_0]^{(i)}\big(V^{(\frac{k}{i})}(\alpha,s)\big)\Big] \,\mathrm{d}s \\
&+ \int_0^t e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} \sum_{i=1}^{6} \sum_{j|(k-2)} [R_i]^{(j)}\big(V^{(\frac{k-2}{j})}(\alpha,s);\alpha\big) \,\mathrm{d}s \\
&+ \int_0^t e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} \sum_{i|(k-1)} [S]^{(i)}\big(V^{(\frac{k-1}{i})}(\alpha,s)\big)[\hat{T}_\alpha \mathrm{d}W_s^Q],
\end{aligned}$$

where $\chi(k) = 1$ if $k$ is even and $\chi(k) = 0$ otherwise.

### 2.3.4 The combined system approximation

We now combine the expansion of $\alpha$ in $v$ and the expansion of $v$ in $\sigma$ to construct our full approximations to the coupled system (2.67). We define for $k \geq 0$ approximations $\alpha_k$ to $\alpha$ as

$$\alpha_k(t) = A_k\big(V_1(\alpha_{k-1},\cdot),\ldots,V_k(\alpha_{k-1},\cdot),t\big),$$

and approximations $\Omega_k$ to $\Omega$ as

$$\Omega_k(t) = \bar{\Omega}_k\big(V_1(\alpha_k,\cdot),\ldots,V_k(\alpha_k,\cdot),\alpha_k,t\big).$$

For $k \geq 1$ we furthermore introduce the approximations

$$v_k(t) = V_k(\alpha_{k-1},t),$$

to $v$.

The soliton amplitude directly follows from the rescaling process $\alpha$ via the relation $c(t) = c_*\alpha^{-2}(t)$. We can therefore define approximations $c_0, c_1, c_2, \ldots$ to $c$ by directly writing

$$c_k = c_*\alpha_k^{-2} \quad \text{for} \quad k \geq 0.$$

We now examine what these approximation constructions produce for the examples discussed in §2.2.

### 2.3.5 Example I: Soliton dynamics for scalar noise

We first turn to the setting of Example I outlined in §2.2.1. Here, $v, \alpha$ and $\xi$ follow the modulation system (2.44)–(2.46). We observe that the fully decoupled approximation in this setting satisfies the SDE

$$\begin{aligned} d\alpha_0 &= \overline{\gamma}_{d;I}(0)\sigma^2\alpha_0 \, dt - \overline{\gamma}_{s;I}(0)\sigma\alpha_0 \, d\beta_t \\ &= (\tfrac{74}{135} + \tfrac{4\pi^2}{405})\sigma^2\alpha_0 \, dt - \tfrac{2}{3}\sigma\alpha_0 \, d\beta_t \end{aligned}$$

with $\alpha_0(0) = 1$. Here we have used Table 2.1 to evaluate the constants. This SDE admits the explicit solution

$$\alpha_0(t) = e^{(\overline{\gamma}_{d;I}(0) - \frac{1}{2}\overline{\gamma}^2_{s;I}(0))\sigma^2 t - \overline{\gamma}_{s;I}(0)\sigma\beta_t} = e^{(\frac{44}{135} + \frac{4\pi^2}{405})\sigma^2 t - \frac{2}{3}\sigma\beta_t},$$

which is a geometric Brownian motion.

At first order, the perturbation is given by

$$v_1(t) = \sigma \int_0^t e^{\int_s^t \alpha_0^{-3}(t')dt' \mathcal{L}_{c*}} S_I(0) \, d\beta_s$$

where we remark that

$$S_I(0) = -\tfrac{1}{3}\phi_{c_*} - \tfrac{2}{3}x\partial_x\phi_{c_*} + \tfrac{2}{3}c_*^{-1/2}\partial_x\phi_{c_*}.$$

The subsequent approximation $\alpha_1$ to the rescaling process is the geometric Brownian motion

$$d\alpha_1 = \sigma^2 K_I^{1,1}(t)\alpha_1 \, dt - \sigma K_I^{2,1}(t)\alpha_1 \, d\beta_t,$$

with random coefficients $K_I^{1,1}(t), K_I^{2,1}(t)$ that are given explicitly by

$$\begin{aligned} K_I^{1,1}(t) &= \overline{\gamma}_{d;I}(0) + [\overline{\gamma}_{d;I}]^{(1)}\big(v_1(t)\big), \\ K_I^{2,1}(t) &= \overline{\gamma}_{s;I}(0) + [\overline{\gamma}_{s;I}]^{(1)}\big(v_1(t)\big). \end{aligned}$$

The second order approximation for the perturbation is given explicitly by

$$v_2(t) = \sigma V^{(1)}(\alpha_1, t) + \sigma^2 V^{(2)}(\alpha_1, t),$$

using

$$V^{(1)}(\alpha, t) = \int_0^t e^{\int_s^t \alpha^{-3}(t')dt' \mathcal{L}_{c*}} S_I(0) \, d\beta_s$$

and

$$
\begin{aligned}
V^{(2)}(\alpha,t) &= \int_0^t \alpha^{-3}(s) e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} \Big[ N\big(V^{(1)}(\alpha,s)\big) + [R_0]^{(2)}\big(V^{(1)}(\alpha,s)\big)\Big] \,\mathrm{d}s \\
&+ \int_0^t e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} \sum_{i=1}^{6} R_{i;I}(0)\,\mathrm{d}s \\
&+ \int_0^t e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t' \mathcal{L}_{c_*}} [S_I]^{(1)}\big(V^{(1)}(\alpha,s)\big)\,\mathrm{d}\beta_s.
\end{aligned}
$$

This approximation collects the leading-order drift effects in the perturbation $v$. The sample mean of $v$ and $v_2$ are displayed in Figure 2.4, which shows the development of an average radiation field induced by the noise. These sample means are computed by first subtracting the process $v_1$, which has mean zero. This eliminates the leading-order fluctuations and speeds up convergence to the mean. In contrast to wave profiles in the stochastic (FitzHugh)-Nagumo equations analysed in [58], the average perturbation does not localise around the wave profile and does not seem to converge in time. Rather, the noise leads to an average pattern of radiation waves that expands far behind the soliton.

Using $v_2$, the approximation $\alpha_2$ is defined through the scalar SDE

$$
\mathrm{d}\alpha_2 = [-K_I^{0,2}(t)\alpha_2^{-2} + \sigma^2 K_I^{1,2}(t)\alpha_2]\,\mathrm{d}t - \sigma K_I^{2,2}(t)\alpha_2\,\mathrm{d}\beta_t
$$

with random coefficients

$$
\begin{aligned}
K_I^{0,2}(t) &= [\overline{\gamma}_d^0]^{(2)}\big(\sigma V^{(1)}(\alpha_1,t)\big), \\
K_I^{1,2}(t) &= \overline{\gamma}_{d;I}(0) + [\overline{\gamma}_{d;I}]^{(1)}\big(v_2(t)\big) + [\overline{\gamma}_{d;I}]^{(2)}\big(\sigma V^{(1)}(\alpha_1,t)\big), \\
K_I^{2,2}(t) &= \overline{\gamma}_{s;I}(0) + [\overline{\gamma}_{s;I}]^{(1)}\big(v_2(t)\big) + [\overline{\gamma}_{s;I}]^{(2)}\big(\sigma V^{(1)}(\alpha_1,t)\big).
\end{aligned}
$$

**Amplitude** The first approximation for the soliton amplitude $c(t)$ is the geometric Brownian motion

$$
c_0(t) = c_*\alpha_0^{-2}(t) = c_* e^{-(\frac{88}{135}+\frac{8\pi^2}{405})\sigma^2 t + \frac{4}{3}\sigma\beta_t}.
$$

We remark that for small noise strengths $\sigma$, the dynamics of $c_0(t)$ are largely determined by the factor $e^{\frac{4}{3}\sigma\beta_t}$. Note that this factor is also present in (2.10), which heuristically explains how the leading-order stochastic dynamics of $c(t)$ arise. Using the exact expression

$$
\mathrm{Var}[c_0(t)] = c_*^2 e^{(\frac{64}{135}-\frac{16\pi^2}{405})\sigma^2 t}(e^{\frac{16}{9}\sigma^2 t} - 1), \tag{2.74}
$$

we compare the variance of $c_0(t)$ to the sample variance of $c_{\mathrm{fit}}(t)$ in Figure 2.5a. Although the approximation $c_0$ is fully decoupled from the perturbation $v$, we see that its variance already agrees quite well with that of $c(t)$.

Figure 2.6 compares the mean of $c_{\mathrm{fit}}(t)$ with that of the increasingly refined

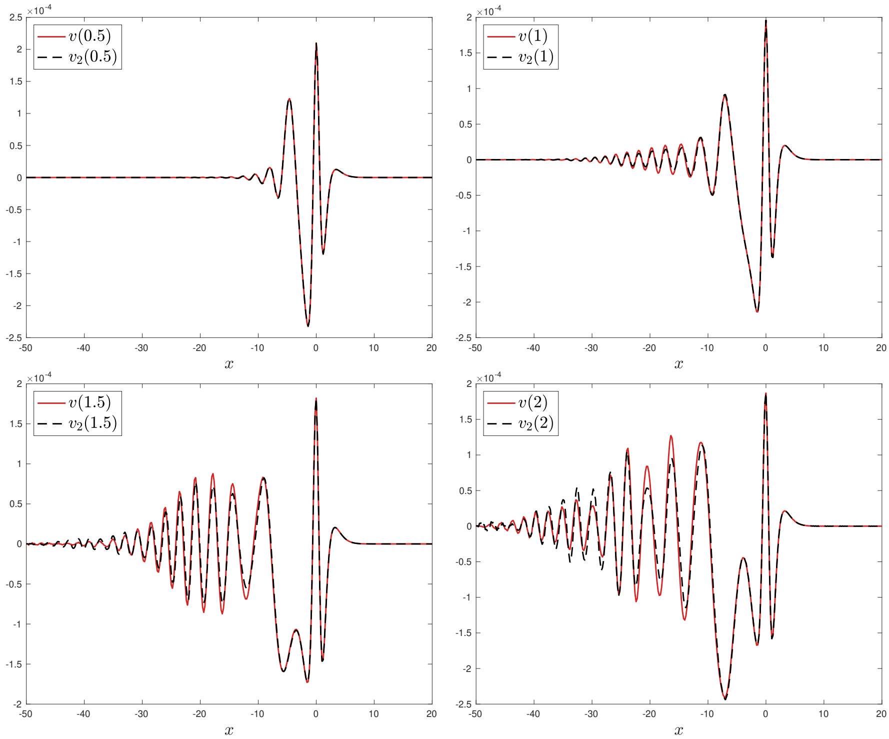



**Figure 2.4:** Sample mean of $v$ (dashed) and the approximation $v_2$ (solid) as the perturbation develops between $t = 0.5$ and $t = 2$. Computed over 3000 realisations for $\sigma = 0.03$.

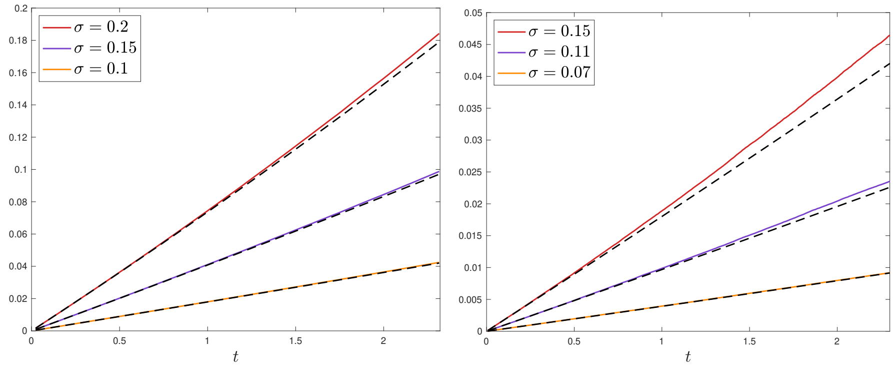
approximations $c_0(t), c_1(t)$ and $c_2(t)$, using

$$\mathbb{E}[c_0(t)] = c_* e^{(\frac{32}{135} - \frac{8\pi^2}{405})\sigma^2 t} \tag{2.75}$$

for $c_0(t)$. The means of $c_1(t)$ and $c_2(t)$ are not as easily computed analytically due to the dependence on $\alpha$ and $v$ in the random coefficients $K_I^{0,2}(t), K_I^{1,2}(t)$ and $K_I^{2,2}(t)$. We therefore consider their sample means. We see in Figure 2.6 that the mean of $c_{\text{fit}}(t)$ is not well-approximated by that of $c_0(t)$ or $c_1(t)$. The sample mean of $c_2(t)$, however, agrees well with that of $c_{\text{fit}}(t)$, indicating that the quadratic terms of $v$ contribute significantly to the evolution of the mean amplitude.

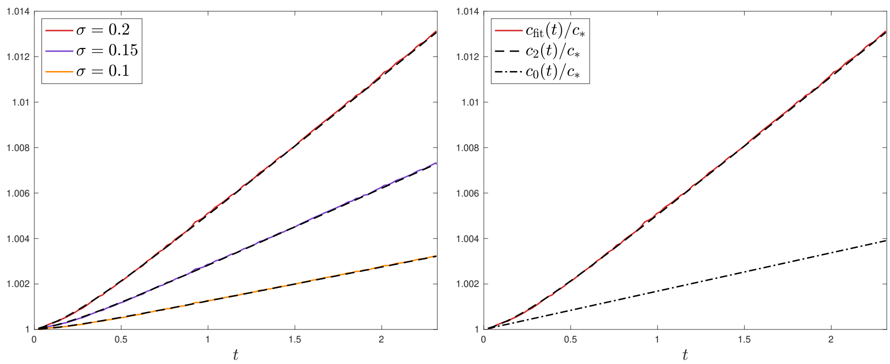
**Phase shift** The first approximation to the phase shift process $\Omega(t)$ defined in (2.66) is given by

$$\Omega_0(t) = \overline{\mu}_{d;I}(0)\sigma^2 \int_0^t \alpha_0(s)\ \mathrm{d}s + \tfrac{2}{3} c_*^{-1/2} \sigma \int_0^t \alpha_0(s)\ \mathrm{d}\beta_s.$$




**(a)** Scalar noise. **(b)** Space-time white noise.

**Figure 2.5:** Sample variance of the process $c_{\text{fit}}(t)/c_*$ for scalar noise and space-time white noise at various noise strengths $\sigma$. Solid lines indicate the sample variance, dashed lines indicate the variance of $c_0(t)$ as in (2.74) and (2.81), respectively. The sample variance is computed over $204 \cdot 10^4$ and $8 \cdot 10^4$ realizations, respectively.




**(a)** Mean of $c_{\text{fit}}(t)/c_*$ at various noise strengths. **(b)** Means of $c_{\text{fit}}(t)/c_*$ and approximations $c_0(t)$ and $c_2(t)$.

**Figure 2.6:** Sample mean of the process $c_{\text{fit}}(t)/c_*$ computed over $2 \cdot 10^4$ realisations for scalar noise. The mean of $c_2(t)$ (dashed) agrees well with that of $c_{\text{fit}}(t)$ at the simulated noise strength values $\sigma \in \{0.1, 0.15, 0.2\}$, whereas the mean of $c_0(t)$ (dash dot) as in (2.75) fails to capture the correct amplitude drift. The mean of $c_1(t)$ provides no improvement, it can not be distinguished from that of $c_0(t)$ at these simulation values.

We remark that numerical computations of $\Omega_{\text{fit}}(t)$ are unsuitable for ensemble simulations, due to a large truncation error (see Figure 2.2). We therefore consider ensemble simulations of the more robust process $\Omega(t)$. Figure 2.7 shows that the soliton position, on average, develops a phase lag from the velocity $c(t)$ which appears to grow almost linearly in time. The mean of the lowest approximation $\Omega_0(t)$

is evaluated as

$$
\begin{aligned}
\mathbb{E}[\Omega_0(t)] &= \overline{\mu}_{d;I}(0)\sigma^2 \int_0^t \mathbb{E}\big[\alpha_0(s)\big] \,\mathrm{d}s \\
&= \frac{(\frac{16\pi^2}{405} - \frac{34}{45})c_0^{-1/2}}{\frac{74}{135} + \frac{4\pi^2}{405}} \big(e^{(\frac{74}{135}+\frac{4\pi^2}{405})\sigma^2 t} - 1\big).
\end{aligned} \tag{2.76}
$$

For the variance, we write

$$
\begin{aligned}
\mathrm{Var}[\Omega_0(t)] = \tfrac{4}{9}c_*^{-1}\sigma^2 \,\mathrm{Var}\Big[\int_0^t \alpha_0(s)\,\mathrm{d}\beta_s\Big] \\
+ \tfrac{4}{3}c_*^{-1/2}\overline{\mu}_{d;I}(0)\sigma^3 \,\mathrm{Cov}\Big(\int_0^t \alpha_0(s)\,\mathrm{d}s, \int_0^t \alpha_0(s)\,\mathrm{d}\beta_s\Big) \\
+ \overline{\mu}_{d;I}(0)^2\sigma^4 \,\mathrm{Var}\Big[\int_0^t \alpha_0(s)\,\mathrm{d}s\Big],
\end{aligned}
$$

and explicitly compute the leading-order term

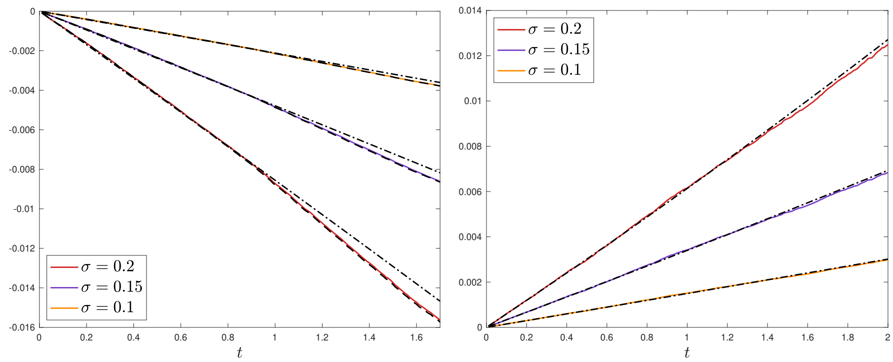
$$
\begin{aligned}
\sigma^2 \,\mathrm{Var}\Big[\int_0^t \alpha_0(s)\,\mathrm{d}\beta_s\Big] &= \tfrac{4}{9}c_*^{-1}\sigma^2 \mathbb{E}\left(\int_0^t \alpha_0(s)\,\mathrm{d}\beta_s\right)^2 \\
&= \tfrac{4}{9}c_*^{-1}\sigma^2 \int_0^t \mathbb{E}\big[\alpha_0^2(s)\big]\,\mathrm{d}s \\
&= \tfrac{45}{2c_*(78+\pi^2)}\big(e^{(\frac{208}{135}+\frac{8\pi^2}{405})\sigma^2 t} - 1\big).
\end{aligned} \tag{2.77}
$$

In Figure 2.7 we compare these statistics of $\Omega_0(t)$ to sample statistics of $\Omega(t)$. The sample variance agrees well with the prediction (2.77). The sample mean, however, differs slightly from the prediction (2.76). We observe that the approximation is significantly improved upon considering the sample mean of $\Omega_2(t)$.

**Remainders** In order to confirm that our approximation procedure only neglects higher-order noise effects, we numerically investigate the resulting error. Figure 2.8 shows the growth of the remainders $\|v(t) - v_1(t)\|_{L^2_a}$ and $|c(t) - c_2(t)|$. Indeed, the size of the remainders $\|v(t) - v_1(t)\|_{L^2_a}$ and $|c(t) - c_2(t)|$ decreases significantly with decreasing values of $\sigma$. An estimation of the order $\beta$ at which these remainders depend on $\sigma$ (see Figure 2.13 in 2.8) reveals that the remainders $\|v(t) - v_1(t)\|_{L^2_a}$ and $|c - c_2|$ scale with a power of $\sigma$ higher than 2 and 3, respectively. This indicates that $c_2$ indeed captures all effects of $\mathcal{O}(\sigma^2)$.

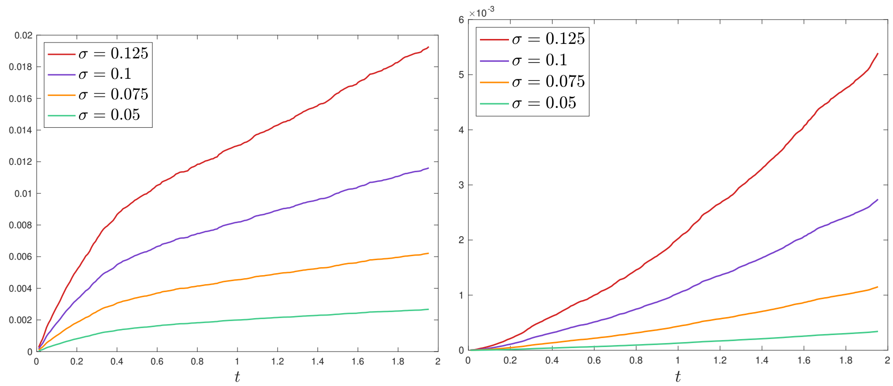
### 2.3.6 Example III: Soliton dynamics for space-time white noise

We now turn to the setting of space-time white noise, introduced in §2.2.1, where $\alpha$ and $\xi$ follow the modulation equations (2.62)–(2.63). A key difference in this example is that the noise is space-dependent. The modulation equations are formulated



**(a)** Sample mean of $\Omega(t)$. **(b)** Sample variance of $\Omega(t)$.

**Figure 2.7:** Sample statistics of the process $\Omega(t)$ for scalar noise. Computed over $3 \cdot 10^3$ realizations for $\sigma \in \{0.1, 0.15, 0.2\}$. Dash-dotted lines indicate the theoretical mean and (leading-order) variance of $\Omega_0(t)$ as in (2.76) and (2.77), respectively. Panel (a) also shows the sample mean of $\Omega_2(t)$ (dotted), which gives significant improvement over the mean of $\Omega_0(t)$.



**(a)** Sample mean of $\sup_{s \leq t} \|v(s) - v_1(s)\|_{L^2_a}$. **(b)** Sample mean of $\sup_{s \leq t} |c(s) - c_2(s)|$.

**Figure 2.8:** The error made by approximating $v$ with the first order expansion $v_1$ and $c$ with the second order expansion $c_2$, for scalar noise. Computed over 200 simulations, for $\sigma \in \{0.05, 0.075, 0.1, 0.125\}$.

in the stochastic co-moving frame, where the noise undergoes a spatial rescaling by $T_\alpha$. We discuss this noise transformation in 2.5, where we show that the process $\tilde{W}_t := \alpha^{1/2} T_\alpha W_t$ generates the same statistics as the white noise $W_t$. In what follows, we formulate the modulation equations (2.62)–(2.63) using the space-time white noise $\tilde{W}_t$.

The first approximation to the rescaling process $\alpha$ is then defined as the process

$\alpha_0$ which satisfies the SDE

$$\mathrm{d}\alpha_0 = \overline{\gamma}_{d;\mathit{III}}(0)\sigma^2 \ \mathrm{d}t - \sigma\alpha_0^{1/2}\big\langle \mathrm{d}\tilde{W}_t, \overline{\gamma}_\diamond(0)\big\rangle_{L^2}, \tag{2.78}$$

with $\alpha_0(0) = 1$. Note that, in distribution, the Brownian motion driving this SDE equals

$$-\sigma\big\langle \tilde{W}_t, \overline{\gamma}_\diamond(0)\big\rangle_{L^2} \stackrel{d}{=} \big\|\overline{\gamma}_\diamond(0)\big\|_{L^2}\sigma\beta_t = \sqrt{\tfrac{4}{35}}c_*^{1/4}\sigma\beta_t,$$

so that (2.78) is of the form

$$\mathrm{d}X(t) = \delta \ \mathrm{d}t + sX^{1/2}(t) \ \mathrm{d}\beta_t. \tag{2.79}$$

The solution to (2.79) with $\delta \geq 0$ is known as a squared Bessel process [52]. The squared Bessel process $X(t)$ remains strictly positive for $\delta \geq \frac{1}{2}s^2$, and therefore the drift component $\overline{\gamma}_{d;\mathit{III}}(0) \approx 0.093c_*^{1/2} \geq \frac{2}{35}c_*^{1/2}$ in (2.78) is large enough to ensure that $\alpha_0(t)$ remains strictly positive. The mean of the approximation $\alpha_0$ is easily computed as[6]

$$\mathbb{E}\big[\alpha_0(t)\big] = 1 + \overline{\gamma}_{d;\mathit{III}}(0)\sigma^2 t \approx 1 + 0.093c_*^{1/2}\sigma^2 t.$$

At first order, the perturbation is given by

$$v_1(t) = \sigma\int_0^t \alpha^{-1/2}(s)e^{\int_s^t \alpha_0^{-3}(t')\mathrm{d}t'\mathcal{L}_{c_*}}S_\diamond(0)[\mathrm{d}W_s],$$

where

$$\begin{aligned} S_\diamond(0)[h] = \phi_{c_*}h - \tfrac{1}{9}c_*^{-3/2}(x\partial_x + 2)\phi_{c_*}\langle h, \phi_{c_*}^2\rangle_{L^2} \\ - \tfrac{2}{9}\partial_x\phi_{c_*}\langle h, c_*^{-2}\phi_{c_*}^2 - c_*^{-1/2}\phi_{c_*}\zeta_{c_*}\rangle_{L^2}. \end{aligned} \tag{2.80}$$

The second order approximation for the perturbation is given explicitly as

$$v_2(t) = \sigma V^{(1)}(\alpha_1, t) + \sigma^2 V^{(2)}(\alpha_1, t),$$

using

$$V^{(1)}(\alpha, t) = \int_0^t \alpha^{-1/2}(s)e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t'\mathcal{L}_{c_*}}S_\diamond(0)[\mathrm{d}W_s]$$

and

$$V^{(2)}(\alpha, t) = \int_0^t \alpha^{-3}(s)e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t'\mathcal{L}_{c_*}}\Big[N\big(V^{(1)}(\alpha, s)\big) + [R_0]^{(2)}\big(V^{(1)}(\alpha, s)\big)\Big] \ \mathrm{d}s$$

[6] Higher order and negative moments can in principle be computed explicitly by using that a Bessel process $X(t)$ as defined by (2.79) has the noncentral Chi-square distribution $\frac{4}{s^2 t}X(t) \sim \chi^2_{4\delta/s^2}\big(\frac{4}{s^2 t}X(0)\big)$.

$$+\int_0^t \alpha^{-1}(s)e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t'\mathcal{L}_{c*}}\sum_{i=1}^{6} R_{i;III}(0)\ \mathrm{d}s$$
$$+\int_0^t \alpha^{-1/2}(s)e^{\int_s^t \alpha^{-3}(t')\mathrm{d}t'\mathcal{L}_{c*}}[S_\diamond]^{(1)}\big(V^{(1)}(\alpha,s)\big)[\mathrm{d}W_s].$$

The subsequent approximation $\alpha_1$ to the rescaling process satisfies the square root SDE

$$\mathrm{d}\alpha_1 = \sigma^2 K_{III}^{1,1}(t)\ \mathrm{d}t - \sigma\alpha_1^{1/2}\big\langle \mathrm{d}\tilde{W}_t, K_{III}^{2,1}(t)\big\rangle_{L^2},$$

with random coefficient

$$K_{III}^{1,1}(t) = \overline{\gamma}_{d;III}(0) + [\overline{\gamma}_{d;III}]^{(1)}\big(v_1(t)\big),$$

where $K_{III}^{2,1}$ is the process

$$K_{III}^{2,1}(t) = \overline{\gamma}_\diamond(0) + [\overline{\gamma}_\diamond]^{(1)}\big(v_1(t)\big).$$

Subsequently, the approximation $\alpha_2$ is given by the SDE

$$\mathrm{d}\alpha_2 = [-K_{III}^{0,2}(t)\alpha_2^{-2} + \sigma^2 K_{III}^{1,2}(t)]\ \mathrm{d}t - \sigma\alpha_2^{1/2}\big\langle \mathrm{d}\tilde{W}_t, K_{III}^{2,2}(t)\big\rangle_{L^2}$$

with random coefficients

$$\begin{aligned} K_{III}^{0,2}(t) &= [\overline{\gamma}_d^0]^{(2)}\big(\sigma V^{(1)}(\alpha_1,t)\big), \\ K_{III}^{1,2}(t) &= \overline{\gamma}_{d;III}(0) + [\overline{\gamma}_{d;III}]^{(1)}\big(v_2(t)\big) + [\overline{\gamma}_{d;III}]^{(2)}\big(\sigma V^{(1)}(\alpha_1,t)\big), \end{aligned}$$

where $K_{III}^{2,2}(t)$ is the process

$$K_{III}^{2,2}(t) = \overline{\gamma}_\diamond(0) + [\overline{\gamma}_\diamond]^{(1)}\big(v_2(t)\big) + [\overline{\gamma}_\diamond]^{(2)}\big(\sigma V^{(1)}(\alpha_1,t)\big).$$

**Amplitude** Using Itô's lemma and (2.78), we find that the first approximation $c_0(t) = c_*\alpha_0^{-2}(t)$ for the amplitude process $c(t)$ has the Itô form

$$\begin{aligned} c_0(t) = c_* &+ 2c_*\sigma\int_0^t \alpha_0^{-5/2}\big\langle \mathrm{d}\tilde{W}_s, \overline{\gamma}_\diamond(0)\big\rangle_{L^2} \\ &+ c_*\sigma^2\big[-2\overline{\gamma}_{d;III}(0) + 3\|\overline{\gamma}_\diamond(0)\|_{L^2}^2\big]\int_0^t \alpha_0^{-3}\ \mathrm{d}s. \end{aligned}$$

We compute the leading-order variance

$$\begin{aligned} \mathrm{Var}[c_0(t)] &= 4\sigma^2 c_*^2\mathbb{E}\Big[\Big(\int_0^t \alpha_0^{-5/2}\big\langle \mathrm{d}W_s, \overline{\gamma}_\diamond(0)\big\rangle_{L^2}\Big)^2\Big] + \mathcal{O}(\sigma^3) \\ &= 4\sigma^2 c_*^2\|\overline{\gamma}_\diamond(0)\|_{L^2}^2\int_0^t \mathbb{E}\big[\alpha_0^{-5}\big]\ \mathrm{d}s + \mathcal{O}(\sigma^3) \end{aligned}$$

$$= \tfrac{16}{35}\sigma^2 c_*^{5/2} \int_0^t \mathbb{E}\big[\alpha_0^{-5}\big]\ \mathrm{d}s + \mathcal{O}(\sigma^3) \tag{2.81}$$

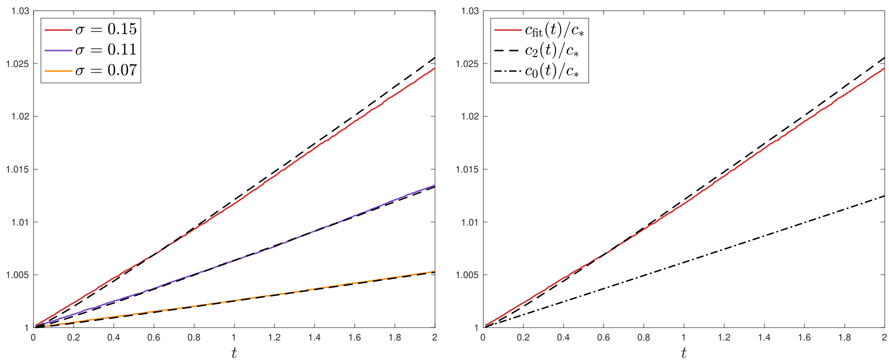
and compare this expression to the sample variance of $c_{\text{fit}}(t)$ in Figure 2.5b.

Figure 2.9 compares the mean of $c_{\text{fit}}(t)$ with that of the increasingly refined approximations $c_0(t), c_1(t)$ and $c_2(t)$. Here we use

$$\mathbb{E}[c_0(t)] = c_* + c_*\sigma^2\Big[-2\overline{\gamma}_{d;\mathit{III}}(0) + 3\big\|\overline{\gamma}_\diamond(0)\big\|_{L^2}^2\Big] \int_0^t \mathbb{E}\big[\alpha_0^{-3}(s)\big]\ \mathrm{d}s \tag{2.82}$$

$$\approx c_* + 0.16 c_*^{3/2}\sigma^2 \int_0^t \mathbb{E}\big[\alpha_0^{-3}(s)\big]\ \mathrm{d}s, \tag{2.83}$$

and numerically compute the negative moment $\mathbb{E}[\alpha_0^{-3}(t)]$[7]. For the means of $c_1(t)$ and $c_2(t)$ we, once more, resort to the sample mean. As is the case for scalar noise, the quadratic terms of $v$ contribute significantly to the mean amplitude, and the mean of $c_{\text{fit}}(t)$ is not well-approximated by that of $c_0(t)$. We see in Figure 2.9 that the amplitude drift is well-approximated by the mean of $c_2(t)$.




**(a)** Mean of $c_{\text{fit}}(t)/c_*$ at various noise strengths.

**(b)** Means of $c_{\text{fit}}(t)/c_*$ and approximations $c_0(t)$ and $c_2(t)$.

**Figure 2.9:** Sample mean of the process $c_{\text{fit}}(t)/c_*$ computed over $36 \cdot 10^3$ realisations for space-time white noise. The mean of $c_2(t)$ (dashed) agrees well with that of $c_{\text{fit}}(t)$ at the simulated noise strength values $\sigma \in \{0.07, 0.11, 0.15\}$, whereas the mean of $c_0(t)$ (dash dot) as in (2.75) fails to capture the correct amplitude drift. The mean of $c_1(t)$ provides no improvement, it can not be distinguished from that of $c_0(t)$ at these simulation values.

[7] We remark that, alternatively, this moment can be evaluated numerically by repeated integration of the moment generating function of $\alpha_0(t)$, which has a noncentral Chi-square distribution.

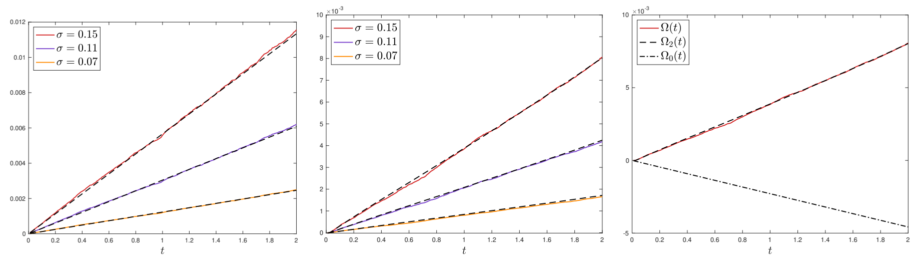
**Phase shift** The first approximation $\Omega_0(t)$ to the phase shift process $\Omega(t)$ defined in (2.66) is given by

$$\Omega_0(t) = \sigma^2 \overline{\mu}_{d;III}(0)t - \sigma \int_0^t \alpha_0^{1/2} \big\langle \mathrm{d}\tilde{W}_s, \overline{\mu}_\diamond(0) \big\rangle_{L^2}. \tag{2.84}$$

We can explicitly compute the variance of the phase shift approximation $\Omega_0(t)$:

$$\begin{aligned}
\mathrm{Var}[\Omega_0(t)] &= \sigma^2 \mathbb{E}\left( \int_0^t \alpha_0^{1/2} \langle \mathrm{d}\tilde{W}_s, \overline{\mu}_\diamond(0) \rangle_{L^2} \right)^2 \\
&= \sigma^2 \big\| \langle \cdot, \overline{\mu}_\diamond(0) \rangle_{L^2} \big\|^2_{\mathrm{HS}(L^2,\mathbb{R})} \int_0^t \mathbb{E}\big[\alpha_0(s)\big] \ \mathrm{d}s \\
&= \sigma^2 \big\| \overline{\mu}_\diamond(0) \big\|^2_{L^2} (t + \tfrac{1}{2} \overline{\gamma}_{d;III}(0) \sigma^2 t^2) \\
&\approx 0.435 c_*^{-1/2} \sigma^2 t + 0.02 \sigma^4 t^2.
\end{aligned} \tag{2.85}$$

This expression agrees well with the sample variance of $\Omega(t)$, as shown in Figure 2.10a.




**(a)** Sample variance of $\Omega(t)$. **(b)** Sample mean of $\Omega(t)$. **(c)** Means of $\Omega_{\mathrm{fit}}(t)$ and approximations $\Omega_0(t)$ and $\Omega_2(t)$.

**Figure 2.10:** Sample statistics (solid) of the process $\Omega(t)$ for space-time white noise, at noise strengths $\sigma \in \{0.07, 0.11, 0.15\}$. Dashed lines indicate the theoretical variance of $\Omega_0(t)$ as in (2.85) and the sample mean of $\Omega_2(t)$, respectively. The dash-dotted line in panel (c) shows the mean of $\Omega_0(t)$, see (2.84), which fails to capture the correct phase drift. The sample variance is computed over 2500 realizations and the sample mean over $10^4$ realizations.

The mean phase shift from the primary velocity $c(t)$ is influenced by quadratic terms of the perturbation $v$. We thus consider the process

$$\begin{aligned}
\Omega_2(t) = &- \int_0^t M_{III}^{0,2}(s) \alpha_2^{-2}(s) \ \mathrm{d}s + \sigma^2 \int_0^t M_{III}^{1,2}(s) \ \mathrm{d}s \\
&- \sigma \int_0^t \alpha_2^{1/2} \big\langle \mathrm{d}\tilde{W}_s, M_{III}^{2,2}(s) \big\rangle_{L^2}
\end{aligned}$$

with random coefficients

$$\begin{aligned}
M_{III}^{0,2}(t) &= [\overline{\mu}_d^0]^{(2)}\big(V_1(\alpha_2,t)\big),\\
M_{III}^{1,2}(t) &= \overline{\mu}_{d;III}(0) + [\overline{\mu}_{d;III}]^{(1)}\big(V_2(\alpha_2,t)\big) + [\overline{\mu}_{d;III}]^{(2)}\big(V_1(\alpha_2,t)\big)
\end{aligned}$$

and where $M_{III}^{2,2}$ is the process

$$M_{III}^{2,2}(t) = \overline{\mu}_\diamond(0) + [\overline{\mu}_\diamond]^{(1)}\big(V_2(\alpha_2,t)\big) + [\overline{\mu}_\diamond]^{(2)}\big(V_1(\alpha_2,t)\big).$$

We compare a numerical evaluation of the mean

$$\mathbb{E}[\Omega_2(t)] = -\int_0^t \mathbb{E}\big[M_{III}^{0,2}(s)\alpha_2^{-2}(s)\big]\ \mathrm{d}s + \sigma^2 \int_0^t \mathbb{E}\big[M_{III}^{1,2}(s)\big]\ \mathrm{d}s$$

to the sample mean of $\Omega(t)$ in Figure 2.10b and observe that they agree quite well.

## 2.4 Frozen-frame transformation

Our goal here is to derive the SPDE (2.31) for the remainder $v = \alpha^2 T_{\alpha,\xi}u - \phi_{c_*}$, where we recall that the processes $\alpha$ and $\xi$ satisfy (2.25) and (2.26), respectively. We define $\Phi_{\alpha,\xi}u = \alpha^2 T_{\alpha,\xi}u$ so that $v = \Phi_{\alpha,\xi}u - \phi_{c_*}$. For an arbitrary test-function $\zeta$, we now characterize the evolution of the real-valued process

$$\langle v,\zeta\rangle_{L^2} = \langle \Phi_{\alpha,\xi}u - \phi_{c_*},\zeta\rangle_{L^2}.$$

To this end, we collect the first and second-order (Fréchet) derivatives of the mapping $(u,\alpha,\xi) \mapsto \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}$. For the derivatives with respect to $u,\alpha$ and $\xi$ we can write

$$\begin{aligned}
\partial_u \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}[v] &= \alpha^2\langle T_{\alpha,\xi}v,\zeta\rangle_{L^2}\\
\partial_\alpha \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2} &= 2\alpha\langle T_{\alpha,\xi}u,\zeta\rangle_{L^2} + \alpha^2\langle xT_{\alpha,\xi}u_x,\zeta\rangle_{L^2}\\
\partial_\xi \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2} &= \alpha^2\langle T_{\alpha,\xi}u_x,\zeta\rangle_{L^2},
\end{aligned}$$

and for the second derivatives we find

$$\begin{aligned}
\partial_u^2 \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}[v,w] &= 0,\\
\partial_\alpha^2 \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2} &= 2\langle T_{\alpha,\xi}u,\zeta\rangle_{L^2} + 4\alpha\langle xT_{\alpha,\xi}u_x,\zeta\rangle_{L^2} + \alpha^2\langle x^2T_{\alpha,\xi}u_{xx},\zeta\rangle_{L^2},\\
\partial_\xi^2 \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2} &= \alpha^2\langle T_{\alpha,\xi}u_{xx},\zeta\rangle_{L^2},\\
\partial_{\alpha\xi} \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2} &= 2\alpha\langle T_{\alpha,\xi}u_x,\zeta\rangle_{L^2} + \alpha^2\langle xT_{\alpha,\xi}u_{xx},\zeta\rangle_{L^2},\\
\partial_{u\alpha} \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}[v] &= 2\alpha\langle T_{\alpha,\xi}v,\zeta\rangle_{L^2} + \alpha^2\langle xT_{\alpha,\xi}v_x,\zeta\rangle_{L^2},\\
\partial_{u\xi} \langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}[v] &= \alpha^2\langle T_{\alpha,\xi}v_x,\zeta\rangle_{L^2}.
\end{aligned}$$

Upon choosing an orthonormal basis $\{e_k\}_{k=0}^{\infty}$ of $\mathcal{H}$, Itô's lemma [23, Theorem 4.32] now gives

$$\mathrm{d}\langle v,\zeta\rangle_{L^2}=\sum_{i=0}^{6}\overline{R}_i^{\sigma}(u,\alpha,\xi,\zeta)\ \mathrm{d}t+\overline{S}^{\sigma}(u,\alpha,\xi,\zeta)[\mathrm{d}W_t^Q],$$

where

$$\begin{aligned}
\overline{R}_0^{\sigma}(u,\alpha,\xi,\zeta)&=\partial_u\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}[-u_{xxx}-2uu_x],\\
\overline{R}_1^{\sigma}(u,\alpha,\xi,\zeta)&=\tfrac{1}{2}\sum_{k=0}^{\infty}\gamma_s^{\sigma}[Q^{1/2}e_k]^2\partial_\alpha^2\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2},\\
\overline{R}_2^{\sigma}(u,\alpha,\xi,\zeta)&=\tfrac{1}{2}\sum_{k=0}^{\infty}\mu_s^{\sigma}[Q^{1/2}e_k]^2\partial_\xi^2\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2},\\
\overline{R}_3^{\sigma}(u,\alpha,\xi,\zeta)&=\sum_{k=0}^{\infty}\gamma_s^{\sigma}[Q^{1/2}e_k]\mu_s^{\sigma}[Q^{1/2}e_k]\partial_{\alpha\xi}\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2},\\
\overline{R}_4^{\sigma}(u,\alpha,\xi,\zeta)&=\sigma\sum_{k=0}^{\infty}\gamma_s^{\sigma}[Q^{1/2}e_k]\partial_{u\alpha}\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}\big[M(u)[Q^{1/2}e_k]\big],\\
\overline{R}_5^{\sigma}(u,\alpha,\xi,\zeta)&=\sigma\sum_{k=0}^{\infty}\mu_s^{\sigma}[Q^{1/2}e_k]\partial_{u\xi}\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}\big[M(u)[Q^{1/2}e_k]\big],\\
\overline{R}_6^{\sigma}(u,\alpha,\xi,\zeta)&=\gamma_d^{\sigma}\partial_\alpha\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}+\mu_d^{\sigma}\partial_\xi\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
\overline{S}^{\sigma}(u,\alpha,\xi,\zeta)[h]&=\sigma\partial_u\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}\big[M(u)[h]\big]+\partial_\alpha\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}\gamma_s^{\sigma}[h]\\
&\quad+\partial_\xi\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}\mu_s^{\sigma}[h].
\end{aligned}$$

In the term $\overline{R}_3^{\sigma}(u,\alpha,\xi,\zeta)$, we simplify the summation as

$$\begin{aligned}
\sum_{k=0}^{\infty}\gamma_s^{\sigma}[Q^{1/2}e_k]\mu_s^{\sigma}[Q^{1/2}e_k]&=\sigma^2\alpha^2\sum_{k=0}^{\infty}\langle\hat{T}_{\alpha,\xi}Q^{1/2}e_k,\overline{\gamma}_s\rangle_{\mathcal{H}}\langle\hat{T}_{\alpha,\xi}Q^{1/2}e_k,\overline{\mu}_s\rangle_{\mathcal{H}}\\
&=\sigma^2\alpha^2\langle Q^{1/2}\hat{T}_{\alpha,\xi}^{*}\overline{\gamma}_s,Q^{1/2}\hat{T}_{\alpha,\xi}^{*}\overline{\mu}_s\rangle_{\mathcal{H}},
\end{aligned}$$

where we have used that $\gamma_s^{\sigma}$ is of the form (2.27). In the mixed derivative

$$\partial_{\alpha\xi}\langle\Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}=2\alpha\langle T_{\alpha,\xi}u_x,\zeta\rangle_{L^2}+\alpha^2\langle xT_{\alpha,\xi}u_{xx},\zeta\rangle_{L^2}$$

we substitute

$$T_{\alpha,\xi}[\partial_x^j u]=\alpha^{-(j+2)}\partial_x^j[\phi_{c_*}+v], \tag{2.86}$$

for $j = 1, 2$ and find

$$\begin{aligned}\overline{R}_3^\sigma(u,\alpha,\xi,\zeta) &= 2\sigma^2 \langle Q^{1/2}\hat{T}^*_{\alpha,\xi}\overline{\gamma}_s, Q^{1/2}\hat{T}^*_{\alpha,\xi}\overline{\mu}_s\rangle_{\mathcal{H}} \langle \partial_x[\phi_{c_*}+v],\zeta\rangle_{L^2} \\ &\quad + \sigma^2 \langle Q^{1/2}\hat{T}^*_{\alpha,\xi}\overline{\gamma}_s, Q^{1/2}\hat{T}^*_{\alpha,\xi}\overline{\mu}_s\rangle_{\mathcal{H}} \langle x\partial_x^2[\phi_{c_*}+v],\zeta\rangle_{L^2}.\end{aligned}$$

An analogous computation shows that

$$\begin{aligned}\overline{R}_1^\sigma(u,\alpha,\xi,\zeta) &= \sigma^2 \|Q^{1/2}\hat{T}^*_{\alpha,\xi}\overline{\gamma}_s\|^2_{\mathcal{H}}\big(2\langle \phi_{c_*}+v,\zeta\rangle_{L^2} \\ &\quad + 4\langle x\partial_x[\phi_{c_*}+v],\zeta\rangle_{L^2} + \langle x^2\partial_x^2[\phi_{c_*}+v],\zeta\rangle_{L^2}\big)\end{aligned}$$

and

$$\overline{R}_2^\sigma(u,\alpha,\xi,\zeta) = \sigma^2 \|Q^{1/2}\hat{T}^*_{\alpha,\xi}\overline{\mu}_s\|^2_{\mathcal{H}} \langle \partial_x[\phi_{c_*}+v],\zeta\rangle_{L^2}.$$

In order to simplify the term $\overline{R}_5^\sigma(u,\alpha,\xi,\zeta)$, we rewrite the summands as

$$\begin{aligned}&\mu_s^\sigma[Q^{1/2}e_k]\partial^2_{u\xi}\langle \Phi_{\alpha,\xi}u,\zeta\rangle_{L^2}\big[M(u)[Q^{1/2}e_k]\big] \\ &\qquad = -\sigma\alpha^3 \langle \hat{T}_{\alpha,\xi}Q^{1/2}e_k, \overline{\mu}_s\rangle_{\mathcal{H}} \langle T_{\alpha,\xi}\partial_x M(u)[Q^{1/2}e_k],\zeta\rangle_{L^2} \\ &\qquad = \sigma\alpha^3 \langle e_k, Q^{1/2}\hat{T}^*_{\alpha,\xi}\overline{\mu}_s\rangle_{\mathcal{H}} \big\langle e_k, Q^{1/2}M^*(u)[\partial_x T^*_{\alpha,\xi}\zeta]\big\rangle_{\mathcal{H}},\end{aligned}$$

so that

$$\begin{aligned}\overline{R}_5^\sigma(u,\alpha,\xi,\zeta) &= \sigma^2\alpha^3 \big\langle Q^{1/2}\hat{T}^*_{\alpha,\xi}\overline{\mu}_s, Q^{1/2}M^*(u)[\partial_x T^*_{\alpha,\xi}\zeta]\big\rangle_{\mathcal{H}} \\ &= -\sigma^2\alpha^3 \langle T_{\alpha,\xi}\partial_x M(u)[Q\hat{T}^*_{\alpha,\xi}\overline{\mu}_s],\zeta\rangle_{L^2} \\ &= -\sigma^2\alpha^3 \langle M(T_{\alpha,\xi}u_x)[\hat{T}_{\alpha,\xi}Q\hat{T}^*_{\alpha,\xi}\overline{\mu}_s],\zeta\rangle_{L^2} \\ &\quad - \sigma^2\alpha^3 \langle M(T_{\alpha,\xi}u)[\hat{T}_{\alpha,\xi}\hat{\partial}_x Q\hat{T}^*_{\alpha,\xi}\overline{\mu}_s],\zeta\rangle_{L^2}.\end{aligned}$$

After substituting (2.86) for $j = 0, 1$ we find

$$\begin{aligned}\overline{R}_5^\sigma(u,\alpha,\xi,\zeta) &= -\sigma^2 \big\langle M\big(\partial_x[\phi_{c_*}+v]\big)[\hat{T}_{\alpha,\xi}Q\hat{T}^*_{\alpha,\xi}\overline{\mu}_s],\zeta\big\rangle_{L^2} \\ &\quad - \sigma^2\alpha \langle M(\phi_{c_*}+v)[\hat{T}_{\alpha,\xi}\hat{\partial}_x Q\hat{T}^*_{\alpha,\xi}\overline{\mu}_s],\zeta\rangle_{L^2}.\end{aligned}$$

An analogous computation shows that

$$\begin{aligned}\overline{R}_4^\sigma(u,\alpha,\xi,\zeta) &= -2\sigma^2 \langle M(\phi_{c_*}+v)[\hat{T}_{\alpha,\xi}Q\hat{T}^*_{\alpha,\xi}\overline{\gamma}_s],\zeta\rangle_{L^2} \\ &\quad - \sigma^2 \big\langle xM\big(\partial_x[\phi_{c_*}+v]\big)[\hat{T}_{\alpha,\xi}Q\hat{T}^*_{\alpha,\xi}\overline{\gamma}_s],\zeta\big\rangle_{L^2} \\ &\quad - \sigma^2\alpha \langle xM(\phi_{c_*}+v)[\hat{T}_{\alpha,\xi}\hat{\partial}_x Q\hat{T}^*_{\alpha,\xi}\overline{\gamma}_s],\zeta\rangle_{L^2}.\end{aligned}$$

In the term $\overline{R}_0^\sigma(u,\alpha,\xi,\zeta)$ we substitute (2.86) for $j = 0, 1, 3$ and

$$-\partial_x^3[\phi_{c_*}+v] - 2[\phi_{c_*}+v]\partial_x[\phi_{c_*}+v] = \mathcal{L}_{c_*}v + N(v) - c_*\partial_x[\phi_{c_*}+v]$$

to obtain

$$\overline{R}_0^\sigma(u,\alpha,\xi,\zeta) = \alpha^{-3}\langle \mathcal{L}_{c_*}v + N(v) - c_*\partial_x[\phi_{c_*} + v], \zeta\rangle_{L^2}.$$

In the martingale component $\overline{S}^\sigma$, we substitute the derivatives and (2.27)–(2.28), which gives

$$\begin{aligned}\overline{S}^\sigma(u,\alpha,\xi,\zeta)[h] &= \sigma\alpha^2\langle M(T_{\alpha,\xi}u)[\hat{T}_{\alpha,\xi}h], \zeta\rangle_{L^2} \\ &\quad - \sigma(2\alpha^2\langle T_{\alpha,\xi}u,\zeta\rangle_{L^2} + \alpha^3\langle xT_{\alpha,\xi}u_x,\zeta\rangle_{L^2})\langle \hat{T}_{\alpha,\xi}h, \overline{\gamma}_s\rangle_{\mathcal{H}} \\ &\quad - \sigma\alpha^3\langle T_{\alpha,\xi}u_x,\zeta\rangle_{L^2}\langle \hat{T}_{\alpha,\xi}h, \overline{\mu}_s\rangle_{\mathcal{H}}.\end{aligned}$$

Substituting (2.86) then yields

$$\overline{S}^\sigma(u,\alpha,\xi,\zeta)[h] = \langle \sigma S(v)[\hat{T}_{\alpha,\xi}h], \zeta\rangle_{L^2},$$

where $S$ is defined in (2.34). We collect also that

$$\overline{R}_0^\sigma(u,\alpha,\xi,\zeta) = \langle \alpha^{-3}\mathcal{L}_{c_*}v, \zeta\rangle_{L^2} + \langle R_0^\sigma(v,\alpha,\xi), \zeta\rangle_{L^2}$$

and

$$\overline{R}_i^\sigma(u,\alpha,\xi,\zeta) = \langle R_i^\sigma(v,\alpha,\xi), \zeta\rangle_{L^2},$$

for $i = 1,\ldots,6$, where $R_0^\sigma,\ldots,R_6^\sigma$ are defined in (2.33). Since the test function $\zeta$ was arbitrary, we conclude that (2.31) follows.

## 2.5 Noise rescaling

In this appendix we collect several useful properties of the family $\{Q_\alpha\}_{\alpha>0}$ defined in (2.22) in relation to the transformation operators $T_{\alpha,\xi}$ defined in (2.15).

**Lemma 2.5.1.** *Let $\alpha,\beta > 0$ and $\xi \in \mathbb{R}$. Denote by $T_{\alpha,\xi}^*$ the $L^2(\mathbb{R})$-adjoint of $T_{\alpha,\xi}$. Then we have the identities*

1. $(Q_\alpha)^{1/2} = (Q^{1/2})_\alpha$;
2. $T_{\alpha,\xi}Q_\beta = Q_{\beta\alpha}T_{\alpha,\xi}$;
3. $T_{\alpha,\xi}^* = \alpha^{-1}T_{\alpha^{-1},-\alpha^{-1}\xi}$.

*Proof.* Recall that $\hat{q}$ denotes the Fourier transform of the convolution kernel $q$, and $q_{1/2} := \mathcal{F}^{-1}\{\sqrt{\hat{q}}\}$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Let $f \in L^2(\mathbb{R})$.

1. We compute

$$\begin{aligned}\alpha^2 q_{1/2}(\alpha\cdot) * q_{1/2}(\alpha\cdot) * f &= \mathcal{F}^{-1}\big\{\sqrt{\hat{q}}(\alpha^{-1}\cdot)\sqrt{\hat{q}}(\alpha^{-1}\cdot)\hat{f}\big\} \\ &= \mathcal{F}^{-1}\big\{\hat{q}(\alpha^{-1}\cdot)\hat{f}\big\} = \alpha q(\alpha\cdot) * f = Q_\alpha f.\end{aligned}$$

2. Substituting $y = \alpha z + \xi$, we have

$$\begin{aligned} T_{\alpha,\xi}[Q_\beta f] &= \beta \int_{\mathbb{R}} q(\beta(\alpha x + \xi - y)) f(y) \,\mathrm{d}y \\ &= \beta\alpha \int_{\mathbb{R}} q(\beta\alpha(x - z)) f(\alpha z + \xi) \,\mathrm{d}z \\ &= \beta\alpha q(\beta\alpha \cdot) * T_{\alpha,\xi}[f] = Q_{\beta\alpha} T_{\alpha,\xi}[f]. \end{aligned}$$

3. Substituting $y = \alpha x + \xi$, we have

$$\begin{aligned} \langle T_{\alpha,\xi}[f], g\rangle_{L^2} &= \int_{\mathbb{R}} f(\alpha x + \xi) g(x) \,\mathrm{d}x \\ &= \alpha^{-1} \int_{\mathbb{R}} f(y) g(\alpha^{-1}(y - \xi)) \,\mathrm{d}y \\ &= \langle f, \alpha^{-1} T_{\alpha^{-1}, -\alpha^{-1}\xi}[g]\rangle_{L^2}. \end{aligned}$$ □



We now discuss the effect of the transformation $T_\alpha$ on the space-time white noise $W_t$. A defining property of the space-time white noise $W_t$ is the isometry

$$\mathbb{E}\big[\int_0^{t_1} \langle \mathrm{d}W_s, w_1\rangle_{L^2} \int_0^{t_2} \langle \mathrm{d}W_s, w_2\rangle_{L^2}\big] = (t_1 \wedge t_2)\langle w_1, w_2\rangle_{L^2}, \tag{2.87}$$

which holds for $t_1, t_2 > 0$ and $w_1, w_2 \in L^2(\mathbb{R})$. From the isometry (2.87) one obtains, upon differentiating with respect to $t_1$ and $t_2$ and choosing $w_1 = \delta_x$ and $w_2 = \delta_y$, the formal covariance identity

$$\mathbb{E}\big[\frac{\mathrm{d}W_{t_1}(x)}{\mathrm{d}t} \frac{\mathrm{d}W_{t_2}(y)}{\mathrm{d}t}\big] = \delta(t_1 - t_2)\delta(x - y).$$

Let us consider the covariance structure that results from rescaling the space-time white noise $W_t$ via $T_\alpha$. Picking an orthonormal basis $\{e_k\}_{k=0}^\infty$ of $L^2(\mathbb{R})$, we compute

$$\langle T_\alpha e_k, T_\alpha e_j\rangle_{L^2} = \int_{\mathbb{R}} e_k(\alpha x) e_j(\alpha x) \,\mathrm{d}x = \alpha^{-1} \int_{\mathbb{R}} e_k(y) e_j(y) \,\mathrm{d}y,$$

which shows that $\{\alpha^{1/2} T_\alpha e_k\}_{k=0}^\infty$ is also an orthonormal basis of $L^2(\mathbb{R})$. We then find via Itô's isometry that,

$$\begin{aligned} &\mathbb{E}\big[\int_0^{t_1} \langle \alpha^{1/2} T_\alpha[\mathrm{d}W_s], w_1\rangle_{L^2} \int_0^{t_2} \langle \alpha^{1/2} T_\alpha[\mathrm{d}W_s], w_2\rangle_{L^2}\big] \\ &= \mathbb{E} \int_0^{t_1\wedge t_2} \sum_{k=0}^\infty \langle \alpha^{1/2} T_\alpha e_k, w_1\rangle_{L^2} \langle \alpha^{1/2} T_\alpha e_k, w_2\rangle_{L^2} \,\mathrm{d}s, \end{aligned}$$

| Constant/Function | Defining equation | Value |
|---|---|---|
| $\overline{\gamma}_{d;I}(0)$ | (2.54) | $\frac{74}{135} + \frac{4\pi^2}{405}$ |
| $\overline{\mu}_{d;I}(0)$ | (2.54) | $(\frac{16\pi^2}{405} - \frac{34}{45})c_*^{-1/2}$ |
| $\overline{\gamma}_{d;III}(0)$ | (2.64) | $\approx 0.093c_*^{1/2}$ |
| $\overline{\mu}_{d;III}(0)$ | (2.64) | $\approx -0.1$ |
| $\overline{\gamma}_{s;I}(0)$ | (2.53) | $\frac{2}{3}$ |
| $\overline{\mu}_{s;I}(0)$ | (2.53) | $-\frac{2}{3}c_*^{-1/2}$ |
| $\overline{\gamma}_{\diamond}(0)$ | (2.59) | $\frac{1}{9}c_*^{-3/2}\phi_{c_*}^2$ |
| $\Vert\overline{\gamma}_{\diamond}(0)\Vert_{L^2}^2$ | (2.59) | $\frac{4}{35}c_*^{1/2}$ |
| $\overline{\mu}_{\diamond}(0)$ | (2.59) | $\frac{2}{9}c_*^{-2}\phi_{c_*}^2 - \frac{2}{9}c_*^{-1/2}\phi_{c_*}\zeta_{c_*}$ |
| $\Vert\overline{\mu}_{\diamond}(0)\Vert_{L^2}^2$ | (2.59) | $\approx 0.435c_*^{-1/2}$ |

**Table 2.1:** Values of various constants that appear in §2.3.

which via (2.87) gives the covariance identity

$$\mathbb{E}\Big[\int_0^{t_1} \langle \alpha^{1/2} T_\alpha \mathrm{d}W_s, w_1\rangle_{L^2} \int_0^{t_2} \langle \alpha^{1/2} T_\alpha \mathrm{d}W_s, w_2\rangle_{L^2}\Big] = (t_1 \wedge t_2)\langle w_1, w_2\rangle_{L^2}. \quad (2.88)$$

Here we have used that $\{\alpha^{1/2}T_\alpha e_k\}_{k=0}^\infty$ is an orthonormal basis of $L^2(\mathbb{R})$. We thus observe that the process $\tilde{W}_t := \alpha^{1/2}T_\alpha W_t$ generates the same statistics as the white noise $W_t$.

## 2.6 Expansions

Table 2.1 collects evaluations the various constants that appear in the expansion of the modulation system in §2.3. The evaluations denoted with the symbol '$\approx$' have been computed numerically. For the exact evaluations, we have used that the defining equations consist of inner products between $\phi_{c_*}, \zeta_{c_*}$ and derivatives thereof. These can be written as integrals over hyperbolic functions, for which exact evaluations are available.

We proceed by outlining how the expansions of the functionals $\overline{\gamma}_d^0, \overline{\mu}_d^0, \overline{\gamma}_d, \overline{\mu}_d$ and mappings $\overline{\gamma}_s, \overline{\mu}_s, R_0$ and $S$ that appear in §2.3 can be computed. Since these mappings are defined as products with the matrix $K^{-1}(v)$, we first derive an expansion

$$K^{-1}(v) = K^{-1}(0) + [K^{-1}]^{(1)}(v) + [K^{-1}]^{(2)}(v) + \mathcal{O}(v^3).$$

We therefore write

$$K(v) = \begin{bmatrix} 9c_*^{3/2} & 0 \\ 9 & -\frac{9}{2}c_*^{1/2} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},$$

where $b_1, b_2, b_3$ and $b_4$ denote the functionals

$$\begin{aligned} b_1 &= \big\langle (x\partial_x + 2)v, \phi_{c_*} \big\rangle_{L^2}, \\ b_2 &= \big\langle \partial_x v, \phi_{c_*} \big\rangle_{L^2}, \\ b_3 &= \big\langle (x\partial_x + 2)v, \zeta_{c_*} \big\rangle_{L^2}, \\ b_4 &= \big\langle \partial_x v, \zeta_{c_*} \big\rangle_{L^2}. \end{aligned}$$

We furthermore write

$$K^{-1} = \frac{1}{\det K}\left( 9\begin{bmatrix} -\frac{1}{2}c_*^{1/2} & 0 \\ -1 & c_*^{3/2} \end{bmatrix} + \begin{bmatrix} b_4 & -b_2 \\ -b_3 & b_1 \end{bmatrix} \right)$$

and compute

$$\det K = -\tfrac{81}{2}c_*^2 + 9c_*^{3/2}b_4 - \tfrac{9}{2}c_*^{1/2}b_1 - 9b_2 + b_1b_4 - b_3b_2.$$

Using $\frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} + \mathcal{O}(x^3)$, we expand $\frac{1}{\det K(v)}$ as

$$\begin{aligned} K^{-1}(0) &= \frac{1}{9}\begin{bmatrix} c_*^{-3/2} & 0 \\ 2c_*^{-2} & -2c_*^{-1/2} \end{bmatrix}, \\ [K^{-1}]^{(1)} &= \frac{2}{81}\begin{bmatrix} c_*^{-2}b_1 + 2c_*^{-5/2}b_2 & c_*^{-2}b_2 \\ c_*^{-2}b_3 + 2c_*^{-5/2}b_4 - c_*^{-7/2}b_1 - 2c_*^{-4}b_2 & -2c_*^{-1}b_4 + 2c_*^{-5/2}b_2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} [K^{-1}]^{(2)} &= \tfrac{2}{729}(-2c_*^{-5/2}b_4 + c_*^{-7/2}b_1 + 2c_*^{-4}b_2)\begin{bmatrix} b_4 & -b_2 \\ -b_3 & b_1 \end{bmatrix} \\ &\quad + \tfrac{2}{729}\Big( -4c_*^{-3}b_4^2 - c_*^{-5}b_1^2 - 4c_*^{-6}b_2^2 + 2c_*^{-4}b_1b_4 \\ &\quad + 8c_*^{-9/2}b_2b_4 - 4c_*^{-11/2}b_1b_2 + 2c_0^{-4}b_3b_2 \Big)\begin{bmatrix} -\frac{1}{2}c_*^{1/2} & 0 \\ -1 & c_*^{3/2} \end{bmatrix}. \end{aligned}$$

We then collect that

$$\begin{bmatrix} [\overline{\gamma}_d^0]^{(2)}(v) \\ [\overline{\mu}_d^0]^{(2)}(v) \end{bmatrix} = K^{-1}(0)\begin{bmatrix} \langle N(v), \phi_{c_*}\rangle_{L^2} \\ \langle N(v), \zeta_{c_*}\rangle_{L^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{9}c_*^{-3/2}\langle v, v\partial_x\phi_{c_*}\rangle_{L^2} \\ \frac{2}{9}\big\langle v, v(c_*^{-2}\partial_x\phi_{c_*} - c_*^{-1/2}\partial_c\phi_{c_*})\big\rangle \end{bmatrix},$$

and

$$[R_0]^{(2)}(v) = -[\overline{\gamma}_d^0]^{(2)}(v)(2 + x\partial_x)\phi_{c_*} - [\overline{\mu}_d^0]^{(2)}(v)\partial_x\phi_{c_*}.$$

For the functionals $\overline{\gamma}_s$ and $\overline{\mu}_s$ we obtain

$$\begin{bmatrix} [\overline{\gamma}_s]^{(1)}(v) \\ [\overline{\mu}_s]^{(1)}(v) \end{bmatrix} = K^{-1}(0)\begin{bmatrix} M^*(v)[\phi_{c_*}] \\ M^*(v)[\zeta_{c_*}] \end{bmatrix} + [K^{-1}]^{(1)}(v)\begin{bmatrix} M^*(\phi_{c_*})[\phi_{c_*}] \\ M^*(\phi_{c_*})[\zeta_{c_*}] \end{bmatrix}$$

and

$$\begin{bmatrix} [\overline{\gamma}_s]^{(2)}(v) \\ [\overline{\mu}_s]^{(2)}(v) \end{bmatrix} = [K^{-1}]^{(1)}(v) \begin{bmatrix} M^*(v)[\phi_{c_*}] \\ M^*(v)[\zeta_{c_*}] \end{bmatrix} + [K^{-1}]^{(2)}(v) \begin{bmatrix} M^*(\phi_{c_*})[\phi_{c_*}] \\ M^*(\phi_{c_*})[\zeta_{c_*}] \end{bmatrix}.$$

Using these expressions, we have

$$\begin{aligned} [S]^{(1)}(v)[h] &= M(v)[h] - (x\partial_x + 2)\phi_{c_*}\big\langle h, [\overline{\gamma_s}]^{(1)}(v)\big\rangle_{\mathcal{H}} \\ &\quad - (x\partial_x + 2)v\big\langle h, \overline{\gamma_s}(0)\big\rangle_{\mathcal{H}} - \partial_x\phi_{c_*}\big\langle h, [\overline{\mu}_s]^{(1)}(v)\rangle_{\mathcal{H}} - \partial_x v\langle h, \overline{\mu}_s(0)\big\rangle_{\mathcal{H}}. \end{aligned}$$

Lastly, we have

$$\begin{aligned} \begin{bmatrix} [\overline{\gamma}_d]^{(1)}(v,\alpha) \\ [\overline{\mu}_d]^{(1)}(v,\alpha) \end{bmatrix} &= -\alpha K^{-1}(0) \sum_{i=1}^{5} \begin{bmatrix} \big\langle [R_i]^{(1)}(v,\alpha), \phi_{c_*}\big\rangle_{L^2} \\ \big\langle [R_i]^{(1)}(v,\alpha), \zeta_{c_*}\big\rangle_{L^2} \end{bmatrix} \\ &\quad - \alpha[K^{-1}]^{(1)}(v) \sum_{i=1}^{5} \begin{bmatrix} \langle R_i(0,\alpha), \phi_{c_*}\rangle_{L^2} \\ \langle R_i(0,\alpha), \zeta_{c_*}\rangle_{L^2} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} [\overline{\gamma}_d]^{(2)}(v,\alpha) \\ [\overline{\mu}_d]^{(2)}(v,\alpha) \end{bmatrix} &= -\alpha K^{-1}(0) \sum_{i=1}^{5} \begin{bmatrix} \big\langle [R_i]^{(2)}(v,\alpha), \phi_{c_*}\big\rangle_{L^2} \\ \big\langle [R_i]^{(2)}(v,\alpha), \zeta_{c_*}\big\rangle_{L^2} \end{bmatrix} \\ &\quad - \alpha[K^{-1}]^{(1)}(v) \sum_{i=1}^{5} \begin{bmatrix} \big\langle [R_i]^{(1)}(v,\alpha), \phi_{c_*}\big\rangle_{L^2} \\ \big\langle [R_i]^{(1)}(v,\alpha), \zeta_{c_*}\big\rangle_{L^2} \end{bmatrix} \\ &\quad - \alpha[K^{-1}]^{(2)}(v) \sum_{i=1}^{5} \begin{bmatrix} \langle R_i(0,\alpha), \phi_{c_*}\rangle_{L^2} \\ \langle R_i(0,\alpha), \zeta_{c_*}\rangle_{L^2} \end{bmatrix}. \end{aligned}$$

## 2.7 Numerical schemes

Here, we describe the numerical schemes that were employed to simulate the stochastic KdV equation (2.12) and the modulation system (2.41)–(2.43), in the cases of scalar noise and space-time white noise. We employ a semi-implicit finite-difference scheme from [75], as was also used in [19].

In space, we use $N+1$ grid points $x_n = n\Delta x - L$ for $n = 1, \ldots, N+1$, where $\Delta x = \frac{2L}{N}$ and $L > 0$ is the right-boundary of the computational domain $[-L, L]$. We denote the numerical solution to (2.12) at time $j\Delta t$ by $U^j = [U_1^j, U_2^j, \ldots, U_n^j, \ldots, U_{N+1}^j]^T$, and denote by $D_1, D_2$ and $D_3$ the $(N+1) \times (N+1)$ centered finite difference matrices of second order for the differential operators $\partial_x, \partial_x^2$ and $\partial_x^3$, respectively. We initialize by using an Euler-Maruyama step as

$$U^1 = U^0 - \Delta t(D_3 * U^0 + 2U^0(D_1 * U^0)) + \sigma U^0 \Delta W,$$

where the noise is discretized as $\Delta W \sim \sqrt{\Delta t} N(0,1)$ in the case of scalar noise, and as

$$\Delta W = [W_1, W_2, \ldots, W_n, \ldots, W_{N+1}]^T$$

with $W_n \overset{i.i.d.}{\sim} \sqrt{\frac{\Delta t}{\Delta x}} N(0,1)$ for $n = 1, \ldots, N+1$ in the case of space-time white noise. Thereafter, for $j \geq 1$, the scheme continues with a semi-implicit step and a two-step Adam-Bashforth discretization for the nonlinear term as

$$\begin{aligned} U^{j+1} = \left(I + \frac{\Delta t}{2} D_3\right)^{-1} \Bigg[ &\left(I - \frac{\Delta t}{2} D_3\right) * U^j + \sigma U^j \Delta W \\ &- 3\Delta t U^j (D_1 * U^j) + \Delta t U^{j-1}(D_1 * U^{j-1}) \Bigg]. \end{aligned}$$

To simulate (2.41), we also employ a semi-implicit method. Denote by $V^j = [V_1^j, V_2^j, \ldots, V_n^j, \ldots, V_{N+1}^j]^T$ and $A^j = [A_1^j, A_2^j, \ldots, A_n^j, \ldots, A_{N+1}^j]^T$ the numerical solutions to (2.41) and (2.42), respectively. For $j \geq 0$, the numerical solution $V^{j+1}$ to (2.41) at time $j\Delta t$ is computed as

$$\begin{aligned} V^{j+1} = \left(I - \frac{\Delta t}{2} L_0\right)^{-1} \Bigg[ &\left(I + \frac{\Delta t}{2} L_0\right) * V^j - 2\Delta t (A^j)^{-3} V^j (D_1 * V^j) \\ &+ \Delta t \sum_{i=0}^{2} R_i^\sigma (V^j, A^j) + \sigma S(V^j) \Delta W \Bigg], \end{aligned}$$

where

$$L_0 = -D_3 + c_* D_1 - 2D_1 * \mathrm{Diag}(\Phi_0)$$

and $\mathrm{Diag}(\Phi_0)$ is a diagonal matrix with entries

$$\phi_{c_*}(-L), \phi_{c_*}(-L + \Delta x), \ldots, \phi_{c_*}(-L + n\Delta x), \ldots, \phi_{c_*}(L)$$

on the diagonal.

The numerical solutions $A^{j+1}$ and $X^{j+1} = [X_1^j, X_2^j, \ldots, X_n^j, \ldots, X_{N+1}^j]^T$ to (2.42) and (2.43) at time $j\Delta t$, are respectively given for $j \geq 0$ by

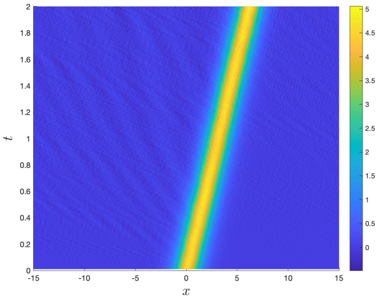
$$\begin{aligned} A^{j+1} &= A^j + [-(A^j)^{-2} \overline{\gamma}_d^0 (V^j) + \sigma^2 A^j \overline{\gamma}_{d;I}(V^j)] \Delta t - \sigma A^j \overline{\gamma}_{s;I}(V^j) \Delta W, \\ X^{j+1} &= X^j + [-(A^j)^{-2} \overline{\mu}_d^0 (V^j) + \sigma^2 A^j \overline{\mu}_{d;I}(V^j)] \Delta t - \sigma A^j \overline{\mu}_{s;I}(V^j) \Delta W, \end{aligned}$$

in the case of scalar noise, and as

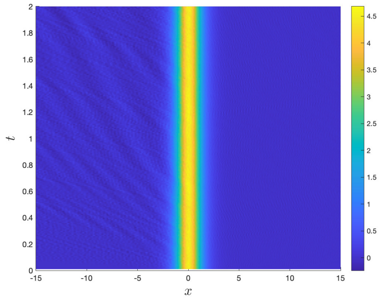
$$\begin{aligned} A^{j+1} &= A^j + [-(A^j)^{-2} \overline{\gamma}_d^0 (V^j) + \sigma^2 \overline{\gamma}_{d;III}(V^j)] \Delta t - \sigma A^j \langle \Delta W, \overline{\gamma}_\diamond (V^j) \rangle, \\ X^{j+1} &= X^j + [-(A^j)^{-2} \overline{\mu}_d^0 (V^j) + \sigma^2 \overline{\mu}_{d;III}(V^j)] \Delta t - \sigma A^j \langle \Delta W, \overline{\mu}_\diamond (V^j) \rangle, \end{aligned}$$

in the case of space-time white noise. We remark that, to obtain path-wise correspondence in this latter case between numerical solutions to (2.12) and the modulation system (2.41)–(2.43), realizations of the noise $\Delta W$ should be shifted and rescaled via the map $T_{\alpha,\xi}$.

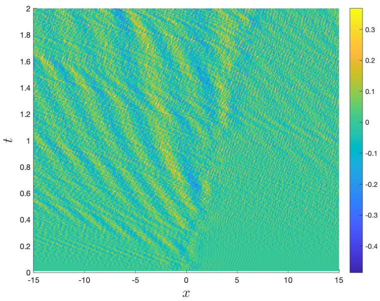
## 2.8 Supplementary figures



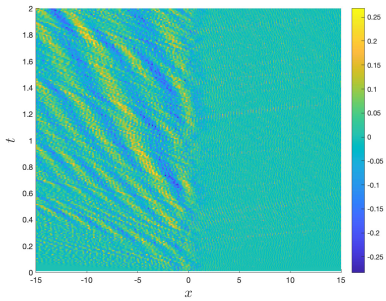
**(a)** Original frame.



**(b)** Stochastic co-moving frame.

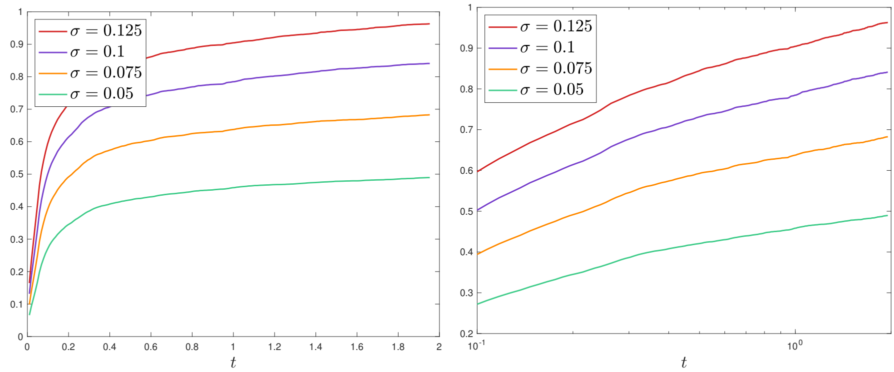


**(c)** Original frame, soliton removed.



**(d)** Stochastic co-moving frame, soliton removed.

**Figure 2.11:** Simulation of the KdV equation with space-time white noise of strength $\sigma = 0.05$. The original frame realization shows $u(t, x)$, from a simulation of (2.12). The frozen frame simulation shows $\phi_{c_*}(x) + v(t, x)$, from simulation of (2.41)–(2.43) with the same realization of the noise. Figure 2.11c and Figure 2.11d show the perturbation with respect to the soliton. The original frame realization shows $u(t, x) - \phi_{c(t)}(x - \xi(t))$ with the phase-definitions (2.65), and the frozen frame simulation shows $v(t, x)$.




**(a)** Perturbation size over time.

**(b)** Perturbation size over time, log-scale.

**Figure 2.12:** Sample mean of the process $\sup_{s\leq t} \|v(s)\|_{L^2_a([-40,10])}$ for space-time white noise, computed over 200 realisations for $\sigma \in \{0.05, 0.075, 0.1, 0.125\}$. This simulation was computed on the computational domain $[-40, 40]$, with values $c_* = 3$ and $a = 0.15$.

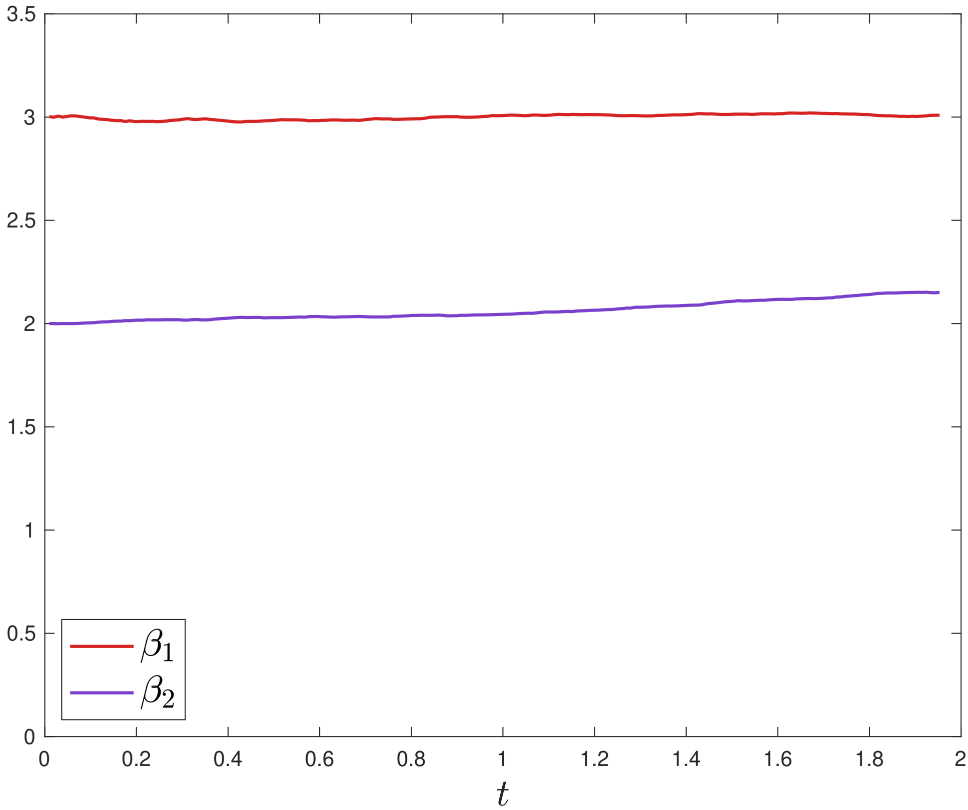



**Figure 2.13:** Estimation of the orders $\beta_1$ and $\beta_2$ at which the remainders $|c - c_2|$ and $\|v(t)-v_1(t)\|_{L^2_a}$ depend on the noise strength $\sigma$. Here, $\beta_1(t)$ is obtained from a least squares fit of $\mathbb{E}\sup_{s\le t}|c(s) - c_2(s)|$ (as in Figure 2.8) to $k_1(t)\sigma^{\beta_1(t)}$. Similarly, the exponent $\beta_2(t)$ is obtained from a least squares fit of $\mathbb{E}\sup_{s\le t}\|v(s) - v_1(s)\|_{L^2_a}$ to $k_2(t)\sigma^{\beta_2(t)}$.

# CHAPTER 3
# Deterministic stability

We study the stability and dynamics of solitons in the Korteweg-de Vries (KdV) equation with small multiplicative forcing. Forcing breaks the conservative structure of the KdV equation, leading to substantial changes in energy over long times. We show that, for small forcing, the inserted energy is almost fully absorbed by the soliton, resulting in a drastically changed amplitude and velocity. We decompose the solution to the forced equation into a modulated soliton and an infinite dimensional perturbation. Assuming slow exponential decay of the forcing, we show that the perturbation decays at the same exponential rate in a weighted Sobolev norm centered around the soliton.



## 3.1 Introduction

In this chapter[1], we study the forced Korteweg-de Vries equation

$$u_t = -\partial_x^3 u - 2u\partial_x u + \epsilon f(\epsilon t/E)u, \tag{3.1}$$

where $u$ is a real-valued function on $(t, x) \in (\mathbb{R}^+, \mathbb{R})$ and $f$ is an integrable time-dependent forcing term. The small parameter $\epsilon > 0$ controls the amplitude of the forcing, while the (potentially large) parameter $E > 0$ is a measure for the total supplied energy. Our main goal is to understand the effect of this forcing on the family of soliton solutions to the unforced system.

Forced KdV equations such as (3.1) appear in the study of wave-phenomena subject to external disturbing mechanisms. Motivated by physical considerations (such as pressure inhomogeneities or bottom topographies), various types of forcing have been considered; see for instance [40, 2, 108, 55, 33, 53, 54]. The multiplicative form of the forcing term in (3.1) can be thought of as a generic mechanism to modify the amount of energy present in the system. For our purposes, (3.1) constitutes a

[1]The contents of this chapter have been published as R.W.S. Westdorp, H.J. Hupkes, *Soliton Amplification in the Korteweg-de Vries Equation by Multiplicative Forcing*, Communications on Pure and Applied Analysis, see [103]

toy model that facilitates the rigorous study of perturbed waved phenomena. In particular, we view this work as a step towards establishing rigorous long-time stability of the KdV solitary waves under stochastic forcing, extending the preliminary results in Chapter 2.

In the absence of forcing ($f(t) \equiv 0$), (3.1) is the well-known KdV equation: a well-studied dispersive PDE that first appeared in the description of shallow water waves in a longitudinal canal [68]. Among its most notable features is the existence of soliton solutions $u(t,x) = \phi_c(x - ct)$ of the form

$$\phi_c(x) = \tfrac{3c}{2}\,\mathrm{sech}^2(\sqrt{c}x/2), \quad c > 0, \tag{3.2}$$

which mark a balance between dispersive and nonlinear effects. As seen in (3.2), the solitary waves $\phi_c$ satisfy the self-similarity property $\phi_c(x) = c\phi_1(\sqrt{c}x)$, owing to the scaling invariance

$$u(t,x) \mapsto \alpha^2 u(\alpha^3 t, \alpha x) \tag{3.3}$$

of the KdV equation.

Another celebrated quality of the KdV equation is that it is completely integrable, which means that it enjoys an infinite amount of conserved quantities. In particular, the KdV flow conserves the $L^2$-norm



$$\mathcal{N}[u] = \int_{\mathbb{R}} u^2 \mathrm{d}x$$

and the Hamiltonian

$$\mathcal{H}[u] = \int_{\mathbb{R}} \tfrac{1}{2}(\partial_x u)^2 - \tfrac{1}{3}u^3 \mathrm{d}x.$$

Introducing the forcing term in (3.1) breaks this conservative structure. The $L^2$-norm, for instance, evolves as

$$\mathcal{N}[u(t)] = \mathcal{N}[u(0)]e^{2E\int_0^{\epsilon t/E} f(s)\mathrm{d}s}. \tag{3.4}$$

With $\mathcal{N}$, $\mathcal{H}$, and other KdV-invariants undergoing slow (but eventually large) changes, we may expect significant consequences for the propagation of the solitons (3.2). Using (3.4) and the relation $\mathcal{N}[\phi_c] = 6c^{3/2}$, we can heuristically predict that the amplitude of a soliton starting at $c(0) > 0$ will approximately evolve according to

$$c_{\mathrm{ap}}(t) = c(0)e^{\frac{4}{3}E\int_0^{\epsilon t/E} f(s)\mathrm{d}s}. \tag{3.5}$$

We will show that this description is valid to leading-order in the small parameter $\epsilon$. If, for instance, $f(t) = e^{-t}$ and $E = \frac{3}{4}\ln 2$, then the soliton amplitude roughly doubles in size over time. Letting $c(t)$ denote the evolution of the soliton amplitude over time, we furthermore derive that the soliton phase, starting at a position

$\xi(0) \in \mathbb{R}$, evolves according to

$$\xi_{\mathrm{ap}}(t) = \xi(0) + \int_0^t c(s)\mathrm{d}s + \tfrac{2}{3}\epsilon \int_0^t \frac{f(\epsilon s/E)}{c^{1/2}(s)}\mathrm{d}s, \tag{3.6}$$

to leading-order in $\epsilon$.

In this work, we establish the orbital stability of the traveling-wave family (3.2) under the influence of multiplicative forcing. We show that solitons evolving via (3.1) remain close to the family (3.2) in the $H^1$-norm, while undergoing a potentially large change in amplitude. In particular, we supply (3.1) with the initial condition

$$u(0,x) = \phi_{c_*}(x) + \overline{v}_*(x) \tag{3.7}$$

for some $c_* > 0$, where $\overline{v}_* \in H^2$ and $e^{wx}\overline{v}_* \in H^1$ are suitably small for some weight $w \in (0, \sqrt{c_*}/3)$. If the forcing term $f$ is assumed to be exponentially decaying, then the solution actually converges to a limiting wave profile in $H^1$ with an exponential weight centered around the soliton. Our main interest in pursuing this line of work is to open up rigorous stability results for solitons undergoing large amplitude changes. As such, we view this work as a step towards understanding the stability of solitons under more general perturbations, such as stochastic forcing (Chapter 2), as well as the stability of KdV-like quasi-solitons such as micropterons and nanopterons in systems/lattices with periodic structure [37, 80, 36, 60, 38]. Indeed, these gradually decrease in amplitude over time when perturbed due to the presence of small oscillatory tails that interact with the perturbation.

3

## Soliton stability

Stability of the soliton family (3.2) under small initial perturbations in the KdV equation has long been established in various forms [11, 91, 82]. The pioneering work [11] by Bona, Souganidis and Strauss proves that the soliton family (3.2) is orbitally stable in $H^1$ via energy methods. Pego and Weinstein expand on this result in [91] by showing that, up to a small change in the speed and the phase of the soliton, small perturbations decay when measured in the exponentially weighted spaces

$$L^2_w(\mathbb{R}) = \{g : e^{wx}g \in L^2(\mathbb{R})\} \quad \text{with} \quad \|g\|_{L^2_w} = \|e^{wx}g\|_{L^2}$$

and

$$H^1_w(\mathbb{R}) = \{g : e^{wx}g \in H^1(\mathbb{R})\} \quad \text{with} \quad \|g\|_{H^1_w} = \|e^{wx}g\|_{H^1} \tag{3.8}$$

for $w \in (0, \sqrt{c_*/3})$.

Our main theorem generalizes this classic stability result to the setting of (3.1). The main new feature is that we are able to track the amplitude and phase changes introduced by the forcing, which can be of arbitrary size.

**Theorem 3.1.1** (See Section 3.7)**.** *Pick $c_*, E_{\max} > 0$, $w \in (0, \sqrt{c_*}/3)$, and $p \in [0, \frac{1}{4})$. There exist a weight $w_\infty \in (0, w)$ and constants $\delta_1, C_1, c_{\min}, c_{\max} > 0$ with*

$c_{\min} < c_* < c_{\max}$ *such that the following holds true. For each* $E \in (0, E_{\max}]$, *each* $\epsilon \in (0, 1]$ *that satisfies*

$$\epsilon^p + \epsilon^{1-4p} \leq \delta_1/E, \quad \epsilon^{1-p} \leq E, \quad \epsilon \leq \delta_1 E,$$

*each* $\overline{v}_* \in H^2 \cap H^1_w$ *for which* $\|\overline{v}_*\|^2_{H^1} + \|\overline{v}_*\|^2_{H^1_w} \leq \delta_1\epsilon/E$, *and each continuous function* $f : \mathbb{R}^+ \to \mathbb{R}$ *that satisfies the bound*

$$|f(t)| \leq e^{-t}, \quad t \geq 0, \tag{3.9}$$

*there exist modulation functions* $c, \xi \in C^1(\mathbb{R}^+; \mathbb{R})$ *associated to the solution* $u$ *of* (3.1) *with* (3.7) *that satisfy the following properties:*

1. *In the weighted space* $H^1_{w_\infty}$, *we have the exponential decay*

$$\sup_{t\geq 0} e^{\epsilon t/E} \|u(t, \cdot + \xi(t)) - \phi_{c(t)}\|_{H^1_{w_\infty}} \leq C_1\Big(\epsilon^{1-2p} + \|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w}\Big).$$

2. *In the unweighted space* $H^1$, *we have the stability bound*

$$\begin{aligned}\sup_{t\geq 0} \|u(t, \cdot + \xi(t)) - \phi_{c(t)}\|^2_{H^1} \leq C_1 E\Big(\epsilon^{1-4p} + \|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w}\Big)\\ + C_1\frac{E}{\epsilon}\big(\|\overline{v}_*\|^2_{H^1} + \|\overline{v}_*\|^2_{H^1_w}\big).\end{aligned}$$

3. *The amplitude function* $c(t)$ *takes values in* $[c_{\min}, c_{\max}]$, *and can be approximated by*

$$\begin{aligned}\sup_{t\geq 0} \big|c(t) - c_{\mathrm{ap}}(t)\big| \leq C_1\|\overline{v}_*\|_{H^1_w} + C_1 E\big(\epsilon^{1-4p} + \|\overline{v}_*\|_{H^1}\big)\\ + C_1\frac{E}{\epsilon}\big(\|\overline{v}_*\|^2_{H^1} + \|\overline{v}_*\|^2_{H^1_w}\big),\end{aligned}$$

   *where* $c_{\mathrm{ap}}(t)$ *is defined in* (3.5). *Moreover,* $c(t)$ *converges as* $t \to \infty$.

4. *The position function* $\xi(t)$ *satisfies*

$$\begin{aligned}\sup_{t\geq 0} |\xi(t) - \xi_{\mathrm{ap}}(t)| \leq C_1\|\overline{v}_*\|_{H^1_w} + C_1 E\big(\epsilon^{1-4p} + \|\overline{v}_*\|_{H^1}\big)\\ + C_1\frac{E}{\epsilon}\big(\|\overline{v}_*\|^2_{H^1} + \|\overline{v}_*\|^2_{H^1_w}\big),\end{aligned}$$

   *where* $\xi_{\mathrm{ap}}(t)$ *is defined in* (3.6).

**Remark 3.1.2.** 1. The classic result in [91] can be retrieved by letting $\epsilon = \delta_1 E \to 0$, up to a small loss in the weight (since $w_\infty < w$), which is further discussed in Section 3.5. In this case, it is required that $p = 0$ through the assumption $\epsilon^p \leq \delta_1/E$. The case of large amplitude modulation ($E \gg 0$) requires that $p > 0$.

2. In case we replace the assumption on $\overline{v}_*$ by the stronger assumption $\|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w} \leq \delta_1\epsilon/E$, the bounds in items 2-4 simplify to

$$\sup_{t\geq 0} \|u(t,\cdot+\xi(t)) - \phi_{c(t)}\|^2_{H^1} \leq C_1 E\epsilon^{1-4p} + C_1(E+\delta_1)\Big(\|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w}\Big),$$

and

$$\begin{aligned}\sup_{t\geq 0}\big|c(t) - c_{\mathrm{ap}}(t)\big| + \sup_{t\geq 0}|\xi(t) - \xi_{\mathrm{ap}}(t)| &\leq 2C_1 E\epsilon^{1-4p} + 3C_1\|\overline{v}_*\|_{H^1_w} \\ &\quad + 2C_1(E+\delta_1)\|\overline{v}_*\|_{H^1}.\end{aligned}$$

3. Property (3.9) is needed to obtain exponential decay of $\|v(t)\|_{H^1_{w_\infty}}$. We believe that this condition can be relaxed to $f \in L^1(0,\infty)\cap L^\infty(0,\infty)$. In this case, exponential decay over time of the weighted norm can not be expected.

4. The integrability of $f$ ensures that $c(t)$ and $c^{-1}(t)$ remain bounded, which prevents technical complications. First, it allows us to construct bounds that do not depend on $c(t)$. Second, it guarantees that we can apply a minimal exponential weight on the modulated soliton $\phi_{c(t)}$.

5. The assumption $\overline{v}_* \in H^2 \cap H^1_w$ fits the well-posedness results established in [91, Appendix A]. It would be interesting to see if this assumption can be relaxed to $\overline{v}_* \in L^2$ by following the arguments of [86].

6. The assumption $w < \frac{1}{3}\sqrt{c_*}$ is slightly stricter than the common assumption $w < \sqrt{c_*/3}$. We require this stricter bound at various points to establish sharper bounds than previous works.

Two related results for different forcing types are available in [61, 110]. The result in [61] deals with finite time stability, and in [110], the authors use a factorization technique that is not available in the current setting.

## Approach

Our approach is based on the stability theory of the solitons $\phi_c$ in the exponentially weighted spaces (3.8) developed in [91]. The exponential weight facilitates stronger stability properties of the operator

$$\mathcal{L}_c = -\partial_x^3 + (c - 2\phi_c)\partial_x - 2\partial_x\phi_c = -\partial_x^3 + c\partial_x - 2\partial_x(\phi_c\cdot), \tag{3.10}$$

which is associated with the linearization of the KdV equation around the soliton $\phi_c$. Pego and Weinstein established that $\mathcal{L}_c$ generates a $C_0$-semigroup $\{e^{\mathcal{L}_c t}\}_{t\geq 0}$ on $L^2_w$ which is exponentially stable on the subspace of functions $v \in L^2_w$ that satisfy the orthogonality conditions

$$\langle v, \zeta_c\rangle_{L^2} = \langle v, \phi_c\rangle_{L^2} = 0, \tag{3.11}$$

where $\zeta_c$ is the primitive

$$\zeta_c(x) = \int_{-\infty}^{x} \partial_c \phi_c(y)\mathrm{d}y \in L^2_{-w};$$

see Section 3.8 for further details. The conditions (3.11) play a central role in our approach.

The main obstacle in proving Theorem 3.1.1 using the linear stability tools developed in [91] is that, due to the large changes in $c(t)$, it is not feasible to linearize (3.1) around a soliton with fixed amplitude. Instead, we move to a co-moving frame where the solution is not only translated, but also rescaled according to the natural scaling of the soliton family (3.2). More precisely, we introduce the remainder

$$v(t,x) = \alpha^2(t)u\big(t, \alpha(t)x + \xi(t)\big) - \phi_{c_0}(x), \tag{3.12}$$

in reference to a soliton $\phi_{c_0}$ with fixed amplitude and position. The remainder $v$ then follows an evolution equation of the form

$$v_t = \alpha^{-3}\mathcal{L}_{c_0}v + \frac{\alpha_t}{\alpha}x\partial_x v + \mathcal{O}(\epsilon + v^2),$$

which allows us to leverage the linear stability properties of $\mathcal{L}_{c_0}$ on exponentially weighted spaces. The term $x\partial_x v$ arising from the dilation of $u$ in (3.12) causes significant technical complications. In order to estimate $x\partial_x v$ in terms of $v$, one needs to obtain some extra control at $|x| \to \infty$. For $v$ supported on $[0,\infty)$, we do so by estimating

$$\|x\partial_x v\|_{L^2_w} \leq C_\beta \|\partial_x v\|_{L^2_{w+\beta}},$$

for some small $\beta > 0$, which establishes control of the troublesome term viewed as an operator between *different* exponentially weighted spaces. We arrive at a consistent argument by continuously decreasing the exponential weight over time. This, however, presents another difficulty on short-time scales, since the constant $C_\beta$ blows up as $\beta \downarrow 0$. We remedy this problem by employing the classical stability argument of [91] on short time-scales where $c(t)$ only undergoes small fluctuations.

As has been argued by Pego & Weinstein, stability in exponentially weighted spaces still requires control of the *unweighted* $H^1$-norm of the perturbation, due to the nonlinearity in (3.1) which would otherwise require double the exponential weight. This is established in [91] by exploiting the fact that $\phi_c$ is a critical point of the conserved functional

$$\mathcal{E}_c[u] = \mathcal{H}[u] + \tfrac{1}{2}c\mathcal{N}[u]. \tag{3.13}$$

We generalize this argument to accommodate for large fluctuations in $c(t)$ by analyzing the evolution of $\mathcal{E}_{c(t)}[u(t)]$ over time. We compute that

$$\partial_t\big(\mathcal{E}_{c(t)}[u(t)] - \mathcal{E}_{c(t)}[\phi_{c(t)}]\big) = \mathcal{O}(\epsilon v + v^2),$$

which allows us to control the $H^1$ norm of the perturbation in terms of itself and the weighted norm.

Our combined argument lifts the restrictions on the size of $|c_* - c(t)|$ inherent to previous approaches to establish stability. This is particularly useful for studying the KdV soliton in settings where $c(t)$ naturally undergoes large fluctuations on short time-scales. Indeed, we are pursuing the techniques developed in this chapter in order to achieve stability on long time-scales in the setting of stochastic multiplicative forcing (Chapter 2).

### Outline

The chapter is organized as follows. In Section 3.2, we derive a system of modulation equations that governs the behavior of the soliton amplitude $c(t)$, the position $\xi(t)$ and remainder $v(t)$. Then, in Section 3.3, we introduce the function spaces that are central to our stability argument, and assert stability and smoothing properties of the operator $\mathcal{L}_c$ on these spaces. We proceed by establishing control of the remainder $v$ over short-timescales by adapting a Duhamel argument of [91], and introduce the notion of time-varying weights. Thereafter, in Section 3.5, we show that the remainder $v$ can be controlled over long time-scales in a weighted norm. The evolution of unweighted norms of $v$ is then analyzed in Section 3.6. We finally provide the proof of Theorem 3.1.1 in Section 3.7.



## 3.2 Modulation system

In this section, we introduce our decomposition of solutions to (3.1), which forms the basis for our arguments. For convenience and brevity, we introduce the parameter $\gamma = \epsilon/E$ and recast (3.1) in the form

$$u_t = -\partial_x^3 u - 2u\partial_x u + \epsilon f(\gamma t)u. \tag{3.14}$$

In order to track how constants depend on the system parameters and the various choices that we make, we collect the various assumptions that we make throughout the chapter in a number of labeled ‘settings’. The first of these relates to the global parameters $(c_*, E_{\max}, w)$ and the initial condition for (3.14).

**S1** *We have $c_* > 0$ together with $E_{\max} > 0$ and $w \in (0, \sqrt{c_*}/3)$. The initial condition for* (3.14) *satisfies*

$$u(0,x) = \phi_{c_*}(x) + \overline{v}_*(x), \quad \overline{v}_* \in H^2 \cap H^1_w,$$

*and the forcing term $f$ is continuous and lies in the space $L^\infty(0,\infty)$.*

We now start by making an observation regarding the regularity of solutions to (3.14).

**Lemma 3.2.1.** *Assuming S1 and letting $\epsilon, \gamma > 0$, the solution $u$ to (3.14) has regularity*

$$u \in C([0,T], H^2) \cap C^1([0,T], H^{-1}), \tag{3.15}$$
$$e^{wx}u \in C([0,T], H^1) \cap C^1([0,T], H^{-3}), \tag{3.16}$$

*for any $T > 0$.*

*Proof.* This result is established in [91, Appendix A] for $f(t) \equiv 0$ by modifying a well-posedness result of Kato [66]. To see that the arguments for local well-posedness (i.e. small $T > 0$) remain valid upon including the forcing term $\epsilon f(\gamma t)$, we note that (3.14) is equivalent to a time-dependent KdV equation. Indeed, $u(t)$ solves (3.14) if and only if $z(t) = e^{-(\epsilon/\gamma)\int_0^{\gamma t} f(s)\mathrm{d}s}u(t)$ solves

$$z_t = -\partial_x^3 z - 2e^{(\epsilon/\gamma)\int_0^{\gamma t} f(s)\mathrm{d}s} z\partial_x z.$$

The arguments for global well-posedness (i.e. arbitrary $T > 0$) rely on an a priori bound for the $H^2$-norm. We show here that such a bound remains available upon including the forcing term $\epsilon f(\gamma t)$. Indeed, writing $u(t) = u$ for some $t \geq 0$ and using (3.14) we compute that

$$\partial_t \|u\|_{L^2}^2 \overset{(3.14)}{=} -2\langle u, \partial_x^3 u + 2u\partial_x u\rangle + 2\epsilon f(\gamma t)\langle u, u\rangle = 2\epsilon f(\gamma t)\|u\|_{L^2}^2,$$

leading to the identity

$$\|u(t)\|_{L^2}^2 = e^{2(\epsilon/\gamma)\int_0^{\gamma t} f(s)\mathrm{d}s}\|u(0)\|_{L^2}^2.$$

Moving on to the first derivative, an application of the Gagliardo-Nirenberg inequality yields

$$\|\partial_x u\|_{L^2}^2 = 2\mathcal{H}[u] + \tfrac{2}{3}\int_{\mathbb{R}} u^3 \mathrm{d}x \leq 2\Big|\mathcal{H}[u]\Big| + C\|u\|_{L^2}^{5/2}\|\partial_x u\|_{L^2}^{1/2}.$$

This inequality is of the form $x^4 \leq A + B|x|$ with $A, B \geq 0$, from which we may conclude $x^4 \leq 2(A + B^{4/3})$, i.e.

$$\|\partial_x u\|_{L^2}^2 \leq 4\Big|\mathcal{H}[u]\Big| + 2C^{4/3}\|u\|_{L^2}^{10/3}. \tag{3.17}$$

We then compute

$$\begin{aligned}\partial_t \mathcal{H}[u] &= \partial_t \int_{\mathbb{R}} \tfrac{1}{2}(u_x)^2 - \tfrac{1}{3}u^3 \mathrm{d}x = \int_{\mathbb{R}} u_x u_{xt} - u^2 u_t \mathrm{d}x = -\int_{\mathbb{R}} (u_{xx} + u^2)u_t \mathrm{d}x \\ &\overset{(3.14)}{=} \Big\langle u_{xx} + u^2, u_{xxx} + (u^2)_x - \epsilon f(\gamma t)u\Big\rangle = \epsilon f(\gamma t)\|\partial_x u\|^2 - \epsilon f(\gamma t)\int_{\mathbb{R}} u^3 \mathrm{d}x \\ &= \epsilon f(\gamma t)(3\mathcal{H}[u] - \tfrac{1}{2}\|\partial_x u\|_{L^2}^2),\end{aligned}$$

which leads to

$$\Big|\partial_t \mathcal{H}[u]\Big| \leq 5\epsilon |f(\gamma t)| \Big|\mathcal{H}[u]\Big| + \epsilon |f(\gamma t)| C^{4/3} \|u\|_{L^2}^{10/3}$$

and, via an application of Grönwall's inequality, to the a priori bound

$$\Big|\mathcal{H}[u(t)]\Big| \leq \Big(\mathcal{H}[u(0)] + \frac{\epsilon}{\gamma} C^{4/3} \int_0^{\gamma t} |f(s)| \mathrm{d}s \sup_{0 \leq s \leq t} \|u(s)\|_{L^2}^{10/3}\Big) e^{5\epsilon/\gamma \int_0^{\gamma t} |f(s)| \mathrm{d}s}.$$

Via (3.17), this provides an a priori bound on $\|\partial_x u(t)\|_{L^2}^2$. Using the fact that the integral

$$\mathcal{E}_2[u] = \int_{\mathbb{R}} (\partial_x^2 u)^2 - \tfrac{10}{3} u (\partial_x u)^2 + \tfrac{5}{9} u^4 \mathrm{d}x$$

is conserved for the unperturbed KdV flow, we may use the bound $\|u\|_{L^4}^4 \leq 4\|u\|_{H^1}^4$ to estimate

$$\|\partial_x^2 u\|_{L^2}^2 = \mathcal{E}_2[u] + \tfrac{10}{3} \int_{\mathbb{R}} u (\partial_x u)^2 \mathrm{d}x - \tfrac{5}{9} \int_{\mathbb{R}} u^4 \mathrm{d}x \leq \Big|\mathcal{E}_2[u]\Big| + \tfrac{10}{3} \|u\|_{H^1}^3 + \tfrac{20}{9} \|u\|_{H^1}^4.$$

One may furthermore verify that

$$\partial_t \mathcal{E}_2[u] = 2\epsilon f(\gamma t) \mathcal{E}_2[u] + \epsilon f(\gamma t) \int_{\mathbb{R}} -\tfrac{10}{3} u (\partial_x u)^2 + \tfrac{10}{9} u^4 \ \mathrm{d}x,$$

so that

$$\Big|\partial_t \mathcal{E}_2[u]\Big| \leq 2\epsilon |f(\gamma t)| \Big|\mathcal{E}_2[u]\Big| + \epsilon |f(\gamma t)| \Big(\tfrac{10}{3} \|u\|_{H^1}^3 + \tfrac{10}{9} \|u\|_{H^1}^4\Big),$$

through which one arrives at an a priori bound on $\mathcal{E}_2[u(t)]$ and hence $\|u(t)\|_{H^2}$. □

With these preliminaries in place, let us introduce our decomposition of solutions to (3.14), which is based on Lemma 3.8.2. Provided that $\|\overline{v}_*\|_{L^2_w}$ is small enough, there exist unique parameters $\xi_0 \in \mathbb{R}$ and $c_0 > 0$ that allow for the *orthogonal* decomposition

$$u(0, x + \xi_0) = \phi_{c_0}(x) + \overline{v}_0(x) \quad \text{with} \quad \langle \overline{v}_0, \phi_{c_0} \rangle = \langle \overline{v}_0, \zeta_{c_0} \rangle = 0,$$

where $\overline{v}_0 \in H^2 \cap H^1_w$. From there, we decompose the solution $u(t,x)$ to (3.14) for $t \geq 0$ via

$$\overline{v}(t,x) = u(t, x + \xi(t)) - \phi_{c(t)}(x), \tag{3.18}$$

where the perturbation $\overline{v}$ satisfies

$$\langle \overline{v}(t,\cdot), \phi_{c(t)} \rangle = \langle \overline{v}(t,\cdot), \zeta_{c(t)} \rangle = 0, \tag{3.19}$$

and $\xi, c$ are time-dependent modulation parameters. The existence, uniqueness, and continuous time-dependence of this decomposition is guaranteed by Lemma 3.8.2 as long as $\|\overline{v}(t)\|_{L^2_w}$ is kept below some constant $\delta_2 > 0$. Based on this decomposition, we introduce a phase-shift parameter $\Omega$ through

$$\xi(t) = \xi_0 + \int_0^t c(s)\ \mathrm{d}s + \Omega(t).$$

Lastly, we introduce a scaling parameter $\alpha$ and a *rescaled* perturbation $v$ through

$$v(t,x) = \alpha^2(t)u\big(t, \alpha(t)x + \xi(t)\big) - \phi_{c_0}(x) \quad \text{with} \quad c(t) = c_0\alpha^{-2}(t), \tag{3.20}$$

in which $u$ is rescaled in accordance with the scaling symmetry (3.3). Below, we collect various properties of $v$ and $\overline{v}$, including the relation between their (distinct!) weighted norms.

**Lemma 3.2.2.** *Assuming S1, let $\overline{v}(t) \in H^2 \cap H^1_b$ for some $t \geq 0$ and $b > 0$. Furthermore, let $v(t)$ and $\alpha(t)$ be defined through* (3.20)*. It then holds that*

1. $v(t,x) = \alpha^2(t)\overline{v}(t, \alpha(t)x)$;
2. $\langle v(t,\cdot), \phi_{c_0}\rangle = \langle v(t,\cdot), \zeta_{c_0}\rangle = 0$;
3. $v(t) \in H^2 \cap H^1_{\alpha(t)b}$ *with*

$$\|v(t)\|_{L^2_{\alpha(t)b}} = \alpha^{3/2}(t)\|\overline{v}(t)\|_{L^2_b} \quad \text{and} \quad \|\partial_x v(t)\|_{L^2_{\alpha(t)b}} = \alpha^{5/2}(t)\|\partial_x \overline{v}(t)\|_{L^2_b}.$$

*Proof.* Item 1 follows from (3.18) by substituting $y = \alpha(t)x = \sqrt{c_0/c(t)}x$. In the same way, item 2 follows from (3.19). Finally, we compute the norms

$$\begin{aligned}\|v(t)\|^2_{L^2_{\alpha(t)b}} &= \int_{\mathbb{R}} v^2(t,x)e^{2\alpha(t)bx}\mathrm{d}x = \alpha^4(t)\int_{\mathbb{R}} \overline{v}^2(t,\alpha(t)x)e^{2\alpha(t)bx}\mathrm{d}x \\ &= \alpha^3(t)\int_{\mathbb{R}} \overline{v}^2(t,y)e^{2by}\mathrm{d}x = \alpha^3(t)\|\overline{v}(t)\|^2_{L^2_b}\end{aligned}$$

and

$$\begin{aligned}\|\partial_x v(t)\|^2_{L^2_{\alpha(t)b}} &= \int_{\mathbb{R}} (\partial_x v)^2(t,x)e^{2\alpha(t)bx}\mathrm{d}x = \alpha^6(t)\int_{\mathbb{R}} (\partial_x \overline{v})^2(t,\alpha(t)x)e^{2\alpha(t)bx}\mathrm{d}x \\ &= \alpha^5(t)\int_{\mathbb{R}} (\partial_x\overline{v})^2(t,y)e^{2by}\mathrm{d}x = \alpha^5(t)\|\partial_x\overline{v}(t)\|^2_{L^2_b},\end{aligned}$$

which yield item 3. □

## Modulation equations

Below, we derive a system of evolution equations that governs the behavior of the modulation parameters $v(t), \alpha(t)$ and $\Omega(t)$. An application of the chain rule to (3.20) yields

$$v_t = \alpha^{-3}\big(\mathcal{L}_{c_0} v + N(v)\big) + R(t, v; \alpha, \Omega), \tag{3.21}$$

where $N$ is the KdV nonlinearity $N(v) = -2v\partial_x v$ and

$$R(t, v; \alpha, \Omega) = \frac{\alpha_t}{\alpha}(2 + x\partial_x)(\phi_{c_0} + v) + \frac{\Omega_t}{\alpha}\partial_x(\phi_{c_0} + v) + \epsilon f(\gamma t)(\phi_{c_0} + v). \tag{3.22}$$

We claim that the modulation parameters $\alpha$ and $\Omega$ follow the system of equations

$$\begin{bmatrix}\alpha_t \\ \Omega_t\end{bmatrix} = -\alpha\epsilon f(\gamma t) K^{-1}(v) \begin{bmatrix}\langle \phi_{c_0} + v, \phi_{c_0}\rangle \\ \langle \phi_{c_0} + v, \zeta_{c_0}\rangle\end{bmatrix} - \alpha^{-2} K^{-1}(v) \begin{bmatrix}\langle N(v), \phi_{c_0}\rangle \\ \langle N(v), \zeta_{c_0}\rangle\end{bmatrix} \tag{3.23}$$

where

$$K(v) = \begin{bmatrix}\big\langle (x\partial_x + 2)(\phi_{c_0} + v), \phi_{c_0}\big\rangle & \big\langle \partial_x v, \phi_{c_0}\big\rangle \\ \big\langle (x\partial_x + 2)(\phi_{c_0} + v), \zeta_{c_0}\big\rangle & \big\langle \partial_x(\phi_{c_0} + v), \zeta_{c_0}\big\rangle\end{bmatrix} \tag{3.24}$$

and

$$\alpha(0) = 1, \quad \Omega(0) = 0.$$

Indeed, this implies that

$$\big\langle \alpha^{-3} N(v) + R(t, v; \alpha, \Omega), \phi_{c_0}\big\rangle = \big\langle \alpha^{-3} N(v) + R(t, v; \alpha, \Omega), \zeta_{c_0}\big\rangle = 0, \tag{3.25}$$

which is necessary to ensure that $\langle v, \phi_{c_0}\rangle = \langle v, \zeta_{c_0}\rangle = 0$ and equivalently $\langle \overline{v}, \phi_{c(t)}\rangle = \langle \overline{v}, \zeta_{c(t)}\rangle = 0$. The matrix $K(v)$ is invertible in case $\|v\|_{L^2_w}$ is suitably small, since $K(0)$ is invertible and $v \mapsto \det K(v)$ is continuous from $L^2_w$ to $\mathbb{R}$. Consequently, the system (3.21), (3.23) is well-defined as long as $\|v(t)\|_{L^2_w}$ remains suitably bounded.

Setting $v = 0$ reduces (3.23) to

$$\begin{bmatrix}\alpha_t \\ \Omega_t\end{bmatrix} = -\alpha\epsilon f(\gamma t)\frac{1}{9}\begin{bmatrix}c_0^{-3/2} & 0 \\ 2c_0^{-2} & -2c_0^{-1/2}\end{bmatrix}\begin{bmatrix}6c_0^{3/2} \\ 9\end{bmatrix} = -\alpha\epsilon f(\gamma t)\begin{bmatrix}\frac{2}{3} \\ -\frac{2}{3}c_0^{-1/2}\end{bmatrix}$$

and gives rise to the leading-order approximations $c_{\mathrm{ap}}(t)$ and $\xi_{\mathrm{ap}}(t)$ defined in (3.5) and (3.6).

We conclude this section by noting that, as a result of Lemma 3.2.1, the evolution equation (3.21) is initially well-posed in $H_b^{-3}$ on $[0, T]$ for some $T > 0$ and any

$$b \in \big(0, w \min_{t \in [0,T]} \alpha(t)\big).$$

In particular, the term $x\partial_x[\phi_{c_0} + v]$ in (3.22) lies in $L^2_b$ since there exists a constant $C > 0$, for which we have

$$\|x\partial_x(\phi_{c_0} + v)\|_{L^2_b} \overset{(3.20)}{=} \|x\partial_x \alpha^2 u(t, \alpha \cdot + \xi)\|_{L^2_b} = \alpha^{3/2}\|x\partial_x u\|_{L^2_{b/\alpha}}$$

$$\leq C\big((b/\alpha)^{-1}\|\partial_x u\|_{L^2} + (w - b/\alpha)^{-1}\|\partial_x u\|_{L^2_w}\big);$$

see also Lemma 3.3.2 below.

## 3.3 Linear stability on weighted spaces

The orbital stability proof of [91] relies on stability and smoothing properties of the evolution generated by the linear operator $\mathcal{L}_c$ defined in (3.10); see Theorem 3.8.1. These properties hold on exponentially weighted spaces after applying the projection $Q_c = I - P_c$, where $P_c$ is the spectral projection corresponding to the 0-eigenvalue of $\mathcal{L}_c$, given by

$$P_c f = \langle f, \eta_c^1\rangle \partial_x \phi_c + \langle f, \eta_c^2\rangle \partial_c \phi_c.$$

Here, $\eta_c^1$ and $\eta_c^2$ are linear combinations of $\phi_c$ and $\zeta_c$ that are defined in (3.71). As such, the subspace of $L^2_w$ characterized by (3.11) corresponds to $\ker P_c \subseteq L^2_w$. We review this classical result in Section 3.8, where it is stated as Theorem 3.8.1.

The spaces $L^2_w$, however, are unsuitable for controlling the term $x\partial_x v$ present in (3.22). One can for instance not expect that

$$\|x\partial_x v\|_{L^2_w} \leq C\|v\|_{H^1_w}$$

for a constant $C$ and all $v \in L^2_w$, due to the unbounded and non-integrable factor $x$. It is, however, true that

$$\|xg\|^2_{L^2_w} = \int_{-\infty}^{\infty} e^{2wx}x^2 g^2(x)\mathrm{d}x \leq C\int_{-\infty}^{0} e^{2b_- x} g^2(x)\mathrm{d}x + C\int_0^{\infty} e^{2b_+ x} g^2(x)\mathrm{d}x$$

for $b_- < w < b_+$, some constant $C > 0$, and functions $g$ for which the above quantity is well-defined. This leads us to introduce the notion of *asymmetrically-weighted spaces*. For every $\mathbf{w} = (w_-, w_+) \in \mathbb{R}^2$, we introduce the weighted space

$$L^2_{\mathbf{w}} = \big\{g : e^{w_- x} g \in L^2(-\infty, 0) \quad \text{and} \quad e^{w_+ x} g \in L^2(0,\infty)\big\}$$

with norm

$$\|g\|^2_{L^2_{\mathbf{w}}} := \int_{-\infty}^{0} e^{2w_- x} g^2(x)\mathrm{d}x + \int_0^{\infty} e^{2w_+ x} g^2(x)\mathrm{d}x. \tag{3.26}$$

Writing $g_+(x) = g(x)\chi_{x\geq 0}(x)$ and $g_-(x) = g(x)\chi_{x\leq 0}(x)$, we have

$$\|g\|^2_{L^2_{\mathbf{w}}} = \|g_-\|^2_{L^2_{w_-}} + \|g_+\|^2_{L^2_{w_+}}. \tag{3.27}$$

The following result asserts that the stability and smoothing properties of $\{e^{\mathcal{L}_c t}\}_{t\geq 0}$ provided in Theorem 3.8.1 extend to the asymmetrically-weighted spaces.

**Proposition 3.3.1.** *Let $c > 0$ and $w_-, w_+ \in (0, \sqrt{c})$. For all $\beta > 0$ that satisfy*

$$\beta < \min\{w_-(c - w_-^2), w_+(c - w_+^2)\},$$

*there exists a constant $M > 0$ such that for all $g \in L^2_{\mathbf{w}}$, $t > 0$, and $k \in \{0, 1\}$ we have*

$$\|\partial_x^k e^{\mathcal{L}_c t} Q_c g\|_{L^2_{\mathbf{w}}} \leq M t^{-k/2} e^{-\beta t} \|g\|_{L^2_{\mathbf{w}}}. \tag{3.28}$$

Proposition 3.3.1 is easily proved using some elementary observations regarding the norm (3.26). Items 2 and 3 will be used in later sections.

**Lemma 3.3.2.** *If $\mathbf{w}, \mathbf{b} \in \mathbb{R}^2$ satisfy $b_- < w_-$, $b_+ > w_+$ and $g \in L^2_{\mathbf{b}}$, then $g, xg \in L^2_{\mathbf{w}}$ and*

1. $\|g\|_{L^2_{\mathbf{w}}} \leq \|g\|_{L^2_{w_+}} + \|g\|_{L^2_{w_-}}$;
2. $\|g\|_{L^2_{\mathbf{w}}} \leq \|g\|_{L^2_{\mathbf{b}}}$;
3. $\|xg\|^2_{L^2_{\mathbf{w}}} \leq e^{-2}(b_- - w_-)^{-2}\|g_-\|^2_{L^2_{b_-}} + e^{-2}(b_+ - w_+)^{-2}\|g_+\|^2_{L^2_{b_+}}$.

*Proof.* Item 1 follows directly from (3.27). For item 2, we estimate

$$\|g\|^2_{L^2_{\mathbf{w}}} = \|g_-\|^2_{L^2_{w_-}} + \|g_+\|^2_{L^2_{w_+}} \leq \|g_-\|^2_{L^2_{b_-}} + \|g_+\|^2_{L^2_{b_+}} = \|g\|^2_{L^2_{\mathbf{b}}}.$$

Similarly, we prove item 3 by estimating

$$\begin{aligned}
\|xg\|^2_{L^2_{\mathbf{w}}} &= \|xg_-\|^2_{L^2_{w_-}} + \|xg_+\|^2_{L^2_{w_+}} \\
&\leq \sup_{x \leq 0} x^2 e^{2(w_- - b_-)x} \|g_-\|^2_{L^2_{b_-}} + \sup_{x \geq 0} x^2 e^{2(w_+ - b_+)x} \|g_+\|^2_{L^2_{b_+}} \\
&= e^{-2}(b_- - w_-)^{-2}\|g_-\|^2_{L^2_{b_-}} + e^{-2}(b_+ - w_+)^{-2}\|g_+\|^2_{L^2_{b_+}}. \qquad \square
\end{aligned}$$

*Proof of Proposition 3.3.1.* For $g \in L^2_{\mathbf{w}}$, we may use item 1 of Lemma 3.3.2 and Theorem 3.8.1 to compute

$$\begin{aligned}
\|\partial_x^k e^{\mathcal{L}_c t} Q_c g\|_{L^2_{\mathbf{w}}} &\leq \|\partial_x^k e^{\mathcal{L}_c t} Q_c g_-\|_{L^2_{\mathbf{w}}} + \|\partial_x^k e^{\mathcal{L}_c t} Q_c g_+\|_{L^2_{\mathbf{w}}} \\
&\leq \|\partial_x^k e^{\mathcal{L}_c t} Q_c g_-\|_{L^2_{w_-}} + \|\partial_x^k e^{\mathcal{L}_c t} Q_c g_-\|_{L^2_{w_+}} \\
&\quad + \|\partial_x^k e^{\mathcal{L}_c t} Q_c g_+\|_{L^2_{w_-}} + \|\partial_x^k e^{\mathcal{L}_c t} Q_c g_+\|_{L^2_{w_+}} \\
&\overset{(3.72)}{\leq} M t^{-k/2} e^{-\beta t} (\|g_-\|_{L^2_{w_-}} + \|g_+\|_{L^2_{w_+}}) \\
&\leq M t^{-k/2} e^{-\beta t} \|w\|_{L^2_{\mathbf{w}}}. \qquad \square
\end{aligned}$$

## 3.4 Short-time control

In this section, we establish control over the perturbation in the original frame in asymmetrically-weighted spaces over short time-scales. Using our rescaled frame description (3.21) on *short* time-scales leads to problematic complications arising from the term $x\partial_x$ in (3.22). Instead, we rely on classical results valid for small amplitude fluctuations. We follow the argument of Pego & Weinstein [91, Proposition 6.1], which uses the evolution

$$\overline{v}_t = \mathcal{L}_{c(t)}\overline{v} + N(\overline{v}) - c_t\partial_c\phi_{c(t)} + \Omega_t\partial_x(\overline{v} + \phi_{c(t)}) + \epsilon f(\gamma t)(\overline{v} + \phi_{c(t)}),$$

for the perturbation in the original frame $\overline{v}$ (initially justified in $H^{-1}$ via (3.15)). This argument relies heavily on an approximation of the form

$$\mathcal{L}_{c(t)} = \mathcal{L}_{c_0} + O\big(|c_0 - c(t)|\big).$$

In our setting, we pick $t_\diamond > 0$ and linearize around the fixed soliton $\phi_{c(t_\diamond)}$ by writing

$$\overline{v}_t(t_\diamond + s) = \mathcal{L}_{c(t_\diamond)}\overline{v}(t_\diamond + s) + Y(t_\diamond, s, \overline{v}; c, \Omega), \tag{3.29}$$

where

$$\begin{aligned} Y(t_\diamond, s, \overline{v}; c, \Omega) = {} & \big(\mathcal{L}_{c(t_\diamond+s)} - \mathcal{L}_{c(t_\diamond)}\big)\overline{v}(t_\diamond + s) + N(\overline{v}(t_\diamond + s)) - c_t(t_\diamond + s)\partial_c\phi_{c(t_\diamond+s)} \\ & + \Omega_t(t_\diamond + s)\partial_x\big(\overline{v}(t_\diamond + s) + \phi_{c(t_\diamond+s)}\big) \\ & + \epsilon f(\gamma(t_\diamond + s))\big(\overline{v}(t_\diamond + s) + \phi_{c(t_\diamond+s)}\big). \end{aligned} \tag{3.30}$$

We recall that $c(t)$ is related to the rescaling process through $c(t) = c_0\alpha^{-2}(t)$, so that $c(t)$ follows the modulation equation $c_t = -2c_0\alpha^{-3}\alpha_t$ where $\alpha_t$ is given by (3.23). Throughout this section, we will assume that both $\alpha$ and $c$ can be bounded away from zero. More precisely, we make the following assumptions S2 and S3, and formulate a condition C1 that underlies most of the results in this section.

**S2** *The constants* $\alpha_{\min}, \alpha_{\max} \in \mathbb{R}$ *satisfy* $0 < \alpha_{\min} < 1 < \alpha_{\max}$.

**S3** *The constant* $w_{\min} \in \mathbb{R}$ *satisfies* $0 < w_{\min} < w$.

**C1** *Given* $T > 0$*, the function* $\overline{v} \in C([0,T], H^2) \cap C^1([0,T], H^{-1})$ *satisfies* (3.29) *while the function* $\alpha \in C^1([0,T], \mathbb{R})$ *solves* (3.23)*. In addition, we have the inclusion*

$$\alpha(t) \in [\alpha_{\min}, \alpha_{\max}], \quad t \in [0,T].$$

Our main result in this section provides short-time control over $\overline{v}$. More precisely, on time intervals where $\overline{v}$ and the fluctuations of $c$ are small enough, the growth of $\overline{v}$ measured in the $H^1_{\mathbf{w}}$-norm can be explicitly controlled by the small forcing amplitude $\epsilon > 0$ and the length of the time interval $\delta > 0$.

**Proposition 3.4.1** (Short-time control)**.** *Assuming S1–S3, there exist constants* $\delta_4, C_4 > 0$ *such that the following holds true for each* $\epsilon, \gamma > 0$. *If C1 holds for some* $T > 0$, *then for each* $t, \delta > 0$ *with* $t + \delta \le T$, *and each* $\overline{\mathbf{w}} = (\overline{w}_-, \overline{w}_+) \in \mathbb{R}^2$ *with*

$$\overline{w}_-, \overline{w}_+ \in \Big[\frac{w_{\min}}{\alpha(t+s)}, \frac{\sqrt{c_0}/3}{\alpha(t+s)}\Big], \quad s \in [0, \delta],$$

*the bound*

$$\Big(\epsilon + \sup_{s\in[0,\delta]} \big(\|\overline{v}(t+s)\|_{H^1_{\overline{\mathbf{w}}}} + \|\overline{v}(t+s)\|_{H^1} + |c(t+s) - c(t)|\big)\Big)(\sqrt{\delta} + \delta^2) \le \delta_4 \tag{3.31}$$

*implies*

$$\sup_{s\in[0,\delta]} \|\overline{v}(t+s)\|_{H^1_{\overline{\mathbf{w}}}} \le C_4\Big(\|\overline{v}(t)\|_{H^1_{\overline{\mathbf{w}}}} + \epsilon \sup_{s\in[0,\delta]} |f(\gamma(t+s))|(\sqrt{\delta} + \delta^2)\Big). \tag{3.32}$$

In order to apply this result to the perturbation $v$ in the rescaled frame, it is essential to note that (3.32) transforms into an estimate between *different* weighted spaces due to the time-dependent rescaling in the $x$-direction via $\alpha(t)$. To remedy this, and to deal with the problematic $x\partial_x v$ term in (3.49), we introduce time-dependent weights $\mathbf{w}(t) = \big(w_-(t), w_+(t)\big)$ that increase/decrease at a rate sufficient to compensate rescaling by $\alpha$. More precisely, we assume the following.

**C2** *Given* $T \ge \delta > 0$, *the increasing function* $w_- : \mathbb{R}^+ \to \mathbb{R}^+$ *and decreasing function* $w_+ : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfy*

$$\frac{w_-(t+s/2)}{w_-(t+s)} \le \frac{\alpha(t)}{\alpha(t+s)} \le \frac{w_+(t+s/2)}{w_+(t+s)} \tag{3.33}$$

*for each* $s \in [0, \delta]$ *and* $t \in [0, T-s]$. *Furthermore,*

$$w_{\min} \le w_\pm(t) \le w, \quad t \in [0, T].$$

Assumption C2 implies that $t \to \frac{w_+(t)}{\alpha(t)}$ is decreasing and $t \to \frac{w_-(t)}{\alpha(t)}$ is increasing on $[0, T]$, which essentially means that the weight-functions retain their monotonicity after rescaling. Since $w_-$ and $w_+$ are evaluated at $t+s/2$ in (3.33), it furthermore follows that there is a lower bound on their absolute growth rate in the sense that

$$\log\big(\tfrac{w_-}{\alpha}\big)'(t) \ge \tfrac{1}{2}\log(w_-)'(t) \quad \text{and} \quad \log\big(\tfrac{w_+}{\alpha}\big)'(t) \le \tfrac{1}{2}\log(w_+)'(t), \quad t \in [0, T].$$

With this condition in place, we formulate the following corollary to Proposition 3.4.1.

**Corollary 3.4.2.** *Assuming S1–S3, there exist constants* $\delta_5, C_5 > 0$ *so that the following holds true for each* $\epsilon, \gamma > 0$. *If C1 and C2 hold for some* $T \ge \delta > 0$, *then* *for each* $t \in [0, T-\delta]$, *the bound*

$$\Big(\epsilon + \sup_{s\in[0,\delta]} \big(\|v(t+s)\|_{H^1_{\mathbf{w}(t+s)}} + \|v(t+s)\|_{H^1} + |c(t+s) - c(t)|\big)\Big)(\sqrt{\delta} + \delta^2) \leq \delta_5$$

*implies*

$$\sup_{s\in[0,\delta]} \|v(t+s)\|_{H^1_{\mathbf{w}(t+s)}} \leq C_5\Big(\|v(t)\|_{H^1_{\mathbf{w}(t+\delta/2)}} + \epsilon \sup_{s\in[0,\delta]} |f(\gamma(t+s))|(\sqrt{\delta}+\delta^2)\Big).$$

In preparation for the proof of Proposition 3.4.1, we examine (3.23) and show that $\alpha_t, \Omega_t$ can be controlled by the perturbation $v$. In contrast to [91], we deal with the presence of the forcing term and require slightly sharper control on the modulation parameters.

**Lemma 3.4.3.** *Assuming S1 and S2, there exist constants* $\delta_6, C_6 > 0$ *so that the following holds true for each* $\epsilon, \gamma > 0$. *If C1 holds for some* $T > 0$, *then for each* $\mathbf{b} = (b_-, b_+) \in \mathbb{R}^2$ *with* $b_-, b_+ \in (0, \sqrt{c_*}/3]$ *and* $t \in [0, T]$, *the bound*

$$\|v(t)\|_{L^2_{\mathbf{b}}} \leq b_+^{1/2}\delta_6$$

*implies*

$$\begin{aligned} |\alpha_t(t) + \tfrac{2}{3}\alpha(t)\epsilon f(\gamma t)| + |\Omega_t(t) - \tfrac{2}{3}c_0^{-1/2}\alpha(t)\epsilon f(\gamma t)| \leq{}& C_6\epsilon|f(\gamma t)|b_+^{-1/2}\|v(t)\|_{L^2_{\mathbf{b}}} \\ &+ C_6\|v(t)\|^2_{L^2_{\mathbf{b}}}, \end{aligned} \tag{3.34}$$

*and hence*

$$|\alpha_t(t)| + |\Omega_t(t)| \leq C_6\Big(\epsilon|f(\gamma t)|(1 + b_+^{-1/2}\|v(t)\|_{L^2_{\mathbf{b}}}) + \|v(t)\|^2_{L^2_{\mathbf{b}}}\Big). \tag{3.35}$$

*Proof.* Let us write $v = v(t)$ for brevity, and rewrite (3.23) as

$$\begin{aligned} \begin{bmatrix} \alpha_t + \frac{2}{3}\alpha(t)\epsilon f(\gamma t) \\ \Omega_t - \frac{2}{3}c_0^{-1/2}\alpha(t)\epsilon f(\gamma t)\end{bmatrix} ={}& -\alpha\epsilon f(\gamma t)K^{-1}(v)\begin{bmatrix}\langle v, \phi_{c_0}\rangle \\ \langle v, \zeta_{c_0}\rangle\end{bmatrix} \\ &- \alpha f(\gamma t)(K^{-1}(v) - K^{-1}(0))\begin{bmatrix}\langle \phi_{c_0}, \phi_{c_0}\rangle \\ \langle \phi_{c_0}, \zeta_{c_0}\rangle\end{bmatrix} \\ &- \alpha^{-2}K^{-1}(v)\begin{bmatrix}\langle N(v), \phi_{c_0}\rangle \\ \langle N(v), \zeta_{c_0}\rangle\end{bmatrix}. \end{aligned} \tag{3.36}$$

Here we have used that

$$K^{-1}(0)\begin{bmatrix}\langle \phi_{c_0}, \phi_{c_0}\rangle \\ \langle \phi_{c_0}, \zeta_{c_0}\rangle\end{bmatrix} = \begin{bmatrix}\frac{2}{3} \\ -\frac{2}{3}c_0^{-1/2}\end{bmatrix}.$$

Setting out to control $K^{-1}(v)$, we note that

$$K(0) = \begin{bmatrix} 9c_0^{3/2} & 0 \\ 9 & \frac{9}{2}c_0^{1/2} \end{bmatrix}$$

is invertible, so that we can find constants $\tilde{C}_1, \tilde{C}_2 > 0$ that ensure $\|A^{-1}\|_{\text{op}} \le \tilde{C}_2$ for all $A \in \mathbb{R}^{2\times 2}$ which satisfy

$$|A_{ij} - K_{ij}(0)| \le \tilde{C}_1, \quad (i,j) \in \{1,2\}^2.$$

Here, $\|\cdot\|_{\text{op}}$ denotes the operator-norm on $(\mathbb{R}^2, \|\cdot\|_1)$, chosen for convenience in the computations below. Now note that

$$\begin{aligned} |K_{ij}(v) - K_{ij}(0)| &\overset{(3.24)}{\le} \Big((2c_0+1)\|\partial_c\phi_{c_0}\|_{L^2_{-\mathbf{b}}} + \|\partial_x\phi_{c_0}\|_{L^2_{-\mathbf{b}}}\Big)\|v\|_{L^2_{\mathbf{b}}} \\ &\quad + \big(\|\zeta_{c_0}\|_{L^2_{-\mathbf{b}}} + \|x\partial_c\phi_{c_0}\|_{L^2_{-\mathbf{b}}}\big)\|v\|_{L^2_{\mathbf{b}}} \end{aligned}$$

for all $(i,j) \in \{1,2\}^2$. Since $\partial_c\phi_{c_0}$ and $\partial_x\phi_{c_0}$ decay exponentially as $|x| \to \infty$, we can estimate their weighted norm by a constant that does not depend on $\mathbf{b}$. The function $\zeta_{c_0}$, however, tends to $\int_{\mathbb{R}} \partial_c\phi_{c_0}\mathrm{d}x = 3c_0^{-1/2}$ as $x \to \infty$, and is not an $L^2$-function. We therefore estimate

$$\begin{aligned} \|\zeta_{c_0}\|^2_{L^2_{-\mathbf{b}}} &= \int_{-\infty}^0 e^{-2b_-x}\zeta^2_{c_0}(x)\mathrm{d}x + \int_0^\infty e^{-2b_+x}\zeta^2_{c_0}(x)\mathrm{d}x \\ &\le \int_{-\infty}^0 \zeta^2_{c_0}(x)\mathrm{d}x + \frac{\|\zeta_{c_0}\|^2_{L^\infty}}{2b_+} \end{aligned}$$

and we thus have

$$|K_{ij}(v) - K_{ij}(0)| \le \tilde{C}_3 b_+^{-1/2}\|v\|_{L^2_{\mathbf{b}}} \tag{3.37}$$

for all $(i,j) \in \{1,2\}^2$ and some constant $\tilde{C}_3 > 0$. In case $\delta_6 \le b_+^{1/2}\frac{\tilde{C}_1}{\tilde{C}_3}$, it follows via (3.37) that $\|K^{-1}(v)\|_{\text{op}} \le \tilde{C}_2$. Turning to the term $K^{-1}(v) - K^{-1}(0)$ in (3.36), we note that

$$\begin{aligned} \|K^{-1}(v) - K^{-1}(0)\|_{\text{op}} &= \|K^{-1}(0)\big(K(v) - K(0)\big)K^{-1}(v)\|_{\text{op}} \\ &\le \|K^{-1}(0)\|_{\text{op}}\|K(v) - K(0)\|_{\text{op}}\|K^{-1}(v)\|_{\text{op}} \\ &\le \tilde{C}_4 b_+^{-1/2}\|v\|_{L^2_{\mathbf{b}}}. \end{aligned}$$

We proceed by estimating

$$\begin{aligned} |\langle N(v), \phi_{c_0}\rangle| + |\langle N(v), \zeta_{c_0}\rangle| &= |\langle \partial_x(v^2), \phi_{c_0}\rangle| + |\langle \partial_x(v^2), \zeta_{c_0}\rangle| \\ &= |\langle v^2, \partial_x\phi_{c_0}\rangle| + |\langle v^2, \partial_c\phi_{c_0}\rangle| \end{aligned}$$



3

$$
\begin{aligned}
&\le \left\| e^{-2b_-\cdot}\chi_{x\le 0}(|\partial_x\phi_{c_0}|+|\partial_c\phi_{c_0}|)\right\|_{L^\infty}\|v\|^2_{L^2_{\mathbf{b}}}\\
&\le \tilde{C}_5\|v\|^2_{L^2_{\mathbf{b}}},
\end{aligned}
$$

together with

$$
|\langle v,\phi_{c_0}\rangle|+|\langle v,\zeta_{c_0}\rangle|\le \|\phi_{c_0}\|_{L^2_{-\mathbf{b}}}\|v\|_{L^2_{\mathbf{b}}}+\tilde{C}_3 b_+^{-1/2}\|v\|_{L^2_{\mathbf{b}}}.
$$

We conclude via (3.36) that

$$
\begin{aligned}
&|\alpha_t+\tfrac{2}{3}\alpha(t)\epsilon f(\gamma t)|+|\Omega_t-\tfrac{2}{3}c_0^{-1/2}\alpha(t)\epsilon f(\gamma t)|\\
&\quad\le \alpha_{\max}\epsilon|f(\gamma t)|\tilde{C}_2\Big(\|\phi_{c_0}\|_{L^2_{-\mathbf{b}}}\|v\|_{L^2_{\mathbf{b}}}+\tilde{C}_3 b_+^{-1/2}\|v\|_{L^2_{\mathbf{b}}}\Big)\\
&\quad\quad+\alpha_{\max}\epsilon|f(\gamma t)|\tilde{C}_4\Big(\|\phi_{c_0}\|^2_{L^2}+|\langle\phi_{c_0},\zeta_{c_0}\rangle|\Big)b_+^{-1/2}\|v\|_{L^2_{\mathbf{b}}}\\
&\quad\quad+\alpha_{\min}^{-2}\tilde{C}_2\tilde{C}_5\|v\|^2_{L^2_{\mathbf{b}}}.
\end{aligned}
$$

□

Using Lemma 3.4.3, it is now straightforward to control the term $Y(t_\diamond,s,\overline{v};c,\Omega)$ introduced in (3.30) in terms of $\overline{v}$. We note, though, that Lemma 3.4.3 provides control over $\alpha_t$ and $\Omega_t$ in terms of a weighted norm of $v$, instead of $\overline{v}$. We remedy this using item 1 of Lemma 3.2.2, taking care to apply a rescaled weight.

**Corollary 3.4.4.** *Assuming S1–S3, there exists a constant $C_7>0$ so that the following holds true for each $\epsilon,\gamma>0$. If C1 holds for some $T>0$, then for any $t_\diamond,s\ge 0$ with $t_\diamond+s\in[0,T]$ and $\overline{\mathbf{w}}=(\overline{w}_-,\overline{w}_+)\in\mathbb{R}^2$ with*

$$
\overline{w}_-,\overline{w}_+\in\Big[\frac{w_{\min}}{\alpha(t_\diamond+s)},\frac{\sqrt{c_0}/3}{\alpha(t_\diamond+s)}\Big],
$$

*the bound*

$$
\|\overline{v}(t_\diamond+s)\|_{L^2_{\overline{\mathbf{w}}}}\le\alpha(t_\diamond+s)^{-1}\delta_6\overline{w}_+^{1/2}
$$

*implies*

$$
\begin{aligned}
\|Y(t_\diamond,s,\overline{v};c,\Omega)\|_{L^2_{\overline{\mathbf{w}}}}\le C_7\Big(&|c(t_\diamond+s)-c(t_\diamond)|+\epsilon|f(\gamma(t_\diamond+s))|(1+\|\overline{v}(t_\diamond+s)\|_{H^1_{\overline{\mathbf{w}}}})\\
&+\|\overline{v}(t_\diamond+s)\|^2_{H^1_{\overline{\mathbf{w}}}}+\|\overline{v}(t_\diamond+s)\|_{H^1}\Big)\|\overline{v}(t_\diamond+s)\|_{H^1_{\overline{\mathbf{w}}}}\\
&+C_6\epsilon\big|f\big(\gamma(t_\diamond+s)\big)\big|.
\end{aligned}\tag{3.38}
$$

*Proof.* We write $\overline{v}=\overline{v}(t_\diamond+s)$ and estimate the various components in (3.30). Firstly, we have

$$
\begin{aligned}
\|(\mathcal{L}_{c(t_\diamond+s)}-\mathcal{L}_{c(t_\diamond)})\overline{v}\|_{L^2_{\overline{\mathbf{w}}}}&\overset{(3.10)}{=}\big\|(c(t_\diamond+s)-c(t_\diamond))\partial_x\overline{v}-2\partial_x((\phi_{c(t_\diamond+s)}-\phi_{c(t_\diamond)})\overline{v})\big\|_{L^2_{\overline{\mathbf{w}}}}\\
&\le|c(t_\diamond+s)-c(t_\diamond)|\|\partial_x\overline{v}\|_{L^2_{\overline{\mathbf{w}}}}\\
&\quad+2\|\partial_x(\phi_{c(t_\diamond+s)}-\phi_{c(t_\diamond)})\|_{L^\infty}\|\overline{v}\|_{L^2_{\overline{\mathbf{w}}}}
\end{aligned}
$$

$$+ 2\|\phi_{c(t_\diamond+s)} - \phi_{c(t_\diamond)}\|_{L^\infty}\|\partial_x\overline{v}\|_{L^2_{\overline{\mathbf{w}}}}.$$

Hence, we see that

$$\|(\mathcal{L}_{c(t_\diamond+s)} - \mathcal{L}_{c(t_\diamond)})\overline{v}\|_{L^2_{\overline{\mathbf{w}}}} \leq \tilde{C}_1|c(t_\diamond+s) - c(t_\diamond)|\|\overline{v}\|_{H^1_{\overline{\mathbf{w}}}}$$

for some constant $\tilde{C}_1 > 0$, since $c \mapsto \phi_c + \partial_x\phi_c$ is Lipschitz from $\mathbb{R}$ to $L^\infty$. Next, we apply (3.35) at $t_\diamond + s$ with weight $\alpha(t_\diamond+s)\overline{\mathbf{w}}$ to obtain

$$\begin{aligned}|\Omega_t(t_\diamond+s)| \leq\ & C_6\epsilon|f(\gamma(t_\diamond+s))|\big(1+\alpha(t_\diamond+s)^{-1/2}\overline{w}_+^{-1/2}\|v(t_\diamond+s)\|_{L^2_{\alpha(t_\diamond+s)\overline{\mathbf{w}}}}\big)\\ &+ C_6\|v(t_\diamond+s)\|^2_{L^2_{\alpha(t_\diamond+s)\overline{\mathbf{w}}}}.\end{aligned}$$

Substituting $v(t_\diamond,x) = \alpha^2(t_\diamond)\overline{v}(t_\diamond,\alpha(t_\diamond)x)$, we then find that

$$\begin{aligned}&\|\Omega_t(t_\diamond+s)\partial_x(\overline{v}(t_\diamond+s)+\phi_{c(t_\diamond+s)})\|_{L^2_{\overline{\mathbf{w}}}}\\ &\leq \tilde{C}_2\Big(\epsilon|f(\gamma(t_\diamond+s))|(1+\|\overline{v}(t_\diamond+s)\|_{L^2_{\overline{\mathbf{w}}}}) + \|\overline{v}(t_\diamond+s)\|^2_{L^2_{\overline{\mathbf{w}}}}\Big)(1+\|\partial_x\overline{v}(t_\diamond+s)\|_{L^2_{\overline{\mathbf{w}}}})\end{aligned}$$

for some constant $\tilde{C}_2 > 0$. Clearly, the term $\|c_t(t_\diamond+s)\partial_c\phi_{c(t_\diamond+s)}\|_{L^2_{\overline{\mathbf{w}}}}$ satisfies the same bound upon increasing $\tilde{C}_2 > 0$ if necessary. Lastly, we estimate

$$\|N(\overline{v})\|_{L^2_{\overline{\mathbf{w}}}} = 2\|\overline{v}\partial_x\overline{v}\|_{L^2_{\overline{\mathbf{w}}}} \leq 2\sqrt{2}\|\overline{v}\|_{H^1}\|\partial_x\overline{v}\|_{L^2_{\overline{\mathbf{w}}}}, \tag{3.39}$$

using the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. □

With this, we are equipped to prove Proposition 3.4.1. Another slight complication compared to [91] is that $\overline{v}_t$ is not completely in the stable subspace characterized by (3.19). This is a result of the fact that our construction of the modulation parameters ensures that $v_t$ satisfies the orthogonality conditions (see (3.25)), whereas $\overline{v}_t$ does not. We therefore decompose $\overline{v}_t$ using $I = P_c + Q_c$, where we recall that $P_c$ is the spectral projection corresponding to the 0-eigenvalue of $\mathcal{L}_c$ as defined in (3.70).

*Proof of Proposition 3.4.1.* The unscaled perturbation satisfies

$$\overline{v}(t_\diamond+r) = e^{r\mathcal{L}_{c(t_\diamond)}}\overline{v}(t_\diamond) + \int_0^r e^{(r-s)\mathcal{L}_{c(t_\diamond)}}\big(P_{c(t_\diamond)} + Q_{c(t_\diamond)}\big)Y(t_\diamond,s,\overline{v};c,\Omega)\,\mathrm{d}s \tag{3.40}$$

for $r \in [0,\delta]$ and $t_\diamond \in [0, T-\delta]$, which is the mild form of (3.29). Since the orthogonality condition (3.19) ensures $P_{c(t_\diamond)}\overline{v}(t_\diamond) = 0$, we have

$$\|e^{r\mathcal{L}_{c(t_\diamond)}}\overline{v}(t_\diamond)\|_{H^1_{\overline{\mathbf{w}}}} \overset{(3.28)}{\leq} Me^{-\beta r}\|\overline{v}(t_\diamond)\|_{H^1_{\overline{\mathbf{w}}}},$$

for $\beta = \frac{4}{9} c_0 \alpha_{\max}^{-2} w_{\min}$, where we use that

$$\tfrac{8}{9} c_0 \alpha_{\max}^{-2} w_{\min} \leq \min\{\overline{w}_-(c(t_\diamond) - \overline{w}_-^2), \overline{w}_+(c(t_\diamond) - \overline{w}_+^2)\}.$$

To account for the projection onto the stable subspace, we use the stability and smoothing properties of $e^{t\mathcal{L}_{c(t_\diamond)}} Q_{c(t_\diamond)}$ to obtain

$$\begin{aligned}
&\left\| \int_0^r e^{(r-s)\mathcal{L}_{c(t_\diamond)}} Q_{c(t_\diamond)} Y(t_\diamond, s, \overline{v}; c, \Omega)\, \mathrm{d}s \right\|_{H^1_{\mathbf{w}}} \\
&\overset{(3.28)}{\leq} M \int_0^\delta e^{-\beta(\delta - s)} (\delta - s)^{-1/2} \| Y(t_\diamond, s, \overline{v}; c, \Omega) \|_{L^2_{\mathbf{w}}} \mathrm{d}s.
\end{aligned}$$

Writing

$$\epsilon_1 = \sup_{s \in [0,\delta]} \Big( \|\overline{v}(t_\diamond + s)\|_{H^1_{\mathbf{w}}} + \|\overline{v}(t_\diamond + s)\|_{H^1} + |c(t_\diamond + s) - c(t_\diamond)| \Big),$$

Corollary 3.4.4 can be used to derive that

$$\begin{aligned}
\sup_{s \in [0,\delta]} \|Y(t_\diamond, s, \overline{v}; c, \Omega)\|_{L^2_{\mathbf{w}}} \leq C_7 \big( \epsilon_1 + \epsilon \|f\|_\infty (1 + \epsilon_1) + \epsilon_1^2 \big) &\sup_{s \in [0,\delta]} \|\overline{v}(t_\diamond + s)\|_{H^1_{\mathbf{w}}} \\
+ C_6 \epsilon \sup_{s \in [0,\delta]} \big| f\big(\gamma(t_\diamond + s)\big) \big|. &\qquad (3.41)
\end{aligned}$$

Using furthermore that $\int_0^\delta (\delta - s)^{-1/2} \mathrm{d}s = 2\sqrt{\delta}$, we find that

$$\begin{aligned}
\left\| \int_0^r e^{(r-s)\mathcal{L}_{c(t_\diamond)}} Q_{c(t_\diamond)} Y(t_\diamond, s, \overline{v}; c, \Omega)\, \mathrm{d}s \right\|_{H^1_{\mathbf{w}}} &\leq \tilde{C}_1 (\epsilon + \epsilon_1) \sqrt{\delta} \sup_{s \in [0,\delta]} \|\overline{v}(t_\diamond + s)\|_{H^1_{\mathbf{w}}} \\
&\quad + \tilde{C}_1 \epsilon \sqrt{\delta} \sup_{s \in [0,\delta]} \big| f\big(\gamma(t_\diamond + s)\big) \big|
\end{aligned}$$

for some constant $\tilde{C}_1 > 0$. To estimate the component in (3.40) with the projection $P_{c(t_\diamond)}$, we recall that

$$\begin{aligned}
P_{c(t_\diamond)} Y(t_\diamond, s, \overline{v}; c, \Omega) &= \big\langle Y(t_\diamond, s, \overline{v}; c, \Omega), \eta^1_{c(t_\diamond)} \big\rangle \partial_x \phi_{c(t_\diamond)} \\
&\quad + \big\langle Y(t_\diamond, s, \overline{v}; c, \Omega), \eta^2_{c(t)} \big\rangle \partial_c \phi_{c(t_\diamond)}
\end{aligned}$$

where $\mathcal{L}_{c(t_\diamond)} \partial_x \phi_{c(t_\diamond)} = 0$ and $\mathcal{L}_{c(t_\diamond)} \partial_c \phi_{c(t_\diamond)} = \partial_x \phi_{c(t_\diamond)}$. Thus,

$$\begin{aligned}
e^{(r-s)\mathcal{L}_{c(t_\diamond)}} P_{c(t_\diamond)} Y(t_\diamond, s, \overline{v}; c, \Omega) &= \big\langle Y(t_\diamond, s, \overline{v}; c, \Omega), \eta^1_{c(t_\diamond)} \big\rangle \partial_x \phi_{c(t_\diamond)} \\
&\quad + \big\langle Y(t_\diamond, s, \overline{v}; c, \Omega), \eta^2_{c(t_\diamond)} \big\rangle \partial_c \phi_{c(t_\diamond)} \\
&\quad + \big\langle Y(t_\diamond, s, \overline{v}; c, \Omega), \eta^2_{c(t_\diamond)} \big\rangle (\delta - s) \partial_x \phi_{c(t_\diamond)},
\end{aligned}$$

and we may estimate

$$\begin{aligned}
&\|e^{(r-s)\mathcal{L}_{c(t_\diamond)}} P_{c(t_\diamond)} Y(t_\diamond, s, \overline{v}; c, \Omega)\|_{H^1_{\mathbf{w}}} \\
&\leq \|Y(t_\diamond, s, \overline{v}; c, \Omega)\|_{L^2_{\mathbf{w}}} \|\eta^1_{c(t_\diamond)}\|_{L^2_{-\mathbf{w}}} \|\partial_x \phi_{c(t_\diamond)}\|_{H^1_{\mathbf{w}}} \\
&\quad + \|Y(t_\diamond, s, \overline{v}; c, \Omega)\|_{L^2_{\mathbf{w}}} \|\eta^2_{c(t_\diamond)}\|_{L^2_{-\mathbf{w}}} \|\partial_c \phi_{c(t_\diamond)}\|_{H^1_{\mathbf{w}}} \\
&\quad + \|Y(t_\diamond, s, \overline{v}; c, \Omega)\|_{L^2_{\mathbf{w}}} \|\eta^2_{c(t_\diamond)}\|_{L^2_{-\mathbf{w}}} (r-s) \|\partial_x \phi_{c(t_\diamond)}\|_{H^1_{\mathbf{w}}} \\
&\leq \tilde{C}_2 \|Y(t_\diamond, s, \overline{v}; c, \Omega)\|_{L^2_{\mathbf{w}}} (1 + \delta - s),
\end{aligned}$$

for some constant $\tilde{C}_2 > 0$. Integrating this inequality, it follows via (3.41) that there exists a constant $\tilde{C}_3 > 0$ for which

$$\begin{aligned}
\left\| \int_0^r e^{(r-s)\mathcal{L}_{c(t_\diamond)}} P_{c(t_\diamond)} Y(t_\diamond, s, \overline{v}; c, \Omega) \, \mathrm{d}s \right\|_{H^1_{\mathbf{w}}} &\leq \tilde{C}_3 (\epsilon + \epsilon_1)(\delta + \delta^2) \sup_{s \in [0,\delta]} \|\overline{v}(t_\diamond + s)\|_{H^1_{\mathbf{w}}} \\
&\quad + \tilde{C}_3 \epsilon (\delta + \delta^2) \sup_{s \in [0,\delta]} \left| f\big(\gamma(t_\diamond + s)\big) \right|.
\end{aligned}$$

We collect that

$$\begin{aligned}
\|\overline{v}(t_\diamond + s)\|_{H^1_{\mathbf{w}}} \leq M e^{-\beta s} \|\overline{v}(t_\diamond)\|_{H^1_{\mathbf{w}}} &+ \epsilon \big(\tilde{C}_1 \sqrt{\delta} + \tilde{C}_3 \delta + \tilde{C}_3 \delta^2\big) \sup_{s \in [0,\delta]} \left| f\big(\gamma(t_\diamond + s)\big) \right| \\
&+ (\epsilon + \epsilon_1) \big(\tilde{C}_1 \sqrt{\delta} + \tilde{C}_3 \delta + \tilde{C}_3 \delta^2\big) \sup_{s \in [0,\delta]} \|\overline{v}(t_\diamond + s)\|_{H^1_{\mathbf{w}}}
\end{aligned}$$

for $s \in [0, \delta]$. The result follows by taking a supremum over $s \in [0, \delta]$, and choosing $\delta_4$ small enough such that $\sup_{s \in [0,\delta]} \|\overline{v}(t_\diamond + s)\|_{H^1_{\mathbf{w}}}$ can be brought to the left. □

We conclude this section with the proof of Corollary 3.4.2. Essentially, we apply Lemma 3.2.2 to translate Proposition 3.4.1 to the rescaled frame. Recall that C2 implies that $t \to \frac{w_+(t)}{\alpha(t)}$ is decreasing and $t \to \frac{w_-(t)}{\alpha(t)}$ is increasing on $[0, T]$. This guarantees that the weight-functions are also monotone in the original frame.

*Proof of Corollary 3.4.2.* Setting $\overline{\mathbf{b}} = \frac{\mathbf{w}(t+\delta)}{\alpha(t+\delta)}$, we have

$$\sup_{s \in [0,\delta]} \|\overline{v}(t+s)\|_{H^1_{\overline{\mathbf{b}}}} \leq \sup_{s \in [0,\delta]} \|\overline{v}(t+s)\|_{H^1_{\mathbf{w}(t+s)/\alpha(t+s)}} \leq \tilde{C}_1 \sup_{s \in [0,\delta]} \|v(t+s)\|_{H^1_{\mathbf{w}(t+s)}},$$

for some constant $\tilde{C}_1 > 0$, where we have used Lemma 3.2.2 in the last step. If $\delta_5$ is small enough, then we can apply Proposition 3.4.1 to obtain

$$\sup_{s \in [0,\delta]} \|\overline{v}(t+s)\|_{H^1_{\overline{\mathbf{b}}}} \leq C_4 \Big( \|\overline{v}(t)\|_{H^1_{\overline{\mathbf{b}}}} + \epsilon \sup_{s \in [0,\delta]} \left| f\big(\gamma(t+s)\big) \right| (\sqrt{\delta} + \delta^2) \Big).$$

Choosing $C_5$ large enough, we have

$$\sup_{s\in[0,\delta]} \|v(t+s)\|_{H^1_{\alpha(t+s)\overline{\mathbf{b}}}} \leq C_5\Big(\|v(t)\|_{H^1_{\alpha(t)\overline{\mathbf{b}}}} + \epsilon \sup_{s\in[0,\delta]} \big|f\big(\gamma(t+s)\big)\big|(\sqrt{\delta}+\delta^2)\Big) \tag{3.42}$$

via Lemma 3.2.2. It follows that

$$\begin{aligned} \|v(t+\delta)\|_{H^1_{\mathbf{w}(t+\delta)}} &= \|v(t+\delta)\|_{H^1_{\alpha(t+\delta)\overline{\mathbf{b}}}} \\ &\overset{(3.42)}{\leq} C_5\Big(\|v(t)\|_{H^1_{\alpha(t)\overline{\mathbf{b}}}} + \epsilon \sup_{s\in[0,\delta]} \big|f\big(\gamma(t+s)\big)\big|(\sqrt{\delta}+\delta^2)\Big) \end{aligned}$$

and the conclusion follows via (3.33) and Lemma 3.3.2. □

## 3.5 Long-time control

In this section, we establish control of the perturbation $v$ over long time intervals under the assumption that the forcing term is exponentially bounded. As a preparation, we fix the minimum weight $w_{\min}$ in S3 and an intermediate weight $w_\infty \in (w_{\min}, w)$ by writing[2]

$$w_{\min} = we^{-4C_6 E_{\max}(2+\delta_6)e^{1/2}} \quad \text{and} \quad w_\infty = we^{-2C_6 E_{\max}(2+\delta_6)e^{1/2}}, \tag{3.43}$$

using the constants $C_6$ and $\delta_6$ from Lemma 3.4.3. In particular, this choice only depends on S1 and S2. In addition, we introduce functions $W_- : \mathbb{R}^+ \to [w_{\min}, w_\infty)$ and $W_+ : \mathbb{R}^+ \to (w_\infty, w]$ through

$$W_-(t) = w_{\min}(\tfrac{w_\infty}{w_{\min}})^{1-e^{-t}}, \quad W_+(t) = w(\tfrac{w_\infty}{w})^{1-e^{-t}}; \tag{3.44}$$

see Figure 3.1. We then claim that the weight-function $\mathbf{w}(t) = \big(w_-(t), w_+(t)\big)$ defined by

$$w_-(t) = W_-(\gamma t), \quad w_+(t) = W_+(\gamma t) \tag{3.45}$$

satisfies C2, provided that the following conditions are met.

**C3** *We have $p \in [0, \frac{1}{4})$ together with $\epsilon \in (0,1]$ and $\gamma > 0$. In addition, we have $\delta = \epsilon^{-p}$ together with the inequalities*

- $\delta \leq \gamma^{-1}$;
- $\gamma \leq \frac{1}{3}\alpha_{\max}^{-3} w_{\min}(c_0 - w_{\min}^2)$;
- $\epsilon/\gamma \leq E_{\max}$.

[2]In principle, the constant $E_{\max}$ in (3.43) can be replaced by $E = \epsilon/\gamma$. For clarity, however, we work with $E_{\max}$ so that the weights do not depend on $\epsilon$ and $\gamma$.

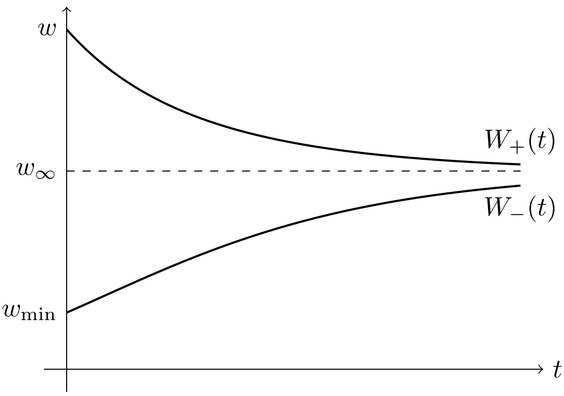



**Figure 3.1:** Graph of the weight-functions $W_-$ and $W_+$ defined in (3.44). We remark that $w_{\min}$ and $w_\infty$ defined in (3.43) approach $w$ upon letting $E_{\max} \downarrow 0$.

**C4** *The forcing term $f$ is continuous and bounded as*

$$|f(t)| \leq e^{-t}, \quad t \geq 0.$$

**C5** *Given $T > 0$, the function $v \in C([0,T], H^2) \cap C^1([0,T], H^{-1})$ satisfies* (3.21) *while $\alpha \in C^1([0,T], \mathbb{R})$ solves* (3.23). *In addition, we have the inclusion*

$$\alpha(t) \in [\alpha_{\min}, \alpha_{\max}], \quad t \in [0,T].$$

**Lemma 3.5.1** (See Section 3.9)**.** *Assume S1, S2 and the choice* (3.43) *for S3. If C3 and C4 are satisfied, and C5 holds for some $T > 0$, then the bound*

$$e^{\gamma t}\|v(t)\|_{L^2_{w_\infty}} \leq \min\{\delta_6 w_\infty^{1/2}, (E_{\max}\gamma)^{1/2}\}, \quad t \in [0,T], \tag{3.46}$$

*implies that the functions $w_-$ and $w_+$ defined in* (3.45) *satisfy C2.*

Our main result here provides exponential control for the perturbation $v$ measured with respect to the time-varying spatial weight-functions $\mathbf{w}(t)$ defined in (3.45). It requires a priori control[3] over the terms in the expression

$$K_{\epsilon,\gamma}(t) = \epsilon + \frac{\epsilon^{1+p}}{\gamma} + \sup_{s\in[0,t]} \Big( e^{\gamma s}\|v(s)\|_{H^1_{\mathbf{w}(s)}} + \|v(s)\|_{H^1} + \gamma^{-1}e^{2\gamma s}\|v(s)\|^2_{H^1_{\mathbf{w}(s)}} \Big), \tag{3.47}$$

which we have introduced for notational convenience. We note in particular that the condition (3.46) is satisfied if $K_{\epsilon,\gamma}(t)$ is sufficiently small.

**Proposition 3.5.2** (Long-time control)**.** *Assuming S1, S2 and the choice* (3.43) *for S3, there exist constants $\delta_8, C_8 > 0$ so that the following holds true. If C3 and*

[3]This control is established in the proof of Theorem 3.1.1.

*C4 are satisfied, and C5 holds for some $T > 0$, then the bound*

$$K_{\epsilon,\gamma}(T) \le \delta_8 \tag{3.48}$$

*implies*

$$\sup_{0\le t\le T} e^{\gamma t}\|v(t)\|_{H^1_{\mathbf{w}(t)}} \le C_8\big(\|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w} + \epsilon^{1-2p}\big).$$

The proof of Proposition 3.5.2 is based on the evolution equation (3.21), in which we isolate the precarious term $\frac{\alpha_t}{\alpha} x\partial_x v$ as

$$v_t = \alpha^{-3}\mathcal{L}_{c_0}v + \frac{\alpha_t}{\alpha}x\partial_x v + M(t, v; \alpha, \Omega) \tag{3.49}$$

with

$$\begin{aligned} M(t, v; \alpha, \Omega) = \alpha^{-3}N(v) &+ \frac{\alpha_t}{\alpha}\big((2 + x\partial_x)\phi_{c_0} + 2v\big) + \frac{\Omega_t}{\alpha}\partial_x(\phi_{c_0} + v) \\ &+ \epsilon f(\gamma t)(\phi_{c_0} + v). \end{aligned}$$

In preparation, we show that $M(t, v; \alpha, \Omega)$ can be controlled in terms of $v$. Additionally, we bound the fluctuations of $c(t)$ over a time-step $\delta$ in terms of $v$ and $\delta$. Recalling that $\delta_6$ is the constant introduced in Lemma 3.4.3, we prove the following.

**Lemma 3.5.3.** *Assuming S1, S2, and the choice* (3.43) *for S3, there exists a constant $C_9 > 0$ so that the following holds true for each $\epsilon, \gamma > 0$. If C5 holds for some $T > 0$, then for each $\mathbf{b} = (b_-, b_+) \in \mathbb{R}^2$ with $b_-, b_+ \in [w_{\min}, \sqrt{c_*}/3]$ and $t \in [0, T]$, the bound $\|v(t)\|_{L^2_{\mathbf{b}}} \le \delta_6 b_+^{1/2}$ implies*

$$\begin{aligned} \|M(t, v(t); \alpha, \Omega)\|_{L^2_{\mathbf{b}}} \le\; & C_9\epsilon|f(\gamma t)|(1 + \|v(t)\|_{H^1_{\mathbf{b}}})\|v(t)\|_{H^1_{\mathbf{b}}} \\ &+ C_9\Big(\|v(t)\|_{H^1_{\mathbf{b}}} + \|v(t)\|^2_{H^1_{\mathbf{b}}} + \|v(t)\|_{H^1}\Big)\|v(t)\|_{H^1_{\mathbf{b}}} \\ &+ C_9\epsilon|f(\gamma t)|. \end{aligned} \tag{3.50}$$

*If, furthermore, $\delta > 0$ satisfies $t + \delta \le T$ and*

$$\sup_{s\in[0,\delta]} \|v(t+s)\|_{L^2_{\mathbf{b}}} \le \delta_6 b_+^{1/2},$$

*then*

$$\sup_{0\le s\le\delta} |c(t+s) - c(t)| \le C_9\delta\Big(\epsilon(1 + \sup_{s\in[0,\delta]}\|v(t+s)\|_{L^2_{\mathbf{b}}}) + \sup_{s\in[0,\delta]}\|v(t+s)\|^2_{L^2_{\mathbf{b}}}\Big). \tag{3.51}$$

*Proof.* Writing $v(t) = v$, the estimate (3.50) follows by computing

$$\|M(t, v; \alpha, \Omega)\|_{L^2_{\mathbf{b}}} \le \alpha_{\min}^{-3}\|N(v)\|_{L^2_{\mathbf{b}}} + 2\left|\frac{\alpha_t}{\alpha}\right| \big\|(1 + \tfrac{1}{2}x\partial_x)\phi_{c_0} + v\big\|_{L^2_{\mathbf{b}}}$$

$$+\left|\frac{\Omega_t}{\alpha}\right| \|\partial_x(\phi_{c_0}+v)\|_{L^2_{\mathbf{b}}} + \epsilon|f(\gamma t)|\|\phi_{c_0}+v\|_{L^2_{\mathbf{b}}},$$

estimating $\|N(v)\|_{L^2_{\mathbf{b}}}$ as in (3.39), and applying the estimate (3.35). To prove (3.51), we estimate

$$\sup_{s\in[0,\delta]} |c(t+s)-c(t)| \le \int_t^{t+\delta} |c_t(t')|\mathrm{d}t',$$

and write

$$\epsilon_1 = \sup_{s\in[0,\delta]} \|v(t+s)\|_{L^2_{\mathbf{b}}}.$$

Using $c_t = -2c_0\alpha^{-3}\alpha_t$ and (3.35), we obtain

$$\begin{aligned}\sup_{s\in[0,\delta]} |c(t+s)-c(t)| &\le 2c_0\alpha_{\min}^{-3}C_6 \int_t^{t+\delta} \Big(\epsilon|f(\gamma t')|(1+b_+^{-1/2}\epsilon_1)+\epsilon_1^2\Big)\mathrm{d}t' \\ &\le 2c_0\alpha_{\min}^{-3}C_6\delta\big(\|f\|_\infty\epsilon(1+w_{\min}^{-1/2}\epsilon_1)+\epsilon_1^2\big).\end{aligned}$$ □

As a final preparation for the proof of Proposition 3.5.2, we establish control of $v(t)$ in the space $H^1_{\mathbf{w}(t+\delta/2)}$. We do so based on the integral equation

$$\begin{aligned}v(t) &= e^{\mathcal{L}_{c_0}\int_0^t \alpha^{-3}(t')\mathrm{d}t'}\overline{v}_0 \\ &\quad + \int_0^t e^{\mathcal{L}_{c_0}\int_s^t \alpha^{-3}(t')\mathrm{d}t'}\Big(\frac{\alpha_t(s)}{\alpha(s)}x\partial_x v(s) + M(s,v(s);\alpha,\Omega)\Big)\mathrm{d}s, \qquad (3.52)\end{aligned}$$

which holds for all $t \ge 0$ and is the mild form of (3.49). Note that the weight is evaluated at time $t+\delta/2$, whereas $v$ is evaluated at $t$. This gap allows us to estimate the $x\partial_x v$ term in (3.52) without introducing a singularity.

**Lemma 3.5.4.** *Assuming S1, S2 and the choice* (3.43) *for S3, there exists a constant* $C_{10} > 0$ *so that the following holds true. If C3 and C4 are satisfied, and C5 holds for some* $T > 0$*, then for each* $t \in [0,T]$ *the bound*

$$\sup_{s\in[0,t]} e^{\gamma s}\|v(s)\|_{H^1_{\mathbf{w}(s)}} \le \min\{\delta_6 w_\infty^{1/2}, 1\}$$

*implies*

$$e^{\gamma t}\|v(t)\|_{H^1_{\mathbf{w}(t+\delta/2)}} \le C_{10}K_{\epsilon,\gamma}(t) \sup_{s\in[0,t]} e^{\gamma s}\|v(s)\|_{H^1_{\mathbf{w}(s)}} + C_{10}\big(\|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w} + \epsilon\big),$$

*where* $K_{\epsilon,\gamma}(t)$ *is defined in* (3.47).

*Proof.* Pick $t \in [0,T]$ and set

$$\epsilon_1 = \sup_{s\in[0,t]} e^{\gamma s}\|v(s)\|_{H^1_{\mathbf{w}(s)}} \quad \text{and} \quad \epsilon_2 = \sup_{s\in[0,t]} \|v(s)\|_{H^1}.$$

Using (3.52), we estimate

$$\|v(t)\|_{H^1_{\mathbf{w}(t+\delta/2)}} \overset{(3.28)}{\leq} Me^{-\beta t}\|\overline{v}_0\|_{H^1_{\mathbf{w}(t+\delta/2)}} + M\alpha_{\min}^{3/2}(I_1+I_2), \tag{3.53}$$

for $\beta = \frac{1}{2}\alpha_{\max}^{-3}w_{\min}(c_0 - w_{\min}^2)$ with

$$I_1 = \int_0^t e^{-\beta(t-s)}(t-s)^{-1/2}\left|\frac{\alpha_t(s)}{\alpha(s)}\right| \|x\partial_x v(s)\|_{L^2_{\mathbf{w}(t+\delta/2)}}\mathrm{d}s; \tag{3.54}$$

$$I_2 = \int_0^t e^{-\beta(t-s)}(t-s)^{-1/2}\|M(s,v(s);\alpha,\Omega)\|_{L^2_{\mathbf{w}(t+\delta/2)}}\mathrm{d}s. \tag{3.55}$$

Here we have used that $\int_s^t \alpha^{-3}(t')\mathrm{d}t' \geq \alpha_{\max}^{-3}(t-s)$. Using Lemma 3.3.2 and Lemma 3.8.2, we estimate

$$\|\overline{v}_0\|_{H^1_{\mathbf{w}(t+\delta/2)}} \leq \|\overline{v}_0\|_{H^1_{\mathbf{w}(0)}} \leq \|\overline{v}_0\|_{H^1_w} + \|\overline{v}_0\|_{H^1} \leq C_2(\|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w})$$

in (3.53). Focusing on $I_1$, we use Lemma 3.3.2 to estimate

$$\|x\partial_x v(s)\|_{L^2_{\mathbf{w}(t+\delta/2)}} \leq e^{-1}q(t,s,\delta)\|v(s)\|_{H^1_{\mathbf{w}(s)}}, \tag{3.56}$$

where

$$q(t,s,\delta) := \tfrac{1}{w_-(t+\delta/2)-w_-(s)} + \tfrac{1}{w_+(s)-w_+(t+\delta/2)}. \tag{3.57}$$

We furthermore use Lemma 3.4.3 to estimate

$$\begin{aligned}\left|\frac{\alpha_t(s)}{\alpha(s)}\right| &\overset{(3.35)}{\leq} \alpha_{\min}^{-1}C_6\Big(\epsilon|f(\gamma s)|(1+w_\infty^{-1/2}\|v(s)\|_{L^2_{\mathbf{w}(s)}}) + \|v(s)\|^2_{L^2_{\mathbf{w}(s)}}\Big)\\ &\leq \alpha_{\min}^{-1}C_6\big(\epsilon e^{-\gamma s}(1+w_\infty^{-1/2}\epsilon_1) + \epsilon_1^2 e^{-2\gamma s}\big)\\ &\leq \tilde{C}_1(\epsilon+\epsilon_1^2)e^{-\gamma s}\end{aligned} \tag{3.58}$$

for some constant $\tilde{C}_1 > 0$. Substituting (3.56) and (3.58) into (3.54) yields

$$e^{\gamma t}I_1 \leq \tilde{C}_1 e^{-1}(\epsilon+\epsilon_1^2)\int_0^t e^{-\gamma s}q(t,s,\delta)e^{-(\beta-\gamma)(t-s)}(t-s)^{-1/2}\mathrm{d}s \sup_{s\in[0,t]} e^{\gamma s}\|v(s)\|_{H^1_{\mathbf{w}(s)}},$$

where we have used

$$e^{\gamma t}e^{-\beta(t-s)} = e^{-(\beta-\gamma)(t-s)}e^{\gamma s}. \tag{3.59}$$

Invoking Lemma 3.9.1 to estimate

$$\sup_{s\in[0,t]} e^{-\gamma s}|q(t,s,\delta)| \leq \frac{\tilde{C}_2}{\gamma\delta}$$

for some $\tilde{C}_2 > 0$, we find

$$e^{\gamma t} I_1 \leq \tilde{C}_3 (\gamma\delta)^{-1}(\epsilon + \epsilon_1^2) \sup_{s\in[0,t]} e^{\gamma s} \|v(s)\|_{H^1_{\mathbf{w}(s)}}$$

where

$$\tilde{C}_3 = \tilde{C}_1 \tilde{C}_2 e^{-1} \int_0^\infty e^{-(\beta-\gamma)s} s^{-1/2} \mathrm{d}s < \infty.$$

Recalling that $\delta = \epsilon^{-p}$ per C3, we have thus shown that

$$e^{\gamma t} I_1 \leq \tilde{C}_3 K_{\epsilon,\gamma}(t) \sup_{s\in[0,t]} e^{\gamma s} \|v(s)\|_{H^1_{\mathbf{w}(s)}}.$$

Turning to $I_2$, we find via (3.59) and Lemma 3.3.2 that

$$e^{\gamma t} I_2 \leq \int_0^t e^{-(\beta-\gamma)(t-s)} (t-s)^{-1/2} e^{\gamma s} \|M(s, v(s); \alpha, \Omega)\|_{L^2_{\mathbf{w}(s)}} \mathrm{d}s.$$

Applying Lemma 3.5.3 yields

$$\begin{aligned} \|M(s, v(s); \alpha, \Omega)\|_{L^2_{\mathbf{w}(s)}} &\leq C_9 \big(\epsilon(1+\epsilon_1) + \epsilon_1 + \epsilon_2 + \epsilon_1^2\big) \|v(s)\|_{H^1_{\mathbf{w}(s)}} \\ &\quad + C_9 \epsilon e^{-\gamma s} \end{aligned}$$

so that

$$e^{\gamma t} I_2 \leq \tilde{C}_4 (\epsilon + \epsilon_1 + \epsilon_2) \sup_{s\in[0,t]} e^{\gamma s} \|v(s)\|_{H^1_{\mathbf{w}(s)}} + \tilde{C}_4 \epsilon$$

for some $\tilde{C}_4 > 0$. The conclusion follows by setting $C_{10} = M \alpha_{\min}^{3/2} (\tilde{C}_3 + \tilde{C}_4)$. □

We finally move on to the proof of Proposition 3.5.2. We first control the fluctuations of $c(t)$ during the time-step $\delta$ through Lemma 3.5.3. This allows us to use the short-time result Corollary 3.4.2, after which we apply the long-time result Lemma 3.5.4.

*Proof of Proposition 3.5.2.* We collect from (3.48) that

$$\Big(\epsilon + \sup_{s\in[0,\delta]} \big(\|v(t+s)\|_{H^1_{\mathbf{w}(t+s)}} + \|v(t+s)\|_{H^1}\big)\Big) \delta^2 \leq \delta_8^{1-2p}.$$

Furthermore, Lemma 3.5.3 applied with weight $w_\infty$ provides the bound

$$\begin{aligned} \sup_{s\in[0,\delta]} |c(t+s) - c(t)| \delta^2 &\leq C_9 \delta^3 \Big(\epsilon(1 + \sup_{s\in[0,\delta]} \|v(t+s)\|_{L^2_{w_\infty}}) + \sup_{s\in[0,\delta]} \|v(t+s)\|^2_{L^2_{w_\infty}}\Big) \\ &\leq C_9 \big(\delta_8^{1-3p}(1+\delta_8) + \delta_8^{2-3p}\big). \end{aligned}$$

Picking $\delta_8$ small enough, we may apply Corollary 3.4.2 at time $t$, which yields

$$e^{\gamma(t+\delta)} \sup_{s\in[0,\delta]} \|v(t+s)\|_{H^1_{\mathbf{w}(t+s)}} \leq C_5\big(e^{\gamma(t+\delta)}\|v(t)\|_{H^1_{\mathbf{w}(t+\delta/2)}} + 2\epsilon e^{\gamma(t+\delta)}e^{-\gamma t}\delta^2\big).$$

Applying Lemma 3.5.4, we hence find

$$\begin{aligned} e^{\gamma(t+\delta)} \sup_{s\in[0,\delta]} \|v(t+s)\|_{H^1_{\mathbf{w}(t+s)}} \leq{}& \tilde{C}_1 K_{\epsilon,\gamma}(t) \sup_{s\in[0,t]} e^{\gamma s}\|v(s)\|_{H^1_{\mathbf{w}(s)}} \\ &+ \tilde{C}_1\big(\|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w} + \epsilon + \epsilon\delta^2\big), \end{aligned}$$

where $\tilde{C}_1 = C_5(C_{10}+2)e$. Taking the supremum over $t \in [0, T-\delta]$ and extending the supremum on the right-hand side shows that

$$\begin{aligned} \sup_{0\leq t\leq T} e^{\gamma t}\|v(t)\|_{H^1_{\mathbf{w}(t)}} \leq{}& \tilde{C}_1 K_{\epsilon,\gamma}(T) \sup_{0\leq t\leq T} e^{\gamma t}\|v(t)\|_{H^1_{\mathbf{w}(t)}} \\ &+ \tilde{C}_1\big(\|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w} + \epsilon + \epsilon\delta^2\big). \end{aligned}$$

For $\delta_8$ small enough, we have $\tilde{C}_1 K_{\epsilon,\gamma}(T) < 1/2$ and hence

$$\sup_{0\leq t\leq T} e^{\gamma t}\|v(t)\|_{H^1_{\mathbf{w}(t)}} \leq 2\tilde{C}_1\big(\|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w} + \epsilon + \epsilon^{1-2p}\big). \qquad \square$$

3

## 3.6 Energy evolution

In this section, we establish control of the perturbation in the unweighted spaces $L^2$ and $H^1$. Since the latter norm appears explicitly in (3.47), this is a crucial step towards applying the long-time results developed in Section 3.5.

Using the decomposition (3.18) and the fact that $\langle \overline{v}(t), \phi_{c(t)}\rangle = 0$, we observe via Pythagoras' theorem that

$$\|\overline{v}(t)\|_{L^2}^2 = \|u(t,\cdot+\xi(t))\|_{L^2}^2 - \|\phi_{c(t)}\|_{L^2}^2 = \|u(t)\|_{L^2}^2 - \|\phi_{c(t)}\|_{L^2}^2. \tag{3.60}$$

We set out to control the $L^2$-norm of the perturbation $\overline{v}$ by estimating the time-derivative of (3.60). The key point is that, in the right-hand side of (3.61) below, the term $\|\overline{v}(t)\|_{L^2}^2$ comes with an integrable factor $\epsilon|f(\gamma t)|$, and the remaining terms are all integrable and small. Invoking Grönwall's inequality, this can subsequently be used to establish the desired control. We recall that $\delta_6$ is the constant introduced in Lemma 3.4.3.

**Lemma 3.6.1.** *Assuming S1, S2 and the choice* (3.43) *for S3, there exists a constant $C_{11} > 0$ so that the following holds true for each $\epsilon, \gamma > 0$. If C5 holds for some $T > 0$ and $t \in [0, T]$, then the bound*

$$\|v(t)\|_{L^2_{w\infty}} \leq \delta_6 a_\infty^{1/2},$$

*implies*

$$|\partial_t \|\overline{v}(t)\|_{L^2}^2| \le C_{11}\Big(\epsilon|f(\gamma t)|\|\overline{v}(t)\|_{L^2}^2 + \epsilon|f(\gamma t)|\|v(t)\|_{L^2_{w\infty}} + \|v(t)\|_{L^2_{w\infty}}^2\Big). \quad (3.61)$$

*Proof.* Let us write $u = u(t)$, $v = v(t)$, and $\overline{v} = \overline{v}(t)$ for brevity. In the presence of forcing via $\epsilon f(\gamma t)$, we observe that

$$\begin{aligned}\partial_t \|u\|_{L^2}^2 &\overset{(3.14)}{=} -2\langle u, \partial_x^3 u + 2u\partial_x u\rangle + 2\epsilon f(\gamma t)\langle u, u\rangle \\ &= 2\epsilon f(\gamma t)\|u\|_{L^2}^2 \overset{(3.18)}{=} 2\epsilon f(\gamma t)\big(\|\phi_{c(t)}\|_{L^2}^2 + \|\overline{v}\|_{L^2}^2\big) \\ &= 12\epsilon f(\gamma t) c^{3/2}(t) + 2\epsilon f(\gamma t)\|\overline{v}\|_{L^2}^2, \end{aligned} \quad (3.62)$$

where we have used that $\|\phi_{c(t)}\|_{L^2}^2 = 6c^{3/2}(t)$ in the last step. On the other hand, we may compute

$$\partial_t \|\phi_{c(t)}\|_{L^2}^2 = 6\partial_t c^{3/2}(t) = 9c_t(t)c^{1/2}(t) = -2c_0^{3/2}\alpha^{-4}(t)\alpha_t(t). \quad (3.63)$$

Combining (3.62) and (3.63), and estimating $|\alpha_t(t) + \frac{2}{3}\alpha(t)\epsilon f(\gamma t)|$ via (3.34) leads to the estimate

$$\begin{aligned}|\partial_t \|\overline{v}\|_{L^2}^2| &= |\partial_t \|u\|_{L^2}^2 - \partial_t \|\phi_{c(t)}\|_{L^2}^2| \\ &\le \tilde{C}_1\Big(\epsilon|f(\gamma t)|\|\overline{v}\|_{L^2}^2 + \epsilon|f(\gamma t)|\|v\|_{L^2_{w\infty}} + \|v\|_{L^2_{w\infty}}^2\Big),\end{aligned}$$

for some $\tilde{C}_1 > 0$. □

**Lemma 3.6.2.** *Assuming S1, S2 and the choice* (3.43) *for S3, there exists a constant* $C_{12} > 0$ *so that the following holds true for each* $\epsilon, \gamma > 0$. *If C4 is satisfied, C5 holds for some* $T > 0$, *and* $\epsilon/\gamma \le E_{\max}$, *then the bound*

$$e^{\gamma t}\|v(t)\|_{L^2_{w\infty}} \le \delta_6 a_\infty^{1/2}, \quad 0 \le t \le T,$$

*implies*

$$\sup_{0\le t\le T} \|\overline{v}(t)\|_{L^2}^2 \le \|\overline{v}_0\|_{L^2}^2 + C_{12}\Big(\frac{\epsilon}{\gamma}\sup_{0\le t\le T} e^{\gamma t}\|v(t)\|_{H^1_{w\infty}} + \gamma^{-1}\sup_{0\le t\le T} e^{2\gamma t}\|v(t)\|_{H^1_{w\infty}}^2\Big).$$

*Proof.* Writing

$$\epsilon_1 = \sup_{0\le t\le T} e^{\gamma t}\|v\|_{L^2_{w\infty}},$$

we have via Lemma 3.6.1

$$|\partial_t \|\overline{v}\|_{L^2}^2| \le C_{11}\big(\epsilon e^{-\gamma t}\|\overline{v}\|_{L^2}^2 + (\epsilon\epsilon_1 + \epsilon_1^2)e^{-2\gamma t}\big),$$

and applying Grönwall's inequality yields

$$\|\overline{v}\|_{L^2}^2 \leq \Big(\|\overline{v}_0\|_{L^2}^2 + \frac{C_{11}}{2\gamma}(\epsilon\epsilon_1 + \epsilon_1^2)\Big)e^{C_{11}\epsilon/\gamma}. \qquad \square$$

Control in $H^1$ is established using the well-known fact that the soliton $\phi_c$ is a critical point of the functional $\mathcal{E}_c[u] = \frac{1}{2}c\|u\|_{L^2}^2 + \mathcal{H}[u]$. This is reflected in the equality

$$\mathcal{E}_c[\phi_c + z] - \mathcal{E}_c[\phi_c] = \tfrac{1}{2}c\|z\|_{L^2}^2 + \tfrac{1}{2}\|\partial_x z\|_{L^2}^2 + \int_{\mathbb{R}} -\phi_c z^2 + \tfrac{1}{3}z^3 \mathrm{d}x,$$

which is $\mathcal{O}(z^2)$. Substituting our decomposition (3.18) into

$$\mathcal{J}(t) = \mathcal{E}_{c(t)}[u(t)] - \mathcal{E}_{c(t)}[\phi_{c(t)}],$$

we arrive at

$$\begin{aligned}\mathcal{J}(t) &= \mathcal{E}_{c(t)}[\phi_{c(t)} + \overline{v}(t)] - \mathcal{E}_{c(t)}[\phi_{c(t)}] \\ &= \tfrac{1}{2}c(t)\|\overline{v}(t)\|_{L^2}^2 + \tfrac{1}{2}\|\partial_x\overline{v}(t)\|_{L^2}^2 + \int_{\mathbb{R}} -\phi_{c(t)}\overline{v}^2(t) + \tfrac{1}{3}\overline{v}^3(t)\mathrm{d}x, \end{aligned} \tag{3.64}$$

which generalizes (3.60) in the sense that the right-hand side is $\mathcal{O}(\overline{v}^2)$. In fact, we note that $\|\overline{v}(t)\|_{H^1}^2$ can be bounded in terms of $|\mathcal{J}(t)|$ and vice versa.

**Lemma 3.6.3.** *Assuming S1, S2 and the choice* (3.43) *for S3, there exists a constant $C_{13} > 0$ so that the following holds true for each $\epsilon, \gamma > 0$. If C5 holds for some $T > 0$ and $t \in [0, T]$, then*

$$\|\overline{v}(t)\|_{H^1}^2 \leq C_{13}\Big(|\mathcal{J}(t)| + \|v(t)\|_{L^2_{w_\infty}}^2 + \|\overline{v}(t)\|_{H^1}\|\overline{v}(t)\|_{L^2}^2\Big), \tag{3.65}$$

*and*

$$|\mathcal{J}(t)| \leq C_{13}\Big(\|\overline{v}(t)\|_{H^1}^2 + \|v(t)\|_{L^2_{w_\infty}}^2 + \|\overline{v}(t)\|_{H^1}\|\overline{v}(t)\|_{L^2}^2\Big). \tag{3.66}$$

*Proof.* Estimating

$$\left|\int_{\mathbb{R}} \phi_{c(t)}\overline{v}^2(t)\mathrm{d}x\right| = \left|\alpha^{-5}(t)\int_{\mathbb{R}} \phi_{c_0}v^2(t)\mathrm{d}x\right| \leq \alpha_{\min}^{-5}\|e^{-2w_\infty x}\phi_{c_0}\|_{L^\infty}\|v(t)\|_{L^2_{w_\infty}}^2 \tag{3.67}$$

in (3.64) yields

$$\begin{aligned}|\mathcal{J}(t)| \leq{}& \tfrac{1}{2}\max\{c_0\alpha_{\min}^{-2}, 1\}\|\overline{v}(t)\|_{H^1}^2 + \alpha_{\min}^{-5}\|e^{-2w_\infty x}\phi_{c_0}\|_{L^\infty}\|v(t)\|_{L^2_{w_\infty}}^2 \\ &+ \tfrac{\sqrt{2}}{3}\|\overline{v}(t)\|_{H^1}\|\overline{v}(t)\|_{L^2}^2,\end{aligned}$$

which yields (3.66). On the other hand, rewriting (3.64) as

$$\tfrac{1}{2}c(t)\|\overline{v}(t)\|_{L^2}^2 + \tfrac{1}{2}\|\partial_x\overline{v}(t)\|_{L^2}^2 = -\mathcal{J}(t) + \int_{\mathbb{R}} \phi_{c(t)}\overline{v}^2(t) - \tfrac{1}{3}\overline{v}^3(t)\mathrm{d}x,$$

and applying (3.67) provides

$$k\|\overline{v}(t)\|_{H^1}^2 \leq |\mathcal{J}(t)| + \alpha_{\min}^{-5}\|e^{-2w_\infty x}\phi_{c_0}\|_{L^\infty}\|v(t)\|_{L^2_{w_\infty}}^2 + \tfrac{\sqrt{2}}{3}\|\overline{v}(t)\|_{H^1}\|\overline{v}(t)\|_{L^2}^2,$$

in case $k \leq \min\{\frac{1}{2}, \frac{1}{2}c_0\alpha_{\min}^{-2}\}$, which establishes (3.65). □

We now establish an estimate of $\partial_t\mathcal{J}(t)$, which we use to control $\|\overline{v}\|_{H^1}$ via a Grönwall argument, analogous to the proof of Lemma 3.6.2. Note also in (3.68) below, that the term $\|\overline{v}(t)\|_{H^1}^2$ carries a factor $\epsilon|f(\gamma t)|$.

**Lemma 3.6.4.** *Assuming S1, S2 and the choice* (3.43) *for S3, there exists a constant $C_{14} > 0$ so that the following holds true for each $\epsilon, \gamma > 0$. If C5 holds for some $T > 0$ and $t \in [0, T]$, then the bounds*

$$\|v(t)\|_{L^2_{w_\infty}} \leq \min\{1, \delta_6 a_\infty^{1/2}\} \quad \textit{and} \quad \|\overline{v}(t)\|_{L^2} \leq 1,$$

*imply*

$$|\partial_t\mathcal{J}(t)| \leq C_{14}\Big(\epsilon|f(\gamma t)|\|\overline{v}(t)\|_{H^1}^2 + \epsilon|f(\gamma t)|\|v(t)\|_{L^2_{w_\infty}} + C_{14}\|v(t)\|_{L^2_{w_\infty}}^2\Big). \tag{3.68}$$

*Proof.* We once more abbreviate $u = u(t)$, $v = v(t)$, and $\overline{v} = \overline{v}(t)$, and compute that

$$\begin{aligned}\partial_t\mathcal{H}[u] &= \partial_t\int_{\mathbb{R}} \tfrac{1}{2}(u_x)^2 - \tfrac{1}{3}u^3\mathrm{d}x = \int_{\mathbb{R}} u_x u_{xt} - u^2u_t\mathrm{d}x = -\int_{\mathbb{R}}(u_{xx} + u^2)u_t\mathrm{d}x \\ &\overset{(3.14)}{=} \langle u_{xx} + u^2, u_{xxx} + (u^2)_x - \epsilon f(\gamma t)u\rangle = \epsilon f(\gamma t)\|\partial_x u\|^2 - \epsilon f(\gamma t)\int_{\mathbb{R}} u^3\mathrm{d}x.\end{aligned}$$

Substituting (3.18), we have

$$\begin{aligned}\partial_t\mathcal{H}[u] = \epsilon f(\gamma t)\|\partial_x u\|_{L^2}^2 - \epsilon f(\gamma t)\int u^3\mathrm{d}x &= \epsilon f(\gamma t)\big(\|\partial_x\phi_{c(t)}\|_{L^2}^2 - \int_{\mathbb{R}}\phi_{c(t)}^3\mathrm{d}x\big) \\ &+ \epsilon f(\gamma t)\big(2\langle\partial_x\phi_{c(t)}, \partial_x\overline{v}\rangle - 3\langle\phi_{c(t)}^2, \overline{v}\rangle\big) \\ &+ \epsilon f(\gamma t)\big(\|\partial_x\overline{v}\|_{L^2}^2 - \int_{\mathbb{R}} 3\phi_{c(t)}\overline{v}^2 + \overline{v}^3\mathrm{d}x\big),\end{aligned}$$

where the leading-order term evaluates to

$$\|\partial_x\phi_{c(t)}\|_{L^2}^2 - \int_{\mathbb{R}}\phi_{c(t)}^3\mathrm{d}x = -6c^{5/2}(t).$$

On the other hand,

$$\partial_t \mathcal{H}[\phi_{c(t)}] = -\tfrac{9}{5}\partial_t c^{5/2}(t) = -\tfrac{9}{2}c_t(t)c^{3/2}(t) = 9c_0^{5/2}\alpha^{-6}(t)\alpha_t(t)$$

and we note via (3.23) that $\partial_t \mathcal{H}[u] = \partial_t \mathcal{H}[\phi_{c(t)}]$ in case $v = 0$. Moreover, estimating

$$|\alpha_t(t) + \tfrac{2}{3}\alpha(t)\epsilon f(\gamma t)| \leq \tilde{C}_1\big(\epsilon|f(\gamma t)|\|v\|_{L^2_{w_\infty}} + \|v\|^2_{L^2_{w_\infty}}\big)$$

for some $\tilde{C}_1 > 0$ leads to

$$\begin{aligned}|\partial_t \mathcal{H}[u] - \partial_t \mathcal{H}[\phi_{c(t)}]| \leq \epsilon|f(\gamma t)|\big(2\langle \partial_x \phi_{c(t)}, \partial_x \overline{v}\rangle - 3\langle \phi^2_{c(t)}, \overline{v}\rangle\big] \\ + \epsilon|f(\gamma t)|\big(\|\partial_x \overline{v}\|^2_{L^2} - \int_{\mathbb{R}} 3\phi_{c(t)}\overline{v}^2 + \overline{v}^3 \mathrm{d}x\big) \\ + \tilde{C}_1\big(\epsilon|f(\gamma t)|\|v\|_{L^2_{w_\infty}} + \|v\|^2_{L^2_{w_\infty}}\big).\end{aligned}$$

Rescaling the terms that contain the soliton and introducing an exponential weight yields

$$\begin{aligned}|2\langle \partial_x \phi_{c(t)}, \partial_x \overline{v}\rangle - 3\langle \phi^2_{c(t)}, \overline{v}\rangle| &\leq \tilde{C}_2|2\langle \partial_x \phi_{c_0}, \partial_x v\rangle - 3\langle \phi^2_{c_0}, v\rangle| \\ &\leq \tilde{C}_2\|\partial_x^2 \phi_{c_0} + \phi^2_{c_0}\|_{L^2_{-w_\infty}}\|v\|_{L^2_{w_\infty}}\end{aligned}$$

and

$$\left|\int_{\mathbb{R}} 3\phi_{c(t)}\overline{v}^2 \mathrm{d}x\right| \leq \tilde{C}_2\left|\int_{\mathbb{R}} 3\phi_{c_0} v^2 \mathrm{d}x\right| \leq \tilde{C}_2\|e^{-2w_\infty x}\phi_{c_0}\|_{L^\infty}\|v\|^2_{L^2_{w_\infty}}$$

for some constant $\tilde{C}_2 > 0$. The cubic term is controlled by estimating

$$\left|\int_{\mathbb{R}} \overline{v}^3 \mathrm{d}x\right| \leq \|\overline{v}\|_{L^\infty}\|\overline{v}\|^2_{L^2} \leq \sqrt{2}\|\overline{v}\|^2_{H^1},$$

where we have used $\|\overline{v}\|_{L^2} \leq 1$. Collecting our results so far yields

$$\begin{aligned}\big|\partial_t \mathcal{H}[u] - \partial_t \mathcal{H}[\phi_{c(t)}]\big| \leq (1+\sqrt{2})\epsilon|f(\gamma t)|\|\overline{v}\|^2_{H^1} + \tilde{C}_3\epsilon|f(\gamma t)|\big(\|v\|_{L^2_{w_\infty}} + \|v\|^2_{L^2_{w_\infty}}\big) \\ + \tilde{C}_3\|v\|^2_{L^2_{w_\infty}}\end{aligned}$$

for some $\tilde{C}_3 > 0$. Computing

$$\partial_t \mathcal{J}(t) = c_t(t)\|\overline{v}\|^2_{L^2} + c(t)\partial_t\|\overline{v}\|^2_{L^2} + \partial_t \mathcal{H}[u] - \partial_t \mathcal{H}[\phi_{c(t)}],$$

applying Lemma 3.6.1, and estimating $|c_t|$ via Lemma 3.4.3 then gives the result. □

We are now ready to state and prove the main result of this section, which establishes control over the $H^1$-norm of the perturbation.

**Proposition 3.6.5.** *Assuming S1, S2 and the choice* (3.43) *for S3, there exists a constant* $C_{15} > 0$ *so that the following holds true for each* $\epsilon, \gamma > 0$. *If C4 is satisfied, C5 holds for some* $T > 0$, *and* $\epsilon/\gamma \leq E_{\max}$, *then the bounds*

$$\sup_{0 \leq t \leq T} e^{\gamma t} \|v(t)\|_{L^2_{w_\infty}} \leq \min\{\delta_6 w_\infty^{1/2}, 1\},$$
$$\sup_{0 \leq t \leq T} \|\overline{v}(t)\|_{H^1} \leq 1,$$

*imply*

$$\sup_{0 \leq t \leq T} \|v(t)\|^2_{H^1} \leq C_{15}(\|\overline{v}_*\|^2_{H^1} + \|\overline{v}_*\|^2_{H^1_w}) + C_{15}\Big(\frac{\epsilon}{\gamma} \sup_{0 \leq t \leq T} e^{\gamma t}\|v(t)\|_{H^1_{w_\infty}} + \gamma^{-1} \sup_{0 \leq t \leq T} e^{2\gamma t}\|v(t)\|^2_{H^1_{w_\infty}}\Big).$$

*Proof.* We again write

$$\epsilon_1 = \sup_{0 \leq t \leq T} e^{\gamma t}\|v\|_{L^2_{w_\infty}} \quad \text{and} \quad \epsilon_2 = \sup_{0 \leq t \leq T} \|\overline{v}\|^2_{L^2}.$$

Applying (3.65), we obtain

$$\begin{aligned}\|\overline{v}\|^2_{H^1} &\leq C_{13}|\mathcal{J}(t)| + C_{13}\|v\|^2_{L^2_{w_\infty}} + C_{13}\|\overline{v}\|_{H^1}\|\overline{v}\|^2_{L^2} \\ &\leq C_{13}|\mathcal{J}(0)| + C_{13}\int_0^t |\partial_t \mathcal{J}(s)|\mathrm{d}s + C_{13}(\epsilon_1^2 + \epsilon_2),\end{aligned}$$

which, using Lemma 3.6.4 leads to the bound

$$\begin{aligned}\|\overline{v}\|^2_{H^1} \leq{}& C_{13}|\mathcal{J}(0)| + C_{13}C_{14}\int_0^t \epsilon e^{-\gamma s}\|\overline{v}(s)\|^2_{H^1}\mathrm{d}s + C_{13}(\epsilon_1^2 + \epsilon_2) \\ &+ C_{13}C_{14}\big(\frac{\epsilon}{\gamma}(\epsilon_1 + \epsilon_2) + \frac{\epsilon_1^2}{\gamma}\big).\end{aligned}$$

Applying Grönwall's inequality finally yields

$$\|\overline{v}\|^2_{H^1} \leq \Big(C_{13}|\mathcal{J}(0)| + C_{13}(\epsilon_1^2 + \epsilon_2) + C_{13}C_{14}\big(\frac{\epsilon}{\gamma}(\epsilon_1 + \epsilon_2) + \frac{\epsilon_1^2}{\gamma}\big)\Big)e^{C_{13}C_{14}\frac{\epsilon}{\gamma}}.$$

The result now follows by controlling $\epsilon_2$ via Lemma 3.6.2, controlling $|\mathcal{J}(0)|$ via (3.66) and applying item 1 of Lemma 3.2.2. □

## 3.7 Proof of main result

Building upon the results of the previous sections, we set out to prove Theorem 3.1.1. Assuming S1, we fix the constants appearing in S2 by writing

$$\alpha_{\min} = \tfrac{1}{2} \inf_{t \geq 0} e^{\frac{2}{3} E_{\max} \int_0^t f(s) \mathrm{d}s} \quad \text{and} \quad \alpha_{\max} = 2 \sup_{t \geq 0} e^{\frac{2}{3} E_{\max} \int_0^t f(s) \mathrm{d}s}. \tag{3.69}$$

Recall that the weight $w_{\min}$ of S3 and the asymptotic weight $w_\infty$ are then determined through (3.43), in which $\delta_6$ and $C_6$ are the constants introduced in Lemma 3.4.3, which depend only on S1 and S2. As a final preparation, we estimate the deviation of the modulation parameters from their leading-order approximations. Recall that $\xi(t) - \xi_0 - \int_0^t c(s)\ \mathrm{d}s = \Omega(t)$.

**Lemma 3.7.1.** *Assume S1 and the choices* (3.69) *and* (3.43) *for S2 and S3. If C4 is satisfied and C5 holds for some $T > 0$, then for each $\epsilon, \gamma > 0$, the bound*

$$e^{\gamma t} \|v(t)\|_{L^2_{w_\infty}} \leq \delta_6 w_\infty^{1/2}, \quad 0 \leq t \leq T$$

*implies*

$$\begin{aligned} \sup_{t \in [0,T]} \left| \log(\alpha(t)) + \tfrac{2}{3}\epsilon \int_0^t f(\gamma s) \mathrm{d}s \right| + \sup_{t \in [0,T]} \left| \Omega(t) - \tfrac{2}{3}\epsilon \int_0^t \frac{f(\gamma s)}{c^{1/2}(s)} \mathrm{d}s \right| \\ \leq C_6 w_\infty^{-1/2} \frac{\epsilon}{\gamma} \sup_{t \in [0,T]} e^{\gamma t} \|v(t)\|_{L^2_{w_\infty}} + \frac{C_6}{\gamma} \sup_{t \in [0,T]} e^{2\gamma t} \|v(t)\|^2_{L^2_{w_\infty}}. \end{aligned}$$

*Proof.* Estimating (3.23) for $t \in [0,T]$ as in (3.34) yields

$$\begin{aligned} \left| \partial_t \log(\alpha(t)) + \tfrac{2}{3}\epsilon f(\gamma t) \right| + \left| \partial_t \Omega(t) - \tfrac{2}{3}\epsilon \frac{f(\gamma t)}{c^{1/2}(t)} \right| \leq C_6 w_\infty^{-1/2} \epsilon e^{-\gamma t} \|v(t)\|_{L^2_{w_\infty}} \\ + C_6 \|v(t)\|^2_{L^2_{w_\infty}}. \end{aligned}$$

The result now follows by integrating. □

For each $\epsilon, E > 0$ and $\overline{v}_* \in H^2 \cap H^1_w$, we now introduce the sets

$$\begin{aligned} O_1 &= \big\{ T \geq 0 : e^{\gamma t} \|v(t)\|_{H^1_{\mathbf{w}(t)}} \leq C_8 (\|\overline{v}_*\|_{H^1_w} + \|\overline{v}_*\|_{H^1} + \epsilon^{1-2p}) \text{ for all } 0 \leq t \leq T \big\}, \\ O_2 &= \big\{ T \geq 0 : \|v(t)\|^2_{H^1} \leq \alpha^5_{\min} \text{ for all } 0 \leq t \leq T \big\}, \end{aligned}$$

where $C_8$ is the constant from Proposition 3.5.2 and we recall that the weight-function $\mathbf{w}(t) = \big(w_-(t), w_+(t)\big)$ is defined in (3.45). We then define $T_{\max}(\epsilon, \gamma, \overline{v}_*) \in [0, \infty]$ as

$$T_{\max}(\epsilon, \gamma, \overline{v}_*) = \sup(O_1 \cap O_2).$$

Recalling that $E = \epsilon/\gamma$, the key ingredient toward establishing Theorem 3.1.1 is to show that $T_{\max}(\epsilon, \gamma, \overline{v}_*) = \infty$.

*Proof of Theorem 3.1.1.* Pick $\epsilon \in (0,1]$ and $\gamma > 0$ that satisfies $\epsilon/\gamma \in (0, E_{\max}]$ and

$$\epsilon^{1+p} + \epsilon^{2-4p} \leq \delta_1 \gamma, \quad \epsilon^p \geq \gamma, \quad \gamma \leq \delta_1,$$

and consider an initial condition $\overline{v}_* \in H^2 \cap H^1_w$ that satisfies $\|\overline{v}_*\|^2_{H^1} + \|\overline{v}_*\|^2_{H^1_w} \leq \delta_1 \gamma$. Then, the conditions in Theorem 3.1.1 are met, and we note that

$$\epsilon \leq \delta_1 E_{\max} \quad \text{and} \quad \|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w} \leq \sqrt{2}\delta_1.$$

Suppose, for the sake of contradiction, that $T_{\max} < \infty$. By construction of $O_1$, we have

$$\sup_{t \in [0, T_{\max})} e^{\gamma t} \|v(t)\|_{H^1_{\mathbf{w}(t)}} \leq C_8 \big( \|\overline{v}_*\|_{H^1_w} + \|\overline{v}_*\|_{H^1} + \epsilon^{1-2p} \big) \leq C_8 (\sqrt{2} + 1)\delta_1,$$

and

$$\sup_{t \in [0, T_{\max})} \gamma^{-1} e^{2\gamma t} \|v(t)\|^2_{H^1_{\mathbf{w}(t)}} \leq 3C_8^2 \gamma^{-1} \big( \|\overline{v}_*\|^2_{H^1_w} + \|\overline{v}_*\|^2_{H^1} + \epsilon^{2-4p} \big) \leq 6C_8^2 \delta_1.$$

In particular,

$$\sup_{t \in [0, T_{\max})} e^{\gamma t} \|v(t)\|_{H^1_{\mathbf{w}(t)}} \leq \min\{\delta_6 w_\infty^{1/2}, 1\}$$

for $\delta_1$ sufficiently small. Thus, we may use Lemma 3.7.1 to obtain

$$\begin{aligned}
\sup_{t \in [0, T_{\max})} \left| \log(\alpha(t)) + \tfrac{2}{3}\epsilon \int_0^t f(\gamma s) \mathrm{d}s \right| &\leq C_6 C_8 w_\infty^{-1/2} \frac{\epsilon}{\gamma} \big( \|\overline{v}_*\|_{H^1_w} + \|\overline{v}_*\|_{H^1} + \epsilon^{1-2p} \big) \\
&\quad + C_6 C_8^2 \gamma^{-1} \big( \|\overline{v}_*\|_{H^1_w} + \|\overline{v}_*\|_{H^1} + \epsilon^{1-2p} \big)^2 \\
&\leq C_6 C_8 w_\infty^{-1/2} (\sqrt{2} E_{\max} \delta_1 + \delta_1) + 6C_6 C_8^2 \delta_1
\end{aligned}$$

and in particular

$$\alpha_{\min} < \alpha(t) < \alpha_{\max}, \quad t \in [0, T_{\max}],$$

for $\delta_1$ sufficiently small. By construction of $O_2$ and Lemma 3.2.2, we have

$$\sup_{t \in [0, T_{\max})} \|\overline{v}(t)\|^2_{H^1} \leq 1.$$

Via Proposition 3.6.5, we may improve this bound to

$$\begin{aligned}
\sup_{t \in [0, T_{\max})} \|v(t)\|^2_{H^1} &\leq C_{15} (\|\overline{v}_*\|^2_{H^1} + \|\overline{v}_*\|^2_{H^1_{w_\infty}}) + C_{15} C_8 \frac{\epsilon}{\gamma} \big( \|\overline{v}_*\|_{H^1_w} + \|\overline{v}_*\|_{H^1} + \epsilon^{1-2p} \big) \\
&\quad + C_{15} C_8^2 \gamma^{-1} \big( \|\overline{v}_*\|_{H^1_w} + \|\overline{v}_*\|_{H^1} + \epsilon^{1-2p} \big)^2 \\
&\leq C_{15} \delta_1^2 + C_{15} C_8 (\sqrt{2} E_{\max} \delta_1 + \delta_1) + 6 C_{15} C_8^2 \delta_1.
\end{aligned}$$

Combining our results so far, we find via (3.47) that

$$\begin{aligned} K_{\epsilon,\gamma}(T_{\max}-q) \leq & \sup_{s\in[0,T_{\max})} \big(\|v(s)\|_{H^1} + e^{\gamma s}\|v(s)\|_{H^1_{\mathbf{w}(s)}} + \gamma^{-1}e^{2\gamma s}\|v(s)\|^2_{H^1_{\mathbf{w}(s)}}\big) \\ & + E_{\max}\delta_1 + \delta_1 \\ < & \delta_8 \end{aligned}$$

for any $q \in (0, T_{\max}]$, decreasing the size of $\delta_1 > 0$ if necessary. By continuity of

$$t \mapsto e^{\gamma t}\|v(t)\|_{H^1_{\mathbf{w}(t)}} \quad \text{and} \quad t \mapsto \|v(t)\|_{H^1},$$

there must be a small $r > 0$ for which

$$K_{\epsilon,\gamma}(T_{\max} - q + r) \leq \delta_8.$$

Having established a priori control on $K_{\epsilon,\gamma}$, we may apply Proposition 3.5.2 to obtain

$$\sup_{t\in[0,T_{\max}-q+r]} e^{\gamma t}\|v(t)\|_{H^1_{\mathbf{w}(t)}} \leq C_8\big(\|\overline{v}_*\|_{H^1_w} + \|\overline{v}_*\|_{H^1} + \epsilon^{1-2p}\big).$$

Choosing $q < r$ shows that $T_{\max}$ is not maximal, allowing us to conclude that, indeed, $T_{\max} = \infty$.

To complete the proof, we now observe that

$$\sup_{t\geq 0} e^{\gamma t}\|v(t)\|_{H^1_{w\infty}} \leq \sup_{t\geq 0} e^{\gamma t}\|v(t)\|_{H^1_{\mathbf{w}(t)}} \leq C_8\big(\|\overline{v}_*\|_{H^1_w} + \|\overline{v}_*\|_{H^1} + \epsilon^{1-2p}\big),$$

which establishes item 2. Items 3 and 4 follow by applying Lemma 3.7.1, while item 1 follows from Proposition 3.6.5. □

## 3.8 Decompositions

Our work makes frequent use of the linear stability theory of the operator $\mathcal{L}_c$ developed in [91]. Let us briefly review these results here, based on the exposition in [86]. Introducing the exponential weight $e^{wx}$ on $L^2$ moves the essential spectrum of the operator $\mathcal{L}_c$ from the imaginary axis into the stable halfplane, leaving a double eigenvalue at 0. This 0-eigenvalue has a two-dimensional generalized kernel spanned by $\partial_x\phi_c$ and $\partial_c\phi_c$. In particular, it is easily verified that $\mathcal{L}_c\partial_x\phi_c = 0$ and $\mathcal{L}_c\partial_c\phi_c = \partial_x\phi_c$. The operator $\mathcal{L}_c$ is related to its (formal) adjoint $\mathcal{L}_c^*$ on the space $L^2_{-w}$ via the relation $\partial_x\mathcal{L}_c^* = -\mathcal{L}_c\partial_x$. In particular, $\mathcal{L}_c^*$ has a two-dimensional kernel spanned by $\phi_c$ and the primitive

$$\zeta_c(x) = \int_{-\infty}^{x} \partial_c\phi_c(y)\mathrm{d}y,$$

which satisfies $\zeta_c \in L^2_{-w}$. The spectral projection onto the generalized kernel of $\mathcal{L}_c$ is given by

$$P_c f = \langle f, \eta_c^1 \rangle \partial_x \phi_c + \langle f, \eta_c^2 \rangle \partial_c \phi_c, \tag{3.70}$$

where

$$\eta_c^1 = \tfrac{2}{9} c^{-1/2} \zeta_c + \tfrac{2}{9} c^{-2} \phi_c \quad \text{and} \quad \eta_c^2 = \tfrac{2}{9} c^{-1/2} \phi_c, \tag{3.71}$$

and we write $Q_c = I - P_c$ for the complementary projection. Based on the spectral properties of $\mathcal{L}_c$, the following can be concluded about the flow generated by this operator.

**Theorem 3.8.1** ([91]). *Let $c > 0$ and $0 < w < \sqrt{c}$. Then, $\mathcal{L}_c$ is the generator of a $C_0$-semigroup on $H^s_w$ for any real $s$. For all $\beta > 0$ which satisfy $\beta < w(c - w^2)$, there exists a constant $C > 0$ such that, for all $g \in L^2_w$, $t > 0$ and $k \in \{0, 1\}$, we have*

$$\| \partial_x^k e^{\mathcal{L}_c t} Q_c g \|_{L^2_w} \leq C t^{-k/2} e^{-\beta t} \| g \|_{L^2_w}. \tag{3.72}$$

We conclude here with a result regarding the orthogonality conditions arising from the projection $P_c$. This result ensures that the decomposition (3.18) underlying the arguments in this chapter is uniquely defined.

**Lemma 3.8.2** ([86]). *Let $c_* > 0$ and $w \in (0, \sqrt{c_*})$. Then, there exist constants $\delta_2, C_2 > 0$ such that, for each $\overline{v}_* \in H^1_w$, the bound $\|\overline{v}_*\|_{L^2_w} \leq \delta_2$ implies that there exist unique parameters $c_0 > 0$ and $\xi_0 \in \mathbb{R}$ such that*

$$\phi_{c_*}(x) + \overline{v}_*(x) = \phi_{c_0}(x - \xi_0) + \overline{v}_0(x - \xi_0) \quad \textit{with} \quad \langle \overline{v}_0, \phi_{c_0} \rangle = \langle \overline{v_0}, \zeta_{c_0} \rangle = 0$$

*and*

$$\begin{aligned} \|\overline{v}_0\|_{H^1_w} + |\xi_0| + |c_* - c_0| &\leq C_2 \|\overline{v}_*\|_{H^1_w}, \\ \|\overline{v}_0\|_{H^1} &\leq C_2 (\|\overline{v}_*\|_{H^1} + \|\overline{v}_*\|_{H^1_w}). \end{aligned}$$

## 3.9 Time-varying weights

Our goal here is to establish several properties of the time-dependent weights $\mathbf{w}(t) = \big(w_-(t), w_+(t)\big)$ given by

$$w_-(t) = w_{\min} \left(\tfrac{w_\infty}{w_{\min}}\right)^{1 - e^{-\gamma t}}, \quad w_+(t) = w \left(\tfrac{w_\infty}{w}\right)^{1 - e^{-\gamma t}}$$

that were used to control the perturbation over long time-scales in Section 3.5. In particular, we prove Lemma 3.5.1.

*Proof of Lemma 3.5.1.* Writing

$$\epsilon_1 = \sup_{t\in[0,T]} e^{\gamma t}\|v(t)\|_{L^2_{w_\infty}},$$

we note that for $s \in [0,\delta]$, we may compute

$$\begin{aligned}\big|\log\big(\alpha(t)\big) - \log\big(\alpha(t+s)\big)\big| &= \left|\int_t^{t+s} \log(\alpha(r))'\mathrm{d}r\right| \\ &\overset{(3.35)}{\le} C_6 \int_t^{t+s} \epsilon e^{-\gamma r}(1 + w_\infty^{-1/2}\epsilon_1 e^{-\gamma r}) + \epsilon_1^2 e^{-2\gamma r}\mathrm{d}r \\ &\le C_6\big(\epsilon(1+w_\infty^{-1/2}\epsilon_1) + \epsilon_1^2\big)\int_t^{t+s} e^{-\gamma r}\mathrm{d}r \\ &= C_6\gamma^{-1}\big(\epsilon(1+w_\infty^{-1/2}\epsilon_1)+\epsilon_1^2\big)e^{-\gamma t}(1-e^{-\gamma s}) \\ &\le 2C_6 E_{\max}(2+\delta_6)e^{1/2}(e^{-\gamma(t+s/2)} - e^{-\gamma(t+s)}),\end{aligned}$$

where we have used that $1 - e^{-x} \le 2(1-e^{-x/2})$ for all $x \ge 0$ and applied the assumptions $\epsilon_1 \le \min\{\delta_6 w_\infty^{1/2}, (\gamma E_{\max})^{1/2}\}$ and $\frac{\epsilon}{\gamma} \le E_{\max}$. Taking an exponential then gives

$$\frac{\alpha(t)}{\alpha(t+s)} \le \frac{\big(e^{2C_6E_{\max}(2+\delta_6)e^{1/2}}\big)^{e^{-\gamma(t+s/2)}}}{\big(e^{2C_6E_{\max}(2+\delta_6)e^{1/2}}\big)^{e^{-\gamma(t+s)}}} = \frac{w_+(t+s/2)}{w_+(t+s)},$$

and

$$\frac{\alpha(t+s)}{\alpha(t)} \le \frac{\big(e^{2C_6E_{\max}(2+\delta_6)e^{1/2}}\big)^{-e^{-\gamma(t+s)}}}{\big(e^{2C_6E_{\max}(2+\delta_6)e^{1/2}}\big)^{-e^{-\gamma(t+s/2)}}} = \frac{w_-(t+s)}{w_-(t+s/2)}.$$ □

We finally establish a bound for the quantity

$$q(t,s,\delta) = \tfrac{1}{w_-(t+\delta/2)-w_-(s)} + \tfrac{1}{w_+(s)-w_+(t+\delta/2)},$$

which is used in the proof of Lemma 3.5.4.

**Lemma 3.9.1.** *Assuming S1 and S3, there exists a constant $C_{15} > 0$ such that*

$$\sup_{s\in[0,t]} e^{-\gamma s}|q(t,s,\delta)| \le C_{15}\frac{e^{\gamma\delta/2}}{\gamma\delta},$$

*for all $t, \gamma, \delta > 0$.*

*Proof.* We first remark that

$$\sup_{s\in[0,t]} \frac{e^{-\gamma s}}{w_+(s) - w_+(t+\delta/2)} = \frac{w^2}{w_\infty}\sup_{x\in[0,\gamma t]}\frac{1}{B(x)},$$

where

$$B(x) = e^x (\tfrac{w}{w_\infty})^{e^{-x}} - e^x (\tfrac{w}{w_\infty})^{e^{-\gamma(t+\delta/2)}}.$$

We now claim that $B$ is decreasing on $[0, \gamma t]$, so that its infimum is attained at $\gamma t$. To see this, we compute the derivative

$$B'(x) = e^x \Big( (1 + \log(\tfrac{w_\infty}{w}) e^{-x}) (\tfrac{w}{w_\infty})^{e^{-x}} - (\tfrac{w}{w_\infty})^{e^{-\gamma(t+\delta/2)}} \Big)$$

and use $1 + \log(\frac{w_\infty}{w}) e^{-x} = 1 + \log((\frac{w_\infty}{w})^{e^{-x}}) \leq (\frac{w_\infty}{w})^{e^{-x}}$ to find

$$B'(x) \leq e^x \Big( 1 - (\tfrac{w}{w_\infty})^{e^{-\gamma(t+\delta/2)}} \Big) < 0,$$

since $\frac{w}{w_\infty} > 1$.

Using our claim, we may now estimate

$$\begin{aligned} w_+(t) - w_+(t+\delta/2) &\geq \tfrac{\delta}{2} |w_+'(t+\delta/2)| \\ &= \tfrac{\delta}{2} \gamma \log(\tfrac{w_{\min}}{w}) (\tfrac{w_\infty}{w})^{1 - e^{-\gamma(t+\delta/2)}} e^{-\gamma(t+\delta/2)}, \end{aligned}$$

which yields

$$\frac{e^{-\gamma t}}{w_+(t) - w_+(t+\delta/2)} \leq 2 \frac{(\frac{w_{\min}}{w})}{\log(\frac{w_{\min}}{w})} \frac{e^{\gamma\delta/2}}{\gamma\delta}.$$

A similar bound for the remaining term involving $w_-$ can be established analogously, completing the proof. □

# CHAPTER 4
# Stochastic stability

We study the stability and dynamics of solitons in the Korteweg-de Vries (KdV) equation in the presence of noise and deterministic forcing. The noise is space-dependent and statistically translation-invariant. We show that, for small forcing, solitons remain close to the family of traveling waves in a weighted Sobolev norm, with high probability. We study the effective dynamics of the soliton amplitude and position via their variational phase, for which we derive explicit modulation equations. The stability result holds on a time scale where the deterministic forcing induces significant amplitude modulation.

## 4.1 Introduction

This chapter[1] studies the stochastic KdV equation

$$\mathrm{d}u = -(\partial_x^3 u + 2u\partial_x u)\ \mathrm{d}t + \epsilon f(t)u\ \mathrm{d}t + \sigma u\ \mathrm{d}W_t^Q, \tag{4.1}$$

where $u$ is a real-valued process on $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. The scalar parameters $\epsilon > 0$ and $\sigma > 0$ introduce deterministic and random multiplicative forcing to the KdV dynamics, respectively. The noise process $W_t^Q$ is a Wiener process taking values in $L^2(\mathbb{R})$. The noise is white in time and colored in space with translation-invariant statistics given by the formal identity

$$\mathbb{E}\big[W^Q(t,x)W^Q(t,y)\big] = q(x-y)(s \wedge t), \quad x, y \in \mathbb{R}, \quad s, t > 0,$$

with $q \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$. The deterministic forcing depends only on time, prescribed by the scalar-valued function $f : \mathbb{R}^+ \to \mathbb{R}$. We study the effect of this forcing on the family of solitary waves associated to the unforced KdV equation ((4.1) with $\epsilon = \sigma = 0$). In particular, we establish the stability of forced solitons on time

[1]The contents of this chapter have been submitted for publication and are available as R.W.S. Westdorp, H.J. Hupkes, *Stability of Stochastically Forced Solitons in the Korteweg-de Vries Equation*, see [104].

scales where the forcing $\epsilon, \sigma > 0$ causes drastic amplitude modulation. This chapter complements our formal analysis in Chapter 2 with a rigorous stability result, and extends our deterministic results in Chapter 3 to space-dependent noise.

The Korteweg-de Vries equation is a well-known nonlinear dispersive PDE originally introduced as a model for shallow-water waves [15, 68]. It famously possesses a family of right-traveling wave solutions $u(t,x) = \phi_c(x - ct)$ with velocity dependent wave-profiles explicitly given by

$$\phi_c(x) = \frac{3c}{2} \operatorname{sech}^2(\sqrt{c}x/2), \quad c > 0. \tag{4.2}$$

Note that the velocity dependence of the wave profiles satisfies the simple scaling $\phi_c(x) = c\phi_1(\sqrt{c}x)$. The traveling waves (4.2), or solitons, arise due to a balance of dispersion and nonlinear effects. They are of key importance for the KdV dynamics: inverse scattering theory provides exact solutions to the KdV equation in terms of right-traveling solitons and a remaining dispersive component [48].

Many variations on (4.1) have been introduced in the literature to incorporate various (random) forcing mechanisms. In particular, we mention [59], which first considered the stochastic KdV equation. We refer to Chapter 2, Chapter 3 and references therein for more background on stochastic forcing and deterministic forcing, respectively, in the context of the KdV equation.

**Deterministic stability** The stability of solitons (4.2) with respect to an initial perturbation has been extensively studied, for instance, by energy methods [82]. These rely on the conservative nature of the KdV dynamics: an infinite number of 'integrals of motion', such as the $L^2$-norm, is conserved under the KdV flow [85]. The family (4.2) is, as a consequence of dispersion, only marginally linearly stable in $L^2$. Indeed, the linearized dynamics around the soliton $\phi_c$ are detailed by the operator

$$\mathcal{L}_c = -\partial_x^3 + c\partial_x - 2\partial_x(\phi_c \cdot), \tag{4.3}$$

which, viewed as an operator on $L^2$, has spectrum $i\mathbb{R}$. In this chapter we rely heavily on the stability theory for the exponentially weighted spaces

$$L^2_w(\mathbb{R}) = \left\{g : e^{wx}g \in L^2(\mathbb{R})\right\} \quad \text{with} \quad \|g\|_{L^2_w} = \|e^{wx}g\|_{L^2}, \tag{4.4}$$

$$H^1_w(\mathbb{R}) = \left\{g : e^{wx}g \in H^1(\mathbb{R})\right\} \quad \text{with} \quad \|g\|_{H^1_w} = \|e^{wx}g\|_{H^1} \tag{4.5}$$

that was developed by Pego and Weinstein in the classic work [91]. The exponential weight $e^{wx}$ $(w > 0)$ shifts the continuous spectrum of the operator $\mathcal{L}_c$ with $\sqrt{c} > w$ into the stable half-plane, leaving a spectral gap of size $w(c - w^2)$ and a double eigenvalue at 0. Physically, the exponential weight dampens persisting disturbances outrun by the soliton. The linearized dynamics contain two neutral modes, associated with infinitesimal changes of $\phi_c$ with respect to the space variable $x$ and the amplitude parameter $c$. These spectral properties ensure that the linear flow $\{e^{\mathcal{L}_c t}\}_{t \geq 0}$ on $L^2_w$ generated by the operator (4.3) is exponentially stable on a subspace where the neutral eigenvalue is avoided. This subspace of $L_w^2$ consists of functions $v$ that satisfy the orthogonality conditions

$$\langle v, \phi_c \rangle = \langle v, \zeta_c \rangle = 0, \tag{4.6}$$

where

$$\zeta_c(x) = \int_{-\infty}^{x} \partial_c \phi_c(y) \mathrm{d}y.$$

Based on these properties, Pego and Weinstein [91] established the orbital stability of the traveling wave family (4.2) with respect to a small initial perturbation in $H^1 \cap H_w^1$. In fact, such an initial perturbation only causes a small asymptotic phase-shift and amplitude change in the soliton.

**Stochastic stability** Stability of the wave family (4.2) with respect to small stochastic forcing has been previously considered in [25]. Based on energy methods, de Bouard and Debussche show that solutions to (4.1) with $\epsilon = 0$ and small $\sigma > 0$ stay close to the wave family for times small with respect to $\sigma^{-2}$. Their method relies on the fact that the soliton amplitude remains close to its starting value on this time scale. This is a significant restriction, as the result does not cover any (significant) modulation of the soliton. These stochastic modulations have, however, been explored to some degree in a formal sense. For example, the work [19] uses a collective coordinate approach to treat the case where $W_t^Q$ is replaced by a scalar Brownian motion. The numerical and analytical results in Chapter 2 cover both scalar and space-dependent noise and provide explicit Taylor expansions for the behavior of the modulation parameters.

**Present work** In the present work, we consider deterministic and stochastic forcing of the soliton family (4.2), leading to stochastic modulations of the amplitude and position over time. We establish the orbital stability of (4.2) in a setting where the stochastic amplitude modulations can have arbitrary size. This is primarily due to the deterministic forcing mechanism ($\epsilon > 0$) present in (4.1), which increases/decreases the energy present in the system after time $t$ by a factor $e^{\epsilon \int_0^t f(s) \mathrm{d}s}$ (Chapter 3). In this setting, it is vital to explicitly account for the dynamics of the soliton amplitude. In particular, we will show that, to leading order in the small parameters $\epsilon$ and $\sigma$, this amplitude evolves according to the SDE[2]

$$\mathrm{d}c_{\mathrm{ap}} = \left[\tfrac{4}{3} c_{\mathrm{ap}} \epsilon f(t) + \sigma^2 g_Q(c_{\mathrm{ap}})\right] \mathrm{d}t + \tfrac{2}{9} c_{\mathrm{ap}}^{-1/2} \sigma \langle \phi_{c_{\mathrm{ap}}}^2, \mathrm{d}W_t^Q \rangle, \tag{4.7}$$

in which the function $g_Q : \mathbb{R} \to \mathbb{R}$ will be made explicit in Section 4.3. This allows us to demonstrate that stochastic stability is not limited to trivial changes in amplitude.

[2]The process $c_{\mathrm{ap}}$ introduced in (4.7) approximates the soliton amplitude only in distribution. As will be made clear later on, the soliton amplitude is driven by a translated version of $W_t^Q$.

**Main result** We study solutions $u$ to (4.1) in the (relatively standard) setting S1—see Section 4.2 ahead—through the modulation Ansatz

$$u\big(t, x + \xi(t)\big) = \phi_{c(t)}(x) + v(t, x).$$

Here, $c(t)$ and $\xi(t)$ are stochastic processes that track the soliton amplitude and position, respectively, which we fully define later on. In particular, we supply (4.1) with the initial condition

$$u(0, x) = \phi_{c_*}(x),$$

for some $c_* > 0$. The remainder $v(t, x)$ constitutes the error resulting from our modulation approach. The main contribution of this chapter concerns the probabilistic behavior of the exit time

$$t_{\mathrm{st}}(\eta) = \sup\big\{t \geq 0 : \|v(t)\|_{H^1_w} \leq \eta\big\},$$

which signals a deviation from the modulated soliton description.

The parameter $E$ in our result below provides a-priori (but unlimited) control over the total potential impact of the deterministic forcing. This is related to the fact that the linear stability properties of the soliton family $\phi_c$ are limited by the soliton amplitude $c$. Indeed, the $w(c - w^2)$-wide spectral gap of the linear operator $\mathcal{L}_c$ on $L^2_w$ is at most $\frac{2}{3\sqrt{3}}c^{3/2}$ (take $w = \sqrt{c/3}$). The constants appearing in many of our estimates therefore require $c$ to be kept within a fixed range $[c_{\min}, c_{\max}]$ that is controlled by $E$.

**Theorem 4.1.1** (See Section 4.8)**.** *Pick $c_*, E > 0$ and $w \in (0, \sqrt{c_*}/3]$. Assuming S1, there exist constants $\eta_0, C, \delta > 0$ such that the following holds true. For all $\eta \in [0, \eta_0]$, $C\sigma, C\epsilon \in [0, \eta]$, each $T \geq 1$, and each continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ for which*

$$\sup_{t\geq 0} |f(t)| \leq 1, \quad \textit{and} \quad \epsilon \int_0^\infty |f(t)|\mathrm{d}t \leq E, \tag{4.8}$$

*the exit time $t_{\mathrm{st}}$ satisfies*

$$\mathbb{P}\big[t_{\mathrm{st}}(\eta) < T\big] \leq CT\sigma^2 \log(1/\sigma) + CTe^{-\delta\eta^2/\sigma^2}. \tag{4.9}$$

The probability bound (4.9) contains two contributions. The exponential part $Te^{-\delta\eta^2/\sigma^2}$ stems from the linear stability properties of the wave family (4.2) on weighted spaces, and matches modern phase-tracking results valid on time scales that are exponentially long with respect to the ratio $\eta^2/\sigma^2$ [12, 13, 58, 77, 51]. The contribution $T\sigma^2 \log(1/\sigma)$, however, limits the time-scale on which stability can be obtained to times $T$ that are *small* with respect to $\sigma^{-2} \log(1/\sigma)^{-1}$, but independent of the size-parameter $\eta$.

The limiting factor is that we rely on a-priori control of the remainder $v$ in the unweighted space $L^2$. In this space, the solitons $\phi_c$ are not linearly stable, and the norm $\|v(t)\|^2_{L^2}$ grows linearly in time. The factor $\sigma^2 \log(1/\sigma)$ stems from an Itô drift term proportional to $\sigma^2$, combined with a factor $\log(1/\sigma)$ that we need to account for the potentially large fluctuations of the soliton amplitude.

We emphasize that within the timescale discussed above, the deterministic forcing ($\epsilon > 0$) can cause significant modulation of the soliton amplitude $c(t)$. Thus, the main point of our results is that they demonstrate that the stability of the wave family (4.2) is not limited to small fluctuations. To showcase this, we obtain a result on the exit time

$$t_{\mathrm{ap}}(\lambda) = \sup\big\{t \geq 0 : |c(t) - c_{\mathrm{ap}}(t)| \leq \lambda\big\},$$

which describes the validity of the approximation (4.7).

**Theorem 4.1.2** (See Section 4.9)**.** *Assuming the setting of Theorem 4.1.1, for each $\lambda > 0$ that satisfies*

$$\lambda \leq \min\{\tfrac{1}{2}c_* e^{-3E}, c_* e^{3E}\},$$

*the exit times $t_{\mathrm{ap}}$ and $t_{\mathrm{st}}$ satisfy*

$$\mathbb{P}\big[t_{\mathrm{ap}}(\lambda) < T\big] \leq C\frac{\sigma^2}{\lambda}\log(1/\sigma). \tag{4.10}$$

**Approach** The field of stochastic traveling waves has witnessed rapid development in recent years. Several approaches have emerged to study stochastic traveling waves in various PDE settings. These typically feature a decomposition of the solution in terms of a traveling wave modulated by a stochastic phase-shift. The exact definition of this phase-shift is where the various methods differ. Let us mention the phase-lag method [69, 31], variational-phase [12, 13, 58, 77] and isochronal-phase [1], all applied in the context of reaction-diffusion equations. In dispersive settings, (adaptations of) the variational phase have been applied in [25, 24, 29, 103, 51]. Let us also mention the recent work [105], which presents a phase-tracking mechanism for general symmetry groups.

In the present context, where we study the wave family (4.2), we follow our approach in Chapter 3 and decompose the solution $u$ to (4.1) as

$$u\big(t, x + \xi(t)\big) = \phi_{c(t)}(x) + v(t, x).$$

Here, $c(t)$ and $\xi(t)$ are processes that track the soliton amplitude and position, and ensure the orthogonality conditions

$$\big\langle v(t,\cdot), \phi_{c(t)}\big\rangle = \big\langle v(t,\cdot), \zeta_{c(t)}\big\rangle = 0, \quad t \geq 0. \tag{4.11}$$

We point out a technical but essential difference with Chapter 3, where we decomposed $u$ in terms of a fixed soliton $\phi_{c_*}$ in a rescaled frame. We do not employ this coordinate-transformation here, to avoid Itô correction terms in the evolution equation of the remainder $v$. This leaves the challenge of harnessing the linear stability properties of a time-varying soliton $\phi_{c(t)}$. In spirit of [77], we employ the linear stability properties at some time $T > 0$ of $\mathcal{L}_{c(T)}$ on an interval $[T, T + \Delta T]$. On such intermediate intervals, the soliton amplitude $c(t)$ does not deviate much from $c(T)$. We furthermore introduce local modulation parameters on the interval $[T, T+\Delta T]$ resulting in a local remainder $v^T$ which enjoys exponential damping by the linear flow $\{e^{\mathcal{L}_{c(T)}t}\}_{t\geq 0}$. For $t \in [T, T+\Delta T]$, the local remainder can be used to control the global remainder $v(t)$.

**Challenges** Classically, analysis of the $L^2_w$-norm of $v(t)$ requires complementary control on the unweighted $L^2$-norm of $v(t)$. This is due to the nonlinearity $u\partial_x u$ present in (4.1), which is typically estimated in the weighted $L^2$-norm as

$$\int_{\mathbb{R}} e^{2wx} u^2(x) u_x^2(x)\mathrm{d}x \leq \|u\|_\infty^2 \int_{\mathbb{R}} e^{2wx} u_x^2(x)\mathrm{d}x \leq 2\|u\|_{H^1}^2 \|u_x\|_{L^2_w}^2. \tag{4.12}$$

The unweighted $H^1$-norm of $v(t)$ can in turn be analyzed via energy methods, see Section 3.6. A formal application of the Itô lemma shows that, in the presence of stochastic forcing ($\sigma > 0$), the energy of solutions to (4.1) evolves as

$$\mathrm{d}\|u\|_{L^2}^2 = \sigma^2\|u\|_{L^2}^2 \ \mathrm{d}t + 2\epsilon f(t)\|u\|_{L^2}^2 \ \mathrm{d}t + 2\sigma\langle u, u \ \mathrm{d}W_t^Q\rangle.$$

As we will see later on, due to an Itô drift correction, the energy in $v(t)$ grows proportionally to $\sigma^2 t$ at leading order. We incur a further $\log(1/\sigma)$ penalty as a consequence of the $\mathcal{O}(v^2)$ terms, leading to the first term in (4.9).

At present, we can hence not carry out our arguments on a time-interval $[0, T_{\max}\sigma^{-2}]$ for *any* $T_{\max} > 0$, which we believe should be attainable with more refined arguments in the special case $\epsilon = 0$. Indeed, inspecting the approximation (4.7) with $\epsilon = 0$, we see that the effective soliton amplitude $c$ is driven by the noise $\sigma W_t^Q$. We can hence expect $c$ to be kept within the range $[c_{\min}, c_{\max}]$ on time scales proportional to $\sigma^{-2}$ and the (arbitrary) size of this range.

Furthermore, the bound (4.12) shows that it is in fact sufficient to control the *supremum* norm of $v(t)$, instead of an energy norm. Preliminary numerical investigations suggest that this norm remains under control for timescales longer than $\sigma^{-2}$, showing that the dynamics of $c$ are indeed dominant. However, bounds of this nature will require us to depart from energy-based methods, which are only available in $L^2$-based spaces. We envision that future work in this direction will center on a careful (and challenging) direct pointwise analysis of the dispersive dynamics that drive the remainder $v(t)$ in the wake of the soliton. We emphasize however that the framework developed here to control the weighted norm of $v$ can readily accommodate such a refinement.

**Outline** This chapter is organized as follows. In Section 4.2, we outline the setting of (4.1) in more detail and recall classical linear stability results regarding the operator $\mathcal{L}_c$ defined in (4.3). Then, in Section 4.3, we analyze the modulation system brought forth by the conditions (4.11). Following this, in Section 4.4, we introduce local approximations to this system. In Section 4.5, we carry out time-discretized stability arguments for the local remainder $v^T$ in the weighted spaces $H^1_w$. We supplement this with an analysis of the local modulation parameters in

Section 4.6. In Section 4.7, we develop global control of the unweighted $L^2$-norm of the remainder $v$, as well as the global amplitude parameter $c(t)$. Finally, we combine our results in Section 4.8 and Section 4.9 for the proofs of Theorem 4.1.1 and Theorem 4.1.2.

## 4.2 Preliminaries

Below, we describe the setting of (4.1) in more detail and collect several results regarding the linear stability of the soliton family (4.2) as developed in [91].

**Stochastic set-up** We work in the following setting, which we recall from Section 2.2.1.

**S1** *We have $c_* > 0$, and supply* (4.1) *with the initial condition*

$$u(0,x) = \phi_{c_*}(x). \tag{4.13}$$

*The operator $Q \in \mathcal{L}(L^2)$ is defined as the convolution with an even function $q \in H^1 \cap L^1$ through*

$$Qf = \int_{\mathbb{R}} q(x-y)f(y)\mathrm{d}x.$$

*Moreover, the Fourier transform $\hat{q}$ of the kernel $q$ is non-negative. Lastly, the forcing term $f : \mathbb{R}^+ \to \mathbb{R}$ is continuous and bounded as*

$$|f(t)| \leq 1, \quad t \geq 0.$$

By Young's convolution inequality, the integrability assumption $q \in L^1$ ensures that $Q$ is a bounded operator on $L^2(\mathbb{R})$. The non-negativity of the Fourier transform $\hat{q}$ furthermore ensures that $Q$ is a symmetric and non-negative operator. Thus, $Q$ can be used to construct an $L^2$-valued cylindrical Brownian motion $W_t^Q$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ cf. [23, 74], which satisfies the formal covariance identity

$$\mathbb{E}\big[\mathrm{d}W^Q(x,t)\mathrm{d}W^Q(y,s)\big] = \delta(t-s)q(x-y), \quad x, y \in \mathbb{R}, \quad s, t > 0.$$

Since $q$ is an even function, the spatial correlation of $W_t^Q$ depends only on the distance $|x-y|$ between two points $x, y \in \mathbb{R}$ and preserves the translation-invariance of (4.1). We recall furthermore from Section 2.2.1 that the non-negative operator $Q$ has a square-root $Q^{1/2}$ that acts as the convolution

$$Q^{1/2}f(x) = \int_{\mathbb{R}} q_{1/2}(x-y)f(y)\ \mathrm{d}y,$$

where $q_{1/2}$ is the inverse Fourier transform of $\sqrt{\hat{q}}$.

Let us now introduce some notation related to stochastic integration with respect to the noise $W_t^Q$. Letting $\{e_k\}_{k=0}^{\infty}$ be an orthonormal basis of $L^2(\mathbb{R})$, and introducing the space $L^2_Q := Q^{1/2}(L^2)$ equipped with the inner product

$$\langle v, w\rangle_{L^2_Q} = \langle Q^{-1/2}v, Q^{-1/2}w\rangle_{L^2},$$

we note that $L^2_Q$ is a separable Hilbert space for which $\{Q^{1/2}e_k\}_{k=0}^{\infty}$ is an orthonormal basis. We furthermore denote by $\mathrm{HS}(L^2_Q, \mathcal{H})$ the space of Hilbert-Schmidt operators between $L^2_Q$ and a Hilbert space $\mathcal{H}$, equipped with the inner-product

$$\langle A, B\rangle_{\mathrm{HS}(L^2_Q,\mathcal{H})} = \sum_{k=0}^{\infty} \big\langle A[Q^{1/2}e_k], B[Q^{1/2}e_k]\big\rangle_{\mathcal{H}}.$$

In the setting S1, de Bouard and Debussche [26] showed that (4.1) admits unique mild solutions for purely stochastic forcing ($\epsilon = 0$). For deterministic forcing ($\sigma = 0$), the well-posedness of (4.1) has been established in Lemma 3.2.1. Combining these results yields the following well-posedness property, in which $\{e^{-\partial_x^3 t}\}_{t\in\mathbb{R}}$ denotes the $C_0$-group of isometries on $L^2$ associated to the Airy equation.

**Lemma 4.2.1.** *[[24, 103]] Assuming S1,* (4.1) *has a unique mild solution $u \in L^2(\Omega; C(\mathbb{R}^+; L^2(\mathbb{R}))$, that satisfies*

$$\begin{aligned} u(t) = e^{-\partial_x^3 t}u_0 - \int_0^t e^{-\partial_x^3(t-t')}\partial_x\big(u^2(t')\big)\mathrm{d}t' + \epsilon\int_0^t f(t')e^{-\partial_x^3(t-t')}u(t')\mathrm{d}t' \\ + \sigma\int_0^t e^{-\partial_x^3(t-t')}u(t')\mathrm{d}W_{t'}^Q, \end{aligned}$$

*and has paths that are almost surely in $C(\mathbb{R}^+; H^1(\mathbb{R}))$.*

4

**Linear stability tools** Our arguments rely heavily on the linear stability properties of the operator $\mathcal{L}_c$ defined in (4.3), that were developed in [91]. We recall from this work that $\mathcal{L}_c$ satisfies the eigenvalue relations $\mathcal{L}_c\partial_x\phi_c = 0$ and $\mathcal{L}_c\partial_c\phi_c = \partial_x\phi_c$, giving rise to a two-dimensional generalized kernel. The (formal) adjoint $\mathcal{L}_c^*$ also has a two-dimensional kernel, spanned by $\phi_c$ and the primitive

$$\zeta_c(x) = \int_{-\infty}^{x} \partial_c\phi_c(y)\mathrm{d}y.$$

This function satisfies $\zeta_c \in L^2_{-w}$ (but not $\zeta_c \in L^2$). With this notation in place, we note that the projection onto the generalized kernel of $\mathcal{L}_c$ is given by

$$P_c f = \langle f, \tfrac{2}{9}c^{-1/2}\zeta_c + \tfrac{2}{9}c^{-2}\phi_c\rangle\partial_x\phi_c + \langle f, \tfrac{2}{9}c^{-1/2}\phi_c\rangle\partial_c\phi_c. \tag{4.14}$$

We write $Q_c = I - P_c$ for the complementary projection, and note that $\mathrm{Ker}(P_c)$ coincides with the subspace where the conditions (4.6) hold. We collect the following properties of the flow generated by $\mathcal{L}_c$, which demonstrate the parabolic nature of $-\partial_x^3$ on the weighted spaces $L^2_w$, after projecting out the neutral modes. First, let us fix limits $c_{\min}, c_{\max}$ and a weight $w$ as follows.

**S2** *The constants* $c_{\min}, c_{\max}, w \in \mathbb{R}$ *satisfy*

$$0 < c_{\min} < c_* < c_{\max} \quad \textit{and} \quad 0 < w < \sqrt{c_{\min}}/3.$$

**Theorem 4.2.2** ([91, 86]). *Let* $c \in [c_{\min}, c_{\max}]$. *Then,* $\mathcal{L}_c$ *is the generator of a* $C_0$-*semigroup* $\{e^{\mathcal{L}_c t}\}_{t\geq 0}$ *on* $H^s_w$ *for any real* $s$. *For all* $b > 0$ *which satisfy* $b < w(c - w^2)$, *there exists a constant* $M > 0$ *such that, for all* $g \in L^2_w$, $t > 0$ *and* $k \in \{0, 1\}$, *we have*

$$\|\partial_x^k e^{\mathcal{L}_c t} Q_c g\|_{L^2_w} \leq M t^{-k/2} e^{-bt} \|g\|_{L^2_w}, \tag{4.15}$$

*while for all* $g \in L^1_w$ *we have*

$$\|\partial_x^k e^{\mathcal{L}_c t} Q_c g\|_{L^2_w} \leq M t^{-(2k+1)/4} e^{-bt} \|g\|_{L^1_w}. \tag{4.16}$$

## 4.3 Global modulation system

With these preliminaries in place, we are in shape to introduce the decomposition

$$u\big(t, x + \xi(t)\big) = \phi_{c(t)}(x) + v(t, x) \tag{4.17}$$

characterized by the orthogonality conditions

$$\big\langle v(t, \cdot), \phi_{c(t)} \big\rangle = \big\langle v(t, \cdot), \zeta_{c(t)} \big\rangle = 0. \tag{4.18}$$

As a consequence of Lemma 4.2.1, the remainder $v$ introduced through (4.17) has paths that are almost surely in $C(\mathbb{R}^+; H^1(\mathbb{R}))$. The decomposition (4.17) is equivalent to that in Chapter 2 and Chapter 3, albeit phrased in a different frame of reference. The unique existence of decomposition (4.17) is guaranteed as long as $\|v(t)\|_{L^2_w}$ remains sufficiently small, essentially as a result of the implicit function theorem.

**Lemma 4.3.1** ([86]). *Assuming S2, there exist constants* $\delta_1, C_1 > 0$ *such for each* $v_* \in H^1_w \cap H^1$ *with* $\|v_*\|_{L^2_w} \leq \delta_1$ *and* $c_* \in [c_{\min}, c_{\max}]$, *there exist unique parameters* $c > 0, \xi \in \mathbb{R}$ *and a unique function* $v \in H^1_w \cap H^1$ *that together satisfy the identities*

$$\phi_{c_*}(x) + v_*(x) = \phi_c(x - \xi) + v(x - \xi) \quad \textit{with} \quad \langle v, \phi_c \rangle = \langle v, \zeta_c \rangle = 0$$

*and the bounds*

$$\begin{aligned} \|v\|_{H^1_w} + |\xi| + |c_* - c| &\leq C_1 \|v_*\|_{H^1_w}, \\ \|v\|_{H^1} &\leq C_1 \big(\|v_*\|_{H^1} + \|v_*\|_{H^1_w}\big). \end{aligned}$$

For convenience, we introduce a phase-shift function $\Omega(t)$ through $\xi(t) = \int_0^t c(t')\mathrm{d}t' + \Omega(t)$. Our goal in this section is to describe the evolution of the modulation parameters $c(t)$ and $\Omega(t)$ via SDEs of the form

$$\mathrm{d}c = c_d^{\sigma,\epsilon}(v,c,t)\ \mathrm{d}t + \sigma\big\langle c_s(v,c), T_\xi \mathrm{d}W_t^Q\big\rangle, \tag{4.19}$$

$$\mathrm{d}\Omega = \Omega_d^{\sigma,\epsilon}(v,c,t)\ \mathrm{d}t + \sigma\big\langle \Omega_s(v,c), T_\xi \mathrm{d}W_t^Q\big\rangle. \tag{4.20}$$

For $\xi \in \mathbb{R}$, the operator $T_\xi$ above denotes the translation by $\xi$, i.e. $(T_\xi f)(x) = f(x+\xi)$. We thus set out to find mappings $c_d^{\sigma,\epsilon}, c_s, \Omega_d^{\sigma,\epsilon}$ and $\Omega_s$ that ensure the conditions (4.18). Formally applying the Itô lemma [23, Theorem 4.32] to (4.17) with (4.19) and (4.20), yields, after tedious computations

$$\mathrm{d}v = \big[\mathcal{L}_{c(t)}v + Y^{\sigma,\epsilon}(v,c,t)\big]\ \mathrm{d}t + \sigma Z(v,c)\ T_\xi \mathrm{d}W_t^Q \tag{4.21}$$

where

$$\begin{aligned} Y^{\sigma,\epsilon}(v,c,t) = N(v) + \epsilon f(t)(\phi_c + v) + \Omega_d^{\sigma,\epsilon}(v,c,t)\partial_x(\phi_c + v) \\ - c_d^{\sigma,\epsilon}(v,c,t)\partial_c\phi_c + \sigma^2 Y_d(v,c). \end{aligned}$$

Here, $N(v) = -\partial_x(v^2)$ is the KdV nonlinearity, the drift contribution $Y_d$ is given by

$$Y_d(v,c) = \tfrac{1}{2}\big\|Q^{1/2}\Omega_s(v,c)\big\|_{L^2}^2\partial_x^2 v + \tfrac{1}{2}\big\|Q^{1/2}\Omega_s(v,c)\big\|_{L^2}^2\partial_x^2\phi_c + \tfrac{1}{2}\big\|Q^{1/2}c_s(v,c)\big\|_{L^2}^2\partial_c^2\phi_c$$

and the stochastic component is defined by

$$Z(v,c)[h] = \Big(h + \langle\Omega_s(v,c),h\rangle\partial_x\Big)v + \Big(h + \langle\Omega_s(v,c),h\rangle\partial_x - \langle c_s(v,c),h\rangle\partial_c\Big)\phi_c.$$

The formal adjoint of $Z$ is given by

$$Z^*(v,c)[g] = (g+\phi_c)v + \Omega_s(v,c)\big\langle g, \partial_x(\phi_c+v)\big\rangle - c_s(v,c)\langle g, \partial_c\phi_c\rangle.$$

The evolution of $\big\langle v(t), \phi_{c(t)}\big\rangle$ can now formally be obtained by applying the Itô product rule. However, we can not expect (4.21) to hold in the strong sense, as $v$ is not regular enough for $\mathcal{L}_{c(t)}v$ and $\partial_x^2 v$ to be well-defined. We take care of this technical issue in Section 4.11 by resorting to a mild Itô formula. The result is as follows.

**Lemma 4.3.2** (See Section 4.11). *Assume S1 and S2. For each $t \geq 0$, the inequalities*

$$\|v(t')\|_{L^2_w} \leq \delta_1 \quad \textit{and} \quad c(t') \in [c_{\min}, c_{\max}], \quad t' \in [0,t]$$

*imply that*

$$\begin{aligned} \mathrm{d}\big\langle v(t), \phi_{c(t)}\big\rangle = \Big(c_d^{\sigma,\epsilon}(v,c,t)\langle v, \partial_c\phi_{c(t)}\rangle + \big\langle Y^{\sigma,\epsilon}(v,c,t), \phi_{c(t)}\big\rangle\Big)\ \mathrm{d}t \\ + \tfrac{1}{2}\sigma^2\big\|Q^{1/2}c_s(v,c)\big\|_{L^2}^2\langle v, \partial_c^2\phi_{c(t)}\rangle\ \mathrm{d}t \end{aligned}$$



4

$$
\begin{aligned}
&+\sigma^2\big\langle Q^{1/2}Z^*(v,c)[\partial_c\phi_{c(t)}], Q^{1/2}c_s(v,c)\big\rangle \,\mathrm{d}t \\
&+\sigma\langle v, \partial_c\phi_{c(t)}\rangle\big\langle c_s(v,c), T_\xi \mathrm{d}W_t^Q\big\rangle + \sigma\big\langle Z(v,c)[T_\xi \mathrm{d}W_t^Q], \phi_{c(t)}\big\rangle
\end{aligned}
\tag{4.22}
$$

*and*

$$
\begin{aligned}
\mathrm{d}\big\langle v(t), \zeta_{c(t)}\big\rangle = &\Big(c_d^{\sigma,\epsilon}(v,c,t)\langle v, \partial_c\zeta_{c(t)}\rangle + \big\langle Y^{\sigma,\epsilon}(v,c,t), \zeta_{c(t)}\big\rangle\Big)\,\mathrm{d}t \\
&+\tfrac{1}{2}\sigma^2\big\|Q^{1/2}c_s(v,c)\big\|_{L^2}^2\langle v, \partial_c^2\zeta_{c(t)}\rangle \,\mathrm{d}t \qquad (4.23)\\
&+\sigma^2\big\langle Q^{1/2}Z^*(v,c)[\partial_c\zeta_{c(t)}], Q^{1/2}c_s(v,c)\big\rangle \,\mathrm{d}t \\
&+\sigma\langle v, \partial_c\zeta_{c(t)}\rangle\big\langle c_s(v,c), T_\xi \mathrm{d}W_t^Q\big\rangle + \sigma\big\langle Z(v,c)[T_\xi \mathrm{d}W_t^Q], \zeta_{c(t)}\big\rangle \qquad (4.24)
\end{aligned}
$$

*hold $\mathbb{P}$-almost surely.*

**Remark 4.3.3.** In (4.22) and (4.24) above, derivatives on $v$ are interpreted in the weak sense.

Solving for solutions $c_s(v,c)$ and $\Omega_s(v,c)$ to the system

$$
\begin{bmatrix} \langle v, \partial_c\phi_c\rangle\big\langle c_s(v,c), h\big\rangle + \big\langle Z(v,c)[h], \phi_c\big\rangle \\ \langle v, \partial_c\zeta_c\rangle\big\langle c_s(v,c), h\big\rangle + \big\langle Z(v,c)[h], \zeta_c\big\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \forall h \in L_Q^2,
$$

reveals that

$$
K_c(v)\begin{bmatrix} \langle c_s(v,c), h\rangle \\ \langle \Omega_s(v,c), h\rangle \end{bmatrix} = -\begin{bmatrix} \big\langle h(\phi_c + v), \phi_c\big\rangle \\ \big\langle h(\phi_c + v), \zeta_c\big\rangle \end{bmatrix}
$$

for all $h \in L_Q^2$, where $K_c(v)$ is the matrix

$$
K_c(v) = \begin{bmatrix} \langle -\phi_c + v, \partial_c\phi_c\rangle & \langle \partial_x v, \phi_c\rangle \\ \langle v, \partial_c\zeta_c\rangle - \langle \partial_c\phi_c, \zeta_c\rangle & \big\langle \partial_x(\phi_c + v), \zeta_c\big\rangle \end{bmatrix}.
$$

This implies that the mappings $c_s(v,c)$ and $\Omega_s(v,c)$ are given by

$$
\begin{bmatrix} c_s(v,c) \\ \Omega_s(v,c) \end{bmatrix} = -K_c^{-1}(v)\begin{bmatrix} (\phi_c + v)\phi_c \\ (\phi_c + v)\zeta_c \end{bmatrix}.
$$

We solve for $c_d^{\sigma,\epsilon}$ and $\Omega_d^{\sigma,\epsilon}$ by decomposing

$$
\begin{aligned}
c_d^{\sigma,\epsilon}(v,c,t) &= c_d^0(v,c) + \epsilon f(t)c_f(v,c) + \sigma^2 c_d(v,c), \\
\Omega_d^{\sigma,\epsilon}(v,c,t) &= \Omega_d^0(v,c) + \epsilon f(t)\Omega_f(v,c) + \sigma^2 \Omega_d(v,c),
\end{aligned}
$$

and isolating the $\sigma^2$ and $\epsilon$ dependent terms in (4.22) and (4.24). This yields

$$
\begin{bmatrix} c_d^0(v,c) \\ \Omega_d^0(v,c) \end{bmatrix} = -K_c^{-1}(v)\begin{bmatrix} \big\langle N(v), \phi_c\big\rangle \\ \big\langle N(v), \zeta_c\big\rangle \end{bmatrix},
$$

$$\begin{bmatrix} c_f(v,c) \\ \Omega_f(v,c) \end{bmatrix} = -K_c^{-1}(v) \begin{bmatrix} \langle \phi_c + v, \phi_c \rangle \\ \langle \phi_c + v, \zeta_c \rangle \end{bmatrix},$$

together with

$$\begin{bmatrix} c_d(v,c) \\ \Omega_d(v,c) \end{bmatrix} = -K_c^{-1}(v) \begin{bmatrix} \langle Y_d(v,c), \phi_c \rangle \\ \langle Y_d(v,c), \zeta_c \rangle \end{bmatrix} - \tfrac{1}{2} \| Q^{1/2} c_s(v,c) \|_{L^2}^2 K_c^{-1}(v) \begin{bmatrix} \langle v, \partial_c^2 \phi_c \rangle \\ \langle v, \partial_c^2 \zeta_c \rangle \end{bmatrix}$$
$$- K_c^{-1}(v) \begin{bmatrix} \langle Q^{1/2} Z^*(v,c)[\partial_c \phi_c], Q^{1/2} c_s(v,c) \rangle \\ \langle Q^{1/2} Z^*(v,c)[\partial_c \zeta_c], Q^{1/2} c_s(v,c) \rangle \end{bmatrix}.$$

We remark that, in the absence of deterministic forcing ($\epsilon = 0$), the modulation system derived above is equivalent to the system found in Section 2.2.3, and in the absence of stochastic forcing ($\sigma = 0$) to that in Section 3.2. Evaluating the modulation system at $v = 0$ gives rise to the approximation $c_{\mathrm{ap}}$ defined in (4.7), with $g_Q(c) = c_d(0,c)$. Now that the modulation system has taken concrete form, we can establish control on the modulation parameters, which will be important for our stability arguments later on.

**Lemma 4.3.4.** *Assuming S2, there exist constants $\delta_2, C_2 > 0$ such that for all $\tilde{v} \in L^2_w$ that satisfy $\|\tilde{v}\|_{L^2_w} \leq \delta_2$ and each $\tilde{c} \in [c_{\min}, c_{\max}]$ we have*

$$\big\| Q^{1/2} c_s(\tilde{v},\tilde{c}) - Q^{1/2} c_s(0,\tilde{c}) \big\|_{L^2} + \big\| Q^{1/2} \Omega_s(\tilde{v},\tilde{c}) - Q^{1/2} \Omega_s(0,\tilde{c}) \big\|_{L^2} \leq C_2 \|\tilde{v}\|_{L^2_w}, \tag{4.25}$$

$$\big| c_d^0(\tilde{v},\tilde{c}) \big| + \big| \Omega_d^0(\tilde{v},\tilde{c}) \big| \leq C_2 \|\tilde{v}\|_{L^2_w}^2, \tag{4.26}$$

$$\big| c_d(\tilde{v},\tilde{c}) - \tilde{c}_d(0,\tilde{c}) \big| + \big| \Omega_d(\tilde{v},\tilde{c}) - \Omega_d(0,\tilde{c}) \big| \leq C_2 \big(1 + \|\tilde{v}\|_{L^2_w}^2\big) \|\tilde{v}\|_{L^2_w}, \tag{4.27}$$

$$\big| c_f(\tilde{v},\tilde{c}) - c_f(0,\tilde{c}) \big| + \big| \Omega_f(\tilde{v},\tilde{c}) - \Omega_f(0,\tilde{c}) \big| \leq C_2 \|\tilde{v}\|_{L^2_w}. \tag{4.28}$$

*Proof.* Setting out to control $K_{\tilde{c}}^{-1}(\tilde{v})$, we note that

$$K_{\tilde{c}}(0) = -\frac{9}{2} \begin{bmatrix} \tilde{c}^{1/2} & 0 \\ \tilde{c}^{-1} & \tilde{c}^{1/2} \end{bmatrix}$$

is invertible, so that we can find constants $\tilde{C}_1, \tilde{C}_2 > 0$ that ensure $\|A^{-1}\|_{\mathrm{op}} \leq \tilde{C}_2$ for all $A \in \mathbb{R}^{2\times 2}$ which satisfy

$$\big| A_{ij} - [K_{\tilde{c}}(0)]_{ij} \big| \leq \tilde{C}_1, \quad (i,j) \in \{1,2\}^2.$$

Here, $\|\cdot\|_{\mathrm{op}}$ denotes the operator-norm on $\big(\mathbb{R}^2, \|\cdot\|_1\big)$, chosen for convenience in the computations below. Now note that

$$\begin{aligned} \big| [K_{\tilde{c}}(\tilde{v})]_{ij} - [K_{\tilde{c}}(0)]_{ij} \big| &\leq \Big( \big|\langle \tilde{v}, \partial_c \phi_{\tilde{c}} \rangle\big| + \big|\langle \tilde{v}, \partial_x \phi_{\tilde{c}} \rangle\big| + \big|\langle \tilde{v}, \partial_c \zeta_{\tilde{c}} \rangle\big| \Big) \\ &\leq \big( \|\partial_c \phi_{\tilde{c}}\|_{L^2_{-w}} + \|\partial_x \phi_{\tilde{c}}\|_{L^2_{-w}} + \|\partial_c \zeta_{\tilde{c}}\|_{L^2_{-w}} \big) \|\tilde{v}\|_{L^2_w} \end{aligned}$$

for all $(i,j) \in \{1,2\}^2$. Ensuring $\delta_2 \leq \big( \|\partial_c \phi_{\tilde{c}}\|_{L^2_{-w}} + \|\partial_x \phi_{\tilde{c}}\|_{L^2_{-w}} + \|\partial_c \zeta_{\tilde{c}}\|_{L^2_{-w}} \big)^{-1} \tilde{C}_1$, it follows that $\left\|K_{\tilde{c}}^{-1}(\tilde{v})\right\|_{\text{op}} \le \tilde{C}_2$. Estimating

$$\begin{aligned}\left|\left\langle N(\tilde{v}), \phi_{\tilde{c}}\right\rangle\right| + \left|\left\langle N(\tilde{v}), \zeta_{\tilde{c}}\right\rangle\right| &= \left|\left\langle \partial_x(\tilde{v}^2), \phi_{\tilde{c}}\right\rangle\right| + \left|\left\langle \partial_x(\tilde{v}^2), \zeta_{\tilde{c}}\right\rangle\right| \\ &= \left|\langle \tilde{v}^2, \partial_x\phi_{\tilde{c}}\rangle\right| + \left|\langle \tilde{v}^2, \partial_c\phi_{\tilde{c}}\rangle\right| \\ &\le \tilde{C}_5\|\tilde{v}\|_{L^2_w}^2\end{aligned}$$

where

$$\tilde{C}_5 = \sup_{x\le 0}\Big(|e^{-2wx}\partial_x\phi_{\tilde{c}}(x)| + |e^{-2wx}\partial_c\phi_{\tilde{c}}(x)|\Big)$$

establishes (4.26). Writing

$$\begin{bmatrix} c_f(\tilde{v},\tilde{c}) - c_f(0,\tilde{c}) \\ \Omega_f(\tilde{v},\tilde{c}) - \Omega_f(0,\tilde{c})\end{bmatrix} = -K_{\tilde{c}}^{-1}(\tilde{v})\begin{bmatrix}\langle \tilde{v}, \phi_{\tilde{c}}\rangle \\ \langle \tilde{v}, \zeta_{\tilde{c}}\rangle\end{bmatrix} + \big(K_{\tilde{c}}(0)^{-1} - K_{\tilde{c}}^{-1}(\tilde{v})\big)\begin{bmatrix}\langle \phi_{\tilde{c}}, \phi_{\tilde{c}}\rangle \\ \langle \phi_{\tilde{c}}, \zeta_{\tilde{c}}\rangle\end{bmatrix}$$

establishes (4.28). Writing

$$\begin{aligned}\begin{bmatrix} Q^{1/2}c_s(\tilde{v},\tilde{c}) - Q^{1/2}c_s(0,\tilde{c}) \\ Q^{1/2}\Omega_s(\tilde{v},\tilde{c}) - Q^{1/2}\Omega_s(0,\tilde{c})\end{bmatrix} &= -K_{\tilde{c}}^{-1}(\tilde{v})\begin{bmatrix} Q^{1/2}(\tilde{v}\phi_{\tilde{c}}) \\ Q^{1/2}(\tilde{v}\zeta_{\tilde{c}})\end{bmatrix} \\ &\quad + \big(K_{\tilde{c}}^{-1}(0) - K_{\tilde{c}}^{-1}(\tilde{v})\big)\begin{bmatrix} Q^{1/2}(\phi_{\tilde{c}}^2) \\ Q^{1/2}(\phi_{\tilde{c}}\zeta_{\tilde{c}})\end{bmatrix}\end{aligned}$$

together with

$$\|Q^{1/2}g\|_{L^2} = \|\sqrt{\hat{q}}\hat{g}\|_{L^2} \le \|\hat{q}\|_\infty^{1/2}\|\hat{g}\|_{L^2} \le \|q\|_{L^1}^{1/2}\|g\|_{L^2}, \quad g \in L^2$$

and

$$\|\tilde{v}\phi_{\tilde{c}}\|_{L^2} + \|\tilde{v}\zeta_{\tilde{c}}\|_{L^2} \le \big(\|\phi_{\tilde{c}}\|_{L^2_{-w}} + \|\zeta_{\tilde{c}}\|_{L^2_{-w}}\big)\|\tilde{v}\|_{L^2_w}$$

establishes (4.25). Lastly, (4.27) follows in the same way, using

$$\begin{aligned}&\left|\left\langle Y_d(\tilde{v},\tilde{c}) - Y_d(0,\tilde{c}), \phi_{\tilde{c}}\right\rangle\right| + \left|\left\langle Y_d(\tilde{v},\tilde{c}) - Y_d(0,\tilde{c}), \zeta_{\tilde{c}}\right\rangle\right| \\ &\quad\le \tfrac{1}{2}\left|\left\|Q^{1/2}\Omega_s(\tilde{v},\tilde{c})\right\|_{L^2}^2 - \left\|Q^{1/2}\Omega_s(0,\tilde{c})\right\|_{L^2}^2\right|\Big(\left|\langle \partial_x^2\phi_{\tilde{c}}, \phi_{\tilde{c}}\rangle\right| + \left|\langle \partial_x^2\phi_{\tilde{c}}, \zeta_{\tilde{c}}\rangle\right|\Big) \\ &\qquad + \tfrac{1}{2}\left|\left\|Q^{1/2}c_s(\tilde{v},\tilde{c})\right\|_{L^2}^2 - \left\|Q^{1/2}c_s(0,\tilde{c})\right\|_{L^2}^2\right|\Big(\left|\langle \partial_c^2\phi_{\tilde{c}}, \phi_{\tilde{c}}\rangle\right| + \left|\langle \partial_c^2\phi_{\tilde{c}}, \zeta_{\tilde{c}}\rangle\right|\Big) \\ &\qquad + \tfrac{1}{2}\left\|Q^{1/2}\Omega_s(\tilde{v},\tilde{c})\right\|_{L^2}^2\Big(\left|\langle \tilde{v}, \partial_x^2\phi_{\tilde{c}}\rangle\right| + \left|\langle \tilde{v}, \partial_x^2\zeta_{\tilde{c}}\rangle\right|\Big) \\ &\quad\le \tilde{C}_6\big(1 + \|\tilde{v}\|_{L^2_w} + \|\tilde{v}\|_{L^2_w}^2\big)\|\tilde{v}\|_{L^2_w}\end{aligned}$$

and

$$\begin{aligned}&\left\|\big(Z^*(\tilde{v},\tilde{c}) - Z^*(0,\tilde{c})\big)[g]\right\|_{L^2} \\ &\qquad\le \left\|(g+\phi_{\tilde{c}})\tilde{v}\right\|_{L^2} + \left\|\Omega_s(\tilde{v},\tilde{c}) - \Omega_s(0,\tilde{c})\right\|_{L^2}\left|\langle \partial_x g, \phi_{\tilde{c}} + \tilde{v}\rangle\right|\end{aligned}$$

$$
\begin{aligned}
&+ \big\|\Omega_s(\tilde v, \tilde c)\big\|_{L^2} \big|\langle \partial_x g, \tilde v\rangle\big| + \big\|c_s(\tilde v, \tilde c) - c_s(0, \tilde c)\big\|_{L^2} \big|\langle g, \partial_c \phi_{\tilde c}\rangle\big| \\
&\leq \tilde C_7 \big(1 + \|\tilde v\|_{L^2_w}\big) \|\tilde v\|_{L^2_w}. \qquad \square
\end{aligned}
$$

## 4.4 Local modulation system

Now that we have set up the modulation system $\big(v(t), c(t), \xi(t)\big)$, we turn our attention to the main goal of this paper: asserting that the remainder $v(t)$ defined through

$$v(t,x) = u\big(t, x + \xi(t)\big) - \phi_{c(t)}(x)$$

remains small (measured in the norm $H^1_w$). However, we do not base our arguments on the operator $\mathcal{L}_{c(t)}$ that represents the linear part of (4.21). The main reason is that $\mathcal{L}_{c(t)}$ is non-autonomous, which complicates stability arguments based on the stability properties of the flow generated by $\mathcal{L}_c$. A second reason is that $v$ is defined through a stochastic shift of $u$, and hence (4.21) contains the term $\partial_x^2 v$, which presents regularity issues.

Given $T \geq 0$, we therefore introduce a *local* modulation system $\mathbf{m}^T := \big(v^T, c^T, \xi^T\big)$ through

$$v^T(s,x) = u\big(T + s, x + \xi(T) + c(T)s\big) - \Phi^T(\mathbf{m}^T(s), s, x), \tag{4.29}$$

where we have abbreviated

$$\Phi^T(\mathbf{m}^T(s), s, x) := \phi_{c^T(s)}\big(x + \xi(T) + c(T)s - \xi^T(s)\big).$$

The local remainder $v^T(s)$ is defined by shifting $u$ with constant velocity $c(T)$. This freezes the wave around the origin in the absence of forcing, while avoiding an Itô correction term of the form $\partial_x^2$. The parameter $\xi^T(s)$ then has the interpretation as soliton position, and accounts for corrections due to the forcing. The local modulation parameters $c^T(s)$ and $\xi^T(s)$ are uniquely determined through the condition

$$\big\langle v^T(s), \phi_{c(T)}\big\rangle = \big\langle v^T(s), \zeta_{c(T)}\big\rangle = 0. \tag{4.30}$$

In particular, we have

$$\big(v^T(0), c^T(0), \xi^T(0)\big) = \big(v(T), c(T), \xi(T)\big).$$

In the absence of noise and forcing, with $v(T) = 0$, the local modulation parameters keep their constant value $c^T(s) = c(T)$ and $\xi^T(s) = \xi(T)$. The unique existence of the decomposition (4.29) is guaranteed by the following lemma, a variation on Lemma 4.3.1 tailored to the condition (4.30).

**Lemma 4.4.1.** *Assuming S2, there exists a constant $\delta_3 > 0$ so that for each $v_* \in H^1_w \cap H^1$ and $c_*, c_0 \in [c_{\min}, c_{\max}]$ with $\|v_*\|_{L^2_w}, |c_* - c_0| \leq \delta_3$, there exist unique parameters $c > 0, \xi \in \mathbb{R}$ and a unique function $v \in H^1_w \cap H^1$ that together enforce* *the identities*

$$\phi_{c_*}(x) + v_*(x) = \phi_c(x-\xi) + v(x-\xi) \quad \textit{with} \quad \langle v, \phi_{c_0}\rangle = \langle v, \zeta_{c_0}\rangle = 0.$$

Existence of the local decomposition is thus guaranteed as long as the remainder $v^T(s)$ and the difference $|c(T) - c^T(s)|$ remain under $\delta_3$. The advantage of defining the local system through the conditions (4.30) is that it demands orthogonality with respect to *fixed* eigenfunctions $\phi_{c(T)}$ and $\zeta_{c(T)}$, facilitating the use of the stability properties of $\{e^{\mathcal{L}_{c(T)}t}\}_{t\geq 0}$ to control the growth of $v^T(s)$. Rewriting (4.29) as

$$\begin{aligned} v^T(s,x) &= v\big(T+s, x+\xi(T)+c(T)s-\xi(T+s)\big) \\ &\quad + \phi_{c(T+s)}\big(x+\xi(T)+c(T)s-\xi(T+s)\big) - \Phi^T(\mathbf{m}^T(s), s, x), \end{aligned}$$

we observe that as long as the local modulation parameters $c^T(s)$ and $\xi^T(s)$ do not deviate substantially from their global counterparts $c(T+s)$ and $\xi(T+s)$, then neither do $v^T(s)$ and $v(T+s)$. As long as this holds, we can understand the growth of $v(T+s)$ through $v^T(s)$. The following lemma asserts this correspondence.

**Lemma 4.4.2.** *Assuming S1 and S2, there exists a constant $C_4 > 0$ such that the following holds true. For all $T, s, \delta \geq 0$, the inclusions*

$$c^T(s), c(T+s) \in [\tfrac{1}{2}c_{\min}, 2c_{\max}]$$

*and the bound*

$$\big|c^T(s) - c(T+s)\big| + \big|\xi^T(s) - \xi(T+s)\big| \leq \delta,$$

*imply*

$$\Big|\big\|v(T+s)\big\|_{H^1_w} - e^{w(\xi(T)+c(T)s-\xi(T+s))}\big\|v^T(s)\big\|_{H^1_w}\Big| \leq C_4\delta.$$

*Proof.* We compute

$$\begin{aligned} &\Big|\big\|e^{w\cdot}v(T+s,\cdot)\big\|_{H^1} - e^{w(\xi(T)+c(T)s-\xi(T+s))}\big\|e^{w\cdot}v^T(s,\cdot)\big\|_{H^1}\Big| \\ &= \Big|\big\|e^{w\cdot}v(T+s,\cdot)\big\|_{H^1} - \big\|e^{w\cdot}v^T(s,\cdot-\xi(T)-c(T)s+\xi(T+s))\big\|_{H^1}\Big|. \end{aligned}$$

Using the reverse triangle inequality, we get

$$\begin{aligned} &\Big|\big\|e^{w\cdot}v(T+s,\cdot)\big\|_{H^1} - e^{w(\xi(T)+c(T)s-\xi(T+s))}\big\|e^{w\cdot}v^T(s,\cdot)\big\|_{H^1}\Big| \\ &\leq \Big\|v(T+s) - v^T\big(s,\cdot-\xi(T)-c(T)s+\xi(T+s)\big)\Big\|_{H^1_w} \\ &= \Big\|\phi_{c(T+s)} - \phi_{c^T(s)}\big(\cdot+\xi(T+s)-\xi^T(s)\big)\Big\|_{H^1_w}. \end{aligned}$$

The result now follows by exploiting the $O(e^{-\sqrt{c}|x|})$ decay of the wave-profile

$$\phi_c(x) = \tfrac{3c}{2}\operatorname{sech}^2(\sqrt{c}x/2) = \frac{6c}{(e^{-\sqrt{c}x/2}+e^{\sqrt{c}x/2})^2}$$

and its derivatives $\partial_x \phi_c, \partial_x^2 \phi_c, \partial_c \phi_c$ and $\partial^2_{cx} \phi_c$. Indeed, it implies that the map $c \mapsto e^{wx}\phi_c(x) + e^{wx}\partial_x \phi_c(x)$ is Lipschitz from $[0, 2c_{\max}]$ to $L^2$. □

Let us proceed by describing the dynamics of the local modulation system. We once again introduce a phase-shift parameter $\Omega^T(s)$ through

$$\xi^T(s) = \int_0^s c^T(s')\mathrm{d}s' + \Omega^T(s)$$

and will see that $c^T$ and $\Omega^T$ satisfy SDEs of the form

$$\begin{aligned}
\mathrm{d}c^T &= c_d^{\sigma,\epsilon,T}(\mathbf{m}^T(s), s)\ \mathrm{d}s + \sigma\big\langle c_s^T(\mathbf{m}^T(s), s), T_{\xi(T)+c(T)s}\mathrm{d}W^Q_{T+s}\big\rangle, && (4.31)\\
\mathrm{d}\Omega^T &= \Omega_d^{\sigma,\epsilon,T}(\mathbf{m}^T(s), s)\ \mathrm{d}s + \sigma\big\langle \Omega_s^T(\mathbf{m}^T(s), s), T_{\xi(T)+c(T)s}\mathrm{d}W^Q_{T+s}\big\rangle. && (4.32)
\end{aligned}$$

A formal application of Itô's lemma then shows that

$$\mathrm{d}v^T = \mathcal{L}_{c(T)}v^T\ \mathrm{d}s + Y^{\sigma,\epsilon,T}(\mathbf{m}^T(s), s)\ \mathrm{d}s + \sigma Z^T(\mathbf{m}^T(s), s)\ T_{\xi(T)+c(T)s}\mathrm{d}W^Q_{T+s}, \tag{4.33}$$

where

$$Y^{\sigma,\epsilon,T}(\mathbf{m}^T, s) = Y_I^{\sigma,\epsilon,T}(\mathbf{m}^T, s) + \epsilon f(T+s)v^T \tag{4.34}$$

with

$$\begin{aligned}
Y_I^{\sigma,\epsilon,T}(\mathbf{m}^T, s) = {}& 2\partial_x\big((\phi_{c(T)} - \Phi^T(\mathbf{m}^T, s))v^T\big) + N(v^T) + \epsilon f(T+s)\Phi^T(\mathbf{m}^T, s) && (4.35)\\
& - c_d^{\sigma,\epsilon,T}(\mathbf{m}^T, s)\partial_c\Phi^T(\mathbf{m}^T, s) + \Omega_d^{\sigma,\epsilon,T}(\mathbf{m}^T, s)\partial_x\Phi^T(\mathbf{m}^T, s)\\
& + \tfrac{1}{2}\sigma^2\Big[\big\|Q^{1/2}c_s^T(\mathbf{m}^T, s)\big\|^2_{L^2}\partial_c^2 + \big\|Q^{1/2}\Omega_s^T(\mathbf{m}^T, s)\big\|^2_{L^2}\partial_x^2\Big]\Phi^T(\mathbf{m}^T, s)
\end{aligned}$$

and

$$\begin{aligned}
Z^T(\mathbf{m}^T, s)[h] = {}& \big(\Phi^T(\mathbf{m}^T, s) + v^T\big)h\\
& + \Big( - \langle c_s^T(\mathbf{m}^T, s), h\rangle\partial_c + \langle \Omega_s^T(\mathbf{m}^T, s), h\rangle\partial_x\Big)\Phi^T(\mathbf{m}^T, s). && (4.36)
\end{aligned}$$

In (4.34) and (4.36), $\partial_c\Phi^T$ should be interpreted as

$$\partial_c\Phi^T(\mathbf{m}^T, s, x) := \partial_c\phi_{c^T}\big(x + \xi(T) + c(T)s - \xi^T\big).$$

However, we can not rigorously justify (4.33), as $v^T$ is not regular enough for $\mathcal{L}_{c(T)}v^T$ to be well-defined. We therefore pass to a mild formulation with respect to the flow generated by the linear equation $w_t = \mathcal{L}_{c(T)}w$ (see Theorem 4.2.2).

**Proposition 4.4.3** (See Section 4.11)**.** *Assume S1 and S2. For each $T, s \geq 0$, the* *inequalities*

$$\|v^T(s')\|_{L^2_w} \le \delta_3 \quad \textit{and} \quad c^T(s') \in [\tfrac{1}{2}c_{\min}, 2c_{\max}], \quad s' \in [0,s]$$

*imply that*

$$\begin{aligned} v^T(s) &= e^{\mathcal{L}_{c(T)}s}v(T) + \int_0^s e^{\mathcal{L}_{c(T)}(s-s')}Y^{\sigma,\epsilon,T}(\mathbf{m}^T(s'),s')\mathrm{d}s' \\ &\quad + \sigma \int_0^s e^{\mathcal{L}_{c(T)}(s-s')}Z^T(\mathbf{m}^T(s'),s')\ T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}, \end{aligned} \tag{4.37}$$

$\mathbb{P}$*-almost surely.*

It follows straightforwardly that

$$\begin{aligned} \mathrm{d}\big\langle v^T(s), \phi_{c(T)}\big\rangle &= \Big\langle Y^{\sigma,\epsilon,T}\big(\mathbf{m}^T(s),s\big), \phi_{c(T)}\Big\rangle\ \mathrm{d}s \\ &\quad + \sigma\Big\langle Z^T\big(\mathbf{m}^T(s),s\big)[T_{\xi(T)+c(T)s}\mathrm{d}W^Q_{T+s}], \phi_{c(T)}\Big\rangle, \\ \mathrm{d}\big\langle v^T(s), \zeta_{c(T)}\big\rangle &= \Big\langle Y^{\sigma,\epsilon,T}\big(\mathbf{m}^T(s),s\big), \phi_{c(T)}\Big\rangle\ \mathrm{d}s \\ &\quad + \sigma\Big\langle Z^T\big(\mathbf{m}^T(s),s\big)[T_{\xi(T)+c(T)s}\mathrm{d}W^Q_{T+s}], \zeta_{c(T)}\Big\rangle. \end{aligned}$$

For the orthogonality conditions (4.30) to hold, we must have

$$\begin{bmatrix} \big\langle Z^T(\mathbf{m}^T,s)[h], \phi_{c(T)}\big\rangle \\ \big\langle Z^T(\mathbf{m}^T,s)[h], \zeta_{c(T)}\big\rangle \end{bmatrix} = 0, \quad \forall h \in L^2_Q.$$

This shows that

$$\begin{bmatrix} c^T_s(\mathbf{m}^T,s) \\ \Omega^T_s(\mathbf{m}^T,s) \end{bmatrix} = (K^T)^{-1}(\mathbf{m}^T,s) \begin{bmatrix} (\Phi^T(\mathbf{m}^T,s)+v^T)\phi_{c(T)} \\ (\Phi^T(\mathbf{m}^T,s)+v^T)\zeta_{c(T)} \end{bmatrix},$$

where

$$K^T(\mathbf{m}^T,s) = \begin{bmatrix} \langle \partial_c \Phi^T(\mathbf{m}^T,s), \phi_{c(T)}\rangle & -\langle \partial_x \Phi^T(\mathbf{m}^T,s), \phi_{c(T)}\rangle \\ \langle \partial_c \Phi^T(\mathbf{m}^T,s), \zeta_{c(T)}\rangle & -\langle \partial_x \Phi^T(\mathbf{m}^T,s), \zeta_{c(T)}\rangle \end{bmatrix}.$$

The drift components $c^{\sigma,\epsilon,T}_d$, and $\Omega^{\sigma,\epsilon,T}_d$ follow by solving

$$\begin{bmatrix} \big\langle Y^{\sigma,\epsilon,T}(\mathbf{m}^T,s), \phi_{c(T)}\big\rangle \\ \big\langle Y^{\sigma,\epsilon,T}(\mathbf{m}^T,s), \zeta_{c(T)}\big\rangle \end{bmatrix} = 0,$$

leading to

$$\begin{aligned} c^{\sigma,\epsilon,T}_d(\mathbf{m}^T,s) &= c^{0,T}_d(\mathbf{m}^T,s) + \epsilon f(T+s)c^T_f(\mathbf{m}^T,s) + \sigma^2 c^T_d(\mathbf{m}^T,s) \\ \Omega^{\sigma,\epsilon,T}_d(\mathbf{m}^T,s) &= \Omega^{0,T}_d(\mathbf{m}^T,s) + \epsilon f(T+s)\Omega^T_f(\mathbf{m}^T,s) + \sigma^2 \Omega^T_d(\mathbf{m}^T,s) \end{aligned}$$

with

$$\begin{bmatrix} c_d^{0,T}(\mathbf{m}^T, s) \\ \Omega_d^{0,T}(\mathbf{m}^T, s) \end{bmatrix} = (K^T)^{-1}(\mathbf{m}^T, s) \begin{bmatrix} \langle N(v^T), \phi_{c(T)} \rangle \\ \langle N(v^T), \zeta_{c(T)} \rangle \end{bmatrix} + 2(K^T)^{-1}(\mathbf{m}^T, s) \begin{bmatrix} \Big\langle \partial_x\big((\phi_{c(T)} - \Phi^T(\mathbf{m}^T, s))v^T\big), \phi_{c(T)} \Big\rangle \\ \Big\langle \partial_x\big((\phi_{c(T)} - \Phi^T(\mathbf{m}^T, s))v^T\big), \zeta_{c(T)} \Big\rangle \end{bmatrix},$$

$$\begin{bmatrix} c_f^T(\mathbf{m}^T, s) \\ \Omega_f^T(\mathbf{m}^T, s) \end{bmatrix} = (K^T)^{-1}(\mathbf{m}^T, s) \begin{bmatrix} \langle \Phi^T(\mathbf{m}^T, s) + v^T, \phi_{c(T)} \rangle \\ \langle \Phi^T(\mathbf{m}^T, s) + v^T, \zeta_{c(T)} \rangle \end{bmatrix},$$

$$\begin{bmatrix} c_d^T(\mathbf{m}^T, s) \\ \Omega_d^T(\mathbf{m}^T, s) \end{bmatrix} = \tfrac{1}{2}\|Q^{1/2}c_s^T(\mathbf{m}^T, s)\|_{L^2}^2 (K^T)^{-1}(\mathbf{m}^T, s) \begin{bmatrix} \Big\langle \partial_c^2\Phi^T(\mathbf{m}^T, s), \phi_{c(T)} \Big\rangle \\ \Big\langle \partial_c^2\Phi^T(\mathbf{m}^T, s), \zeta_{c(T)} \Big\rangle \end{bmatrix} + \tfrac{1}{2}\|Q^{1/2}\Omega_s^T(\mathbf{m}^T, s)\|_{L^2}^2 (K^T)^{-1}(\mathbf{m}^T, s) \begin{bmatrix} \Big\langle \partial_x^2\Phi^T(\mathbf{m}^T, s), \phi_{c(T)} \Big\rangle \\ \Big\langle \partial_x^2\Phi^T(\mathbf{m}^T, s), \zeta_{c(T)} \Big\rangle \end{bmatrix}.$$

We now establish control on the local modulation parameters, assuming a-priori control of the quantity

$$R_w^T(s) := \|v^T(s)\|_{L_w^2} + \big|c(T) - c^T(s)\big| + \big|\xi(T) + c(T)s - \xi^T(s)\big|. \tag{4.38}$$

**Lemma 4.4.4.** *Assuming S1 and S2, there exist constants $\delta_5, C_5 > 0$ such that following holds true. For all $T, s \geq 0$, the bounds*

$$c(T), c^T(s) \in [\tfrac{1}{2}c_{\min}, 2c_{\max}] \quad \textit{and} \quad R_w^T(s) \leq \delta_5$$

*imply*

$$\big\|Q^{1/2}c_s^T(\mathbf{m}^T(s), s)\big\|_{L^2} + \big\|Q^{1/2}\Omega_s^T(\mathbf{m}^T(s), s)\big\|_{L^2} \leq C_5\big(1 + \|v^T(s)\|_{L_w^2}\big), \tag{4.39}$$

$$\big|c_d^{T,0}(\mathbf{m}^T(s), s)\big| + \big|\Omega_d^{T,0}(\mathbf{m}^T(s), s)\big| \leq C_5 R_w^T(s)\|v^T(s)\|_{L_w^2}, \tag{4.40}$$

$$\big|c_f(\mathbf{m}^T(s), s)\big| + \big|\Omega_f(\mathbf{m}^T(s), s)\big| \leq C_5\big(1 + \|v^T(s)\|_{L_w^2}\big), \tag{4.41}$$

$$\big|c_d(\mathbf{m}^T(s), s)\big| + \big|\Omega_d(\mathbf{m}^T(s), s)\big| \leq C_5\big(1 + \|v^T(s)\|_{L_w^2}\big). \tag{4.42}$$

*Proof.* As in the proof of Lemma 4.3.4, note that

$$K^T(\mathbf{m}^T(0), 0) = \begin{bmatrix} \langle \partial_c\phi_{c(T)}, \phi_{c(T)} \rangle & 0 \\ \langle \partial_c\phi_{c(T)}, \zeta_{c(T)} \rangle & -\langle \partial_x\phi_{c(T)}, \zeta_{c(T)} \rangle \end{bmatrix}.$$

In particular, we can find constants $\tilde{C}_1, \tilde{C}_2 > 0$ that ensure $\|A^{-1}\|_{\mathrm{op}} \leq \tilde{C}_2$ for all $A \in \mathbb{R}^{2\times 2}$ which satisfy

$$\Big|A_{ij} - K_{ij}^T(\mathbf{m}^T(0), 0)\Big| \leq \tilde{C}_1, \quad (i,j) \in \{1,2\}^2.$$

Recalling that $\Phi^T(\mathbf{m}^T(s),s) = \phi_{c^T(s)}\big(\cdot + \xi(T) + c(T)s - \xi^T(s)\big)$, the Lipschitz properties of $\phi_c$ imply

$$\Big|K_{ij}^T(\mathbf{m}^T(s),s) - K_{ij}^T(\mathbf{m}^T(0),0)\Big| \leq \tilde{C}_3\Big(\big|\xi(T) + c(T)s - \xi^T(s)\big| + \big|c^T(s) - c(T)\big|\Big)$$

for all $(i,j) \in \{1,2\}^2$. In case $\delta_5 \leq \tilde{C}_3^{-1}\tilde{C}_1$, it follows that $\big\|(K^T)^{-1}(\mathbf{m}^T,s)\big\|_{\mathrm{op}} \leq \tilde{C}_2$. Estimating

$$\begin{aligned}\Big|\Big\langle \partial_x\big((\phi_{c(T)} - \Phi^T(\mathbf{m}^T,s))v^T(s)\big), \phi_{c(T)}\Big\rangle\Big| + \Big|\Big\langle \partial_x\big((\phi_{c(T)} - \Phi^T(\mathbf{m}^T(s),s))v^T(s)\big), \zeta_{c(T)}\Big\rangle\Big| \\ \leq \big(\|\partial_x\phi_{c(T)}\|_\infty + \|\partial_x\phi_{c(T)}\|_\infty\big)\|\phi_{c(T)} - \Phi^T(\mathbf{m}^T(s),s)\|_{L^2_{-w}}\|v^T(s)\|_{L^2_w}\end{aligned}$$

establishes (4.40). The estimates (4.39), (4.41) and (4.42) follow by applying the Cauchy-Schwarz inequality. □

We conclude this section with a result on the deterministic integral in (4.37). This is provided by Corollary 4.4.6 below and forms the basis of our stability arguments. Our estimates for the deterministic terms ($\sigma = 0$) in (4.37) are similar to those in [91], save for our treatment of the nonlinearity $N(v^T) = -\partial_x(v^T)^2$. Here we follow the approach of Mizumachi and Tzvetkov [86] which uses property (4.16). This allows us to control the nonlinear term based on the condition that $\|v^T\|_{L^2}$ is small. This is an improvement over the standard argument used in [91], which requires control of $\|\partial_x v^T\|_{L^2}$ and consequently a more cumbersome energy argument.

**Lemma 4.4.5.** *Assuming S1 and S2, there exists a constant $C_6 > 0$ such that for each $T, s \geq 0$ the inequalities*

$$c(T), c^T(s) \in [\tfrac{1}{2}c_{\min}, 2c_{\max}] \quad \textit{and} \quad R_w^T(s) \leq \delta_5$$

*imply*

$$\begin{aligned}\Big\|Y_I^{\sigma,\epsilon,T}(\mathbf{m}^T(s),s)\Big\|_{L^1_w} &\leq C_6\sigma^2\big(1 + \|v^T(s)\|_{L^2_w}\big)^2 + C_6\epsilon \\ &\quad + C_6\Big(\|v^T(s)\|_{L^2} + R_w^T(s)\Big)\|v^T(s)\|_{H^1_w}.\end{aligned}$$

*Proof.* The various terms in $Y_I^{\sigma,\epsilon,T}\big(\mathbf{m}^T(s),s\big)$—see (4.35)—can be controlled as follows. The inequality

$$\begin{aligned}&\Big\|2\partial_x\big((\phi_{c(T)} - \Phi^T(\mathbf{m}^T(s),s))v^T\big)\Big\|_{L^1_w} \\ &\qquad \leq 2\big\|\phi_{c(T)} - \Phi^T(\mathbf{m}^T(s),s)\big\|_{H^1}\big\|v^T(s)\big\|_{H^1_w} \\ &\qquad \leq \tilde{C}_1\Big(\big|c(T) - c^T(s)\big| + \big|\xi(T) + c(T)s - \xi^T(s)\big|\Big)\big\|v^T(s)\big\|_{H^1_w}\end{aligned}$$

follows from the fact that $c \mapsto \phi_c(x) + \partial_x\phi_c(x)$ is Lipschitz from $[0, c_{\max}]$ to $L^2$. Via

Lemma 4.4.4 we find

$$\begin{aligned}\Big\|\Big[-c_d^{\sigma,\epsilon,T}(\mathbf{m}^T(s),s)\partial_c+\Omega_d^{\sigma,\epsilon,T}(\mathbf{m}^T(s),s)\partial_x\Big]\Phi^T(\mathbf{m}^T(s),s)\Big\|_{L^1_w}\\ \leq \tilde{C}_2\Big|c_d^{\sigma,\epsilon,T}(\mathbf{m}^T(s),s)+\Omega_d^{\sigma,\epsilon,T}(\mathbf{m}^T(s),s)\Big|\\ \leq \tilde{C}_2C_5R_w^T(s)\|v^T(s)\|_{L^2_w},\end{aligned}$$

where $\tilde{C}_2$ is a constant large enough to ensure

$$\|\partial_c\Phi^T(\mathbf{m}^T(s),s)\|_{L^1_w}+\|\partial_x\Phi^T(\mathbf{m}^T(s),s)\|_{L^1_w}\leq\tilde{C}_2.$$

Similarly,

$$\big\|Q^{1/2}c_s^T(\mathbf{m}^T(s),s)\big\|_{L^2}^2\big\|\partial_c^2\Phi^T(\mathbf{m}^T(s),s)\big\|_{L^1_w}\leq\tilde{C}_3\big(1+\|v^T(s)\|_{L^2_w}\big)^2,$$

and

$$\big\|Q^{1/2}\Omega_s^T(\mathbf{m}^T(s),s)\big\|_{L^2}^2\big\|\partial_x^2\Phi^T(\mathbf{m}^T(s),s)\big\|_{L^1_w}\leq\tilde{C}_3\big(1+\|v^T(s)\|_{L^2_w}\big)^2.$$

Lastly,

$$\Big\|\partial_x\big((v^T(s))^2\big)\Big\|_{L^1_w}\leq 2\|v^T(s)\|_{L^2}\|v^T(s)\|_{H^1_w},$$

and

$$\big\|\epsilon f(T+s)\Phi^T(\mathbf{m}^T(s),s)\big\|_{L^1_w}\leq\tilde{C}_4\epsilon.$$ □

**Corollary 4.4.6.** *Assuming S1 and S2, there exists a constant $C_7>0$ such that for each $T,s\geq 0$ the inequalities*

$$c_{\min}\leq c^T(s')\leq c_{\max}\quad\textit{and}\quad R_w^T(s')\leq\delta_5,\quad s'\in[0,s]$$

*imply that*

$$\begin{aligned}\Big\|\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}Y^{\sigma,\epsilon,T}\big(\mathbf{m}^T(s'),s'\big)\mathrm{d}s'\Big\|_{H^1_w}\leq C_7 s\sup_{0\leq s'\leq s}\Big(\|v^T(s')\|_{L^2}+R_w^T(s')\Big)\big\|v^T(s')\big\|_{H^1_w}\\ +C_7(\sigma^2+\epsilon)s,\end{aligned}\tag{4.43}$$

*holds $\mathbb{P}$-almost surely.*

*Proof.* Consider the decomposition

$$Y^{\sigma,\epsilon,T}\big(\mathbf{m}^T(s'),s'\big)=\underbrace{Y_I^{\sigma,\epsilon,T}\big(\mathbf{m}^T(s'),s'\big)}_{\text{in }L^1_w}+\underbrace{\epsilon f(T+s')v^T(s')}_{\text{in }L^2_w}.$$

Both terms are contained in the stable subspace of $\left\{e^{\mathcal{L}_{c(T)}t}\right\}_{t\geq 0}$. We have via (4.15)

$$\left\| \int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\left[f(T+s')v^T(s')\right]\mathrm{d}s'\right\|_{H^1_w} \leq M\int_0^s e^{-b(s-s')}(s-s')^{-1/2}\mathrm{d}s' \sup_{s'\in[0,s]}\left\|v^T(s')\right\|_{L^2_w}.$$

Here, $b$ and $M$ are the constants appearing in the semigroup-bounds (4.15) and (4.16). We remark also that

$$\sup_{s\geq 0}\int_0^s e^{-b(s-s')}(s-s')^{-1/2}\mathrm{d}s' < \infty.$$

Using (4.16), on the other hand, we have

$$\begin{aligned}
&\left\| \int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\left[Y_I^{\sigma,\epsilon,T}\left(\mathbf{m}^T(s'),s'\right)\right]\mathrm{d}s'\right\|_{H^1_w} \\
&\leq M\int_0^s e^{-b(s-s')}(s-s')^{-3/4}\mathrm{d}s \sup_{s'\in[0,s]}\left\|Y_I^{\sigma,\epsilon,T}\left(\mathbf{m}^T(s'),s'\right)\right\|_{L^1_w},
\end{aligned}$$

where also

$$\sup_{s\geq 0}\int_0^s e^{-b(s-s')}(s-s')^{-3/4}\mathrm{d}s < \infty.$$

The result now follows by applying Lemma 4.4.5. □

## 4.5 Weighted norm control

In this section, we control the local remainder $v^T$ on time intervals $[0,\Delta T]$. We do so by exploiting the stability properties of the linear flow $\{e^{\mathcal{L}_{c(T)}t}\}_{t\geq 0}$ on weighted spaces. As pointed out, control of the local remainder $v^T$ transfers to the global remainder $v$ via Lemma 4.4.2. The main result of this section is Proposition 4.5.1 below, which bounds the probability that $v^T$ grows large on a time interval $[0,\Delta T]$. We show that $\|v^T\|_{H^1_w}$ only grows large with small probability, provided that

- the soliton amplitude remains within fixed bounds;
- the unweighted $L^2$-norm of $v^T$ remains small;
- the difference between global and local modulation parameters remains small.

An important ingredient for ensuring that we can repeat our stability argument on time intervals of size $\Delta T$ is the exponential decay of the semigroup after time $\Delta T$ in the first term of (4.46). We thus require that $\Delta T$ is large enough to guarantee significant decay. We formulate this in the following condition, and fix constants $\delta_*, \eta_0$ for later use.

**C1** *The constants $\Delta T, \delta_* > 0$ and $\eta_0 > 0$ satisfy*

- $\Delta T = \log(6M)/b$;

– $\delta_* \leq \min\left\{\frac{1}{216MC_7(\Delta T+\Delta T^2)}, \frac{1}{4}c_{\min}, \frac{1}{2}c_{\max}, \frac{1}{e^2+3C_4}\right\}$;
– $\eta_0 \leq \min\left\{\delta_*, \delta_5, 1, \frac{1}{8C_5(\Delta T+\Delta T^2)}\right\}$.

The constants $C_4, C_5, \delta_5$ and $C_7$ have been introduced in Lemma 4.4.2, Lemma 4.4.4 and Corollary 4.4.6, respectively. The constants $b, M$ are introduced in the semigroup-bounds (4.15). To keep track of the weighted norm $\|v^T(s)\|_{H^1_w}$, we define for each $T, \eta > 0$ the stopping time $\tau^T_{\mathrm{st}}$ as

$$\tau^T_{\mathrm{st}}(\eta) = \sup\left\{s \geq 0 : \left\|v^T(s)\right\|_{H^1_w} \leq \eta\right\}.$$

We furthermore introduce stopping times $\tau^T_c, \tau^T_{\mathrm{en}}, \tau^T_{\mathrm{amp}}$ and $\tau^T_{\mathrm{pos}}$ which encode the conditions for stability:

$$\begin{aligned}
\tau^T_c &= \sup\left\{s \geq 0 : c(T+s) \in \left[\tfrac{1}{2}c_{\min}, 2c_{\max}\right]\right\};\\
\tau^T_{\mathrm{en}}(\eta) &= \sup\left\{s \geq 0 : \left\|v^T(s)\right\|_{L^2} \leq \eta\right\};\\
\tau^T_{\mathrm{amp},1}(\eta) &= \sup\left\{s \geq 0 : \left|c(T) - c^T(s)\right| \leq \delta_*\eta\right\};\\
\tau^T_{\mathrm{amp},2}(\eta) &= \sup\left\{s \geq 0 : \left|c(T+s) - c^T(s)\right| \leq \delta_*\eta\right\};\\
\tau^T_{\mathrm{pos},1}(\eta) &= \sup\left\{s \geq 0 : \left|\xi(T) + c(T)s - \xi^T(s)\right| \leq 2\Delta T\delta_*\eta\right\};\\
\tau^T_{\mathrm{pos},2}(\eta) &= \sup\left\{s \geq 0 : \left|\xi(T+s) - \xi^T(s)\right| \leq 2\Delta T\delta_*\eta\right\},
\end{aligned}$$

and we define

$$\tau^T_{\mathrm{mod}}(\eta) = \tau^T_{\mathrm{amp},1}(\eta) \wedge \tau^T_{\mathrm{amp},2}(\eta) \wedge \tau^T_{\mathrm{pos},1}(\eta) \wedge \tau^T_{\mathrm{pos},2}(\eta).$$

Our result is then as follows.

**Proposition 4.5.1** (Short-time control). *Assuming S1, S2 and C1, there exist constants $\delta_9 > 0$ and $C_9 \geq 1$ such that the following holds true. For all $\eta \in [0, \eta_0]$, $C_9\sigma, C_9\epsilon \in [0, \eta]$ and $T \geq 0$ the events*

$$\mathcal{E}_1 = \left\{\tau^T_{\mathrm{st}}(\eta) \leq \Delta T \wedge \tau^T_{\mathrm{en}}(\delta_*) \wedge \tau^T_{\mathrm{mod}}(\eta) \wedge \tau^T_c\right\}$$

*and*

$$\mathcal{E}_2 = \left\{\left\|v^T(\Delta T)\right\|_{H^1_w} \geq \tfrac{\eta}{9M}\right\} \cap \left\{\tau^T_{\mathrm{st}}(\eta) \wedge \tau^T_{\mathrm{en}}(\delta_*) \wedge \tau^T_{\mathrm{mod}}(\eta) \wedge \tau^T_c \geq \Delta T\right\}$$

*satisfy*

$$\mathbb{P}\left[\mathcal{E}_1 \cap \left\{\|v(T)\|_{H^1_w} \leq \tfrac{\eta}{3M}\right\}\right] + \mathbb{P}\left[\mathcal{E}_2 \cap \left\{\left\|v(T)\right\|_{H^1_w} \leq \tfrac{\eta}{3M}\right\}\right] \leq e^{-\delta_9\eta^2/\sigma^2}. \tag{4.44}$$

Our main tool for establishing Proposition 4.5.1 is based on the results of Section 4.4. We recall that the constant $C_7$ was introduced in Corollary 4.4.6.

**Lemma 4.5.2.** *Assume S1 and S2. For all $\sigma, \epsilon, T, s, \delta \geq 0$, each $\delta_* > 0$ and*

$\eta \in [0, \min\{\delta_*, \delta_5, 1\}]$ *that satisfy*

$$s + s^2 \leq \frac{\delta}{6C_7\delta_*} \quad \textit{and} \quad \max\{\sigma^2 s, \epsilon s\} \leq \frac{\delta\eta}{3C_7}, \tag{4.45}$$

*the inequalities*

$$\left\|v^T(s')\right\|_{H^1_w} \leq \eta \quad \textit{and} \quad \|v^T(s')\|_{L^2} + R^T_w(s') \leq \delta_*(1+2s), \quad s' \in [0,s]$$

*imply*

$$\begin{aligned}\left\|v^T(s)\right\|_{H^1_w} &\leq Me^{-bs}\left\|v(T)\right\|_{H^1_w} + \delta\eta \\ &\quad + \sigma\left\| \int_0^s e^{\mathcal{L}_{c(T)}(s-s')} Z^T\big(\mathbf{m}^T(s'), s'\big)\ T_{\xi(T)+c(T)s'} \mathrm{d}W^Q_{T+s'}\right\|_{H^1_w}. \end{aligned} \tag{4.46}$$

*Proof.* We apply Corollary 4.4.6 to estimate (4.37):

$$\begin{aligned}\left\|v^T(s)\right\|_{H^1_w} &\leq Me^{-bs}\left\|v(T)\right\|_{H^1_w} + C_7(\sigma^2+\epsilon)s + 2C_7\delta_*(s+s^2)\eta \\ &\quad + \sigma\left\| \int_0^s e^{\mathcal{L}_{c(T)}(s-s')} Z^T\big(\mathbf{m}^T(s'), s'\big)\ T_{\xi(T)+c(T)s'} \mathrm{d}W^Q_{T+s'}\right\|_{H^1_w}.\end{aligned}$$

The result hence follows from the assumptions (4.45). □

We remark that the constants $\delta_*, \eta_0$ introduced in C1 ensure that Lemma 4.5.2 may be applied on the interval $[0, \Delta T]$. With Lemma 4.5.2 established, we furthermore require control of the stochastic convolution present in (4.46). Our main tool for doing so is the following.

**Theorem 4.5.3** (Gaussian tails of stochastic convolution, see Section 4.11)**.** *There exists a constant $K > 0$ such that the following holds true. Suppose that $\big\{S(t)\big\}_{t\geq 0}$ is a $C_0$-semigroup on a Hilbert space $\mathcal{H}$ satisfying*

$$\sup_{t\geq 0} \left\|S(t)\right\|_{\mathcal{L}(\mathcal{H})} \leq M$$

*for some $M \geq 1$, and $g \in L^p(\Omega; L^p(0,T; \mathrm{HS}(L^2_Q, \mathcal{H}))$ satisfies*

$$\int_0^T \mathbb{E}\Big[\left\|g(t)\right\|^p_{\mathrm{HS}(L^2_Q,\mathcal{H})}\Big] \mathrm{d}t \leq B^p T, \quad p > 2$$

*for some $B \geq 0$. Then, for $\lambda > eBKM\sqrt{T}$ we have the inequality*

$$\mathbb{P}\Big[\sup_{t\in[0,T]} \Big\| \int_0^t S(t-s)g(s)\mathrm{d}W^Q_s\Big\|_{\mathcal{H}} \geq \lambda\Big] \leq e^{-(eBKM)^{-2}\lambda^2/T}.$$

In the sequel, we will apply Theorem 4.5.3 to stochastic convolutions with respect to the semigroup $\{e^{\mathcal{L}_{c(T)}t}\}_{t\geq 0}$, as well as ordinary stochastic integrals. In the latter

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case, we take the semigroup in Theorem 4.5.3 to be the trivial semigroup, i.e. the identity operator. Below, we explicitly compute the norms of various Hilbert-Schmidt operators used in the sequel.

**Lemma 4.5.4.** *Assuming S1 and S2, there exists a constant* $C_{10}$ *such that for all* $\tilde{\xi} \in \mathbb{R}$, $g \in L^2$ *and* $h \in H^1_w$, *we have*

$$\begin{aligned}
\|hT_{\tilde{\xi}} \cdot \|_{\mathrm{HS}(L^2_Q, H^1_w)} &\leq C_{10} \|h\|_{H^1_w}, \\
\big\| \langle g, T_{\tilde{\xi}} \cdot \rangle h \big\|_{\mathrm{HS}(L^2_Q, H^1_w)} &= \|Q^{1/2} g\|_{L^2} \|h\|_{H^1_w}, \\
\big\| \langle g, T_{\tilde{\xi}} \cdot \rangle \big\|_{\mathrm{HS}(L^2_Q, \mathbb{R})} &= \|Q^{1/2} g\|_{L^2},
\end{aligned}$$

*while for all* $g \in L^1$ *we have*

$$\big\| \langle g, T_{\tilde{\xi}} \cdot \rangle \big\|_{\mathrm{HS}(L^2_Q, \mathbb{R})} \leq C_{10} \|g\|_{L^1}.$$

*Proof.* We compute that pointwise multiplication with a function $h \in H^1_w$ leads to the identity

$$\begin{aligned}
\|hT_{\tilde{\xi}} \cdot \|^2_{\mathrm{HS}(L^2_Q, H^1_w)} &= \sum_{k=0}^{\infty} \|hQ^{1/2} e_k\|^2_{H^1_w} \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{R}} e^{2wx} \big(h^2(x) + h_x^2(x)\big) \big\langle q_{1/2}(x - \cdot), e_k \big\rangle^2 \mathrm{d}x \\
&\quad + \sum_{k=0}^{\infty} \int_{\mathbb{R}} + e^{2wx} h^2(x) \big\langle q'_{1/2}(x - \cdot), e_k \big\rangle^2 \mathrm{d}x \\
&= \|q_{1/2}\|^2_{L^2} \|h\|^2_{H^1_w} + \|q'_{1/2}\|^2_{L^2} \|h\|^2_{L^2_w}.
\end{aligned}$$

Inner products against a function $g \in L^2$ lead to

$$\big\| \langle g, T_{\tilde{\xi}} \cdot \rangle \big\|^2_{\mathrm{HS}(L^2_Q, \mathbb{R})} = \sum_{k=0}^{\infty} \big| \langle g, T_{\tilde{\xi}} Q^{1/2} e_k \rangle \big|^2 = \|Q^{1/2} g\|^2_{L^2},$$

and hence also

$$\begin{aligned}
\big\| \langle g, T_{\tilde{\xi}} \cdot \rangle h \big\|^2_{\mathrm{HS}(L^2_Q, H^1_w)} &= \sum_{k=0}^{\infty} \big\| \langle g, Q^{1/2} e_k \rangle h \big\|^2_{H^1_w} = \big\| \langle g, T_{\tilde{\xi}} \cdot \rangle \big\|^2_{\mathrm{HS}(L^2_Q, \mathbb{R})} \|h\|^2_{H^1_w} \\
&= \|Q^{1/2} g\|^2_{L^2} \|h\|^2_{H^1_w}.
\end{aligned}$$

For $g \in L^1$:

$$\big\| \langle g, T_{\tilde{\xi}} \cdot \rangle \big\|^2_{\mathrm{HS}(L^2_Q, \mathbb{R})} = \|Q^{1/2} g\|^2_{L^2} \leq \|\sqrt{\hat{q}}\|^2_{L^2} \|\hat{g}\|^2_{\infty} = \|\hat{q}\|_{L^1} \|g\|^2_{L^1}.$$

Here we note that $\hat{q} \in L^1$, since $q$ is assumed to be an element of $H^1$ in S1 and

$$\|\hat{q}\|_{L^1}^2 \leq \int_{\mathbb{R}} (1+|\omega|^2)|\hat{q}(\omega)|^2 \mathrm{d}\omega \times \int_{\mathbb{R}} \frac{1}{1+|\omega|^2} \mathrm{d}\omega \leq \tilde{C}_1 \|q\|_{H^1}^2. \qquad \square$$

With these preliminaries in place, control on the stochastic integral in (4.46) is provided by the following lemma.

**Lemma 4.5.5.** *Assuming S1 and S2, there exists a constant* $C_{11} > 0$ *so that for each* $T, s \geq 0$ *and* $\tilde{\xi} \in \mathbb{R}$*, the bounds*

$$c(T), c^T(s) \in [\tfrac{1}{2}c_{\min}, 2c_{\max}] \quad \textit{and} \quad R_w^T(s) \leq \delta_5$$

*imply*

$$\left\| Z^T(\mathbf{m}^T(s), s) T_{\tilde{\xi}} \right\|_{\mathrm{HS}(L_Q^2, H_w^1)} \leq C_{11} \Big( 1 + \left\| v^T(s) \right\|_{H_w^1} \Big).$$

*Proof.* From (4.36), a straightforward application of the triangle inequality yields the $\mathbb{P}$-a.s. bound

$$\begin{aligned} \left\| Z^T(\mathbf{m}^T(s), s) T_{\tilde{\xi}} \right\|_{\mathrm{HS}(L_Q^2, H_w^1)} &\leq \left\| (\Phi^T(\mathbf{m}^T(s), s) + v^T(s)) T_{\tilde{\xi}} \cdot \right\|_{\mathrm{HS}(L_Q^2, H_w^1)} \\ &\quad + \left\| \langle c_s^T(\mathbf{m}^T(s), s), T_{\tilde{\xi}} \cdot \rangle \partial_c \Phi^T(\mathbf{m}^T(s), s) \right\|_{\mathrm{HS}(L_Q^2, H_w^1)} \\ &\quad + \left\| \langle \Omega_s^T(\mathbf{m}^T(s), s), T_{\tilde{\xi}} \cdot \rangle \partial_x \Phi^T(\mathbf{m}^T(s), s) \right\|_{\mathrm{HS}(L_Q^2, H_w^1)}. \end{aligned}$$

Applying Lemma 4.5.4 yields

$$\begin{aligned} &\left\| Z^T(\mathbf{m}^T(s), s) T_{\tilde{\xi}} \right\|_{\mathrm{HS}(L_Q^2, H_w^1)} \\ &\qquad \leq \tilde{C}_4 \sigma \Big( 1 + \left\| Q^{1/2} c_s^T(\mathbf{m}^T(s), s) \right\|_{L^2} + \left\| Q^{1/2} \Omega_s^T(\mathbf{m}^T(s), s) \right\|_{L^2} \Big) \\ &\qquad\quad + \tilde{C}_4 \sigma \| v^T(s) \|_{H_w^1} \end{aligned}$$

and applying Lemma 4.4.4 provides the result. $\square$

Having established control on $v^T$ via Lemma 4.5.2 and Lemma 4.5.5, we are ready to prove the main result of this section: Proposition 4.5.1.

*Proof of Proposition 4.5.1.* Let us fix $C_9 = 18 C_7 \Delta T$. Writing

$$\tau = \min \left\{ \tau_{\mathrm{st}}^T(\eta), \tau_{\mathrm{en}}^T(\delta_*), \tau_{\mathrm{mod}}^T(\eta), \tau_c^T, \Delta T \right\},$$

we may establish control of $\|v^T(\tau)\|_{H_w^1}$ by applying Lemma 4.5.2 with $\delta = 1/3$:

$$\left\| v^T(\tau) \right\|_{H_w^1} \leq M \left\| v(T) \right\|_{H_w^1} + \eta/3 + \sigma \sup_{0 \leq s \leq \Delta T} I(s),$$

where we have abbreviated

$$I(s) = \Big\| \int_0^s e^{\mathcal{L}_{c(T)}(s-s')} 1_{[0,\tau]}(s') Z^T\big(\mathbf{m}^T(s'), s'\big)\ T_{\xi(T)+c(T)s'} \mathrm{d}W^Q_{T+s'} \Big\|_{H^1_w}.$$

Suppose now that $\tau^T_{\mathrm{st}}(\eta) \leq \Delta T$, meaning that the stopping time $\tau^T_{\mathrm{st}}$ is activated because the bound $\eta$ is reached on $[0, \Delta T]$, while also

$$\tau^T_{\mathrm{st}}(\eta) \leq \min\big\{\tau^T_{\mathrm{en}}(\delta_*), \tau^T_{\mathrm{mod}}(\eta), \tau^T_c\big\}.$$

Then,

$$\eta = \big\|v^T\big(\tau^T_{\mathrm{st}}(\eta)\big)\big\|_{H^1_w} = \big\|v^T(\tau)\big\|_{H^1_w} \leq M\big\|v(T)\big\|_{H^1_w} + \eta/3 + \sigma \sup_{0\leq s\leq \Delta T} I(s).$$

It follows that the event $\mathcal{E}_1 \cap \{\|v(T)\|_{H^1_w} \leq \frac{\eta}{3M}\}$ can only happen if

$$\sigma \sup_{0\leq s\leq \Delta T} I(s) \geq \eta/3,$$

so that

$$\mathbb{P}\Big[\mathcal{E}_1 \cap \big\{\|v(T)\|_{H^1_w} \leq \tfrac{\eta}{3M}\big\}\Big] \leq \mathbb{P}\Big[\sigma \sup_{0\leq s\leq \Delta T} I(s) \geq \eta/3\Big].$$

To control this probability, note that Lemma 4.5.5 implies that $\mathbb{P}$-almost surely

$$1_{[0,\tau]}(s')\Big\|Z^T\big(\mathbf{m}^T(s'), s'\big)T_{\xi(T)+c(T)s'}\Big\|_{\mathrm{HS}(L^2_Q, H^1_w)} \leq C_{11}(1+\eta), \quad s' \in [0, \Delta T].$$

By applying Theorem 4.5.3 with the semigroup $\{e^{\mathcal{L}_{c(T)}t}\}_{t\geq 0}$ restricted to its stable subspace, and by increasing $C_9$ to meet $C_9 \geq 2eC_7K\sqrt{\Delta T}$, we find

$$\mathbb{P}\Big[\sigma \sup_{0\leq s\leq \Delta T} I(s) \geq \eta/3\Big] \leq e^{-\delta_9\eta^2/\sigma^2},$$

with

$$\delta_9 = \big(2eC_7M^{-1}K\big)^{-2}/\Delta T.$$

It remains to establish the same bound on $\mathcal{E}_2$. To this end, suppose that

$$\tau = \Delta T \quad \text{and} \quad \|v(T)\|_{H^1_w} \leq \tfrac{\eta}{3M}, \quad \text{but} \quad \big\|v^T(\Delta T)\big\|_{H^1_w} \geq \tfrac{\eta}{9M}.$$

Upon increasing $C_9$ to ensure that Lemma 4.5.2 may be applied on $[0, \Delta T]$ with $\delta = 1/36M$, we obtain

$$\tfrac{\eta}{9M} \leq \big\|v^T(\Delta T)\big\|_{H^1_w} \leq \tfrac{1}{6}\big\|v(T)\big\|_{H^1_w} + \tfrac{\eta}{36M} + \sigma I(\Delta T),$$

where C1 produces the factor $\frac{1}{6}$ above. The conclusion follows, upon decreasing $\delta_9$

to

$$\delta_9 = \big(72eC_7K\big)^{-2}/\Delta T,$$

via the tail bound

$$\mathbb{P}\Big[\mathcal{E}_2 \cap \big\|v(T)\big\|_{H^1_w} \le \tfrac{\eta}{3M}\Big] \le \mathbb{P}\Big[\sigma I(\Delta T) \ge \tfrac{\eta}{36M}\Big] \le e^{-\delta_9\eta^2/\sigma^2}. \qquad \square$$

## 4.6 Local control of modulation parameters

In this section, we establish several facts regarding the modulation parameters $c, \xi$ and their local counterparts. Our goal is to address one of the conditions for stability formulated in Section 4.5: an estimate on the local modulation parameters, encoded by the stopping time

$$\tau^T_{\mathrm{mod}}(\eta) = \tau^T_{\mathrm{amp},1}(\eta) \wedge \tau^T_{\mathrm{amp},2}(\eta) \wedge \tau^T_{\mathrm{pos},1}(\eta) \wedge \tau^T_{\mathrm{pos},2}(\eta)$$

introduced in Section 4.5, where

$$\begin{aligned}
\tau^T_{\mathrm{amp},1}(\eta) &= \sup\big\{s \ge 0 : \big|c(T) - c^T(s)\big| \le \delta_*\eta\big\};\\
\tau^T_{\mathrm{amp},2}(\eta) &= \sup\big\{s \ge 0 : \big|c(T+s) - c^T(s)\big| \le \delta_*\eta\big\};\\
\tau^T_{\mathrm{pos},1}(\eta) &= \sup\big\{s \ge 0 : \big|\xi(T) + c(T)s - \xi^T(s)\big| \le 2\Delta T\delta_*\eta\big\};\\
\tau^T_{\mathrm{pos},2}(\eta) &= \sup\big\{s \ge 0 : \big|\xi(T+s) - \xi^T(s)\big| \le 2\Delta T\delta_*\eta\big\}.
\end{aligned}$$

We show that the probability that one of the stopping times above is activated on $[0, \Delta T]$, while the local perturbation $v^T$ is small and the global amplitude is within the bounds $[c_{\mathrm{min}}, c_{\mathrm{max}}]$, satisfies an exponential tail estimate. Recall from Section 4.5 that

$$\begin{aligned}
\tau^T_c &= \sup\big\{s \ge 0 : c(T+s) \in [\tfrac{1}{2}c_{\mathrm{min}}, 2c_{\mathrm{max}}]\big\},\\
\tau^T_{\mathrm{st}}(\eta) &= \sup\big\{s \ge 0 : \big\|v^T(s)\big\|_{H^1_w} \le \eta\big\},
\end{aligned}$$

and let us furthermore introduce the stopping time

$$t_c = \sup\{t \ge 0 : c(t) \in [c_{\mathrm{min}}, c_{\mathrm{max}}]\},$$

which signals that $c(t)$ exits its bounds $[c_{\mathrm{min}}, c_{\mathrm{max}}]$.

**Proposition 4.6.1** (Control of modulation parameters)**.** *Assuming S1, S2 and C1, there exist constants $\delta_{12}, C_{12} > 0$ such that the following holds true. For all $\eta \in [0, \eta_0]$, all $C_{12}\sigma, C_{12}\epsilon \in [0, \eta]$ and $T \ge 0$ the stopping times $\tau^T_{\mathrm{mod}}, \tau^T_{\mathrm{st}}$ and $t_c$ satisfy*

$$\mathbb{P}\big[\{\tau^T_{\mathrm{mod}}(\eta) \le \tau^T_{\mathrm{st}}(\eta) \wedge \Delta T\} \cap \{T + \tau^T_{\mathrm{mod}}(\eta) \le t_c\}\big] \le e^{-\delta_{12}\eta^2/\sigma^2}.$$

We first treat the stopping time $\tau^T_{\mathrm{amp},1}$ by estimating the local amplitude $c^T$

through

$$\begin{aligned}\big|c(T) - c^T(s)\big| \le \int_0^s \Big|c_d^{\sigma,\epsilon,T}\big(\mathbf{m}^T(s'),s'\big)\Big|\,\mathrm{d}s' \\ + \sigma\Big|\int_0^s \big\langle c_s^T\big(\mathbf{m}^T(s'),s'\big), T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}\big\rangle\Big|.\end{aligned} \tag{4.47}$$

This is obtained straightforwardly from (4.31). We recall that $\eta_0$ below is introduced in Proposition 4.5.1.

**Lemma 4.6.2.** *Assuming S1, S2 and C1, there exist constants $\delta_{13}, C_{13} > 0$ such that the following holds true. For all $\eta \in [0,\eta_0]$, all $C_{13}\sigma, C_{13}\epsilon \in [0,\eta]$ and $T \ge 0$ the stopping times $\tau^T_{\mathrm{amp},1}, \tau^T_{\mathrm{pos},1}, \tau^T_{\mathrm{st}}$ and $\tau^T_c$ satisfy*

$$\mathbb{P}\big[\tau^T_{\mathrm{amp},1}(\eta) \le \tau^T_{\mathrm{pos},1}(\eta) \wedge \tau^T_{\mathrm{st}}(\eta) \wedge \tau^T_c \wedge \Delta T\big] \le e^{-\delta_{13}\eta^2/\sigma^2}.$$

*Proof.* Writing $\tau = \min\{\tau^T_{\mathrm{amp},1}(\eta), \tau^T_{\mathrm{pos},1}(\eta), \tau^T_{\mathrm{st}}(\eta), \tau^T_c\}$, we apply Lemma 4.4.4 to (4.47) and find

$$\begin{aligned}\big|c(T) - c^T(s)\big| &\le C_5 \int_0^s R^T_w(s')\|v^T(s')\|_{L^2_w}\,\mathrm{d}s' \\ &\quad + C_5\epsilon\int_0^s f(T+s')\big(1+\|v^T(s')\|_{L^2_w}\big)\mathrm{d}s' + C_5\sigma^2\int_0^s \big(1+\|v^T(s')\|_{L^2_w}\big)\mathrm{d}s' \\ &\quad + \sigma\Big|\int_0^s \big\langle c_s^T\big(\mathbf{m}^T(s'),s'\big), T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}\big\rangle\Big|,\end{aligned}$$

$\mathbb{P}$-almost surely for $s' \in [0,\tau]$. We thus have

$$\big|c(T) - c^T(\tau)\big| \le C_5\Delta T\Big(\delta_*(2+2\Delta T)\eta^2 + 2(\epsilon+\sigma^2)\Big) + \sigma \sup_{0\le s\le \Delta T} C(s),$$

where we have abbreviated

$$C(s) := \Big|\int_0^s 1_{[0,\tau]}(s')\big\langle c_s^T\big(\mathbf{m}^T(s'),s'\big), T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}\big\rangle\Big|.$$

We note via Lemma 4.5.4 that $\mathbb{P}$-almost surely for $s' \in [0,\tau]$, the integrand above satisfies

$$\Big\|\big\langle c_s^T\big(\mathbf{m}^T(s'),s'\big), T_{\xi(T)+c(T)s'}\,\cdot\,\big\rangle\Big\|^2_{\mathrm{HS}(L^2_Q,\mathbb{R})} \le C_5^2\big(1+\|v^T(s')\|_{L^2_w}\big)^2 \le C_5^2(1+\eta)^2.$$

Suppose now that $\tau^T_{\mathrm{amp},1}(\eta) \le \tau^T_{\mathrm{pos},1}(\eta) \wedge \tau^T_{\mathrm{st}}(\eta) \wedge \tau^T_c$ and that the stopping time $\tau^T_{\mathrm{amp},1}$ is activated because the bound $\delta_*\eta$ is reached on $[0,\Delta T]$. In this case,

$$\delta_*\eta = \big|c(T) - c^T(\tau)\big| \le \delta_*\eta/4 + 2C_5\Delta T(\epsilon+\sigma^2) + \sigma\sup_{0\le s\le\Delta T} C(s),$$

where we have used that $C_5\Delta T\delta_*(2+2\Delta T)\eta^2 \leq \delta_*\eta/4$ via C1. Ensuring that also $2C_5\Delta T(\epsilon+\sigma^2) \leq \delta_*\eta/4$ via $C_{13}$, this can only happen if

$$\sigma \sup_{0\leq s\leq \Delta T} C(s) \geq \delta_*\eta/2.$$

Ensuring furthermore that $\sigma\sqrt{\Delta T} \leq \frac{\delta_*\eta}{2eC_5(1+\eta_0)K}$, Theorem 4.5.3 implies the tail bound

$$\mathbb{P}\Big[\sigma \sup_{0\leq s\leq \Delta T} C(s) \geq \delta_*/2\Big] \leq e^{-\delta_{13}\eta^2/\sigma^2}, \tag{4.48}$$

with

$$\delta_{13} = \big(2eC_5(1+\eta_0)K\big)^{-2}\delta_*^2/\Delta T.$$

Hence,

$$\mathbb{P}\big[\tau^T_{\mathrm{amp},1}(\eta) \leq \tau^T_{\mathrm{pos},1}(\eta) \wedge \tau^T_{\mathrm{st}}(\eta) \wedge \tau^T_c \wedge \Delta T\big] \leq e^{-\delta_{13}\eta^2/\sigma^2}. \qquad \square$$

Next, we set out to control the stopping time $\tau^T_{\mathrm{pos},1}$ via the estimate

$$\begin{aligned}\big|\xi(T)+c(T)s-\xi^T(s)\big| \leq \int_0^s &\big|c^T(s')-c(T)\big| + \big|\Omega_d^{\sigma,\epsilon,T}\big(\mathbf{m}^T(s'),s'\big)\big|\mathrm{d}s' \\ &+\sigma\Big|\int_0^s \langle\Omega_s^T\big(\mathbf{m}^T(s'),s'\big), T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}\rangle\Big|\end{aligned} \tag{4.49}$$

on the local soliton position $\xi^T$. Note here the dependence on $|c^T(s')-c(T)\big|$, which is under control before time $\tau^T_{\mathrm{amp},1}$.

**Lemma 4.6.3.** *Assuming S1, S2 and C1, there exist constants $\delta_{14}, C_{14} > 0$ such that the following holds true. For all $\eta \in [0,\eta_0]$, all $C_{14}\sigma, C_{14}\epsilon \in [0,\eta]$ and $T \geq 0$ the stopping times $\tau^T_{\mathrm{amp},2}, \tau^T_{\mathrm{pos},2}, t_{\mathrm{st}}$ and $t_c$ satisfy*

$$\mathbb{P}\big[\tau^T_{\mathrm{pos},1}(\eta) \leq \tau^T_{\mathrm{amp},1}(\eta) \wedge \tau^T_{\mathrm{st}}(\eta) \wedge \tau^T_c \wedge \Delta T\big] \leq e^{-\delta_{14}\eta^2/\sigma^2}.$$

*Proof.* Let us once more write $\tau = \min\{\tau^T_{\mathrm{amp},1}(\eta), \tau^T_{\mathrm{pos},1}(\eta), \tau^T_{\mathrm{st}}(\eta), \tau^T_c\}$. From (4.49) we obtain the inequality

$$\begin{aligned}2\Delta T\delta_*\eta &= \big|\xi(T)+c(T)\tau-\xi^T(\tau)\big| \\ &\leq \delta_*\Delta T\eta + C_5\Delta T\big(\delta_*(2+2\Delta T)\eta^2+\epsilon(1+\eta)+\sigma^2(1+\eta)\big) \\ &\quad + \sigma\sup_{0\leq s\leq\Delta T}\Big|\int_0^s 1_{[0,\tau]}(s')\langle\Omega_s^T\big(\mathbf{m}^T(s'),s'\big), T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}\rangle\Big|.\end{aligned}$$

Ensuring $C_5\big(\delta_*(2+2\Delta T)\eta^2+\epsilon(1+\eta)+\sigma^2(1+\eta)\big) \leq \delta_*\eta/2$ as well as $\sigma \leq \frac{\sqrt{\Delta T}\delta_*\eta}{2eC_5(1+\eta_0)K}$ via $C_{14}$, the result follows from the tail bound

$$\mathbb{P}\Big[\sigma\sup_{0\leq s\leq\Delta T}\Big|\int_0^s 1_{[0,\tau]}(s')\langle\Omega_s^T\big(\mathbf{m}^T(s'),s'\big), T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}\rangle\Big| \geq \Delta T\eta/2\Big] \leq e^{-\delta_{14}\eta^2/\sigma^2},$$

where

$$\delta_{14} = \Delta T\big(2eC_5(1+\eta_0)K\big)^{-2}\delta_*^2,$$

analogous to (4.48). □

Control of the global amplitude is established by estimating (4.19) as

$$\begin{aligned}\big|c(T) - c(T+s)\big| \leq &\int_T^{T+s} \Big|c_d^{\sigma,\epsilon}\big(v(t'), c(t'), t'\big)\Big| \,\mathrm{d}t' \\ &+ \sigma\Big|\int_T^{T+s} \big\langle c_s\big(v(t'), c(t')\big), T_{\xi(t')}\mathrm{d}W_{t'}^Q\big\rangle\Big|.\end{aligned}$$

The difference between the local and global positions is controlled via

$$|\xi(T+s) - \xi^T(s)| \leq \big|\xi(T) + c(T)s - \xi^T(s)\big| + \big|\xi(T) + c(T)s - \xi(T+s)\big|,$$

where

$$\begin{aligned}\big|\xi(T) + c(T)s - \xi(T+s)\big| \leq &\int_T^{T+s} |c(T) - c(t')| + \big|\Omega_d^{\sigma,\epsilon}(v(t'), c(t'), t')\big|\mathrm{d}t' \\ &+ \sigma\Big|\int_T^{T+s} \langle\Omega_s(v(t'), c(t')), T_{\xi(t')}\mathrm{d}W_{t'}^Q\rangle\Big|.\end{aligned}$$

This leads to the following estimates on $\tau_{\mathrm{amp},2}^T$ and $\tau_{\mathrm{pos},2}^T$.

**Lemma 4.6.4.** *Assuming S1, S2 and C1, there exist constants $\delta_{15}, C_{15} > 0$ such that the following holds true. For all $\eta \in [0, \eta_0]$, all $C_{15}\sigma, C_{15}\epsilon \in [0, \eta]$ and $T \geq 0$ the stopping times $\tau_{\mathrm{amp},2}^T, \tau_{\mathrm{pos},2}^T, t_{\mathrm{st}}$ and $t_c$ satisfy*

$$\mathbb{P}\big[\{\tau_{\mathrm{amp},2}^T(\eta) \leq \tau_{\mathrm{pos},2}^T(\eta) \wedge \Delta T\} \cap \{T + \tau_{\mathrm{amp},2}^T(\eta) \leq t_{\mathrm{st}}(2\eta) \wedge t_c\}\big] \leq e^{-\delta_{15}\eta^2/\sigma^2},$$

*and*

$$\mathbb{P}\big[\{\tau_{\mathrm{pos},2}^T(\eta) \leq \tau_{\mathrm{amp},2}^T(\eta) \wedge \Delta T\} \cap \{T + \tau_{\mathrm{pos},2}^T(\eta) \leq t_{\mathrm{st}}(2\eta) \wedge t_c\}\big] \leq e^{-\delta_{15}\eta^2/\sigma^2}.$$

*Proof.* These bounds follow from computations fully analogous to those in the proofs of Lemma 4.6.2 and Lemma 4.6.3. □

As a last preparation, we show how the correspondence between the local perturbation $v(T+s)$ and global perturbation $v^T(s)$ manifests in the stopping times $\tau_{\mathrm{st}}^T$ and its global counterpart $t_{\mathrm{st}}$. We recall that

$$t_{\mathrm{st}}(\eta) = \sup\big\{t \geq 0 : \|v(t)\|_{H_w^1} \leq \eta\big\}.$$

**Lemma 4.6.5.** *Assume S1, S2 and C1. For each $T \geq 0$ and $\eta \in [0, \eta_0]$, the stopping times $\tau_{\mathrm{st}}^T, \tau_{\mathrm{mod}}^T, t_c$ and $t_{\mathrm{st}}$ satisfy*

$$\min\{T + \tau_{\mathrm{st}}^T(\eta), T + \tau_{\mathrm{mod}}^T(\delta_*\eta), t_c\} \leq t_{\mathrm{st}}(2\eta),$$

$\mathbb{P}$-*almost surely.*

*Proof.* Writing $\tau = \min\{\tau^T_{\rm st}(\eta), \tau^T_{\rm mod}(\eta), t_c - T\}$, we have

$$c(T+s) \in [c_{\min}, c_{\max}], \quad \text{and} \quad \big|c^T(s) - c(T+s)\big| \leq 2\delta_*\eta, \quad \text{for} \quad s \in [0,\tau].$$

Since $\delta_* \leq \min\{\frac{1}{4}c_{\min}, \frac{1}{2}c_{\max}\}$ via C1, it follows that also

$$c^T(s) \in [\tfrac{1}{2}c_{\min}, 2c_{\max}], \quad s \in [0,\tau].$$

Lemma 4.4.2 now gives

$$\Big|\big\|v(T+s)\big\|_{H^1_w} - e^{w(\xi(T)+c(T)s-\xi(T+s))}\big\|v^T(s)\big\|_{H^1_w}\Big| \leq 3C_4\delta_*\eta.$$

We then obtain the bound

$$\|v(T+s)\|_{H^1_w} \leq e^{w(\xi(T)+c(T)s-\xi(T+s))}\|v^T(s)\|_{H^1_w} + 3C_4\eta \leq e^{2\eta_0}\delta_*\eta + 3C_4\delta_*\eta < 2\eta,$$

for all $s \in [0,\tau]$, where we have used that $\delta_* < (e^2 + 3C_4)^{-1}$ via C1. $\square$

We are then ready to collect our results and establish control of $\tau^T_{\rm mod}$. We note that the event

$$\tau^T_{\rm mod}(\eta) \leq \tau^T_{\rm st}(\eta) \wedge \Delta T \quad \text{while} \quad T + \tau^T_{\rm mod}(\eta) \leq t_c$$

implies that one of the events

$$\begin{aligned}
\mathcal{B}_1 &= \big\{\tau^T_{\rm amp,1}(\eta) \leq \tau^T_{\rm pos,1}(\eta) \wedge \tau^T_{\rm amp,2}(\eta) \wedge \tau^T_{\rm st}(\eta) \wedge \Delta T\big\} \cap \big\{T + \tau^T_{\rm amp,1}(\eta) \leq t_c\big\};\\
\mathcal{B}_2 &= \big\{\tau^T_{\rm pos,1}(\eta) \leq \tau^T_{\rm amp,1}(\eta) \wedge \tau^T_{\rm amp,2}(\eta) \wedge \tau^T_{\rm st}(\eta) \wedge \Delta T\big\} \cap \big\{T + \tau^T_{\rm pos,1}(\eta) \leq t_c\big\};\\
\mathcal{B}_3 &= \big\{\tau^T_{\rm amp,2}(\eta) \leq \tau^T_{\rm mod}(\eta) \wedge \tau^T_{\rm st}(\eta) \wedge \Delta T\big\} \cap \big\{T + \tau^T_{\rm amp,2}(\eta) \leq t_c\big\};\\
\mathcal{B}_4 &= \big\{\tau^T_{\rm pos,2}(\eta) \leq \tau^T_{\rm mod}(\eta) \wedge \tau^T_{\rm st}(\eta) \wedge \Delta T\big\} \cap \big\{T + \tau^T_{\rm pos,2}(\eta) \leq t_c\big\},
\end{aligned}$$

holds, depending on which of $\tau^T_{\rm amp,1}, \tau^T_{\rm amp,2}, \tau^T_{\rm pos,1}$ and $\tau^T_{\rm pos,2}$ is smallest.

*Proof of Proposition 4.6.1.* We bound the probability of the events $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ and $\mathcal{B}_4$. In case $\mathcal{B}_1$ or $\mathcal{B}_2$ holds, note that $t_c - T \wedge \tau^T_{\rm amp,2}(\eta) \leq \tau^T_c$. Indeed,

$$|c^T(s) - c(T+s)| \leq \eta_0 \quad \text{and} \quad c(T+s) \in [c_{\min}, c_{\max}]$$

implies that $c^T(s) \in [\frac{1}{2}c_{\min}, 2c_{\max}]$. Thus, the probability of $\mathcal{B}_1 \cup \mathcal{B}_2$ is controlled by Lemma 4.6.2 and Lemma 4.6.3, respectively.

In case $\mathcal{B}_3$ holds, Lemma 4.6.5 implies $T + \tau^T_{\rm amp,2}(\eta) \leq t_{\rm st}(2\eta)$. Similarly, the event $\mathcal{B}_4$ implies $T + \tau^T_{\rm pos,2}(\eta) \leq t_{\rm st}(2\eta)$. Hence, the probability of $\mathcal{B}_3 \cup \mathcal{B}_4$ is bounded via Lemma 4.6.4. $\square$

## 4.7 Global control

In this section, we establish long-time control of the global modulation system $\big(v(t), c(t), \xi(t)\big)$ introduced in Section 4.3. Our first result controls the probability that the soliton amplitude $c(t)$ exits its bounds $[c_{\min}, c_{\max}]$. Secondly, we analyze the growth of the remainder $v(t)$ in the unweighted space $L^2$. Besides being interesting in its own right, this is a crucial ingredient toward controlling the weighted norm of $v$ via Proposition 4.5.1. Since the traveling-wave operator $\mathcal{L}_c$ is not exponentially stable on $L^2$, it is standard to analyze the unweighted norm via energy arguments [91, 103].

Henceforth, we assume an integrability condition on the forcing term $f$, limiting the total energy contribution of the deterministic forcing in (4.1).

**C2** *The constant $E > 0$ satisfies*

$$c_{\min} \leq c_* e^{-3E} \quad \textit{and} \quad c_{\max} \geq c_* e^{3E}.$$

*The forcing term $f$ lies in $L^1\big([0,\infty)\big)$ and the constant $\epsilon_0 \in (0,\infty]$ is small enough to ensure*

$$\epsilon_0 \int_0^\infty |f(t)| \mathrm{d}t \leq E.$$

With this condition in place, the soliton amplitude $c(t)$ stays within the bounds $[c_{\min}, c_{\max}]$ for times that are small with respect to $\sigma^2$. Recall the notation

$$\begin{aligned} t_c &= \sup\big\{t \geq 0 : c(t) \in [c_{\min}, c_{\max}]\big\}, \\ t_{\mathrm{st}}(\eta) &= \sup\big\{t \geq 0 : \|v(t)\|_{H^1_w} \leq \eta\big\}, \end{aligned}$$

the global counterparts of $\tau_c^T$ and $\tau_{\mathrm{st}}^T(\eta)$ from Section 4.5. The constant $\delta_{17}$ below will be introduced in Lemma 4.7.3.

**Proposition 4.7.1.** *Assume S1, S2, C1 and C2. For each $\eta \in [0, \eta_0]$, each $\epsilon \in [0, \epsilon_0]$, and $\sigma, T \geq 0$ that satisfy $\sigma^2 T, \eta^2 T \leq \delta_{17}$ we have*

$$\mathbb{P}[t_c \leq T \wedge t_{\mathrm{st}}(\eta)] \leq e^{-\delta_{17}/(\sigma^2 T)}.$$

The second main result of this section concerns the stopping time

$$t_{\mathrm{en}}(\eta) = \sup\big\{t \geq 0 : \big\|v(t)\big\|_{L^2} \leq \eta\big\},$$

a global counterpart of $\tau_{\mathrm{en}}^T$ of Section 4.5. The probability that $\|v(t)\|_{L^2}$ remains small on an interval $[0, T]$ is controlled by the forcing parameters $\sigma$ and $\epsilon$.

**Proposition 4.7.2.** *Assume S1, S2, C1 and C2. There exist constants $C_{16}, \delta_{16} > 0$ such that for all $\sigma, T, \lambda > 0$ satisfying $\sigma^2 T \leq \delta_{16}$, each $\eta \in [0, \eta_0]$ and each $\epsilon \in [0, \epsilon_0]$,* *we have*

$$\mathbb{P}\Big[t_{\mathrm{en}}(\lambda) \le T \wedge t_c \wedge t_{\mathrm{st}}(\eta)\Big] \le C_{16}\lambda^{-1}\Big(T\big(\sigma^2 + \epsilon\eta + \eta^2\big) + \sqrt{T}\sigma\eta\Big).$$

We first establish Proposition 4.7.1, making use of the evolution equation

$$\begin{aligned} \mathrm{d}\log(c(t)) = &\Big(\frac{c_d^{\sigma,\epsilon}(v(t),c(t),t)}{c(t)} - \sigma^2\frac{\big\|Q^{1/2}c_s(v(t),c(t))\big\|_{L^2}^2}{2c^2(t)}\Big)\,\mathrm{d}t \\ &+ \sigma\frac{\langle c_s(v(t),c(t)), T_\xi \mathrm{d}W_t^Q\rangle}{c(t)}, \end{aligned} \tag{4.50}$$

which one finds by applying the Itô lemma [23, Theorem 4.32] to (4.19). In particular, as an intermediate result, we control the stopping time

$$t_{\log} = \sup\{t \ge 0 : |\log(c(t)/c_*) - \frac{4}{3}\epsilon\int_0^t |f(t')|\mathrm{d}t'| \le E\}.$$

It is then ensured that $c(t)$ remains within the limits $[c_{\min}, c_{\max}]$ before $t_{\log}$, and consequently $t_c \ge t_{\log}$.

**Lemma 4.7.3.** *Assuming S1, S2, C1 and C2, there exists a constant $\delta_{17} > 0$ such that the following holds true. For each $\eta \in [0,\eta_0]$, each $\epsilon \in [0,\epsilon_0]$, and $\sigma, T \ge 0$ satisfying $\sigma^2 T, \eta^2 T \le \delta_{17}$ we have*

$$\mathbb{P}[t_{\log} \le t_{\mathrm{st}}(\eta) \wedge T] \le e^{-\delta_{17}/\sigma^2 T}.$$

*Proof.* Using the identity $c^{-1}c_f(0,c) = \frac{4}{3}$, we estimate (4.50) as

$$\begin{aligned} \Big|\log(c(t)/c_*) - &\frac{4}{3}\epsilon\int_0^t |f(t')|\mathrm{d}t'\Big| \\ &\le \int_0^t \Big|\frac{c_d^0(v(t'),c(t'))}{c(t')}\Big| + \epsilon|f(t')|\Big|\frac{c_f(v(t'),c(t')) - c_f(0,c(t'))}{c(t')}\Big|\,\mathrm{d}t' \\ &\quad + \sigma^2\int_0^t \Big|\frac{c_d(v(t'),c(t'))}{c(t')}\Big| - \frac{\big\|Q^{1/2}c_s(v(t'),c(t'))\big\|_{L^2}^2}{2c^2(t')}\,\mathrm{d}t' \\ &\quad + \sigma\Big|\int_0^t \frac{\langle c_s(v(t'),c(t')), T_\xi \mathrm{d}W_{t'}^Q\rangle}{c(t')}\Big| \\ &\le C_2 t(\eta^2+\sigma^2) + C_2\epsilon\eta\int_0^t |f(t')|\mathrm{d}t' \\ &\quad + \sigma\Big|\int_0^t \frac{\langle c_s(v(t'),c(t')), T_\xi \mathrm{d}W_{t'}^Q\rangle}{c(t')}\Big|, \end{aligned}$$

for $t \in [0, t_{\log} \wedge t_{\mathrm{st}}(\eta)]$. Condition C2 now gives

$$\epsilon\eta \int_0^t |f(t')|\mathrm{d}t' \leq E\eta.$$

Ensuring $C_2 T(\eta^2 + \sigma^2) + C_2 E\eta \leq L/2$, the inequality $t_{\log} \leq t_{\mathrm{st}}(\eta) \wedge T$ implies

$$\begin{aligned} E &= \Big| \log(c(t_{\log})/c_*) - \frac{4}{3}\int_0^{t_{\log}} |f(t')|\mathrm{d}t' \Big| \\ &\leq E/2 + \sigma \sup_{0\leq t\leq T} \Big| \int_0^t 1_{[0, t_{\log}\wedge t_{\mathrm{st}}(\eta)]}(t') \frac{\langle c_s(v(t'), c(t')), T_\xi \mathrm{d}W^Q_{t'}\rangle}{c(t')} \Big| \end{aligned}$$

and consequently

$$\sigma \sup_{0\leq t\leq T} \Big| \int_0^t 1_{[0, t_{\log}\wedge t_{\mathrm{st}}(\eta)]}(t') \frac{\langle c_s(v(t'), c(t')), T_\xi \mathrm{d}W^Q_{t'}\rangle}{c(t')} \Big| \geq E/2.$$

The probability of this event is bounded by the tail estimate Theorem 4.5.3. □

*Proof of Proposition 4.7.1.* The result now follows as an immediate corollary to Lemma 4.7.3, in view of the fact that $t_c \geq t_{\log}$. □

We now turn our attention to the energy result, Proposition 4.7.2. Our first result regarding the global modulation system outlined in Section 4.3 is as follows.

**Lemma 4.7.4.** *Assume S1 and S2. For each $t \geq 0$, the inequalities*

$$\|v(t')\|_{L^2_w} \leq \delta_1 \quad \text{and} \quad c(t') \in [c_{\min}, c_{\max}], \quad t' \in [0, t]$$

*imply that*

$$\begin{aligned} \big\|v(t)\big\|^2_{L^2} &= \sigma^2 \int_0^t \Big(6c^{3/2} - 9c_d(v,c)c^{1/2} - \tfrac{9}{4}\big\|Q^{1/2}c_s(v,c)\big\|^2_{L^2}c^{-1/2} + \big\|v\big\|^2_{L^2}\Big)\mathrm{d}t' \\ &\quad - 9\int_0^t c_d^0(v,c)c^{1/2}\mathrm{d}t' + \epsilon\int_0^t f(t')\Big(2\|v\|^2_{L^2} - 9\big(c_f(v,c) - c_f(0,c)\big)\Big)\mathrm{d}t' \\ &\quad - 9\sigma\int_0^t c^{1/2}\big\langle c_s(v,c) - c_s(0,c), \mathrm{d}W^Q_{t'}\big\rangle + 2\sigma\int_0^t \big\langle 2\phi_c v + v^2, T_\xi \mathrm{d}W^Q_{t'}\big\rangle \end{aligned} \tag{4.51}$$

*holds $\mathbb{P}$-almost surely.*

*Proof.* We observe that

$$\big\|v(t)\big\|^2_{L^2} = \big\|u(t)\big\|^2_{L^2} - \big\|\phi_{c(t)}\big\|^2_{L^2} \tag{4.52}$$

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by Pythagoras' theorem and the orthogonality condition (4.18). Via Itô's lemma [22, Theorem 1] applied to the functional $M(u) = \|u\|_{L^2}^2$, we obtain

$$\begin{aligned} \mathrm{d}\|u\|_{L^2}^2 &= \sigma^2\|u\|_{L^2}^2 \ \mathrm{d}t + 2\epsilon f(t)\|u\|_{L^2}^2 \ \mathrm{d}t + 2\sigma\langle u, u \ \mathrm{d}W_t^Q\rangle \\ &= \big(\sigma^2 + 2\epsilon f(t)\big)\big(6c^{3/2}(t) + \|v\|_{L^2}^2\big) \ \mathrm{d}t + 2\sigma\big\langle \phi_c^2 + 2\phi_c v + v^2, \ T_\xi \mathrm{d}W_t^Q\big\rangle. \end{aligned} \tag{4.53}$$

Here we have used that $M : L^2 \to \mathbb{R}$ is twice differentiable with Fréchet derivatives

$$\mathrm{d}M(u)[v] = 2\langle u, v\rangle, \quad \mathrm{d}^2M(u)[v, w] = 2\langle v, w\rangle,$$

and that the Airy group $\{e^{-\partial_x^3 t}\}_{t\in\mathbb{R}}$ is a $C_0$-group of isometries on $L^2$, in the sense that

$$M(e^{-\partial_x^3 t}u) = \|u\|_{L^2}^2, \quad \mathrm{d}M(e^{-\partial_x^3 t}u)[e^{-\partial_x^3 t}v] = 2\langle u, v\rangle$$

and

$$\mathrm{d}^2M(e^{-\partial_x^3 t}u)[e^{-\partial_x^3 t}v, e^{-\partial_x^3 t}w] = 2\langle v, w\rangle.$$

On the other hand, we can also employ Itô's lemma to compute

$$\begin{aligned} \mathrm{d}\|\phi_{c(t)}\|_{L^2}^2 = \mathrm{d}\big(6c^{3/2}(t)\big) &= \Big(9c_d^{\sigma,\epsilon}(v, c, t)c^{1/2} + \tfrac{9}{4}\sigma^2\big\|Q^{1/2}c_s(v, c)\big\|_{L^2}^2 c^{-1/2}\Big) \ \mathrm{d}t \\ &\quad + 9\sigma c^{1/2}\big\langle c_s(v, c), T_\xi \mathrm{d}W_t^Q\big\rangle, \end{aligned} \tag{4.54}$$

where we recall that

$$c_d^{\sigma,\epsilon}(v, c, t) = c_d^0(v, c) + \epsilon f(t)c_f(v, c) + \sigma^2 c_d(v, c).$$

Note here that $9c^{1/2}c_s(0, c) = 2\phi_c^2$, thus the leading-order diffusion term in $\mathrm{d}\|u\|_{L^2}^2$ equals that in $\mathrm{d}\|\phi_{c(t)}\|_{L^2}^2$. Similarly,

$$c_d^{\sigma,\epsilon}(0, c, t) = \tfrac{4}{3}\epsilon f(t)c + \sigma^2 c_d(0, c)$$

shows that the leading-order $\epsilon$-dependent term in $\mathrm{d}\|u\|_{L^2}^2$ equals that in $\mathrm{d}\|\phi_{c(t)}\|_{L^2}^2$. The result now follows by subtracting (4.54) from (4.53).

□

Control on the stochastic integrals in (4.51) is provided in the following lemma.

**Lemma 4.7.5.** *Assuming S1 and S2, there exists a constant $C_{18} > 0$ such that for all $\tilde{v} \in L_w^2$ satisfying $\|\tilde{v}\|_{L_w^2} \le \delta_2$, all $\tilde{\xi} \in \mathbb{R}$ and $\tilde{c} \in [c_{\min}, c_{\max}]$ we have the inequalities*

$$\big\|\langle 2\phi_{\tilde{c}}\tilde{v} + \tilde{v}^2, T_{\tilde{\xi}}\cdot\rangle\big\|_{\mathrm{HS}(L_Q^2,\mathbb{R})} \le C_{18}\Big(\|\tilde{v}\|_{L_w^2} + \|\tilde{v}\|_{L^2}^2\Big)$$

*and*

$$\left\|\langle c_s(\tilde{v},\tilde{c}) - c_s(0,\tilde{c}), T_{\tilde{\xi}}\cdot\rangle\right\|_{\mathrm{HS}(L^2_Q,\mathbb{R})} \leq C_{18}\|\tilde{v}\|_{L^2_w}.$$

*Proof.* Applying Lemma 4.5.4, we have

$$\begin{aligned}\left\|\langle 2\phi_{\tilde{c}}\tilde{v} + \tilde{v}^2, T_{\tilde{\xi}}\cdot\rangle\right\|_{\mathrm{HS}(L^2_Q,\mathbb{R})} &\leq 2\|Q^{1/2}(\phi_{\tilde{c}}\tilde{v})\|_{L^2} + C_{10}\|\tilde{v}^2\|_{L^1} \\ &\leq \|q\|_{L^1}^{1/2}\|\phi_{\tilde{c}}\|_{L^2_{-w}}\|\tilde{v}\|_{L^2_w} + C_{10}\|\tilde{v}\|_{L^2}^2,\end{aligned}$$

and

$$\left\|\langle c_s(\tilde{v},\tilde{c}) - c_s(0,\tilde{c}), T_{\tilde{\xi}}\cdot\rangle\right\|_{\mathrm{HS}(L^2_Q,\mathbb{R})} = \left\|Q^{1/2}c_s(\tilde{v},\tilde{c}) - Q^{1/2}c_s(0,\tilde{c})\right\|_{L^2}.$$

The result follows by applying Lemma 4.3.4. □

*Proof of Proposition 4.7.2.* Using (4.51), we may $\mathbb{P}$-almost surely estimate

$$\begin{aligned}\|v(t)\|_{L^2}^2 \leq{}& C_2\int_0^t \left(\sigma^2 + 2\epsilon|f(t')|\right)\|v(t')\|_{L^2}^2\mathrm{d}t' + C_2 t\left(\sigma^2 + \epsilon\eta + \eta^2\right) \\ &+ 9\sigma\left|\int_0^t \langle c_s(v,c) - c_s(0,c), \mathrm{d}W^Q_{t'}\rangle\right| + 2\sigma\left|\int_0^t \langle 2\phi_c v + v^2, T_\xi \mathrm{d}W^Q_{t'}\rangle\right|,\end{aligned}$$

for $t \in [0, t_c \wedge t_{\mathrm{st}}(\eta)]$, where we have controlled the modulation parameters in the deterministic integrals via Lemma 4.3.4. After taking a supremum and expectations,

$$\begin{aligned}\mathbb{E}\sup_{0\leq t'\leq T\wedge t_c\wedge t_{\mathrm{st}}(\eta)}\|v(t')\|_{L^2}^2 \leq{}& C_2\int_0^T \left(\sigma^2 + 2\epsilon|f(t)|\right)\mathbb{E}\sup_{0\leq t'\leq t\wedge t_c\wedge t_{\mathrm{st}}(\eta)}\|v(t')\|_{L^2}^2\mathrm{d}t \\ &+ C_2 T\left(\sigma^2 + \epsilon\eta + \eta^2\right) + \sigma I(T),\end{aligned} \tag{4.55}$$

where we have abbreviated

$$\begin{aligned}I(T) ={}& 9\mathbb{E}\sup_{0\leq t\leq T\wedge t_c\wedge t_{\mathrm{st}}(\eta)}\left|\int_0^t \langle c_s(v,c) - c_s(0,c), \mathrm{d}W^Q_{t'}\rangle\right| \\ &+ 2\mathbb{E}\sup_{0\leq t\leq T\wedge t_c\wedge t_{\mathrm{st}}(\eta)}\left|\int_0^t \langle 2\phi_c v + v^2, T_\xi \mathrm{d}W^Q_{t'}\rangle\right|.\end{aligned}$$

The Burkholder-Davis-Gundy inequality [98, proposition 2.1] together with Lemma 4.7.5 yields the control

$$\begin{aligned}I(T) &\leq C_{18}\mathbb{E}\left[\left(\int_0^{T\wedge t_c\wedge t_{\mathrm{st}}(\eta)}\|v(t)\|_{L^2_w}^2 + \|v(t)\|_{L^2}^4\mathrm{d}t\right)^{1/2}\right] \\ &\leq C_{18}\eta\sqrt{T} + C_{18}\sqrt{T}\mathbb{E}\sup_{0\leq t\leq T\wedge t_c\wedge t_{\mathrm{st}}(\eta)}\|v(t)\|_{L^2}^2.\end{aligned}$$

If $\delta_{16}$ is small enough to ensure $C_{18}\sigma\sqrt{T} \leq 1/2$, we may bring this last term to the left in (4.55):

$$\tfrac{1}{2}\mathbb{E} \sup_{0\leq t\leq T\wedge t_c\wedge t_{\mathrm{st}}(\eta)} \|v(t)\|_{L^2}^2 \leq C_2 \int_0^T \left(\sigma^2 + 2\epsilon|f(t)|\right)\mathbb{E} \sup_{0\leq t'\leq t\wedge t_c\wedge t_{\mathrm{st}}(\eta)} \|v(t')\|_{L^2}^2 \mathrm{d}t$$
$$+ C_2 T\left(\sigma^2 + \epsilon\eta + \eta^2\right) + C_2\sigma\sqrt{T}\eta.$$

Grönwall's inequality then yields

$$\mathbb{E} \sup_{0\leq t\leq T\wedge t_c\wedge t_{\mathrm{st}}(\eta)} \|v(t)\|_{L^2}^2 \leq 2C_2 e^{C_2\sigma^2 T} e^{2C_2\epsilon\int_0^T |f(t)|\mathrm{d}t}\Big(T\left(\sigma^2 + \epsilon\eta + \eta^2\right) + \sigma\sqrt{T}\eta\Big).$$

Noting that C2 implies

$$\epsilon\int_0^T |f(t)|\mathrm{d}t \leq E,$$

the result now follows via Markov's inequality:

$$\mathbb{P}\Big[\sup_{0\leq t\leq T\wedge t_c\wedge t_{\mathrm{st}}(\eta)} \|v(t)\|_{L^2}^2 \geq \lambda\Big] \leq 2C_2 e^{C_2\delta_{16}+2C_2 E}\lambda^{-1}\Big(T\big(\sigma^2 + \epsilon\eta + \eta^2\big) + \sigma\sqrt{T}\eta\Big).$$

□

## 4.8 Nonlinear stability

In this section, we collect our results and prove the stability result Theorem 4.1.1. Our goal here is to show that the event $t_{\mathrm{st}}(\eta) \geq T$ occurs with high probability, by ensuring that

- the unweighted $L^2$-norm of $v(t)$ remains under control on $[0, T]$ (Proposition 4.7.2);
- the difference between the local and global modulation parameters remains under control on $[0, T]$ (Proposition 4.6.1);
- the weighted norm of $v(t)$ remains under control on $[0, T]$ (Proposition 4.5.1).

Although our primary interest lies in the latter, our proof requires all of the above to hold. Since the stability results of Section 4.5 and Section 4.6 hold on intervals of length $\Delta T$, we partition the interval $[0, T]$ into

$$[0, T] \subseteq \bigcup_{n=0}^{\lceil T/\Delta T\rceil - 1} [n\Delta T, (n+1)\Delta T),$$

and seek to establish stability on each intermediate interval $[n\Delta T, (n+1)\Delta T)$. Consider, therefore, for each $n \in \mathbb{N}_0$ the stability event $\mathcal{S}_n(\eta) \subseteq \Omega$, defined as the set of $\omega \in \Omega$ for which:

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(i) the local stopping times at $n\Delta T$ satisfy

$$\tau_{\mathrm{mod}}^{n\Delta T}(\eta), \tau_{\mathrm{en}}^{n\Delta T}(\delta_*), \tau_{\mathrm{st}}^{n\Delta T}(\eta) \geq \Delta T;$$

(ii) at the end point $(n+1)\Delta T$ we have

$$\|v^{n\Delta T}(\Delta T)\|_{H^1_w} \leq \tfrac{\eta}{9M}, \quad \|v((n+1)\Delta T)\|_{H^1_w} \leq \tfrac{\eta}{3M}.$$

The following result allows us to establish high probability of the local stability events in a recursive manner: we bound the probability that $\mathcal{S}_n(\eta)$ fails to hold, provided $\mathcal{S}_{n-1}(\eta)$ occurred. We do so under the condition that the global modulation system is under control. In particular, for each $n \in \mathbb{N}$, we define the stability event $\mathcal{G}_n(\eta)$ as the set of $\omega \in \Omega$ for which the global stopping times $t_c$ and $t_{\mathrm{en}}$ reach

$$t_c, t_{\mathrm{en}}(\delta_*/2) \geq n\Delta T + \min\{\tau_{\mathrm{mod}}^{n\Delta T}(\eta), \tau_{\mathrm{en}}^{n\Delta T}(\delta_*), \tau_{\mathrm{st}}^{n\Delta T}(\eta)\}. \tag{4.56}$$

Below, $S_n^c$ denotes the complement $\Omega \setminus S_n$.

**Proposition 4.8.1.** *Assuming S1, S2 and C1, there exist constants $C_{19}, \delta_{19} > 0$, such that for each $n \in \mathbb{N}$ and each $\eta \in [0, \eta_0]$ the stability events $\mathcal{S}_{n-1}, \mathcal{S}_n(\eta)$ and $\mathcal{G}_n(\eta)$ satisfy*

$$\mathbb{P}\Big[\mathcal{S}_{n-1}(\eta) \cap \mathcal{G}_n(\eta) \cap \mathcal{S}_n^c(\eta)\Big] \leq C_{19} e^{-\delta_{19}\eta^2/\sigma^2}.$$

*Proof.* Let us write

$$\bar{t} = t_c \wedge t_{\mathrm{en}}(\delta_*/2) \quad \text{and} \quad \tau = \tau_{\mathrm{mod}}^{n\Delta T}(\eta) \wedge \tau_{\mathrm{en}}^{n\Delta T}(\delta_*) \wedge \tau_{\mathrm{st}}^{n\Delta T}(\eta).$$

Assuming $\mathcal{S}_{n-1}$, we distinguish three scenarios through which condition (i) in $\mathcal{S}_n$ can fail to hold:

$$\begin{aligned}
\mathcal{A}_1 &= \big\{\tau_{\mathrm{mod}}^{n\Delta T}(\eta) < \Delta T\big\} \cap \{n\Delta T + \tau_{\mathrm{mod}}^{n\Delta T}(\eta) \leq \bar{t}\} \cap \{\tau_{\mathrm{mod}}^{n\Delta T}(\eta) \leq \tau\};\\
\mathcal{A}_2 &= \big\{\tau_{\mathrm{en}}^{n\Delta T}(\delta_*) < \Delta T\big\} \cap \{n\Delta T + \tau_{\mathrm{en}}^{n\Delta T}(\delta_*) \leq \bar{t}\} \cap \{\tau_{\mathrm{en}}^{n\Delta T}(\delta_*) \leq \tau\};\\
\mathcal{A}_3 &= \big\{\tau_{\mathrm{st}}^{n\Delta T}(\eta) < \Delta T\big\} \cap \{n\Delta T + \tau_{\mathrm{st}}^{n\Delta T}(\eta) \leq \bar{t}\} \cap \{\tau_{\mathrm{st}}^{n\Delta T}(\eta) \leq \tau\}.
\end{aligned}$$

The events $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ categorize which of the stopping times was first activated before time $(n+1)\Delta T$. Thus, the event that item (i) of $S_n$ fails to hold coincides with

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3.$$

We proceed by estimating the probabilities of the events $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$.

1. The event $\mathcal{A}_1$ while also $\mathcal{S}_{n-1}(\eta)$ implies

$$\tau_{\mathrm{mod}}^{n\Delta T}(\eta) \leq \tau_{\mathrm{st}}^{n\Delta T}(\eta) \wedge \Delta T \quad \text{and} \quad n\Delta T + \tau_{\mathrm{mod}}^{T}(\eta) \leq t_c.$$

Proposition 4.6.1 then gives

$$\mathbb{P}\Big[\mathcal{S}_{n-1}(\eta) \cap \mathcal{G}_n(\eta) \cap \mathcal{A}_1\Big] \leq e^{-\delta_{12}\eta^2/\sigma^2}.$$

2. The event $\mathcal{A}_2$ while also $\mathcal{S}_{n-1}(\eta)$ implies

$$0 < \tau_{\mathrm{en}}^{n\Delta T}(\delta_*) \leq \tau_{\mathrm{mod}}^{n\Delta T}(\eta) \quad \text{and} \quad n\Delta T + \tau_{\mathrm{en}}^{n\Delta T}(\delta_*) \leq t_{\mathrm{en}}(\delta_*/2) \wedge t_c. \tag{4.57}$$

This event has probability zero. Indeed, we have

$$c^T(s), c(T+s) \in [\tfrac{1}{2}c_{\min}, 2c_{\max}], \quad \text{for} \quad s \in [0, \tau_{\mathrm{mod}}^{n\Delta T}(\eta) \wedge (t_c - n\Delta T)].$$

Lemma 4.4.2 then gives

$$\|v^T(s)\|_{H^1} \leq 2\eta + \|v(T+s)\|_{H^1} < \delta_*,$$

which contradicts (4.57).

3. The event $\mathcal{A}_3$ while also $\mathcal{S}_{n-1}(\eta)$ implies

$$\tau_{\mathrm{st}}^{n\Delta T}(\eta) \leq \Delta T \quad \text{while} \quad \|v(n\Delta T)\|_{H^1_w} \leq \tfrac{\eta}{3M}$$

and

$$\tau_{\mathrm{st}}^{n\Delta T}(\eta) \leq \min\big\{\tau_{\mathrm{en}}^{n\Delta T}(\delta_*), \tau_{\mathrm{mod}}^{n\Delta T}(\eta), \tau_c^{n\Delta T}\big\}.$$

Via Proposition 4.5.1, we then obtain

$$\mathbb{P}\Big[\mathcal{S}_{n-1}(\eta) \cap \mathcal{G}_n(\eta) \cap \mathcal{A}_3\Big] \leq e^{-\delta_9\eta^2/\sigma^2}.$$

Summarizing our results so far, we have shown that

$$\mathbb{P}\Big[\mathcal{S}_{n-1}(\eta) \cap \mathcal{G}_n(\eta) \cap \mathcal{A}\Big] \leq e^{-\delta_{12}\eta^2/\sigma^2} + e^{-\delta_9\eta^2/\sigma^2}.$$

To complete the proof, we turn our attention to item (ii) of $\mathcal{S}_n$, which fails to hold in the event that

$$\mathcal{B} = \{\|v^{n\Delta T}(\Delta T)\|_{H^1_w} > \tfrac{\eta}{9M}\} \cup \{\|v((n+1)\Delta T)\|_{H^1_w} > \tfrac{\eta}{3M}\}.$$

Note that the event

$$\|v^{n\Delta T}(\Delta T)\|_{H^1_w} > \tfrac{\eta}{9M} \quad \text{while} \quad \mathcal{S}_{n-1}(\eta) \quad \text{and} \quad \mathcal{A}^c \quad \text{hold}$$

occurs with probability less than $e^{-\delta_9\eta^2/\sigma^2}$ through Proposition 4.5.1. In the likely case that

$$\|v^{n\Delta T}(\Delta T)\|_{H^1_w} \leq \tfrac{\eta}{9M},$$

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it follows via Lemma 4.4.2 that also

$$\|v((n+1)\Delta T)\|_{H^1_w} \leq \tfrac{\eta}{3M}.$$

Hence,

$$\mathbb{P}\Big[\mathcal{S}_{n-1}(\eta) \cap \mathcal{G}_n(\eta) \cap \mathcal{B} \cap \mathcal{A}^c\Big] \leq e^{-\delta_9 \eta^2/\sigma^2}.$$

The proof is then completed by noting that

$$\mathcal{S}_{n-1}(\eta) \cap \mathcal{G}_n(\eta) \cap \mathcal{S}_n^c = \Big(\mathcal{S}_{n-1}(\eta) \cap \mathcal{G}_n(\eta) \cap \mathcal{A}\Big) \cup \Big(\mathcal{S}_{n-1}(\eta) \cap \mathcal{G}_n(\eta) \cap \mathcal{B}\Big),$$

where the probability of the latter can be estimated as

$$\mathbb{P}\Big[\mathcal{S}_{n-1}(\eta)\cap\mathcal{G}_n(\eta)\cap\mathcal{B}\Big] \leq \mathbb{P}\Big[\mathcal{S}_{n-1}(\eta)\cap\mathcal{G}_n(\eta)\cap\mathcal{A}\Big] + \mathbb{P}\Big[\mathcal{S}_{n-1}(\eta)\cap\mathcal{G}_n(\eta)\cap\mathcal{B}\cap\mathcal{A}^c\Big]. \qquad \square$$

We are now in shape to prove Theorem 4.1.1. Given $c_*, E > 0$, we fix $c_{\min}$ and $c_{\max}$ via C2. Picking $w \in (0, \sqrt{c_{\min}/3})$ then ensures that we are in the setting of S2. Lastly, let the constants $\Delta T, \delta_*$ and $\eta_0$ satisfy C1. We set out to control the probability of the slightly larger event

$$\mathcal{C}(\eta) = \{\min\{t_{\mathrm{st}}(2\eta), t_c, t_{\mathrm{en}}(\delta_*/2)\} < T\}.$$

Writing $\bar{t} = \min\{t_{\mathrm{st}}(2\eta), t_c, t_{\mathrm{en}}(\delta_*/2)\}$, we categorize $\mathcal{C}(\eta)$ into three scenarios:

$$\begin{aligned}
\mathcal{C}_1(\eta) &= \{t_c < T\} \cap \{t_c \leq \bar{t}\}; \\
\mathcal{C}_2(\eta) &= \{t_{\mathrm{en}}(\delta_*/2) < T\} \cap \{t_{\mathrm{en}}(\delta_*/2) \leq \bar{t}\}; \\
\mathcal{C}_3(\eta) &= \{t_{\mathrm{st}}(2\eta) < T\} \cap \{t_{\mathrm{st}}(2\eta) \leq \bar{t}\},
\end{aligned}$$

corresponding to which of the stopping times is hit first. We now subdivide the event $\mathcal{C}_3(\eta)$ by noting that for each realisation $\omega \in \mathcal{C}_3(\eta)$, the stopping time $t_{\mathrm{st}}(2\eta)(\omega)$ is contained in an interval

$$\big[n(\omega)\Delta T, n(\omega)\Delta T + \Delta T\big), \quad n(\omega) \in \{0, 1, \ldots, \lceil T/\Delta T\rceil - 1\}.$$

In view of Lemma 4.6.5, we in turn find that

$$n(\omega)\Delta T + \min\{\tau_{\mathrm{st}}^{n(\omega)\Delta T}(\eta)(\omega), \tau_{\mathrm{mod}}^{n(\omega)\Delta T}(\eta)(\omega)\} \leq t_{\mathrm{st}}(2\eta)(\omega).$$

Hence, $\mathcal{C}_3(\eta)$ implies that either

$$\mathcal{C}_{3;i}(\eta) := \bigcup_{n=1}^{\lceil T/\Delta T\rceil - 1} \Big(\mathcal{S}_{n-1}(\eta) \cap \mathcal{G}_n(\eta) \cap \mathcal{S}_n^c(\eta)\Big);$$

i.e. the 'chain' of stable events was interrupted, or stability failed on the first interval

$$\mathcal{C}_{3;ii}(\eta) := \mathcal{S}_0^c(\eta) \cap \mathcal{G}_0(\eta).$$

In summary, we have obtained

$$\mathcal{C}(\eta) \subseteq \mathcal{C}_1(\eta) \cup \mathcal{C}_2(\eta) \cup \mathcal{C}_{3;i}(\eta) \cup \mathcal{C}_{3;ii}(\eta). \tag{4.58}$$

Each of these events can be readily controlled using our prior results.

*Proof of Theorem 4.1.1.* For ease of exposition, we consider $T \geq 1$ for which $T/(\Delta T) \in \mathbb{N}$. We proceed by bounding the probability of each of the events $\mathcal{C}_1(\eta), \mathcal{C}_2(\eta), \mathcal{C}_{3;i}(\eta)$ and $\mathcal{C}_{3;ii}(\eta)$ in (4.58).

1. Proposition 4.7.1 gives

$$\mathbb{P}[\mathcal{C}_1(\eta)] \leq e^{-\delta_{17}/(\sigma^2 T)}. \tag{4.59}$$

2. Proposition 4.7.2 gives

$$\mathbb{P}\Big[\mathcal{C}_2(\eta)\Big] \leq C_{16}(\delta_*/2)^{-1}\Big(T\big(\sigma^2 + 2\epsilon\eta + 4\eta^2\big) + 2\sqrt{T}\sigma\eta\Big). \tag{4.60}$$

3. Applying Proposition 4.8.1 on the $T/\Delta T - 1$ intervals yields

$$\mathbb{P}[\mathcal{C}_{3;i}(\eta)] \leq \Big(\frac{T}{\Delta T} - 1\Big) C_{19} e^{-\delta_{19}\eta^2/\sigma^2}. \tag{4.61}$$

   Similarly,

$$\mathbb{P}[\mathcal{C}_{3;ii}(\eta)] \leq C_{19} e^{-\delta_{19}\eta^2/\sigma^2}. \tag{4.62}$$

The bounds (4.59), (4.60), (4.61) and (4.62) together with the union bound

$$\mathbb{P}[\mathcal{C}(\eta)] \leq \mathbb{P}[\mathcal{C}_1(\eta)] + \mathbb{P}[\mathcal{C}_2(\eta)] + \mathbb{P}[\mathcal{C}_{3;i}(\eta)] + \mathbb{P}[\mathcal{C}_{3;ii}(\eta)]$$

imply that

$$\mathbb{P}[t_{\mathrm{st}}(2\eta) < T] \leq N(\eta, \sigma, T), \tag{4.63}$$

where

$$N(\eta, \sigma, T) = \tilde{C} T(\eta^2 + e^{-\delta_{19}\eta^2/\sigma^2}),$$

for a sufficiently large constant $\tilde{C} > 0$. Observe now that for any $\tilde{\eta} \in [0, \eta]$, it $\mathbb{P}$-almost surely holds that $t_{\mathrm{st}}(2\tilde{\eta}) \leq t_{\mathrm{st}}(2\eta)$. We hence have the inclusion

$$\{t_{\mathrm{st}}(2\eta) < T\} \subseteq \{t_{\mathrm{st}}(2\tilde{\eta}) < T\},$$

allowing (4.63) to be improved to

$$\mathbb{P}[t_{\mathrm{st}}(2\eta) < T] \leq \inf_{0 \leq \tilde{\eta} \leq \eta} N(\eta, \sigma, T).$$

For $\sigma^2 < \delta_{19}$, we compute that $N(\eta,\sigma,T)$ is minimized at $\tilde{\eta}^2 = \frac{\sigma^2}{\delta_{19}}\log(\frac{\delta_{19}}{\sigma^2})$, and

$$\inf_{\tilde{\eta}\geq 0} N(\eta,\sigma,T) = \frac{\tilde{C}}{\delta_{19}}T\sigma^2\Big(\log\Big(\frac{\delta_{19}}{\sigma^2}\Big)+1\Big).$$

Hence, in case $\eta^2 \leq \frac{\sigma^2}{\delta_{19}}\log(\frac{\delta_{19}}{\sigma^2})$, we have

$$\mathbb{P}[t_{\mathrm{st}}(2\eta) < T] \leq N(\eta,\sigma,T) \leq \tilde{C}T\Big(\frac{\sigma^2}{\delta_{19}}\log\Big(\frac{\delta_{19}}{\sigma^2}\Big) + e^{-\delta_{19}\eta^2/\sigma^2}\Big).$$

On the other hand, for $\eta^2 > \frac{\sigma^2}{\delta_{19}}\log(\frac{\delta_{19}}{\sigma^2})$, we find

$$\mathbb{P}[t_{\mathrm{st}}(2\eta) < T] \leq \inf_{\tilde{\eta}\geq 0} N(\eta,\sigma,T) = \frac{\tilde{C}}{\delta_{19}}T\sigma^2\Big(\log\Big(\frac{\delta_{19}}{\sigma^2}\Big)+1\Big).$$

Both estimates can be absorbed by the bound (4.9), completing the proof. □

## 4.9 Validity of reduced dynamics

Our goal here is to establish the remaining approximation result Theorem 4.1.2. This result concerns the validity of the approximation $c_{\mathrm{ap}}(t)$ defined in (4.7) for the soliton amplitude $c(t)$. We thus set out to analyze the evolution equation

$$\begin{aligned} \mathrm{d}\big(c(t) - c_{\mathrm{ap}}(t)\big) = {} & c_d^0(v,c)\ \mathrm{d}t + \epsilon f(t)\big(c_f(v,c) - \tfrac{4}{3}c_{\mathrm{ap}}\big)\ \mathrm{d}t + \sigma^2\big(c_d(v,c) - c_d(0,c_{\mathrm{ap}})\big)\ \mathrm{d}t \\ & + \sigma\big\langle c_s(v,c) - \tfrac{2}{9}c_{\mathrm{ap}}^{-1/2}\phi_{c_{\mathrm{ap}}}^2, T_\xi \mathrm{d}W_t^Q\big\rangle, \end{aligned} \tag{4.64}$$

which one finds by subtracting (4.7) from (4.19). Recalling the constants $C_2$ and $\delta_2$ introduced in Lemma 4.3.4, we obtain the following useful bounds on the terms above.

**Lemma 4.9.1.** *Assuming S1 and S2, there exists a constant $C_{20} > 0$ so that for all $c_1, c_2 \in [\frac{1}{2}c_{\min}, 2c_{\max}]$ and all $v \in L^2_w$ that satisfy $\|v\|_{L^2_w} \leq \delta_2$, we have*

$$\begin{aligned} \big|c_f(v,c_1) - \tfrac{4}{3}c_2\big| &\leq C_2\|v\|_{L^2_w} + \tfrac{4}{3}|c_1 - c_2|, \\ \big|c_d(v,c_1) - c_d(0,c_2)\big| &\leq C_2\big(1 + \|v\|_{L^2_w} + \|v\|^2_{L^2_w}\big)\|v\|_{L^2_w} + C_{20}|c_1 - c_2|, \\ \big\|Q^{1/2}c_s(v,c_1) - Q^{1/2}[\tfrac{2}{9}c_2^{-1/2}\phi^2_{c_2}]\big\|_{L^2} &\leq C_2\|v\|_{L^2_w} + C_{20}|c_1 - c_2|. \end{aligned}$$

*Proof.* Recalling that $c_f(0,c) = \frac{4}{3}c$, we estimate

$$\big|c_f(v,c_1) - \tfrac{4}{3}c_2\big| \leq \big|c_f(v,c_1) - c_f(0,c_1)\big| + \big|\tfrac{4}{3}c_1 - \tfrac{4}{3}c_2\big|.$$

Applying Lemma 4.3.4 to estimate the first term yields

$$\big|c_f(v,c_1) - \tfrac{4}{3}c_2\big| \leq C_2\|v\|_{L^2_w} + \tfrac{4}{3}|c_1 - c_2|.$$

The remaining inequalities follow analogously through the Lipschitz bound

$$|c_d(0,c_1) - c_d(0,c_2)| + \|Q^{1/2}c_s(0,c_1) - Q^{1/2}c_s(0,c_2)\|_{L^2} \leq C_{20}|c_1 - c_2|$$

on $[\frac{1}{2}c_{\min}, 2c_{\max}]$. $\square$

As a further preparation, we establish control on the stochastic integral in (4.64). Recall the notation

$$t_{\mathrm{ap}}(\lambda) = \sup\big\{t \geq 0 : |c(t) - c_{\mathrm{ap}}(t)| \leq \lambda\big\},$$

and let $\lambda_0 = \min\{\frac{1}{2}c_{\min}, c_{\max}\}$.

**Lemma 4.9.2.** *Assuming S1, S2 and C1, there exists a constant $C_{21} > 0$ so that for each $T \geq 0$ and $\eta \in [0, \delta_2]$ we have*

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T \wedge t_c \wedge t_{\mathrm{ap}}(\lambda_0) \wedge t_{\mathrm{st}}(\eta)} & \Big| \int_0^t \langle c_s(v,c) - \tfrac{2}{9}c_{\mathrm{ap}}^{-1/2}\phi_{c_{\mathrm{ap}}}^2, T_\xi \mathrm{d}W_{t'}^Q \rangle \Big| \\
&\leq C_{21}\sqrt{T}\mathbb{E} \sup_{0 \leq t' \leq T \wedge t_{\mathrm{st}}(\eta)} \|v(t')\|_{L^2_w} \\
&\quad + C_{21}\sqrt{T}\mathbb{E} \sup_{0 \leq t' \leq T \wedge t_c \wedge t_{\mathrm{ap}}(\lambda_0)} |c(t') - c_{\mathrm{ap}}(t')|.
\end{aligned}$$

*Proof.* Let us write $\bar{t} = t_c \wedge t_{\mathrm{ap}}(\lambda_0) \wedge t_{\mathrm{st}}(\eta)$. The Burkholder-Davis-Gundy inequality [98, proposition 2.1] provides control of the stochastic integral:

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq t \leq T \wedge \bar{t}} \Big| \int_0^t \langle c_s(v,c) - \tfrac{2}{9}c_{\mathrm{ap}}^{-1/2}\phi_{c_{\mathrm{ap}}}^2, T_\xi \mathrm{d}W_{t'}^Q \rangle \Big| \\
&\leq \tilde{C}_1 \mathbb{E}\Big[\Big(\int_0^{T \wedge \bar{t}} \|Q^{1/2}c_s(v,c) - Q^{1/2}[\tfrac{2}{9}c_{\mathrm{ap}}^{-1/2}\phi_{c_{\mathrm{ap}}}^2]\|_{L^2}^2 \mathrm{d}t'\Big)^{1/2}\Big].
\end{aligned}$$

Lemma 4.9.1 then yields

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq t \leq T \wedge \bar{t}} \Big| \int_0^t \langle c_s(v,c) - \tfrac{2}{9}c_{\mathrm{ap}}^{-1/2}\phi_{c_{\mathrm{ap}}}^2, T_\xi \mathrm{d}W_{t'}^Q \rangle \Big| \\
&\leq \tilde{C}_2 \mathbb{E}\Big[\Big(\int_0^{T \wedge \bar{t}} \|v\|_{L^2_w}^2 + |c - c_{\mathrm{ap}}|^2 \mathrm{d}t\Big)^{1/2}\Big] \\
&\leq \tilde{C}_2\sqrt{T}\mathbb{E} \sup_{0 \leq t \leq T \wedge t_{\mathrm{st}}(\eta)} \|v(t)\|_{L^2_w} + \tilde{C}_2\sqrt{T}\mathbb{E} \sup_{0 \leq t \leq T \wedge t_c \wedge t_{\mathrm{ap}}(\lambda_0)} |c(t) - c_{\mathrm{ap}}(t)|. \quad \square
\end{aligned}$$

We are now ready to control $|c(t) - c_{\mathrm{ap}}(t)|$ via a Grönwall argument, resulting in conditional control of the stopping time $t_{\mathrm{ap}}$.

**Lemma 4.9.3.** *Assuming S1, S2, C1 and C2, there exists a constant $C_{22} > 0$ so* *that for each $T \geq 1, \eta \in (0, \delta_2]$, each $\sigma, \epsilon \in [0, \eta]$ and each $\lambda \in (0, \lambda_0]$, we have*

$$\mathbb{P}\big[t_{\mathrm{ap}}(\lambda) < T \cap t_{\mathrm{st}}(\eta) \wedge t_c \geq T\big] \leq C_{22} T e^{\sigma^2 T} \frac{\eta^2}{\lambda}.$$

*Proof.* Applying Lemma 4.9.1 and Lemma 4.9.2 to the SDE (4.64) yields

$$\begin{aligned}
|c(t) - c_{\mathrm{ap}}(t)| \leq C_2 \int_0^t \|v\|_{L^2_w}^2 \mathrm{d}t' + C_2 \epsilon \int_0^t \|v\|_{L^2_w} \mathrm{d}t' + \tfrac{4}{3}\epsilon \int_0^t |f(t')||c - c_{\mathrm{ap}}| \mathrm{d}t' \\
+ C_2 \sigma^2 \int_0^t (1 + \|v\|_{L^2_w}^2) \|v\|_{L^2_w} \mathrm{d}t' + C_{20}\sigma^2 \int_0^t |c - c_{\mathrm{ap}}| \; \mathrm{d}t' \\
+ \sigma \Big| \int_0^t \langle c_s(v, c) - \tfrac{2}{9} c_{\mathrm{ap}}^{-1/2} \phi_{c_{\mathrm{ap}}}^2, T_\xi \mathrm{d}W_{t'}^Q \rangle \Big|,
\end{aligned}$$

for all $t \in [0, t_c \wedge t_{\mathrm{ap}}(\lambda_0)]$. Let us once more write $\bar{t} = t_c \wedge t_{\mathrm{ap}}(\lambda_0) \wedge t_{\mathrm{st}}(\eta)$. In addition, we introduce the notation

$$E(t) := \mathbb{E} \sup_{0 \leq t' \leq t \wedge \bar{t}} |c(t') - c_{\mathrm{ap}}(t')|, \quad t \geq 0.$$

Inspecting the bound above implies

$$\begin{aligned}
E(T) \leq C_2 T \eta^2 + C_2 \epsilon T \eta + \tfrac{4}{3}\epsilon \int_0^T |f(t)| E(t) \mathrm{d}t \\
+ C_2 \sigma^2 T (1 + \eta^2)\eta + C_{20}\sigma^2 \int_0^T E(t) \mathrm{d}t \\
+ \sigma C_{21} \sqrt{T} \Big( \eta + E(T) \Big).
\end{aligned} \tag{4.65}$$

Imposing the restriction $\sigma\sqrt{T} \leq \frac{1}{2C_{21}}$, we may bring the last term in (4.65) to the left to find

$$\begin{aligned}
\tfrac{1}{2} E(T) \leq C_2 T \eta^2 + C_2 \epsilon T \eta + \tfrac{4}{3}\epsilon \int_0^T |f(t)| E(t) \mathrm{d}t \\
+ C_2 \sigma^2 T (1 + \eta^2)\eta + C_{20}\sigma^2 \int_0^T E(t) \mathrm{d}t + C_{21} \sigma \eta \sqrt{T}.
\end{aligned}$$

Grönwall's inequality then yields

$$\begin{aligned}
\tfrac{1}{2} E(T) \leq C_2 \exp\Big( \int_0^T C_{20}\sigma^2 + \frac{4}{3}\epsilon |f(t)| \mathrm{d}t \Big) \\
\times \Big( T\eta^2 + \epsilon T \eta + \sigma^2 T (1 + \eta^2)\eta + \frac{C_{21}}{C_2} \sigma \eta \sqrt{T} \Big),
\end{aligned}$$

in which C2 ensures

$$\exp\Big(\int_0^T C_{20}\sigma^2 + \frac{4}{3}\epsilon|f(t)|\mathrm{d}t\Big) \leq \exp\Big(C_{20}\sigma^2 T + \frac{4}{3}E\Big).$$

Markov's inequality finally yields

$$\mathbb{P}\Big[\sup_{0\leq t'\leq T\wedge\bar{t}} |c(t') - c_{\mathrm{ap}}(t')| \geq \lambda\Big] \leq \frac{2}{\lambda} C_2 \exp\Big(C_{20}\sigma^2 T + \frac{4}{3}E\Big) \\ \times \Big(T\eta^2 + \epsilon T\eta + \sigma^2 T(1+\eta^2)\eta + \frac{C_{21}}{C_2}\sigma\eta\sqrt{T}\Big).$$

To complete the proof, note that $t_{\mathrm{ap}}(\lambda) < T$ while $t_{\mathrm{st}}(\eta) \wedge t_c \geq T$ implies

$$\sup_{0\leq t'\leq T\wedge\bar{t}} |c(t') - c_{\mathrm{ap}}(t')| > \lambda. \qquad \square$$

*Proof of Theorem 4.1.2.* For any $\eta \geq 0$, we have the union bound

$$\mathbb{P}[t_{\mathrm{ap}}(\lambda) < T] \leq \mathbb{P}[t_{\mathrm{ap}}(\lambda) < T \cap t_{\mathrm{st}}(\eta) \wedge t_c \geq T] + \mathbb{P}[t_{\mathrm{st}}(\eta) \wedge t_c < T]$$

and hence

$$\mathbb{P}[t_{\mathrm{ap}}(\lambda) < T] \leq \inf_{\eta\geq 0} \Big(\mathbb{P}[t_{\mathrm{ap}}(\lambda) < T \cap t_{\mathrm{st}}(\eta) \wedge t_c \geq T] + \mathbb{P}[t_{\mathrm{st}}(\eta) \wedge t_c < T]\Big).$$

Applying Theorem 4.1.1 and Lemma 4.9.3 now gives

$$\mathbb{P}[t_{\mathrm{ap}}(\lambda) < T] \leq (C + C_{22})T \inf_{\eta\geq 0} \Big(\frac{\eta^2}{\lambda} + e^{-\delta\eta^2/\sigma^2}\Big) + CT\sigma^2\log(1/\sigma).$$

In case $\sigma^2 < \lambda\delta$, the infimum is attained at

$$\eta^2 = \frac{\sigma^2}{\delta}\log\Big(\frac{\lambda\delta}{\sigma^2}\Big)$$

yielding

$$\mathbb{P}[t_{\mathrm{ap}}(\lambda) < T] \leq (C + C_{22})T\frac{\sigma^2}{\lambda\delta}\Big(\log\Big(\frac{\lambda\delta}{\sigma^2}\Big) + 1\Big) + CT\sigma^2\log(1/\sigma).$$

Upon increasing the constant $C$ if necessary, we obtain the result (4.10). $\square$

## 4.10 Stopping times

Here, we provide an overview of the various stopping times introduced throughout the chapter. In relation to the global modulation system $\big(v(t), c(t), \xi(t)\big)$ introduced in Section 4.3, we use the stopping times

$$
\begin{aligned}
t_c &= \sup\{t \geq 0 : c(t) \in [c_{\min}, c_{\max}]\};\\
t_{\log} &= \sup\{t \geq 0 : |\log(c(t)/c_*) - \frac{4}{3}\epsilon \int_0^t |f(t')| \mathrm{d}t'| \leq E\};\\
t_{\mathrm{st}}(\eta) &= \sup\big\{t \geq 0 : \|v(t)\|_{H^1_w} \leq \eta\big\};\\
t_{\mathrm{en}}(\eta) &= \sup\big\{t \geq 0 : \big\|v(t)\big\|_{L^2} \leq \eta\big\};\\
t_{\mathrm{ap}}(\eta) &= \sup\big\{t \geq 0 : |c(t) - c_{\mathrm{ap}}(t)| \leq \eta\big\}.
\end{aligned}
$$

For each $T \geq 0$, the following stopping times are related to the local modulation system $\big(v^T(s), c^T(s), \xi^T(s)\big)$ introduced in Section 4.4:

$$
\begin{aligned}
\tau^T_{\mathrm{st}}(\eta) &= \sup\big\{s \geq 0 : \big\|v^T(s)\big\|_{H^1_w} \leq \eta\big\};\\
\tau^T_c &= \sup\big\{s \geq 0 : c(T+s) \in [\tfrac{1}{2}c_{\min}, 2c_{\max}]\big\};\\
\tau^T_{\mathrm{en}}(\eta) &= \sup\big\{s \geq 0 : \big\|v^T(s)\big\|_{L^2} \leq \eta\big\};\\
\tau^T_{\mathrm{amp},1}(\eta) &= \sup\big\{s \geq 0 : \big|c(T) - c^T(s)\big| \leq \delta_*\eta\big\};\\
\tau^T_{\mathrm{amp},2}(\eta) &= \sup\big\{s \geq 0 : \big|c(T+s) - c^T(s)\big| \leq \delta_*\eta\big\};\\
\tau^T_{\mathrm{pos},1}(\eta) &= \sup\big\{s \geq 0 : \big|\xi(T) + c(T)s - \xi^T(s)\big| \leq 2\Delta T\delta_*\eta\big\};\\
\tau^T_{\mathrm{pos},2}(\eta) &= \sup\big\{s \geq 0 : \big|\xi(T+s) - \xi^T(s)\big| \leq 2\Delta T\delta_*\eta\big\},
\end{aligned}
$$

and

$$
\tau^T_{\mathrm{mod}}(\eta) = \tau^T_{\mathrm{amp},1}(\eta) \wedge \tau^T_{\mathrm{amp},2}(\eta) \wedge \tau^T_{\mathrm{pos},1}(\eta) \wedge \tau^T_{\mathrm{pos},2}(\eta).
$$

## 4.11 Technical proofs

Here, we provide the proofs of various lemmas which were omitted in Section 4.3, Section 4.4 and Section 4.5.

*Proof of Lemma 4.3.2.* We derive the evolution equation for $\big\langle v(t), \phi_c(t)\big\rangle$, noting that the remaining evolution equation for $\big\langle v(t), \zeta_c(t)\big\rangle$ follows analogously. First, we introduce the notation

$$
\begin{aligned}
\big\langle v(t), \phi_{c(t)}\big\rangle &= \big\langle u(t, \cdot + \xi(t)), \phi_{c(t)}\big\rangle - \big\langle \phi_{c(t)}, \phi_{c(t)}\big\rangle = \big\langle u(t), \phi_{c(t)}(\cdot - \xi(t))\big\rangle - 6c^{3/2}(t)\\
&=: F\big(u(t), c(t), \xi(t)\big).
\end{aligned}
$$

We now apply the mild Itô formula to the functional $F : H^1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We therefore interpret the tuple $(u, c, \xi)$ as a mild process with respect to the $C_0$-group

$\{S(t)\}_{t\in\mathbb{R}}$ given by

$$S(t) = \begin{bmatrix} e^{-\partial_x^3 t} & 0 & 0 \\ 0 & I_{\mathbb{R}} & 0 \\ 0 & 0 & I_{\mathbb{R}} \end{bmatrix}, \quad t \in \mathbb{R}.$$

We collect that $F$ is twice Fréchet differentiable, with first derivatives

$$\begin{aligned} \mathrm{d}_u F(u,c,\xi)[v] &= \big\langle v, \phi_c(\cdot - \xi)\big\rangle, \\ \mathrm{d}_c F(u,c,\xi) &= \big\langle u, \partial_c \phi_c(\cdot - \xi)\big\rangle - 9c^{1/2}, \\ \mathrm{d}_\xi F(u,c,\xi) &= -\big\langle u, \partial_x \phi_c(\cdot - \xi)\big\rangle, \end{aligned}$$

and second derivatives

$$\begin{aligned} \mathrm{d}^2_{uu} F(u,c,\xi)[v,w] &= 0, \\ \mathrm{d}^2_{cc} F(u,c,\xi) &= \big\langle u, \partial_c^2 \phi_c(\cdot - \xi)\big\rangle - \tfrac{9}{2}c^{-1/2}, \\ \mathrm{d}^2_{\xi\xi} F(u,c,\xi) &= \big\langle u, \partial_x^2 \phi_c(\cdot - \xi)\big\rangle, \\ \mathrm{d}^2_{uc} F(u,c,\xi)[v] &= \big\langle v, \partial_c \phi_c(\cdot - \xi)\big\rangle, \\ \mathrm{d}^2_{u\xi} F(u,c,\xi)[v] &= -\big\langle v, \partial_x \phi_c(\cdot - \xi)\big\rangle, \\ \mathrm{d}^2_{c\xi} F(u,c,\xi) &= -\big\langle u, \partial_{cx}^2 \phi_c(\cdot - \xi)\big\rangle. \end{aligned}$$

For any orthonormal basis $\{e_k\}_{k=0}^\infty$ of $L^2$, [22, Theorem 1] then yields:

$$\begin{aligned} F\big(u(t),c(t),\xi(t)\big) = F\big(e^{-\partial_x^3 t}\phi_{c_*}, c_*, 0\big) &+ \int_0^t F_1(t,t')\mathrm{d}t' + \sigma^2 \int_0^t F_2(t,t')\mathrm{d}t' \\ &+ \sigma^2 \sum_{k=0}^\infty \int_0^t F_3(t,t',k)\mathrm{d}t' + \sigma \int_0^t F_4(t,t')\mathrm{d}W^Q_{t'}, \end{aligned} \tag{4.66}$$

with

$$\begin{aligned} F_1(t,t') &= \mathrm{d}_u F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)\big[e^{-\partial_x^3(t-t')}(-\partial_x(u^2) + \epsilon f(t')u)\big] \\ &\quad + \mathrm{d}_c F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)c_d^{\sigma,\epsilon} + \mathrm{d}_\xi F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)\xi_d^{\sigma,\epsilon}, \\ F_2(t,t') &= \tfrac{1}{2}\mathrm{d}^2_{cc} F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)\|Q^{1/2}c_s\|^2_{L^2} + \tfrac{1}{2}\mathrm{d}^2_{\xi\xi} F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)\|Q^{1/2}\xi_s\|^2_{L^2} \\ &\quad + \mathrm{d}^2_{c\xi} F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)\langle Q^{1/2}\xi_s, Q^{1/2}c_s\rangle, \\ F_3(t,t',k) &= \mathrm{d}^2_{uc} F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)[e^{-\partial_x^3(t-t')}uQ^{1/2}e_k]\langle c_s, T_\xi Q^{1/2}e_k\rangle \\ &\quad + \mathrm{d}^2_{u\xi} F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)[e^{-\partial_x^3(t-t')}uQ^{1/2}e_k]\langle \xi_s, T_\xi Q^{1/2}e_k\rangle, \\ F_4(t,t')[h] &= \mathrm{d}_u F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)[e^{-\partial_x^3(t-t')}uh] \\ &\quad + \mathrm{d}_c F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)\langle c_s, T_\xi h\rangle + \mathrm{d}_\xi F\big(e^{-\partial_x^3(t-t')}u,c,\xi\big)\langle \xi_s, T_\xi h\rangle. \end{aligned}$$

Above, we have suppressed the dependence of $c_d^{\sigma,\epsilon}, \xi_d^{\sigma,\epsilon}$ on $(v(t'), c(t'), t')$ and of $c_s, \xi_s$ on $(v(t'), c(t'))$. Substituting the derivatives, this becomes

$$
\begin{aligned}
F\big(e^{-\partial_x^3 t}\phi_{c_*}, c_*, 0\big) &= \langle e^{-\partial_x^3 t}\phi_{c_*}, \phi_{c_*}\rangle - 6c_*^{3/2},\\
F_1(t,t') &= \big\langle e^{-\partial_x^3(t-t')}(-\partial_x(u^2) + \epsilon f(t')u), \phi_c(\cdot-\xi)\big\rangle\\
&\quad + \big(\langle e^{-\partial_x^3(t-t')}u, \partial_c\phi_c(\cdot-\xi)\rangle - 9c^{1/2}\big)c_d^{\sigma,\epsilon}\\
&\quad - \big\langle e^{-\partial_x^3(t-t')}u, \partial_x\phi_c(\cdot-\xi)\big\rangle\xi_d^{\sigma,\epsilon},\\
F_2(t,t') &= \tfrac{1}{2}\big(\langle e^{-\partial_x^3(t-t')}u, \partial_c^2\phi_c(\cdot-\xi)\rangle - \tfrac{9}{2}c^{-1/2}\big)\|Q^{1/2}c_s\|_{L^2}^2\\
&\quad + \tfrac{1}{2}\big\langle e^{-\partial_x^3(t-t')}u, \partial_x^2\phi_c(\cdot-\xi)\big\rangle\|Q^{1/2}\xi_s\|_{L^2}^2\\
&\quad - \big\langle e^{-\partial_x^3(t-t')}u, \partial_{cx}^2\phi_c(\cdot-\xi)\big\rangle\big\langle Q^{1/2}\xi_s, Q^{1/2}c_s\big\rangle,\\
F_3(t,t',k) &= \big\langle e^{-\partial_x^3(t-t')}uQ^{1/2}e_k, \partial_c\phi_c(\cdot-\xi)\big\rangle\big\langle c_s, T_\xi Q^{1/2}e_k\big\rangle\\
&\quad - \langle e^{-\partial_x^3(t-t')}uQ^{1/2}e_k, \partial_x\phi_c(\cdot-\xi)\rangle\langle\xi_s, T_\xi Q^{1/2}e_k\rangle,\\
F_4(t,t')[h] &= \big\langle e^{-\partial_x^3(t-t')}uh, \phi_c(\cdot-\xi)\big\rangle\\
&\quad + \big(\langle e^{-\partial_x^3(t-t')}u, \partial_c\phi_c(\cdot-\xi)\rangle - 9c^{1/2}\big)\langle c_s, T_\xi h\rangle\\
&\quad - \big\langle e^{-\partial_x^3(t-t')}u, \partial_x\phi_c(\cdot-\xi)\big\rangle\langle\xi_s, T_\xi h\rangle.
\end{aligned}
$$

We now show how to convert the mild expression (4.66) into a strong form[3], focusing on the case $\sigma = \epsilon = 0$ for ease of exposition. Differentiating $\big\langle v(t), \phi_{c(t)}\big\rangle$ via Leibniz' rule gives

$$
\partial_t\big\langle v(t), \phi_{c(t)}\big\rangle = \partial_t F\big(u(t), c(t), \xi(t)\big) = \partial_t F\big(e^{-\partial_x^3 t}\phi_{c_*}, c_*, 0\big) + F_1(t,t) + \int_0^t \partial_t F_1(t,t')\mathrm{d}t',
$$

and thus

$$
\begin{aligned}
\partial_t\big\langle v(t), \phi_{c(t)}\big\rangle &= \big\langle e^{-\partial_x^3 t}\phi_{c_*}, \partial_x^3\phi_{c_*}\big\rangle - \big\langle\partial_x(u^2(t)), \phi_c(\cdot-\xi)\big\rangle\\
&\quad - \int_0^t \big\langle\partial_x\big(u^2(t')\big), e^{\partial_x^3(t-t')}\partial_x^3\phi_c(\cdot-\xi)\big\rangle\mathrm{d}t'\\
&\quad + \big(\langle u(t), \partial_c\phi_c(\cdot-\xi)\rangle - 9c^{1/2}\big)c_d^0\big(v(t)\big)\\
&\quad + \int_0^t \big\langle u(t'), e^{\partial_x^3(t-t')}\partial_x^3\partial_c\phi_c(\cdot-\xi)\big\rangle c_d^0\big(v(t')\big)\mathrm{d}t'\\
&\quad - \big\langle u(t), \partial_x\phi_c(\cdot-\xi)\big\rangle\xi_d^0\big(v(t)\big)\\
&\quad - \int_0^t \big\langle u(t'), e^{\partial_x^3(t-t')}\partial_x^3\partial_x\phi_c(\cdot-\xi)\big\rangle\xi_d^0\big(v(t')\big)\mathrm{d}t',
\end{aligned}
$$

where we have moved the semigroup to the other side of the inner products via the

[3] This computation resembles the procedure for passing from a mild to strong solution in case sufficient regularity is available.

adjoint relation $(e^{-\partial_x^3 t})^* = e^{\partial_x^3 t}$. Now we recognize the mild formula

$$\begin{aligned}\big\langle u(t), \partial_x^3\phi_c(\cdot - \xi)\big\rangle &= \big\langle e^{-\partial_x^3 t}\phi_{c_*}, \partial_x^3\phi_{c_*}\big\rangle - \int_0^t \big\langle \partial_x\big(u^2(t')\big), e^{\partial_x^3(t-t')}\partial_x^3\phi_c(\cdot - \xi)\big\rangle \mathrm{d}t' \\ &\quad + \int_0^t \big\langle u(t'), e^{\partial_x^3(t-t')}\partial_x^3\partial_c\phi_c(\cdot - \xi)\big\rangle c_d^0\big(v(t')\big)\mathrm{d}t' \\ &\quad - \int_0^t \big\langle u(t'), e^{\partial_x^3(t-t')}\partial_x^3\partial_x\phi_c(\cdot - \xi)\big\rangle \xi_d^0\big(v(t')\big)\mathrm{d}t'.\end{aligned}$$

We thus we arrive at the strong form

$$\begin{aligned}\mathrm{d}\big\langle v(t), \phi_{c(t)}\big\rangle &= \big\langle u(t), \partial_x^3\phi_c(\cdot - \xi)\big\rangle \ \mathrm{d}t - \big\langle \partial_x\big(u^2(t)\big), \phi_c(\cdot - \xi)\big\rangle \ \mathrm{d}t \\ &\quad + \big(\big\langle u(t), \partial_c\phi_c(\cdot - \xi)\big\rangle - 9c^{1/2}\big)c_d^0\big(v(t)\big) \ \mathrm{d}t - \big\langle u(t), \partial_x\phi_c(\cdot - \xi)\big\rangle \xi_d^0\big(v(t)\big) \ \mathrm{d}t.\end{aligned}$$

After substituting $u(t, \cdot + \xi) = \phi_c + v(t)$:

$$\begin{aligned}\mathrm{d}\big\langle v(t), \phi_{c(t)}\big\rangle &= \big\langle v(t), \partial_x^3\phi_c\big\rangle \ \mathrm{d}t - \big\langle \partial_x(\phi_c + v(t))^2, \phi_c\big\rangle \ \mathrm{d}t \\ &\quad + \big(\langle \phi_c + v(t), \partial_c\phi_c\rangle - 9c^{1/2}\big)c_d^0\big(v(t)\big) \ \mathrm{d}t - \big\langle v(t), \partial_x\phi_c\big\rangle \xi_d^0\big(v(t)\big) \ \mathrm{d}t\end{aligned}$$

where rewriting

$$\big\langle \partial_x(\phi_c + v(t))^2, \phi_c\big\rangle = -2\big\langle v(t), \phi_c\partial_x\phi_c\big\rangle - \big\langle N\big(v(t)\big), \phi_c\big\rangle$$

leads to

$$\begin{aligned}\mathrm{d}\big\langle v(t), \phi_{c(t)}\big\rangle &= \big\langle v(t), \partial_x^3\phi_c + 2\phi_c\partial_x\phi_c - c\partial_x\phi_c\big\rangle \ \mathrm{d}t + \big\langle N\big(v(t)\big), \phi_c\big\rangle \ \mathrm{d}t \\ &\quad + \big(\langle \phi_c + v(t), \partial_c\phi_c\rangle - 9c^{1/2}\big)c_d^0\big(v(t)\big) \ \mathrm{d}t - \big\langle v(t), \partial_x\phi_c\big\rangle \Omega_d^0\big(v(t)\big) \ \mathrm{d}t.\end{aligned}$$

Using the traveling wave identity $\partial_x^3\phi_c + 2\phi_c\partial_x\phi_c - c\partial_x\phi_c = 0$, we arrive at the result

$$\begin{aligned}\mathrm{d}\big\langle v(t), \phi_{c(t)}\big\rangle &= \big\langle N\big(v(t)\big), \phi_c\big\rangle \ \mathrm{d}t + \big(\langle \phi_c + v(t), \partial_c\phi_c\rangle - 9c^{1/2}\big)c_d^0\big(v(t)\big) \ \mathrm{d}t \\ &\quad - \big\langle v(t), \partial_x\phi_c\big\rangle \Omega_d^0\big(v(t)\big) \ \mathrm{d}t.\end{aligned}$$

The $\sigma$- and $\epsilon$-dependent terms in (4.66) can be treated analogously, which completes the proof. □

*Proof of Proposition 4.4.3.* We compute the mild form of

$$v^T(s, x) = u\big(T + s, x + \xi(T) + c(T)s\big) - \Phi^T(\mathbf{m}^T(s), s, x).$$

Recalling that

$$\Phi^T(\mathbf{m}^T(s), s, x) := \phi_{c^T(s)}\big(x + \xi(T) + c(T)s - \xi^T(s)\big),$$

a straightforward application of Itô's lemma yields

$$\begin{aligned}
\mathrm{d}\Phi^T = & \Big[\partial_c\Phi^T c_d^{\sigma,\epsilon,T} + \partial_x\Phi^T\big(c(T) - c^T - \Omega_d^{\sigma,\epsilon,T}\big)\Big]\mathrm{d}s \\
& + \frac{\sigma^2}{2}\Big[\partial_c^2\Phi^T\|Q^{1/2}c_s^T\|_{L^2}^2 + \partial_x^2\Phi^T\|Q^{1/2}\xi_s^T\|_{L^2}^2\Big]\mathrm{d}s \\
& + \sigma^2\partial_{xc}^2\Phi^T\langle Q^{1/2}c_s^T, Q^{1/2}\xi_s^T\rangle\mathrm{d}s \\
& + \sigma\partial_c\Phi^T\langle c_s^T, T_{\xi(T)+c(T)s}\mathrm{d}W_{T+s}^Q\rangle + \sigma\partial_x\Phi^T\langle \xi_s^T, T_{\xi(T)+c(T)s}\mathrm{d}W_{T+s}^Q\rangle
\end{aligned}$$

where we have suppressed the dependence of $\Phi^T, c_d^{\sigma,\epsilon,T}, \xi_d^{\sigma,\epsilon,T}, c_s^T$ and $\xi_s^T$ on $(\mathbf{m}^T(s), s)$. Using the traveling wave identity

$$\begin{aligned}
0 &= -\partial_x^3\Phi^T - \partial_x(\Phi^T)^2 + c^T\partial_x\Phi^T \\
&= \mathcal{L}_{c(T)}\Phi^T + \big(c^T - c(T)\big)\partial_x\Phi^T - \partial_x\big((\Phi^T)^2\big) - 2\partial_x\big(\phi_{c(T)}\Phi^T\big)
\end{aligned}$$

we may pass to the mild form

$$\begin{aligned}
\Phi^T(s) = & e^{\mathcal{L}_{c(T)}s}\phi_{c(T)} + \int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\Big[-\partial_x((\Phi^T)^2) - 2\partial_x(\phi_{c(T)}\Phi^T)\Big]\mathrm{d}s' \qquad (4.67)\\
& + \int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\Big[\partial_c\Phi^T c_d^{\sigma,\epsilon,T} - \partial_x\Phi^T\Omega_d^{\sigma,\epsilon,T}\Big]\mathrm{d}s' \\
& + \frac{\sigma^2}{2}\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\Big[\partial_c^2\Phi^T\|Q^{1/2}c_s^T\|_{L^2}^2 + \partial_x^2\Phi^T\|Q^{1/2}\xi_s^T\|_{L^2}^2\Big]\mathrm{d}s' \\
& + \sigma^2\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\partial_{xc}^2\Phi^T\langle Q^{1/2}c_s^T, Q^{1/2}\xi_s^T\rangle\mathrm{d}s' \\
& + \sigma\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\partial_c\Phi^T\langle c_s^T, T_{\xi(T)+c(T)s'}\mathrm{d}W_{T+s'}^Q\rangle \\
& + \sigma\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\partial_x\Phi^T\langle \xi_s^T, T_{\xi(T)+c(T)s'}\mathrm{d}W_{T+s'}^Q\rangle.
\end{aligned}$$

Next, we derive a mild formula for $u(T+s, x+\xi(T)+c(T)s)$ based on the identity

$$\begin{aligned}
u(T+s) = & e^{-\partial_x^3 s}u(T) - \int_0^s e^{-\partial_x^3(s-s')}\partial_x\big(u^2(T+s')\big)\mathrm{d}s' \\
& + \epsilon\int_0^s f(T+s')e^{-\partial_x^3(s-s')}u(T+s')\mathrm{d}s' \\
& + \sigma\int_0^s e^{-\partial_x^3(s-s')}u(T+s')\mathrm{d}W_{T+s'}^Q.
\end{aligned}$$

The mild Itô formula [22] now yields

$$\begin{aligned}
& u\big(T+s,\, x+\xi(T)+c(T)s\big) \\
& \quad = e^{-\partial_x^3 s}u\big(T, x+\xi(T)\big)
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^s e^{-\partial_x^3(s-s')}\partial_x\big(u^2(T+s',\cdot+\xi(T)+c(T)s')\big)\mathrm{d}s' \\
&+ \epsilon \int_0^s f(T+s')e^{-\partial_x^3(s-s')}u\big(T+s',\cdot+\xi(T)+c(T)s'\big)\mathrm{d}s' \\
&+ c(T) \int_0^s e^{-\partial_x^3(s-s')}u_x\big(T+s',\cdot+\xi(T)+c(T)s'\big)\mathrm{d}s' \\
&+ \sigma \int_0^s e^{-\partial_x^3(s-s')}u\big(T+s',\cdot+\xi(T)+c(T)s'\big)T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}.
\end{aligned}
$$

Using $-\partial_x^3 + c(T)\partial_x = \mathcal{L}_{c(T)} + 2\partial_x(\phi_{c(T)}\cdot)$, we rephrase the formula above in terms of the semigroup $\{e^{\mathcal{L}_{c(T)}s}\}_{s\geq 0}$ to find:

$$
\begin{aligned}
&u\big(T+s,\, x+\xi(T)+c(T)s\big) \\
&\quad = e^{\mathcal{L}_{c(T)}s}u\big(T,x+\xi(T)\big) \qquad (4.68)\\
&\quad + 2\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\partial_x\big(\phi_{c(T)}u(T+s',\cdot+\xi(T)+c(T)s')\big)\mathrm{d}s' \\
&\quad - \int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\partial_x\big(u^2(T+s',\cdot+\xi(T)+c(T)s')\big)\mathrm{d}s' \\
&\quad + \epsilon\int_0^s f(T+s')e^{\mathcal{L}_{c(T)}(s-s')}u\big(T+s',\cdot+\xi(T)+c(T)s'\big)\mathrm{d}s' \\
&\quad + \sigma\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}u\big(T+s',\cdot+\xi(T)+c(T)s'\big)T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}.
\end{aligned}
$$

Subtracting the mild formula (4.67) from (4.68), we arrive at

$$
\begin{aligned}
v^T(s) &= e^{\mathcal{L}_{c(T)}s}v(T) - \int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\big[\partial_x((v^T)^2) + 2\partial_x\big((\phi_{c(T)} - \Phi^T)v^T\big)\big]\mathrm{d}s' \\
&+ \epsilon\int_0^s f(T+s')e^{\mathcal{L}_{c(T)}(s-s')}\big[\Phi^T + v^T\big]\mathrm{d}s' \\
&- \int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\Big[\partial_c\Phi^T c_d^{\sigma,\epsilon,T} - \partial_x\Phi^T\Omega_d^{\sigma,\epsilon,T}\Big]\mathrm{d}s' \\
&- \frac{\sigma^2}{2}\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\Big[\partial_c^2\Phi^T\|Q^{1/2}c_s^T\|_{L^2}^2 + \partial_x^2\Phi^T\|Q^{1/2}\xi_s^T\|_{L^2}^2\Big]\mathrm{d}s' \\
&- \sigma^2\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\partial_{xc}^2\Phi^T\langle Q^{1/2}c_s^T, Q^{1/2}\xi_s^T\rangle\mathrm{d}s' \\
&- \sigma\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\partial_c\Phi^T\langle c_s^T, T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}\rangle \\
&- \sigma\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}\partial_x\Phi^T\langle \xi_s^T, T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'}\rangle \\
&+ \sigma\int_0^s e^{\mathcal{L}_{c(T)}(s-s')}[\Phi^T + v^T]T_{\xi(T)+c(T)s'}\mathrm{d}W^Q_{T+s'},
\end{aligned}
$$

as desired. □

*Proof of Theorem 4.5.3.* Applying [88, Theorem 4.5] gives

$$\mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t S(t-s)g(s)\mathrm{d}W_s^Q \Big\|_{\mathcal{H}}^p \leq (K_p M\sqrt{p})^p T^{\frac{p}{2}-1} \int_0^T \mathbb{E}\Big[\|g(t)\|_{\mathrm{HS}(L_Q^2,\mathcal{H})}^p\Big]\mathrm{d}t,$$

for $p \in (2,\infty)$, where $\limsup_{p\to\infty} K_p < \infty$. Assume without loss of generality that $K_p \leq K$ for $p > 2$. By our assumption,

$$\mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t S(t-s)g(s)\mathrm{d}W_s^Q \Big\|_{\mathcal{H}}^p \leq (BKM\sqrt{p}\sqrt{T})^p,$$

for $p > 2$. Markov's inequality then gives

$$\mathbb{P}\Big[\sup_{t\in[0,T]} \Big\| \int_0^t S(t-s)g(s)\mathrm{d}W_s^Q \Big\|_{\mathcal{H}}^p \geq \lambda^p\Big] \leq (\lambda^{-1}BKM\sqrt{p}\sqrt{T})^p.$$

For $\lambda > eBKM\sqrt{T}$ we may choose $p = (eBKM)^{-2}\lambda^2/T$ to conclude

$$\begin{aligned}\mathbb{P}\Big[\sup_{t\in[0,T]} \Big\| \int_0^t S(t-s)g(s)\mathrm{d}W_s^Q \Big\|_{\mathcal{H}} \geq \lambda\Big] &= \mathbb{P}\Big[\sup_{t\in[0,T]} \Big\| \int_0^t S(t-s)g(s)\mathrm{d}W_s^Q \Big\|_{\mathcal{H}}^p \geq \lambda^p\Big] \\ &\leq e^{-(eBKM)^{-2}\lambda^2/T}.\end{aligned}$$

□

4

# CHAPTER 5
# Lattice waves

We study the propagation of solitary waves in a Fermi–Pasta–Ulam–Tsingou (FPUT) lattice with small random heterogeneity in the linear spring force. Perturbed by the random environment, solitary waves lose energy through a radiative tail, resulting in gradual amplitude attenuation. As long as the wave remains coherent, we track its position and amplitude via a modulation approach. An expansion of the resulting modulation equations provides explicit predictions for the slow average amplitude decay, which we verify through numerical simulations.

## 5.1 Introduction

When solitary waves in dispersive systems interact with spatial inhomogeneity, numerical simulations demonstrate that the waves emit a radiative tail [3, 5, 7, 9, 106]. This is decidedly so in Fermi–Pasta–Ulam–Tsingou (FPUT) lattices where the phenomenon has been observed in a variety of settings [92, 79, 72, 50, 89]. The study of lattices with spatially varying material coefficients has a long history [78, 30], as disorder naturally arises in crystals through mechanisms such as atomic replacement, isotopic variation, or structural defects [6]. It is well-known that homogeneous FPUT lattices support exact solitary wave solutions [41, 42]. In lattices with *periodic* heterogeneity, propagation of localized solitary waves is obstructed by slow energy loss caused by oscillations in the tail [20, 45, 84]. However, periodic lattices *do* support the propagation of generalized solitary waves such as *micropterons* and *nanopterons* [60, 35, 34, 37], which consist of an exponentially localized core accompanied by a (typically) non-vanishing periodic tail. By contrast, lattices with randomly varying coefficients do not necessarily support such coherent propagation, and it is precisely this phenomenon that motivates the present study.

In this chapter[1], we explore the generation of radiating tails caused by random heterogeneity in FPUT lattices. Recent work [81, 80] shows that long waves in such lattices will remain coherent over (somewhat) long time scales, depending on

[1]The contents of this chapter have been submitted for publication and are available as H.J. Hupkes, J.A. McGinnis, R.W.S. Westdorp, and J.D. Wright, *Radiating Solitary Waves in an FPUT Lattice with Random Coefficients*, see [63].

features of the randomness; this is accomplished by proving rigorous approximations by either wave or Korteweg-de Vries (KdV) equations. Nonetheless, over very long times the radiation has substantial effects on the principal solitary wave, causing (by virtue of energy conservation) a marked attenuation of the amplitude. We use techniques devised in the previous chapters to study random perturbations in KdV equations to analyze this attenuation.

**FPUT** We study the FPUT lattice equation

$$\begin{aligned} \dot{r}(j,t) &= \delta_+ p(j,t), \\ \dot{p}(j,t) &= \delta_-[\mathcal{V}'_\sigma(r)](j,t), \qquad j \in \mathbb{Z}, \end{aligned} \tag{5.1}$$

where

$$\delta_+ f(j) = f(j+1) - f(j) \quad \text{and} \quad \delta_- f(j) = f(j) - f(j-1)$$

are the right and left finite-difference operators and

$$\mathcal{V}_\sigma(r) = \frac{1}{2}(1+\sigma\kappa)r^2 + \frac{1}{3}r^3$$

is the spring potential. The random sequence $\kappa$ models variations in the linear spring force, and the parameter $\sigma \geq 0$ controls the strength of this random effect. This terminology stems from the second-order form of the system

$$\ddot{y}(j,t) = \mathcal{V}'_\sigma\big(y(j+1) - y(j)\big) - \mathcal{V}'_\sigma\big(y(j) - y(j-1)\big), \tag{5.2}$$

upon identifying $r(j) = y(j+1) - y(j)$ and $p(j) = \dot{y}(j)$. In this form, the system has the interpretation of a chain of masses connected by springs. The quantity $y(j)$ models the displacement of the $j$-th mass from equilibrium, and (5.2) is Newton's second law of motion.

In the absence of heterogeneity ($\sigma = 0$), the lattice differential equation (5.1) is also known as the FPUT-$\alpha$ lattice, referring to its cubic interaction potential. Upon introducing the Hamiltonian

$$H_{\sigma\kappa}(r,p) = \sum_{j\in\mathbb{Z}} \frac{1}{2}p(j,t)^2 + \frac{1}{2}\big(1+\sigma\kappa(j)\big)r(j,t)^2 + \frac{1}{3}r(j,t)^3$$

and the skew-symmetric operator

$$\mathcal{J} := \begin{bmatrix} 0 & \delta_+ \\ \delta_- & 0 \end{bmatrix},$$

we may alternatively write

$$\begin{bmatrix} \dot{r} \\ \dot{p} \end{bmatrix} = \mathcal{J}H'_{\sigma\kappa}(r,p) = \mathcal{J}H'_0(r,p) + \sigma\mathcal{J}\begin{bmatrix} \kappa r \\ 0 \end{bmatrix}. \tag{5.3}$$

Here, $H'_{\sigma\kappa}$ is the gradient of $H_{\sigma\kappa}$ with respect to the $\ell^2(\mathbb{Z};\mathbb{R}^2)$ inner-product. Thus, the random coefficients we consider respect the Hamiltonian structure of the FPUT system. We assume the following:

**Hypothesis 1.** The coefficients $\kappa(j)$, $j \in \mathbb{Z}$, are i.i.d.[2] random variables with mean zero and variance 1. The support of $\kappa(j)$, $j \in \mathbb{Z}$, is furthermore contained in an interval $[-\alpha, \alpha]$, and the parameter $\sigma \geq 0$ satisfies $\sigma\alpha < 1$.

Probabilities associated with $\kappa$ are represented by $\mathbb{P}$ and expectations by $\mathbb{E}$. The local well-posedness of (5.1) in $\ell^2(\mathbb{Z};\mathbb{R}^2)$ is a straightforward consequence of the fact that $\mathcal{J}H'_{\sigma\kappa}$ is locally Lipschitz on $\ell^2(\mathbb{Z};\mathbb{R}^2)$.

**Solitary waves** Solitary waves in FPUT lattices were extensively studied by Friesecke and Pego in the series [41, 42, 43, 44]. Notably, there exists a constant $c_+ > 1$, such that for all $c \in (1, c_+]$ there exists a smooth, exponentially decaying function $\phi_c = (r_c,\ p_c)^\top : \mathbb{R} \to \mathbb{R}^2$ so that

$$u(j,t) = \phi_c(j - ct) \tag{5.4}$$

solves (5.1) with $\sigma = 0$ [41]. The profile function satisfies the advance-delay system:

$$-c\phi_c' = \mathcal{J}H_0'(\phi_c). \tag{5.5}$$

The invariance of (5.5) with respect to shifts in the profile coordinate $x$ implies that, for any $\xi \in \mathbb{R}$,

$$\phi_{\xi,c} = \begin{bmatrix} r_{\xi,c} \\ p_{\xi,c} \end{bmatrix} := \begin{bmatrix} r_c(\cdot - \xi) \\ p_c(\cdot - \xi) \end{bmatrix}$$

also solves (5.5). Thus, the solitary waves form a two-parameter solution family, whose energy is independent of the phase $\xi$ and increases with the wave speed $c$ [41, Theorem 1.1]:

$$\frac{\mathrm{d}}{\mathrm{d}c} H_0(r_c, p_c) > 0.$$

The FPUT lattice famously shares a strong connection with the KdV equation: in an appropriate continuum limit, the lattice dynamics are governed by an effective KdV equation [109, 94]. In this long-wave limit, the wave profiles $\phi_c$ approach the $\mathrm{sech}^2$ shape of KdV solitons [41, Theorem 1.1]:

$$r_c(x) \approx -p_c(x) \approx \frac{\epsilon^2}{8}\,\mathrm{sech}^2(\epsilon x/2), \tag{5.6}$$

where the small parameter $\epsilon > 0$ is defined through $c = 1 + \epsilon^2/24$. This gives the speed $c$ a second interpretation: $c - 1$ is proportional to the (approximate)

[2]We suspect that we can also handle more general translation-invariant covariance between the random coefficients, of the form

$$\mathbb{E}[\kappa(i)\kappa(j)] = q(|i - j|),$$

assuming appropriate decay conditions on $(q(j))_{j\in\mathbb{N}}$. For simplicity, we restrict ourselves in this chapter to i.i.d. coefficients.

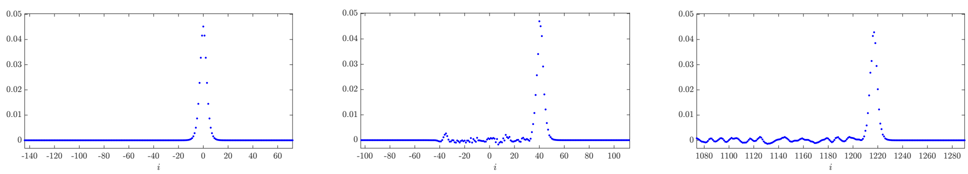
wave-amplitude.

**Random Coefficients** In [80], McGinnis and Wright derived effective KdV equations that govern the dynamics of an FPUT lattice with randomly varying mass coefficients. Central to this approximation is a *transparency condition* on the random variation, enforcing that the random coefficients appear as a perfect discrete Laplacian. Numerical simulations reveal that, if the masses are instead perturbed by i.i.d. random variables, a KdV approximation is no longer appropriate. Notably, solitary waves undergo an amplitude attenuation, inconsistent with KdV dynamics. In this chapter, we consider a random variation in spring coefficients rather than in the masses. This perturbation is somewhat more straightforward, as it arises purely at the linear level. We emphasize that our method can be applied more broadly, but we restrict ourselves here to spring force heterogeneity to keep the exposition transparent.

We study the effect of the random heterogeneity $\sigma\kappa$ in (5.1) on the propagation of the solitary waves by supplying (5.1) with the initial condition

$$\begin{bmatrix} r(j,0) \\ p(j,0) \end{bmatrix} = \begin{bmatrix} r_{c_*}(j) \\ p_{c_*}(j) \end{bmatrix}, \quad j \in \mathbb{Z}, \tag{5.7}$$

for some $c_* \in (1, c_+)$. In the deterministic setting, the solitary wave family is known to be asymptotically stable, meaning that initial conditions close to the wave family asymptotically converge to it. Consequently, when the solitary wave is perturbed by the random coefficients $\kappa$, it does not fully disintegrate. Rather, it slowly loses energy over time and descends to lower speeds/amplitudes. In Figure 5.1, we can observe from a numerical simulation with i.i.d. coefficients (cf. Hypothesis 1) that the loss of energy manifests in the formation of a random radiative 'tail' behind the wave.



**(a)** $t = 0$. **(b)** $t = 40$. **(c)** $t = 1200$.

**Figure 5.1:** A particular realization of the $r$-component of numerical solutions to (5.1) with initial condition (5.7). In this realization, $\sigma = 0.07$, $c_* = 1.015$ and $\kappa(i)$ is drawn from a uniform distribution on $[-\sqrt{3}, \sqrt{3}]$. See Section 5.11 for details regarding the numerical schemes used throughout this chapter.

Figure 5.2 shows how the effective amplitude $c(t)$—which we define more precisely below—gradually decays with time as a consequence of the energy loss. Although the change in amplitude is random, a clear attenuation emerges over time.

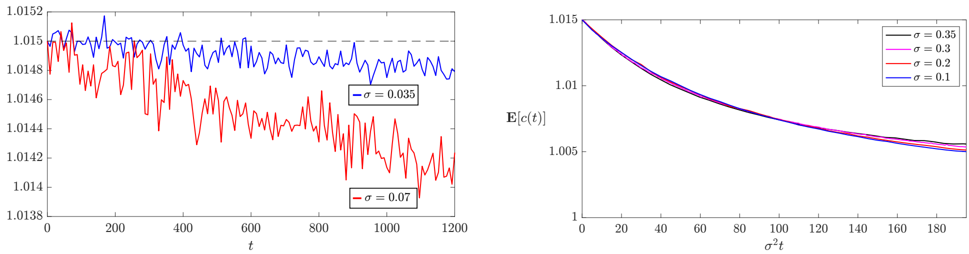



**Figure 5.2:** Left: Amplitude parameter $c(t)$ over time, for two realizations with $\sigma \in \{0.035, 0.07\}$, $c_* = 1.015$, and $\kappa(i)$ drawn from a uniform distribution on $[-\sqrt{3}, \sqrt{3}]$. Right: Sample mean $\mathbf{E}[c(t)]$ computed over 2000 realizations, for $\sigma \in \{0.1, 0.2, 0.3, 0.35\}$ and $c_* = 1.015$. We emphasize the $\sigma^2$-scaling on the temporal axis.

Our main objective in this chapter is to formally derive the attenuation induced by the random coefficients $\sigma\kappa$. Using modulation theory, we obtain an explicit $\mathcal{O}(\sigma^2)$ prediction for the average amplitude decay, $\mathbb{E}[\dot{c}(t)]$, expressed in terms of integrals/sums involving the wave profiles $\phi_c$ and the Green's function associated to the linearized dynamics near $\phi_c$; see (5.9) below. The latter plays a central role in the emission of a radiative tail behind the wave, which is the key mechanism driving amplitude attenuation. Our most explicit (and crudest) approximation is based on the kernel associated with the discrete wave equation, which can be written in closed form using Bessel functions. This leads to an explicitly solvable system that describes the radiation of the solitary wave and is able to reproduce the decay profile that emerges from Figure 5.2 (right) in the limit $\sigma \downarrow 0$; see Figure 5.10 ahead. This outcome is comparable in spirit to [106], but now derived directly from the physically relevant FPUT model. Our method is based on the linear stability theory for the wave profiles $\phi_c$, which we outline below.

**Linear stability** The linearized dynamics around a solitary wave $\phi_c(\cdot - \xi)$ are encoded by the operator $\mathcal{J}\mathcal{L}_{\xi,c}$, where $\mathcal{L}_{\xi,c}$ is the self-adjoint operator that acts on $\eta = (\eta_r,\ \eta_p)^\top \in \ell^2(\mathbb{Z}; \mathbb{R}^2)$ as

$$\mathcal{L}_{\xi,c}\eta = H_0''(\phi_{\xi,c})\eta = \begin{bmatrix} \eta_r + 2r_c(\cdot - \xi)\eta_r \\ \eta_p \end{bmatrix}. \tag{5.8}$$

Indeed, linearizing (5.1) with $\sigma = 0$ around the traveling wave (5.4), we arrive at the system

$$\dot{w}(j,t) = \mathcal{J}\mathcal{L}_{ct,c}w(j,t) = \mathcal{J}H_0''(\phi_c(\cdot - ct))w(j,t). \tag{5.9}$$

Note that the homogeneous part of $\mathcal{J}\mathcal{L}_{ct,c}$ is the discrete wave operator $\mathcal{J}$, whose continuous spectrum is contained in the imaginary axis. Differentiating (5.5) with respect to $x$ and $c$ shows that

$$(\mathcal{J}\mathcal{L}_{\xi,c} + c\partial_x)\partial_\xi\phi_{\xi,c} = 0, \quad \text{and} \quad (\mathcal{J}\mathcal{L}_{\xi,c} + c\partial_x)\partial_c\phi_{\xi,c} = \partial_\xi\phi_{\xi,c}.$$

5

Hence, $\partial_\xi \phi_{\xi,c}$ and $\partial_c \phi_{\xi,c}$ are generalized eigenfunctions of $\mathcal{J}\mathcal{L}_{\xi,c} + c\partial_x$, operating on functions of the real line. Here, $c\partial_x$ generates translations at the wave speed $c$, which are not directly represented in the linearization due to the discrete setting. Next, let $\delta_+^{-1}$ and $\delta_-^{-1}$ be given by

$$\delta_+^{-1} f(j) = \sum_{d=-\infty}^{-1} f(j+d) \quad \text{and} \quad \delta_-^{-1} f(j) = \sum_{d=-\infty}^{0} f(j+d), \quad f \in \ell^1(\mathbb{Z};\mathbb{R}).$$

These are the (formal) inverses of $\delta_\pm$. Then let

$$\mathcal{J}^{-1} := \begin{bmatrix} 0 & \delta_-^{-1} \\ \delta_+^{-1} & 0 \end{bmatrix}.$$

This is the (formal) inverse of $\mathcal{J}$. The bilinear form

$$\Omega(f,g) := \langle \mathcal{J}^{-1} f, g \rangle_{\ell^2(\mathbb{Z};\mathbb{R}^2)}, \qquad f,g \in \ell^1(\mathbb{Z};\mathbb{R}^2),$$

is the *symplectic form* for the FPUT system. This symplectic form characterizes a subspace that avoids the neutral modes $\partial_\xi \phi_{\xi,c}$ and $\partial_c \phi_{\xi,c}$, within which the linear flow defined in (5.9) is exponentially stable.

**Proposition 5.1.1** ([43], Theorem 1.2; [44], Theorem 2.2)**.** *Let $c_- \in (1, c_+)$. Then there exist positive constants $a, b$ and $K$ such that the following holds. For all $c \in [c_-, c_+]$ and $w_0$ satisfying $e^{a\cdot} w_0 \in \ell^2(\mathbb{Z};\mathbb{R}^2)$ with*

$$\Omega(\partial_\xi \phi_c, w_0) = \Omega(\partial_c \phi_c, w_0) = 0, \tag{5.10}$$

*the solution to the linearized evolution equation* (5.9) *with $w(\cdot,0) = w_0$ admits the bound*

$$\|e^{a(\cdot - ct)} w(t)\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \le K e^{-bt} \|e^{a\cdot} w_0\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)}, \quad t \ge 0.$$

Henceforth, we fix $c_- \in (1, c_+)$ and $a > 0$ as above. In [41], the authors compute how the symplectic form $\Omega$ acts on the eigenfunctions. They report that:

$$\begin{aligned} \Omega(\partial_\xi \phi_{\xi,c}, \partial_\xi \phi_{\xi,c}) &= 0 \\ \Omega(\partial_c \phi_{\xi,c}, \partial_c \phi_{\xi,c}) &= \alpha_1(c) \end{aligned}$$

and

$$\Omega(\partial_\xi \phi_{\xi,c}, \partial_c \phi_{\xi,c}) = -\Omega(\partial_c \phi_{\xi,c}, \partial_\xi \phi_{\xi,c}) = \alpha_0(c),$$

where

$$\alpha_0(c) := \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}c} H(\phi_c) > 0 \quad \text{and} \quad \alpha_1(c) := -\left(\frac{\mathrm{d}}{\mathrm{d}c}\int_{\mathbb{R}} r_c(x)\mathrm{d}x\right)\frac{\mathrm{d}}{\mathrm{d}c}\left(c\int_{\mathbb{R}} r_c(x)\mathrm{d}x\right). \tag{5.11}$$

**Approach** Motivated by the linear stability theory, we study the decomposition

$$\begin{bmatrix} r(j,t) \\ p(j,t) \end{bmatrix} = \begin{bmatrix} r_{c(t)}(j-\xi(t)) \\ p_{c(t)}(j-\xi(t)) \end{bmatrix} + \begin{bmatrix} \eta_r(j,t) \\ \eta_p(j,t) \end{bmatrix}, \quad j \in \mathbb{Z} \tag{5.12}$$

characterized by the orthogonality conditions

$$\Omega\Big(\partial_\xi \phi_{c(t)}(\cdot - \xi(t)), \eta(t)\Big) = \Omega\Big(\partial_c \phi_{c(t)}(\cdot - \xi(t)), \eta(t)\Big) = 0. \tag{5.13}$$

Here, $\eta = (\eta_r,\ \eta_p)^\top$ denotes the deviation from the wave manifold

$$\mathcal{M} = \{\phi_c(\cdot - \xi) \mid \xi \in \mathbb{R},\ c \in (c_-, c_+)\},$$

and avoids the neutral modes of the linearized flow. The result is that we can understand (5.1) through the modulation coordinates $(\xi, c)$, where $\xi$ has the interpretation of wave position and $c$ of wave amplitude (see (5.6)). It is furthermore convenient to introduce a phase shift parameter $\gamma(t)$ through

$$\gamma(t) = \xi(t) - \int_0^t c(s)\ \mathrm{d}s. \tag{5.14}$$

In these coordinates, the exact wave solution (5.4) for $\sigma = 0$ takes the form

$$\big(\gamma(t), c(t), \eta(t)\big) = (0, c_*, 0).$$

For general $\sigma \geq 0$, the dynamics deviate from this exact solution and are governed by a coupled system of the form

$$\dot{\gamma}(t) = \Gamma_{\sigma\kappa}(\xi(t), c(t), \eta(t)), \tag{5.15}$$
$$\dot{c}(t) = C_{\sigma\kappa}(\xi(t), c(t), \eta(t)), \tag{5.16}$$
$$\dot{\eta}(t) = \mathcal{J}\mathcal{L}_{\xi(t),c(t)}\eta(t) + T_{\sigma\kappa}(\xi(t), c(t), \eta(t)). \tag{5.17}$$

We study the resulting dynamics through a (formal) expansion of the system in the small parameter $\sigma$ around the deterministic solitary wave. In addition, we provide rigorous results to describe the asymptotic behavior of the expansion functions. We validate our findings through numerical simulations. In summary, we find that quadratic terms in the equation for $\dot{c}$—namely bilinear combinations of $\eta$ and $\kappa$—are the primary cause of the attenuation.

**Outline** First, we derive the modulation equations that govern the evolution of $(\gamma, c, \eta)$ in Section 5.2, and present an expansion in $\sigma$ and $\eta$. Then, we introduce an expansion of the modulation system and resulting explicit approximations in Section 5.3. Next, we construct explicit solutions for linear approximations to the tail $\eta$ in Section 5.4. We then derive concrete predictions for the amplitude attenuation in Section 5.5. Finally, we discuss directions for future research in Section 5.6.

## 5.2 Modulation system

We first establish some fundamental properties of the decomposition (5.12) characterized by the orthogonality conditions (5.13). Our goal in this section is to derive the mappings $\Gamma_{\sigma\kappa}, C_{\sigma\kappa}$ and $T_{\sigma\kappa}$ in the modulation system (5.15)–(5.17). We recall that the phase shift $\gamma(t)$ is related to the position $\xi(t)$ via (5.14), which implies that (5.15) is equivalent to

$$\dot{\xi}(t) = c(t) + \Gamma_{\sigma\kappa}(\xi(t), c(t), \eta(t)). \tag{5.18}$$

We will see that the mapping $T_{\sigma\kappa}$ in (5.17) is given by

$$T_{\sigma\kappa}(\xi, c, \eta) = \mathcal{J}N[\eta_r, \eta_r] - \Gamma_{\sigma\kappa}(\xi, c, \eta)\partial_\xi\phi_{\xi,c} - C_{\sigma\kappa}(\xi, c, \eta)\partial_c\phi_{\xi,c} + \sigma\begin{bmatrix} 0 \\ \delta_-\big(\kappa(r_{\xi,c} + \eta_r)\big)\end{bmatrix},$$

where $N[\eta_r, \eta_r]$ is the quadratic nonlinearity

$$N[\eta_r, \tilde{\eta}_r](j) = \begin{bmatrix} \eta_r(j)\tilde{\eta}_r(j) \\ 0 \end{bmatrix}.$$

The mappings $\Gamma_{\sigma\kappa}$ and $C_{\sigma\kappa}$ are given by

$$\begin{aligned}\begin{bmatrix}\Gamma_{\sigma\kappa}(\xi, c, \eta) \\ C_{\sigma\kappa}(\xi, c, \eta)\end{bmatrix} &= (A(c) - B(\xi, c)[\eta])^{-1}\begin{bmatrix}\Omega(\partial_\xi\phi_{\xi,c}, \mathcal{J}N[\eta_r, \eta_r]) \\ \Omega(\partial_c\phi_{\xi,c}, \mathcal{J}N[\eta_r, \eta_r])\end{bmatrix} \\ &\quad - \sigma(A(c) - B(\xi, c)[\eta])^{-1}\begin{bmatrix}\langle\partial_\xi r_{\xi,c}, \kappa(r_{\xi,c} + \eta_r)\rangle_{\ell^2(\mathbb{Z};\mathbb{R})} \\ \langle\partial_c r_{\xi,c}, \kappa(r_{\xi,c} + \eta_r)\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\end{bmatrix},\end{aligned} \tag{5.19}$$

where $A(c)$ and $B(\xi, c)[\eta]$ are the matrices

$$A(c) = \begin{bmatrix}\Omega(\partial_\xi\phi_c, \partial_\xi\phi_c) & \Omega(\partial_\xi\phi_c, \partial_c\phi_c) \\ \Omega(\partial_c\phi_c, \partial_\xi\phi_c) & \Omega(\partial_c\phi_c, \partial_c\phi_c)\end{bmatrix} = \begin{bmatrix} 0 & \alpha_0(c) \\ -\alpha_0(c) & \alpha_1(c)\end{bmatrix},$$

and

$$B(\xi, c)[\eta] = \begin{bmatrix}\Omega(\partial^2_{\xi\xi}\phi_{\xi,c}, \eta) & \Omega(\partial^2_{c\xi}\phi_{\xi,c}, \eta) \\ \Omega(\partial^2_{\xi c}\phi_{\xi,c}, \eta) & \Omega(\partial^2_{cc}\phi_{\xi,c}, \eta)\end{bmatrix}. \tag{5.20}$$

We remark that for all $c \in (c_-, c_+)$, the coefficient $\alpha_0(c)$ is strictly positive (see (5.11)), so that $A(c)$ is invertible with inverse

$$A^{-1}(c) = \frac{1}{\alpha_0^2(c)}\begin{bmatrix}\alpha_1(c) & -\alpha_0(c) \\ \alpha_0(c) & 0\end{bmatrix}. \tag{5.21}$$

Near the wave family $\mathcal{M}$, the unique existence and differentiability of our decomposition are guaranteed by the following result.

**Proposition 5.2.1.** *Assuming Hypothesis 1, there exists a constant $\delta_* > 0$ such* *that the following holds for all $T > 0$. Let $u(t) = (r(t), p(t))$ be the unique solution to* (5.1) *with initial condition* (5.7) *on* $[0, T]$. *If $u(t)$ satisfies*

$$\inf_{\xi_* \in \mathbb{R}, c_* \in (c_-, c_+)} \|e^{a(\cdot - \xi_*)}\big(u(t) - \phi_{c_*}(\cdot - \xi_*)\big)\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \leq \delta_*, \quad t \in [0, T],$$

*then for each $t \in [0, T]$, there exist unique modulation parameters $(\xi(t), c(t), \eta(t))$ that satisfy* (5.12) *with* (5.13). *Furthermore, the map $t \mapsto (\xi(t), c(t), \eta(t))$ is differentiable on* $[0, T]$ *and satisfies the modulation equations* (5.16), (5.17) *and* (5.18).

*Proof.* The unique existence of the orthogonal decomposition (5.12) with (5.13) near the wave manifold is proved in [42, Proposition 2.2]. We derive (5.17) by differentiating (5.12) with respect to time:

$$\begin{aligned} \dot{\eta}(t) = {} & \mathcal{J} H_0'(\phi_{\xi(t),c(t)} + \eta(t)) - \dot{\xi}(t)\partial_\xi \phi_{\xi(t),c(t)} - \dot{c}(t)\partial_c \phi_{\xi(t),c(t)} \\ & + \sigma \mathcal{J} \begin{bmatrix} \kappa(r_{\xi(t),c(t)} + \eta_r(t)) \\ 0 \end{bmatrix} \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} \mathcal{J} H_0'(\phi_{\xi,c} + \eta) &= \mathcal{J} \begin{bmatrix} r_{\xi,c} + \eta_r + (r_{\xi,c} + \eta_r)^2 \\ p_{\xi,c} + \eta_p \end{bmatrix} \\ &= \mathcal{J} \begin{bmatrix} r_{\xi,c} + r_{\xi,c}^2 \\ p_{\xi,c} \end{bmatrix} + \mathcal{J} \begin{bmatrix} \eta_r + 2 r_{\xi,c} \eta_r \\ \eta_p \end{bmatrix} + \mathcal{J} \begin{bmatrix} \eta_r^2 \\ 0 \end{bmatrix} \\ &= c \partial_\xi \phi_{\xi,c} + \mathcal{J} \mathcal{L}_{\xi,c} \eta + \mathcal{J} N[\eta_r, \eta_r]. \end{aligned}$$

To derive (5.15) and (5.16), we differentiate (5.13):

$$\begin{bmatrix} \Omega(\partial_\xi \phi_{\xi(t),c(t)}, \dot{\eta}(t)) \\ \Omega(\partial_c \phi_{\xi(t),c(t)}, \dot{\eta}(t)) \end{bmatrix} + B(\xi(t), c(t))[\eta(t)] \begin{bmatrix} \Gamma_\kappa(\sigma, \xi(t), c(t), \eta(t)) + c(t) \\ C_\kappa(\sigma, \xi(t), c(t), \eta(t)) \end{bmatrix} = 0. \tag{5.23}$$

We furthermore compute

$$\begin{aligned} \begin{bmatrix} \Omega\Big(\partial_\xi \phi_{\xi(t),c(t)}, \dot{\eta}(t)\Big) \\ \Omega\Big(\partial_c \phi_{\xi(t),c(t)}, \dot{\eta}(t)\Big) \end{bmatrix} \overset{(5.22)}{=} & \begin{bmatrix} \Omega\Big(\partial_\xi \phi_{\xi(t),c(t)}, \mathcal{J}\mathcal{L}_{\xi(t),c(t)}\eta(t) + \mathcal{J}N[\eta_r(t), \eta_r(t)]\Big) \\ \Omega\Big(\partial_c \phi_{\xi(t),c(t)}, \mathcal{J}\mathcal{L}_{\xi(t),c(t)}\eta(t) + \mathcal{J}N[\eta_r(t), \eta_r(t)]\Big) \end{bmatrix} \\ & - A(c(t)) \begin{bmatrix} \dot{\gamma}(t) \\ \dot{c}(t) \end{bmatrix} \\ & + \sigma \begin{bmatrix} \Omega\left(\partial_\xi \phi_{\xi(t),c(t)}, \mathcal{J} \begin{bmatrix} \kappa(r_{\xi(t),c(t)} + \eta_r(t)) \\ 0 \end{bmatrix}\right) \\ \Omega\left(\partial_c \phi_{\xi(t),c(t)}, \mathcal{J} \begin{bmatrix} \kappa(r_{\xi(t),c(t)} + \eta_r(t)) \\ 0 \end{bmatrix}\right) \end{bmatrix}. \end{aligned}$$

The term linear in $\eta$ can be rewritten as

$$\begin{aligned}\begin{bmatrix}\Omega(\partial_\xi\phi_{\xi,c},\mathcal{J}\mathcal{L}_{\xi,c}\eta)\\ \Omega(\partial_c\phi_{\xi,c},\mathcal{J}\mathcal{L}_{\xi,c}\eta)\end{bmatrix} &= -\begin{bmatrix}\langle\partial_\xi\phi_{\xi,c},\mathcal{L}_{\xi,c}\eta\rangle_{\ell^2(\mathbb{Z};\mathbb{R}^2)}\\ \langle\partial_c\phi_{\xi,c},\mathcal{L}_{\xi,c}\eta\rangle_{\ell^2(\mathbb{Z};\mathbb{R}^2)}\end{bmatrix} \overset{(5.8)}{=} -\begin{bmatrix}\langle\partial_\xi H_0'(\phi_{\xi,c}),\eta\rangle_{\ell^2(\mathbb{Z};\mathbb{R}^2)}\\ \langle\partial_c H_0'(\phi_{\xi,c}),\eta\rangle_{\ell^2(\mathbb{Z};\mathbb{R}^2)}\end{bmatrix}\\ &= -\begin{bmatrix}\Omega(\mathcal{J}\partial_\xi H_0'(\phi_{\xi,c}),\eta)\\ \Omega(\mathcal{J}\partial_c H_0'(\phi_{\xi,c}),\eta)\end{bmatrix} \overset{(5.5)}{=} -c\begin{bmatrix}\Omega(\partial^2_{\xi\xi}\phi_{\xi,c},\eta)\\ \Omega(\partial^2_{\xi c}\phi_{\xi,c},\eta)\end{bmatrix}\\ &= -c\begin{bmatrix}B_{11}(\xi,c,\eta)\\ B_{21}(\xi,c,\eta)\end{bmatrix},\end{aligned}$$

and cancels with $B(\xi,c)[\eta](c,\ 0)^\top$ in (5.23). The $\mathcal{O}(\sigma)$ term reduces to

$$\begin{aligned}\begin{bmatrix}\Omega\left(\partial_\xi\phi_{\xi,c},\mathcal{J}\begin{bmatrix}\kappa(r_{\xi,c}+\eta_r)\\0\end{bmatrix}\right)\\ \Omega\left(\partial_c\phi_{\xi,c},\mathcal{J}\begin{bmatrix}\kappa(r_{\xi,c}+\eta_r)\\0\end{bmatrix}\right)\end{bmatrix} &= -\begin{bmatrix}\Omega\left(\mathcal{J}\partial_\xi\phi_{\xi,c},\begin{bmatrix}\kappa(r_{\xi,c}+\eta_r)\\0\end{bmatrix}\right)\\ \Omega\left(\mathcal{J}\partial_c\phi_{\xi,c},\begin{bmatrix}\kappa(r_{\xi,c}+\eta_r)\\0\end{bmatrix}\right)\end{bmatrix}\\ &= -\begin{bmatrix}\left\langle\partial_\xi\phi_{\xi,c},\begin{bmatrix}\kappa(r_{\xi,c}+\eta_r)\\0\end{bmatrix}\right\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\\ \left\langle\partial_c\phi_{\xi,c},\begin{bmatrix}\kappa(r_{\xi,c}+\eta_r)\\0\end{bmatrix}\right\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\end{bmatrix}\\ &= -\begin{bmatrix}\langle\partial_\xi r_{\xi,c},\kappa(r_{\xi,c}+\eta_r)\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\\ \langle\partial_c r_{\xi,c},\kappa(r_{\xi,c}+\eta_r)\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\end{bmatrix}.\end{aligned}$$

Finally, we note

$$\begin{aligned}&\Big(A(c(t))-B(\xi(t),c(t))[\eta(t)]\Big)\begin{bmatrix}\Gamma_\kappa(\sigma,\xi(t),c(t),\eta(t))\\ C_\kappa(\sigma,\xi(t),c(t),\eta(t))\end{bmatrix}\\ &\qquad= \begin{bmatrix}\Omega\Big(\partial_\xi\phi_{\xi(t),c(t)},\mathcal{J}N[\eta_r(t),\eta_r(t)]\Big)\\ \Omega\Big(\partial_c\phi_{\xi(t),c(t)},\mathcal{J}N[\eta_r(t),\eta_r(t)]\Big)\end{bmatrix}\\ &\qquad\quad-\sigma\begin{bmatrix}\Big\langle\partial_\xi r_{\xi(t),c(t)},\kappa(r_{\xi(t),c(t)}+\eta_r(t))\Big\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\\ \Big\langle\partial_c r_{\xi(t),c(t)},\kappa(r_{\xi(t),c(t)}+\eta_r(t))\Big\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\end{bmatrix}.\end{aligned}$$

Following (5.21), the invertibility of $A(c)-B(\xi,c)[\eta]$ is guaranteed for $c\in(c_-,c_+)$ and $\eta$ satisfying $\|e^{a\cdot}\eta\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)}\leq\delta_*$, upon decreasing $\delta_*$ if necessary. □

**Expansion** The modulation system (5.15)–(5.17) describes how a small heterogeneity $\sigma>0$ affects the exact traveling wave $\big(\gamma(t),c(t),\eta(t)\big)=(0,c_*,0)$. It introduces $\mathcal{O}(\sigma)$ fluctuations in $c(t)$, and leads to an $\mathcal{O}(\sigma)$ phase shift $\gamma(t)$. The heterogeneity also develops an $\mathcal{O}(\sigma)$ tail $\eta(t)$, which we expect to remain small for a long time, due to the stabilizing effect of the linearized dynamics. In order to expose the structure of the coupled modulation system (5.15)–(5.17), we expand the system in the parameters $\sigma$ and $\eta$:

$$\begin{aligned}
\Gamma_{\sigma\kappa}(\xi,c,\eta) &= \sigma\Big(\Gamma^{1,0}(\xi,c)[\kappa] + \Gamma^{1,1}(\xi,c)[\kappa,\eta]\Big) + \Gamma^{0,2}(\xi,c)[\eta_r,\eta_r] + \mathcal{O}(\sigma\eta^2) + \mathcal{O}(\eta^3),\\
C_{\sigma\kappa}(\xi,c,\eta) &= \sigma\Big(C^{1,0}(\xi,c)[\kappa] + C^{1,1}(\xi,c)[\kappa,\eta]\Big) + C^{0,2}(\xi,c)[\eta_r,\eta_r] + \mathcal{O}(\sigma\eta^2) + \mathcal{O}(\eta^3),\\
T_{\sigma\kappa}(\xi,c,\eta) &= \sigma\Big(T^{1,0}(\xi,c)[\kappa] + T^{1,1}(\xi,c)[\kappa,\eta]\Big) + T^{0,2}(\xi,c)[\eta_r,\eta_r] + \mathcal{O}(\sigma\eta^2) + \mathcal{O}(\eta^3).
\end{aligned} \tag{5.24}$$

Here, the first index $k \in \{0,1\}$ in $\Gamma^{k,n}$, $C^{k,n}$ and $T^{k,n}$ refers to the accompanying factor $\sigma^k$ in (5.24), whereas the second index $n \geq 0$ indicates that the mapping is purely of order $\mathcal{O}(\eta^n)$. To compute these expansion terms, we note that the matrix inverse in (5.19) can be expanded through a Neumann series:

$$\begin{aligned}
\big(A(c) - B(\xi,c)[\eta]\big)^{-1} &= \big(I - A^{-1}(c)B(\xi,c)[\eta]\big)^{-1}A^{-1}(c)\\
&= \sum_{k=0}^{\infty}\big(A^{-1}(c)B(\xi,c)[\eta]\big)^{k}A^{-1}(c).
\end{aligned}$$

For each $\xi \in \mathbb{R}$ and $c \in (c_-,c_+)$, the maps $\Gamma^{1,0}(\xi,c)$ and $C^{1,0}(\xi,c)$ are linear operators from $\ell^\infty(\mathbb{Z};\mathbb{R})$ to $\mathbb{R}$ that act as

$$\begin{bmatrix}\Gamma^{1,0}(\xi,c)\\ C^{1,0}(\xi,c)\end{bmatrix}[f] = -\tfrac{1}{2}A^{-1}(c)\begin{bmatrix}\langle\partial_\xi r_c^2(\cdot-\xi), f\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\\ \langle\partial_c r_c^2(\cdot-\xi), f\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\end{bmatrix}. \tag{5.25}$$

At order $\mathcal{O}(\eta^2)$, we find that $\Gamma^{0,2}(\xi,c)[\cdot,\cdot]$ and $C^{0,2}(\xi,c)[\cdot,\cdot]$ : are the bilinear maps from $\ell^\infty(\mathbb{Z};\mathbb{R}) \times \ell^\infty(\mathbb{Z};\mathbb{R})$ to $\mathbb{R}$ that act as

$$\begin{bmatrix}\Gamma^{0,2}(\xi,c)\\ C^{0,2}(\xi,c)\end{bmatrix}[f,g] := -A^{-1}(c)\begin{bmatrix}\langle\partial_\xi r_c(\cdot-\xi), fg\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\\ \langle\partial_c r_c(\cdot-\xi), fg\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}\end{bmatrix}. \tag{5.26}$$

In particular, $\Gamma^{0,2}(\xi,c)[\cdot,\cdot]$ and $C^{0,2}(\xi,c)[\cdot,\cdot]$ can be seen as weighted inner products when restricted to $\ell^2(\mathbb{Z};\mathbb{R}) \times \ell^2(\mathbb{Z};\mathbb{R})$. Lastly, $\Gamma^{1,1}(\xi,c)[\cdot,\cdot]$ and $C^{1,1}(\xi,c)[\cdot,\cdot]$ are bilinear maps from $\ell^\infty(\mathbb{Z};\mathbb{R}) \times \ell^\infty(\mathbb{Z};\mathbb{R}^2)$ to $\mathbb{R}$ which act as

$$\begin{bmatrix}\Gamma^{1,1}(\xi,c)\\ C^{1,1}(\xi,c)\end{bmatrix}[f,h] = A^{-1}(c)B(\xi,c)[h]\begin{bmatrix}\Gamma^{1,0}(\xi,c)\\ C^{1,0}(\xi,c)\end{bmatrix}[f] + \begin{bmatrix}\Gamma^{0,2}(\xi,c)\\ C^{0,2}(\xi,c)\end{bmatrix}[f,h_r]. \tag{5.27}$$

The expansion of (5.15)–(5.17) up to second-order is completed by the linear map $T^{1,0}(\xi,c) : \ell^\infty(\mathbb{Z};\mathbb{R}) \to \ell^2(\mathbb{Z};\mathbb{R}^2)$ that acts as

$$T^{1,0}(\xi,c)[f] = -\Gamma^{1,0}(\xi,c)[f]\partial_\xi\phi_{\xi,c} - C^{1,0}(\xi,c)[f]\partial_c\phi_{\xi,c} + \begin{bmatrix}0\\ \delta_-\big(fr_{\xi,c}\big)\end{bmatrix}, \tag{5.28}$$

and the bilinear maps

$$T^{1,1}(\xi,c)[\cdot,\cdot] : \ell^\infty(\mathbb{Z};\mathbb{R}) \times \ell^\infty(\mathbb{Z};\mathbb{R}^2) \to \ell^2(\mathbb{Z};\mathbb{R}^2)$$

and

$$T^{0,2}(\xi,c)[\cdot,\cdot] : \ell^\infty(\mathbb{Z};\mathbb{R}) \times \ell^\infty(\mathbb{Z};\mathbb{R}) \to \ell^2(\mathbb{Z};\mathbb{R}^2)$$

that act as

$$\begin{aligned} T^{1,1}(\xi,c)[f,h] &= -\Gamma^{1,1}(\xi,c)[f,h]\partial_\xi\phi_{\xi,c} - C^{1,1}(\xi,c)[f,h]\partial_c\phi_{\xi,c} + \begin{bmatrix} 0 \\ \delta_-\big(fh_r\big) \end{bmatrix}, \\ T^{0,2}(\xi,c)[f,g] &= -\Gamma^{0,2}(\xi,c)[f,g]\partial_\xi\phi_{\xi,c} - C^{0,2}(\xi,c)[f,g]\partial_c\phi_{\xi,c} + \mathcal{J}N[f,g]. \end{aligned}$$

Although higher-order contributions can be identified in the same manner, we will only make use of the $\mathcal{O}(\sigma)$, $\mathcal{O}(\sigma\eta)$ and $\mathcal{O}(\eta^2)$ contributions identified above. This is because they suffice for our purposes of studying the $\mathcal{O}(\sigma^2)$ amplitude attenuation of $c(t)$, given that $\eta$ develops at order $\mathcal{O}(\sigma)$. See also Figure 5.2. In the following section, we construct explicit approximations based on these second-order terms to facilitate further analysis.

## 5.3 System expansion

Having derived the modulation system (5.15)–(5.17), we turn our attention to the resulting dynamics. We emphasize that the modulation parameters $(\gamma, c, \eta)$ are, through a coordinate transformation, equivalent to the original system $(r,p)$. Hence, solving for $c(t)$ should reveal the observed amplitude attenuation. However, due to its coupled nature, this information is hard to extract directly from (5.15)–(5.17).

Inspired by Section 2.3, this section introduces explicit approximations based on the expansion (5.24). The key issue is that the expansion coefficients $\Gamma^{k,n}$, $C^{k,n}$ and $T^{k,n}$ depend explicitly on the pair $(\xi, c)$. At each step in the expansion procedure one can choose to treat this pair as free functions that must be solved for, but it is also possible to insert an a-priori approximation. The approximations developed in Section 2.3 to describe soliton propagation in the KdV equation were based on the former approach. The main reason is that the random forcing that was considered broke the Hamiltonian structure of the system. In particular, fluctuations appearing at order $\mathcal{O}(\sigma)$ lead to significant deviations from the original amplitude on short time scales, requiring the use of an adaptive approach to account for the short-time fluctuations.

In this chapter however, the random coefficients in (5.1) preserve the Hamiltonian structure of the FPUT system. As a consequence, the dynamics of $c(t)$ induced by terms at $\mathcal{O}(\sigma)$ do not accumulate significantly over time—see Proposition 5.3.1 ahead. In the present context, $c(t)$ remains close to its initial value $c_*$ over time scales proportional to $\sigma^{-2}$, leading to a separation of time scales. In particular, transient dynamics such as the development of the tail $\eta(t)$ occur at an exponential rate independent of $\sigma$—see Corollary 5.5.3 ahead. We therefore depart from an adaptive approach, and simply expand the modulation system (5.15)–(5.17) in the small parameter $\sigma$, around the deterministically propagating soliton $\phi_{c_*}(\cdot - c_* t)$:

$$\gamma(t) = \sigma\gamma_1(t) + \sigma^2\gamma_2(t) + \mathcal{O}(\sigma^3), \tag{5.29}$$

$$c(t) = c_* + \sigma c_1(t) + \sigma^2 c_2(t) + \mathcal{O}(\sigma^3), \tag{5.30}$$

$$\eta(t) = \sigma \eta_1(t) + \mathcal{O}(\sigma^2). \tag{5.31}$$

Since the phase shift $\gamma(t)$ is related to the position $\xi(t)$ through (5.14), we have an equivalent expansion

$$\xi(t) = c_* t + \sigma \xi_1(t) + \sigma^2 \xi_2(t) + \mathcal{O}(\sigma^3),$$

with

$$\xi_i(t) = \gamma_i(t) + \int_0^t c_i(s)\, \mathrm{d}s, \qquad i = 1, 2.$$

In principle, the expansion can be continued up to any desired order in $\sigma$, but we will only consider the explicit terms appearing in (5.29)–(5.31). In Section 5.5 ahead, we compute the expected attenuation of the amplitude approximation $c_* + \sigma c_1(t) + \sigma^2 c_2(t)$. Although $c(t)$ deviates significantly from its initial value after long times, the expansion captures the attenuation occurring over intermediate timescales. In this sense, the attenuation computed from an expansion around $\phi_{c_*}(\cdot - c_* t)$ can be interpreted as representing tangents to the curves in Figure 5.2 at $\mathbf{E}[c(t)] = c_*$. See Figure 5.10 in Section 5.5 for further details.

**Leading-order phase and amplitude** Collecting $\mathcal{O}(\sigma)$ terms in (5.15) and (5.16) yields

$$\begin{bmatrix} \dot{\gamma}_1(t) \\ \dot{c}_1(t) \end{bmatrix} = \begin{bmatrix} \Gamma^{1,0}(c_* t, c_*) \\ C^{1,0}(c_* t, c_*) \end{bmatrix} [\kappa], \quad \text{with} \quad \begin{bmatrix} \gamma_1(0) \\ c_1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.32}$$

These leading-order terms ignore $\eta$ altogether, since it forms an $\mathcal{O}(\eta^2) \sim \mathcal{O}(\sigma^2)$ contribution. We note that

$$\begin{bmatrix} \Gamma^{1,0}(c_* t, c_*) \\ C^{1,0}(c_* t, c_*) \end{bmatrix} [\kappa] = -\tfrac{1}{2} A^{-1}(c_*) \sum_{j \in \mathbb{Z}} \kappa(j) \begin{bmatrix} (\partial_\xi r_{c_*}^2)(j - c_* t) \\ (\partial_c r_{c_*}^2)(j - c_* t) \end{bmatrix},$$

through which we arrive at the explicit solution

$$\begin{aligned} c_1(t) &= \tfrac{1}{2} \alpha_0^{-1}(c_*) \sum_{j \in \mathbb{Z}} \kappa(j) \int_0^t (r_{c_*}^2)'(j - c_* s) \mathrm{d}s \\ &= -\frac{\alpha_0^{-1}(c_*)}{2 c_*} \sum_{j \in \mathbb{Z}} \kappa(j) [r_{c_*}^2(j - c_* t) - r_{c_*}^2(j)]. \end{aligned} \tag{5.33}$$

The bracketed terms in (5.33) form (shifting) windows that select the lattice sites at which the random coefficients contribute. Similarly, we compute that $\gamma_1(t) =$

$\gamma_{1;I}(t) + \gamma_{1;II}(t)$ with

$$\gamma_{1;I}(t) = -\frac{\alpha_0^{-2}(c_*)\alpha_1(c_*)}{2c_*} \sum_{j\in\mathbb{Z}} \kappa(j)[r_{c_*}^2(j - c_*t) - r_{c_*}^2(j)] \tag{5.34}$$

and

$$\gamma_{1;II}(t) = \frac{1}{2\alpha_0(c_*)} \sum_{j\in\mathbb{Z}} \kappa(j) \int_0^t \partial_c r_{c_*}^2(j - c_*s)\mathrm{d}s. \tag{5.35}$$

We note that

$$\int_{-\infty}^x \partial_c r_{c_*}^2(y)\mathrm{d}y \to \int_{\mathbb{R}} \partial_c r_{c_*}^2(y)\mathrm{d}y > 0, \quad \text{as } x \to \infty.$$

Thus, the integral in (5.35) acts as a widening window, selecting an increasing amount of lattice sites with time. Hence, $\gamma_{1;II}$ resembles a continuous random walk, and its variance grows linearly with time. Below, we collect some elementary properties regarding the statistics of $c_1(t)$ and $\gamma_1(t)$.

**Proposition 5.3.1** (See Section 5.7)**.** *Assume Hypothesis 1 and let $c_* \in (c_-, c_+)$. Then*

1. *$c_1(t)$ is a mean-zero random process, whose variance is uniformly bounded in time as*

$$\mathbb{E}[c_1^2(t)] \le \frac{\alpha_0^{-2}(c_*)}{c_*^2} \|r_{c_*}\|_{\ell^4(\mathbb{Z};\mathbb{R})}^4, \quad t \ge 0. \tag{5.36}$$

2. *$\gamma_1(t)$ is a mean-zero random process, whose variance is bounded linearly in time as*

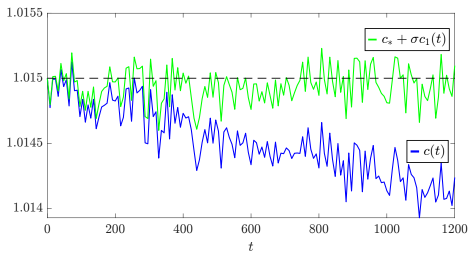
$$\mathbb{E}[\gamma_1^2(t)] \le \big(C_1 + C_2 t\big), \quad t \ge 0,$$

*where the constants $C_1, C_2 > 0$ depend only on $c_*$.*

Figure 5.3 shows simulations of the first-order parameters $(c_1(t), \gamma_1(t))$. The approximation $c_* + \sigma c_1(t)$ captures the leading-order effect of the random coefficients: the solitary wave experiences an $\mathcal{O}(\sigma)$ 'shaking', but the disturbances do not accumulate with time. See the left panel of Figure 5.3: The approximation $c_* + \sigma c_1(t)$ does not capture the attenuation of the soliton.

**Tail approximations** Analogously, we identify the leading-order expansion term $\sigma\eta_1$ of $\eta$ by isolating $\mathcal{O}(\sigma)$ contributions in (5.17). In particular, we find

$$\begin{aligned} \dot{\eta}_1(t) &= \mathcal{J}\mathcal{L}_{c_*t,c_*}\eta_1(t) + T^{1,0}(c_*t, c_*)[\kappa] \\ &= \mathcal{J}\mathcal{L}_{c_*t,c_*}\eta_1(t) + \begin{bmatrix} \partial_\xi \phi_{c_*t,c_*} \\ \partial_c \phi_{c_*t,c_*} \end{bmatrix}^\top A^{-1}(c_*) \begin{bmatrix} \langle \partial_\xi r_{c_*t,c_*}, \kappa r_{c_*t,c_*} \rangle_{\ell^2(\mathbb{Z};\mathbb{R})} \\ \langle \partial_c r_{c_*t,c_*}, \kappa r_{c_*t,c_*} \rangle_{\ell^2(\mathbb{Z};\mathbb{R})} \end{bmatrix} + \mathcal{J} \begin{bmatrix} \kappa r_{c_*t,c_*} \\ 0 \end{bmatrix}, \end{aligned} \tag{5.37}$$



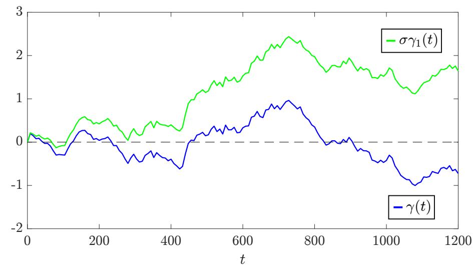




**Figure 5.3:** Comparison of first-order approximations $c_*+\sigma c_1(t)$ and $\sigma\gamma_1(t)$ of amplitude and position, for a particular realization with $\sigma = 0.07$ and $c_* = 1.015$. The random coefficients $\kappa(i)$ are drawn from a uniform distribution on $[-\sqrt{3}, \sqrt{3}]$. The parameters $c_1(t)$ and $\gamma_1(t)$ correspond to a numerical simulation of the approximation (5.32). These processes capture fluctuations, but do not capture the attenuation affecting $c(t)$.

supplemented with the initial condition $\eta_1(0, j) = 0$, $j \in \mathbb{Z}$. This is now a linear equation, forced by the random coefficients $\kappa$. We recall that $\mathcal{L}_{c_*t,c_*}$ is the linearization around the solitary wave $\phi_{c_*}(\cdot - c_*t)$, that acts as

$$\mathcal{L}_{c_*t,c_*}\eta = \begin{bmatrix} \eta_r \\ \eta_p \end{bmatrix} + 2\begin{bmatrix} r_{c_*}(\cdot - c_*t)\eta_r \\ 0 \end{bmatrix}. \tag{5.38}$$

These time-dependent operators generate an exponentially stable evolution in weighted spaces (Proposition 5.1.1), provided the neutral modes are avoided via the orthogonality conditions (5.10). The term $T^{1,0}$ introduces forcing to (5.37) which preserves orthogonality with respect to $\phi_{c_*}(\cdot - c_*t)$. In particular,

$$\Omega\Big(\partial_\xi \phi_{c_*}(\cdot - c_*t), \eta_1(t)\Big) = \Omega\Big(\partial_c \phi_{c_*}(\cdot - c_*t), \eta_1(t)\Big) = 0, \quad t \geq 0. \tag{5.39}$$

Additionally, we consider a cruder 'homogeneous' linear approximation of the tail $\eta(t)$ that neglects the localized linear term:

$$\dot{\eta}_1^{\mathrm{h}}(t) = \mathcal{J}\eta_1^{\mathrm{h}}(t) + T^{1,0}(c_*t, c_*)[\kappa]. \tag{5.40}$$

In comparison to (5.37), the linear operator $\mathcal{L}_{c_*t,c_*}$ has been replaced by $I$. This simplifies the analysis: (5.40) is a forced discrete wave equation, for which we can derive explicit solution formulas (Section 5.4). Despite the seemingly large difference between the definitions (5.37) and (5.40), the resulting 'tails' do not differ much from each other. See Figure 5.4, which shows the difference $\|\eta_1(t) - \eta_1^{\mathrm{h}}(t)\|$ in proportion to $\|\eta_1(t)\| + \|\eta_1^{\mathrm{h}}(t)\|$ over time. Although the wave profile in (5.38) introduces secular modes, the map $T_\kappa$ avoids these by design. This structural aspect is shared by (5.40), since the discrete wave equation $\dot{w} = \mathcal{J}w$ contains no secular modes.

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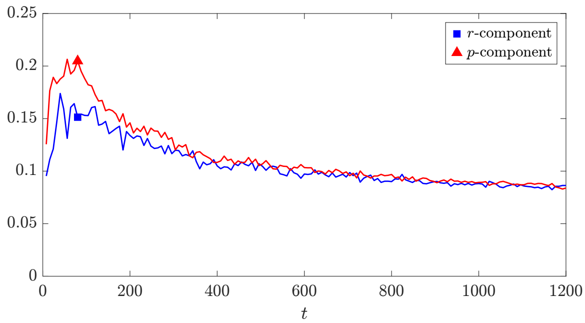


**Figure 5.4:** Relative difference $\frac{\|\eta_1(t)-\eta_1^{\mathrm{h}}(t)\|}{\|\eta_1(t)\|+\|\eta_1^{\mathrm{h}}(t)\|}$, for a particular realization with $c_* = 1.015$ and $\kappa(i) \in \mathrm{Unif}([-\sqrt{3}, \sqrt{3}])$. The differences remain below 0.25, indicating a strong correspondence.

**Second-order Phase and Amplitude Corrections** It has become clear that leading-order effects do not capture any attenuation of the amplitude $c(t)$. Next, we consider second-order contributions in the $\sigma$-expansion of the modulation parameters. We identify $\gamma_2$ and $c_2$ in (5.29) and (5.30) by isolating $\mathcal{O}(\sigma^2)$ contributions in (5.15) and (5.16). In particular, by expanding

$$\begin{aligned}\begin{bmatrix}\Gamma^{1,0}(\xi(t),c(t))\\ C^{1,0}(\xi(t),c(t))\end{bmatrix} = \begin{bmatrix}\Gamma^{1,0}(c_*t,c_*)\\ C^{1,0}(c_*t,c_*)\end{bmatrix} + \sigma\xi_1(t)\partial_\xi \begin{bmatrix}\Gamma^{1,0}(c_*t,c_*)\\ C^{1,0}(c_*t,c_*)\end{bmatrix} + \sigma c_1(t)\partial_c \begin{bmatrix}\Gamma^{1,0}(c_*t,c_*)\\ C^{1,0}(c_*t,c_*)\end{bmatrix}\\ + \mathcal{O}(\sigma^2),\end{aligned}$$

we find that $\dot{\gamma}_2$ and $\dot{c}_2$ satisfy an equation of the form

$$\begin{aligned}\begin{bmatrix}\dot{\overline{\gamma}}_2(t)\\ \dot{\overline{c}}_2(t)\end{bmatrix} &= (\gamma_1(t) + \int_0^t c_1(s)\,\mathrm{d}s)\partial_\xi \begin{bmatrix}\Gamma^{1,0}(c_*t,c_*)\\ C^{1,0}(c_*t,c_*)\end{bmatrix}[\kappa] + c_1(t)\partial_c \begin{bmatrix}\Gamma^{1,0}(c_*t,c_*)\\ C^{1,0}(c_*t,c_*)\end{bmatrix}[\kappa]\\ &\quad + \begin{bmatrix}\Gamma^{1,1}(c_*t,c_*)\\ C^{1,1}(c_*t,c_*)\end{bmatrix}[\kappa,\overline{\eta}(t)] + \begin{bmatrix}\Gamma^{0,2}(c_*t,c_*)\\ C^{0,2}(c_*t,c_*)\end{bmatrix}[\overline{\eta}_r(t),\overline{\eta}_r(t)],\end{aligned} \tag{5.41}$$

with initial conditions $\overline{\gamma}_2(0) = 0$ and $\overline{c}_2(0) = 0$. The system (5.41) defines second-order corrections $(\overline{\gamma}_2, \overline{c}_2)$ based on a general linear approximation $\sigma\overline{\eta}$ to $\eta$. In particular, we retrieve the true second-order corrections $(\gamma_2, c_2)$ in (5.29) and (5.30) upon picking $\overline{\eta} = \eta_1$. Additionally, we obtain corrections $(\gamma_2^{\mathrm{h}}, c_2^{\mathrm{h}})$ based on the homogeneous tail $\overline{\eta} = \eta_1^{\mathrm{h}}$ introduced in (5.40).

As seen in Figure 5.5, the second-order approximation $(\gamma_2, c_2)$ effectively captures the amplitude attenuation. The second-order corrections capture to a large extent how much energy radiates away from the soliton as a consequence of its shaking, leading to an $\mathcal{O}(\sigma^2)$ amplitude attenuation.

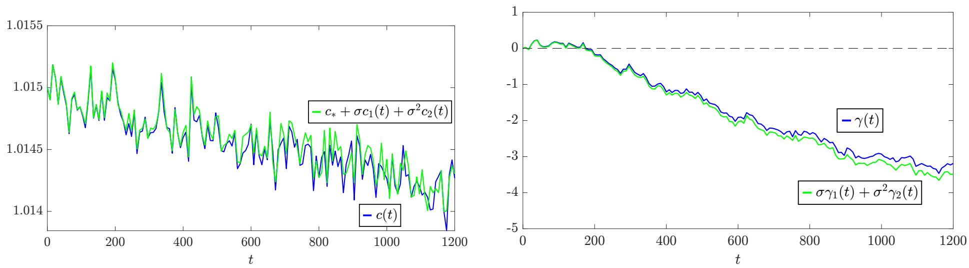


**Figure 5.5:** The second-order approximations $c_* + \sigma c_1(t) + \sigma^2 c_2(t)$ (left) and $\sigma\gamma_1(t) + \sigma^2\gamma_2(t)$ (right) compared to $c(t)$ (left) and $\gamma(t)$ (right), respectively. ($\sigma = 0.07$, $c_* = 1.015$ and $\kappa(i) \in \mathrm{Unif}([-\sqrt{3}, \sqrt{3}])$.)

Although the tail approximation $\eta_1^{\mathrm{h}}$ defined through (5.40) neglects a linear term localized at the soliton location, the second-order corrections $(\gamma_2^{\mathrm{h}}, c_2^{\mathrm{h}})$ based on $\eta_1^{\mathrm{h}}$ are remarkably close to the true corrections $(\gamma_2, c_2)$—see Figure 5.6. We have thus arrived at two tractable approximations to the parameters $(\gamma, c, \eta)$ that capture attenuation. In the next sections, we analyze the linear tails $\overline{\eta}$ and resulting second-order corrections $(\overline{\gamma}_2, \overline{c}_2)$ in more detail.

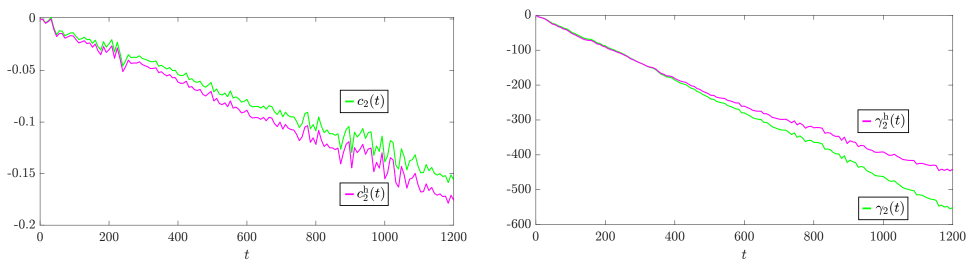


**Figure 5.6:** The second-order amplitude contribution $c_2(t)$ compared to $c_2^{\mathrm{h}}(t)$ (left) and the second-order phase contribution $\gamma_2(t)$ compared to $\gamma_2^{\mathrm{h}}(t)$ (right). ($c_* = 1.015$ and $\kappa(i) \in \mathrm{Unif}([-\sqrt{3}, \sqrt{3}])$.)

## 5.4 Radiative Tail

In this section, we examine the radiative tail that forms behind the solitary wave in more detail. In Section 5.3, we have introduced two approximations to the tail $\eta$ of the form

$$\dot{\overline{\eta}}(j,t) = \mathcal{J}\mathcal{A}_{c_* t, c_*}\, \overline{\eta}(j,t) + T^{1,0}(c_* t, c_*)[\kappa](j), \quad \text{with} \quad \overline{\eta}(j,0) = 0. \tag{5.42}$$

The approximation $\eta_1$ defined in (5.37) features the soliton linearization $\mathcal{A}_{c_* t, c_*} = \mathcal{L}_{c_* t, c_*}$, while $\eta_1^{\mathrm{h}}$ is constructed using the constant-coefficient reduction $\mathcal{A}_{c_* t, c_*} = I$; see (5.40). Our aim here is to characterize the *asymptotic* behavior of such tails $\overline{\eta}(t)$. In particular, we find that $\overline{\eta}(t)$ converges when viewed in a frame that moves along with the solitary wave. Besides being interesting on its own, this can be linked directly to the long-time behavior of $\overline{c}_2(t)$. Indeed, this result allows us to derive a limit for the expected attenuation $\mathbb{E}[\overline{c}_2(t)]$ in Section 5.5. We rely on the following assumptions on $\mathcal{A}_{c_*t,c_*}$, which are met by both $I$ and $\mathcal{L}_{c_*t,c_*}$:

**Hypothesis 2.** We have $c_* \in (c_-, c_+)$ and $\{\mathcal{A}_{c_*t,c_*}\}_{t\in\mathbb{R}}$ is a family of bounded linear operators on $\ell^2(\mathbb{Z};\mathbb{R}^2)$ that satisfies the spatio-temporal shift invariance

$$[\mathcal{A}_{c_*t,c_*}\eta](j+d) = [\mathcal{A}_{c_*t-d,c_*}\eta(\cdot+d)](j), \quad j,d\in\mathbb{Z},\ t\in\mathbb{R}. \tag{5.43}$$

The operators $\{\mathcal{J}\mathcal{A}_{c_*t,c_*}\}_{t\in\mathbb{R}}$ generate an evolution family $\{U_{c_*}(t,s)\}_{t\geq s}$ on $\ell^2(\mathbb{Z};\mathbb{R}^2)$. There exist constants $K, b > 0$ such that

$$\|e^{a(\cdot - c_*t)}U_{c_*}(t,s)w\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \leq Ke^{-b(t-s)}\|e^{a(\cdot-c_*s)}w\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)}, \quad t\geq s, \tag{5.44}$$

provided

$$\Omega(\partial_\xi\phi_{c_*}(\cdot - c_*s), w) = \Omega(\partial_c\phi_{c_*}(\cdot - c_*s), w) = 0. \tag{5.45}$$

We observe numerically (Figure 5.1b) that the tail $\eta(t)$ starts out as zero, and, roughly speaking, develops in the interval $[-c_*t, c_*t]$ as the soliton passes through and excites lattice sites. In particular, when viewed in a co-moving frame traveling rightward with velocity $c_*$, the tail is roughly active in the interval $[-2c_*t, 0]$. This leads to a limiting situation where all lattice sites behind the soliton are excited, while all lattice sites ahead remain almost unperturbed. Our tail approximations $\overline{\eta}$ allow us to characterize this limiting situation in more detail. In terms of the evolution family, (5.42) reads

$$\overline{\eta}(t) = \int_0^t U_{c_*}(t,s)T^{1,0}(c_*s, c_*)[\kappa]\ \mathrm{d}s,$$

in which the forcing term $T^{1,0}(c_*s, c_*)$ is localized around the soliton site $j \approx c_*s$ as it propagates from 0 to $c_*t$. At times

$$t_n = (n+p)/c_*, \quad n\in\mathbb{N},\ p\in[0,1),$$

we shift the tail by an integer $\lfloor c_*t_n\rfloor = n$, which leads to the identity

$$\overline{\eta}(\cdot+n, t_n) = \int_0^{t_n} U_{c_*}(p/c_*, p/c_* - \tau)T^{1,0}(p - c_*\tau, c_*)[\kappa(\cdot+n)]\ \mathrm{d}\tau.$$

We derive such representations using the Green's function associated to $U_{c_*}$, which we analyze in Section 5.4.1 ahead. The only dependence of the integrand on $n$ is through the shift of the random coefficients $\kappa(\cdot+n)$, which does not affect their distribution. Combined with the exponential stability of the evolution family $U_{c_*}$, this allows us to take the limit $n\to\infty$ and show that this shifted tail approaches an equilibrium distribution.

**Proposition 5.4.1.** *Assuming Hypothesis 1 and Hypothesis 2, there exists a (random) function* $\overline{\eta}^\infty : \mathbb{R} \to \mathbb{R}$ *such that for each* $j \in \mathbb{Z}$ *and* $p \in [0,1)$*, the sequence of random variables*

$$\Big(\overline{\eta}(j+n,(n+p)/c_*)\Big)_{n\geq 1}$$

*converges in distribution to* $\overline{\eta}^\infty(j-p)$*.*

**Remark 5.4.2.** A representation of $\overline{\eta}(x)$ is given in (5.59) ahead, which we suspect is (almost surely) continuous in $x$; see Figure 5.7 ahead. We do not attempt to prove this here.

The proof of Proposition 5.4.1 is given at the end of Section 5.4.2. In the limit, the tail is still $1/c_*$-periodic in time in some sense, reflecting that the lattice periodicity is traversed with velocity $c_*$. Here and below, $p \in [0,1)$ denotes the phase of the soliton (for instance, $p=0$ means centered at a lattice point and $p=1/2$ means exactly between lattice points). The convergence in Proposition 5.4.1 holds only in distribution because, roughly speaking, the solitary wave continually encounters different random coefficients. Nevertheless, the distribution of the random coefficients is translation-invariant.

### 5.4.1 Linear theory

Our strategy will be to work with a decomposition of $\overline{\eta}$ in terms of deterministic functions. More precisely, we analyze the asymptotic behavior of tails $\overline{\eta}(t)$ based on a 'response' function $\overline{R} : \mathbb{Z}\times\mathbb{Z}\times\mathbb{R}^+ \to \mathbb{R}^2$, defined through

$$\dot{\overline{R}}(j,m,t) = \mathcal{J}\mathcal{A}_{c_*t,c_*}\overline{R}(j,m,t) + T^{1,0}(c_*t,c_*)[\delta_m](j), \quad \text{with} \quad \overline{R}(j,m,0)=0. \tag{5.46}$$

Here, $\delta_m \in \ell^2(\mathbb{Z};\mathbb{R})$ denotes the sequence with value 1 at $j=m$ and 0 elsewhere. Hence, the response function expresses the effect of a spring heterogeneity at site $m$ on the random tail $\overline{\eta}(t)$. The advantage of this decomposition is that we can analyze the asymptotics of the deterministic components $\overline{R}$ in a co-moving frame, without directly involving the constantly shifting random coefficients $\kappa$. We recall that $T^{1,0}$ is introduced in (5.28), and remark that

$$\begin{aligned} T^{1,0}(c_*t,c_*)[\delta_m](j) = \frac{1}{2}&\begin{bmatrix}\partial_\xi\phi_{c_*}(j-c_*t)\\ \partial_c\phi_{c_*}(j-c_*t)\end{bmatrix}^\top A^{-1}(c_*)\begin{bmatrix}\partial_\xi r^2_{c_*}(m-c_*t)\\ \partial_c r^2_{c_*}(m-c_*t)\end{bmatrix}\\ &+ r_{c_*}(m-c_*t)\begin{bmatrix}0\\ (\delta_m-\delta_{m+1})(j)\end{bmatrix}. \end{aligned} \tag{5.47}$$

Since the wave-profiles and their derivatives are exponentially localized [41, Proposition 5.5], this forcing term satisfies the bound

$$\Big|T^{1,0}(c_*t,c_*)[\delta_m](j)\Big| \leq Ce^{-\beta|j-c_*t|}e^{-\beta|m-c_*t|} + Ce^{-\beta|m-c_*t|}(\delta_0+\delta_1)(j-m) \tag{5.48}$$

for some $C > 0$ and $\beta > a$, where $a$ is the weight of Proposition 5.1.1. Below, we derive the representation of $\overline{\eta}(t)$ as a random series. To this end, we note that the solution to (5.46) can be represented by the Duhamel formula

$$\overline{R}(\cdot, m, t) = \int_0^t U_{c_*}(t,s) T^{1,0}(c_* s, c_*)[\delta_m] \, \mathrm{d}s. \tag{5.49}$$

**Proposition 5.4.3.** *Assume Hypothesis 1 and Hypothesis 2. For each $t \geq 0$, the random sequence*

$$\overline{\eta}(j,t) = \sum_{m\in\mathbb{Z}} \kappa(m) \overline{R}(j,m,t), \quad j \in \mathbb{Z}, \tag{5.50}$$

*lies almost surely in $\ell^2(\mathbb{Z};\mathbb{R}^2)$ and solves the linear system* (5.42).

*Proof.* We first show that the series in (5.50) lies in $\ell^2(\mathbb{Z};\mathbb{R}^2)$. For each $m \in \mathbb{Z}$:

$$\begin{aligned} \|\overline{R}(\cdot, m, t)\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} &= \Big\| \int_0^t U_{c_*}(t,s) T^{1,0}(c_* s, c_*)[\delta_m] \, \mathrm{d}s \Big\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \\ &\leq t \sup_{s\in[0,t]} \|U_{c_*}(t,s)\|_{\mathcal{L}(\ell^2(\mathbb{Z};\mathbb{R}^2))} \|T^{1,0}(c_* s, c_*)[\delta_m]\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)}. \end{aligned}$$

From (5.48) we then get

$$\|T^1(c_* s, c_*)[\delta_m]\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \leq C e^{-\beta|m-c_* s|} \sum_{j\in\mathbb{Z}} e^{-\beta|j-c_* s|} + C e^{-\beta|m-c_* s|} \leq \tilde{C} e^{-\beta|m-c_* s|},$$

which is summable in $m$:

$$\Big\| \sum_{m\in\mathbb{Z}} \kappa(m) \overline{R}(\cdot, m, t) \Big\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \leq \alpha \sum_{m\in\mathbb{Z}} \|\overline{R}(\cdot, m, t)\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} < \infty.$$

The variation of constants formula then shows that (5.42) is solved by

$$\overline{\eta}(j,t) = \int_0^t U_{c_*}(t,s) T^{1,0}(c_* s, c_*)[\kappa] \, \mathrm{d}s.$$

The result follows via (5.49) and the identity

$$T^{1,0}(c_* s, c_*)[\kappa] = \sum_{m\in\mathbb{Z}} \kappa(m) T^{1,0}(c_* s, c_*)[\delta_m].$$

□

We proceed with a result regarding the Green's function associated to $\mathcal{J}\mathcal{A}_{c_* t, c_*}$, which will be key to characterizing the asymptotics of the response function $\overline{R}(j,m,t)$.

**Lemma 5.4.4** (See Section 5.8)**.** *Assuming Hypothesis 2, there exists a Green's function $\mathcal{G}_{c_*} : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ such that the evolution family associated to*

$\mathcal{J}\mathcal{A}_{c_*t,c_*}$ *admits the representation*

$$U_{c_*}(t,s)\begin{bmatrix}u\\v\end{bmatrix}(j)=\sum_{k\in\mathbb{Z}}\mathcal{G}_{c_*}(t-s,j-c_*t,k-c_*s)\begin{bmatrix}u(k)\\v(k)\end{bmatrix},\qquad\begin{bmatrix}u\\v\end{bmatrix}\in\ell^2(\mathbb{Z};\mathbb{R}^2).\tag{5.51}$$

**Remark 5.4.5** (See Lemma 5.8.1)**.** In the constant-coefficient case $\mathcal{J}\mathcal{A}_{c_*t,c_*}=\mathcal{J}$, the evolution family $U_{c_*}$ reduces to the unitary $C_0$-group $\{e^{\mathcal{J}t}\}_{t\in\mathbb{R}}$. In particular, these solution operators admit an explicit kernel representation

$$U_{c_*}(t,s)\begin{bmatrix}u\\v\end{bmatrix}(j)=e^{\mathcal{J}(t-s)}\begin{bmatrix}u\\v\end{bmatrix}(j)=\sum_{k\in\mathbb{Z}}\Phi(j-k,t-s)\begin{bmatrix}u(k)\\v(k)\end{bmatrix},\quad j\in\mathbb{Z},\ t\in\mathbb{R},\tag{5.52}$$

with

$$\Phi(j,t):=\begin{bmatrix}J_{2j}(2t) & -J_{2j+1}(2t)\\ -J_{2j-1}(2t) & J_{2j}(2t)\end{bmatrix}.\tag{5.53}$$

Here, $J_n:\mathbb{R}\to\mathbb{R}$ are Bessel functions of the first kind, and for $n\in\mathbb{Z}^-$ we use the convention that $J_n(x)=(-1)^nJ_{-n}(x)$. Hence, the Green's function associated to the discrete wave operator $\mathcal{J}$ satisfies (5.51), with

$$\mathcal{G}_{c_*}(\alpha,\beta,\gamma)=\Phi(\beta-\gamma+c_*\alpha,\alpha).$$

### 5.4.2 Asymptotic response

We proceed by analyzing the asymptotic behavior of (5.46) in the co-moving frame, based on the Green's function representation (5.51). We will see that at times

$$t_n=(n+p)/c_*,\quad n\in\mathbb{N},\ p\in[0,1),$$

we have a representation of the (shifted) response function $\overline{R}(j+\lfloor c_*t\rfloor,m+\lfloor c_*t\rfloor,t)$ as

$$\begin{aligned}\overline{R}(j+n,m+n,t_n)&=\int_0^{t_n}\Big(U_{c_*}(p/c_*,p/c_*-\tau)T^{1,0}(p-c_*\tau,c_*)[\delta_m]\Big)(j)\ \mathrm{d}\tau\\&=\int_0^{t_n}\sum_{k\in\mathbb{Z}}\mathcal{G}_{c_*}(\tau,j-p,k-p+c_*\tau)T^{1,0}(p-c_*\tau,c_*)[\delta_m](k)\ \mathrm{d}\tau.\end{aligned}$$

Hence, this collects the effect of a heterogeneity at site $m$ on site $j$ as the solitary wave travels from $-n$ to $p$. In this 'co-moving frame', stationary forcing competes with an exponentially stable linear flow. This allows us to take $n\to\infty$ and arrive at the limiting response function $\overline{R}^{\infty} : \mathbb{Z} \times \mathbb{Z} \times [0,1] \to \mathbb{R}^2$, defined as

$$\overline{R}^{\infty}(j,m,p) = \int_0^{\infty} \sum_{k\in\mathbb{Z}} \mathcal{G}_{c_*}(\tau, j-p, k-p+c_*\tau) T^{1,0}(p-c_*\tau, c_*)[\delta_m](k)\ \mathrm{d}\tau. \tag{5.54}$$

Note that in the constant-coefficient case (Lemma 5.8.1), the kernel in (5.54) is explicitly given by

$$\mathcal{G}_{c_*}(\tau, j-p, k-p+c_*\tau) = \Phi(j-k, \tau).$$

**Lemma 5.4.6.** *Assume Hypothesis 2. For each $j, m \in \mathbb{Z}$ and $p \in [0,1)$, we have the pointwise limit*

$$\lim_{n\to\infty} \overline{R}(j+n, m+n, (n+p)/c_*) = \overline{R}^{\infty}(j,m,p). \tag{5.55}$$

*Proof.* Let $p \in [0,1)$ and

$$t_n = (n+p)/c_*, \quad n \in \mathbb{N}.$$

Then (5.49) gives

$$\begin{aligned}
&\overline{R}(j+n, m+n, t_n) \\
&\qquad = \int_0^{t_n} \sum_{k\in\mathbb{Z}} \mathcal{G}_{c_*}(t_n - s, j+n-c_*t_n, k-c_*s) T^{1,0}_{\delta_{m+n}}(c_*s, c_*)(k)\ \mathrm{d}s \\
&\qquad = \int_0^{t_n} \sum_{k'\in\mathbb{Z}} \mathcal{G}_{c_*}(t_n - s, j-p, k'+n-c_*s) T^{1,0}(c_*s, c_*)[\delta_{m+n}](k'+n)\ \mathrm{d}s.
\end{aligned}$$

Using

$$T^{1,0}(c_*s, c_*)[\delta_{m+n}](k'+n) = T^{1,0}(c_*s - n, c_*)[\delta_m](k') = T^{1,0}(p - c_*(t_n - s), c_*)[\delta_m](k'),$$

and substituting $\tau = t_n - s$, we obtain that

$$\begin{aligned}
\overline{R}(j+n, m+n, t_n) &= \int_0^{t_n} \sum_{k'\in\mathbb{Z}} \mathcal{G}_{c_*}(\tau, j-p, k'-p+c_*\tau) T^{1,0}(p-c_*\tau, c_*)[\delta_m](k')\ \mathrm{d}\tau \\
&\to \overline{R}^{\infty}(j,m,p) \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{5.56}$$

Thus, we have established (5.55): pointwise convergence in $j$ and $m$. □

We will in fact require a stronger form of convergence for our purposes of analyzing the asymptotics of $\mathbb{E}[\overline{c}_2(t)]$ in Section 5.5. Recall that the weight $a > 0$ is introduced in Proposition 5.1.1. We then define the exponentially weighted spaces

$$\ell^2_a(\mathbb{Z}^2; \mathbb{R}^2) := \left\{ f : \mathbb{Z}^2 \to \mathbb{R}^2 \ \middle|\ \|f\|_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)} < \infty \right\},$$

with norm

$$\|f\|^2_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)} = \sum_{m\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} e^{2aj}\big(f_r^2(j,m) + f_p^2(j,m)\big).$$

**Proposition 5.4.7.** *Assume Hypothesis 2. The response function* $\overline{R}(\cdot,\cdot,(n+p)/c_*)$ *converges to* $\overline{R}^\infty(\cdot,\cdot,p)$ *in* $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$ *at an exponential rate: there exist constants* $C, q > 0$ *such that*

$$\|\overline{R}(\cdot+n,\cdot+n,(n+p)/c_*) - \overline{R}^\infty(\cdot,\cdot,p)\|_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)} \le Ce^{-qn}, \quad n\in\mathbb{N}. \tag{5.57}$$

Before giving the proof, we prepare two intermediate results. The first regards the weighted norm of the forcing term in (5.54).

**Lemma 5.4.8.** *Assuming Hypothesis 2, there exist constants* $C, \gamma > 0$*, such that for each* $p\in[0,1)$*,* $m\in\mathbb{Z}$ *and* $\tau\ge 0$*, we have*

$$\|e^{a(\cdot+c_*\tau)}T^{1,0}(p-c_*\tau,c_*)[\delta_m]\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \le \tilde{C}e^{-\gamma|m+c_*\tau|}.$$

*Proof.* The estimate (5.48) provides

$$\begin{aligned}\|e^{a(\cdot+c_*\tau)}&T^{1,0}(p-c_*\tau,c_*)[\delta_m]\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)}\\ &\le Ce^{-\beta|m+c_*\tau|}\|e^{a(\cdot+c_*\tau)}e^{-\beta|\cdot+c_*\tau|}\|_{\ell^2(\mathbb{Z};\mathbb{R})}\\ &\quad + Ce^{-\beta|m+c_*\tau|}\|e^{a(\cdot+c_*\tau)}(\delta_0+\delta_1)(\cdot-m)\|_{\ell^2(\mathbb{Z};\mathbb{R})},\end{aligned}$$

where we note that

$$e^{a(\cdot+c_*\tau)}e^{-\beta|\cdot+c_*\tau|}\in\ell^2(\mathbb{Z};\mathbb{R}),$$

since $\gamma := \beta - a > 0$. We furthermore compute

$$\begin{aligned}\|e^{a(\cdot+c_*\tau)}(\delta_0+\delta_1)(\cdot-m)\|_{\ell^2(\mathbb{Z};\mathbb{R})} &= \Big(\sum_{j\in\mathbb{Z}} e^{2a(\cdot+c_*\tau)}(\delta_0+\delta_1)(j-m)\Big)^{1/2}\\ &= (1+e^{2a})^{1/2}e^{a(m+c_*\tau)},\end{aligned}$$

and the result follows. □

We proceed by asserting that both $\overline{R}(\cdot,\cdot,t)$ and $\overline{R}^\infty(\cdot,\cdot,p)$ are well-defined in $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$.

**Lemma 5.4.9.** *Assume Hypothesis 2. For each* $t\ge 0$ *and* $p\in[0,1)$*, the response function* $\overline{R}(\cdot,\cdot,t)$ *and its limit* $\overline{R}^\infty(\cdot,\cdot,p)$ *are contained in* $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$*.*

*Proof.* From (5.54), we compute

$$\begin{aligned}&\|\overline{R}^\infty(\cdot,m,p)\|^2_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)}\\ &\quad = \sum_{m\in\mathbb{Z}}\Big\|\int_0^\infty \sum_{k\in\mathbb{Z}} e^{a\cdot}\mathcal{G}_{c_*}(\tau,j-p,k-p+c_*\tau)T^{1,0}(p-c_*\tau,c_*)[\delta_m](k)\,\mathrm{d}\tau\Big\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^2)}\end{aligned}$$

$$= \sum_{m\in\mathbb{Z}} \Big\| \int_0^\infty e^{a\cdot} U_{c_*}(p/c_*, p/c_* - \tau) T^{1,0}(p - c_*\tau, c_*)[\delta_m] \, \mathrm{d}\tau \Big\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^2)}.$$

We then apply the exponential stability bound (5.44):

$$\begin{aligned}
&\|\overline{R}^\infty(\cdot, m, p)\|^2_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)} \\
&\qquad \le \sum_{m\in\mathbb{Z}} \Big( \int_0^\infty \|e^{a\cdot} U_{c_*}(p/c_*, p/c_* - \tau) T^{1,0}(-c_*\tau, c_*)[\delta_m]\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \, \mathrm{d}\tau \Big)^2 \\
&\qquad \le K^2 \sum_{m\in\mathbb{Z}} \Big( \int_0^\infty e^{-b\tau} \|e^{a(\cdot + c_*\tau)} T^{1,0}(p - c_*\tau, c_*)[\delta_m]\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \, \mathrm{d}\tau \Big)^2.
\end{aligned}$$

Via Lemma 5.4.8, we then obtain

$$\|\overline{R}^\infty(\cdot, m, p)\|^2_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)} \le K^2 \tilde{C}^2 \sum_{m\in\mathbb{Z}} \Big( \int_0^\infty e^{-b\tau} e^{-\gamma|m + c_*\tau|} \mathrm{d}\tau \Big)^2.$$

A straightforward computation shows that for each $t \ge 0$

$$\int_t^\infty e^{-b\tau}\, e^{-\gamma|m+c_*\tau|} \mathrm{d}\tau = \begin{cases} \dfrac{e^{-\gamma(c_*t+m)}\, e^{-bt}}{b/c_* + \gamma}, & c_*t + m \ge 0, \\[2ex] \dfrac{e^{bm/c_*} - e^{\gamma(c_*t+m)} e^{-bt}}{\gamma - b/c_*} + \dfrac{e^{bm/c_*}}{b/c_* + \gamma}, & c_*t + m \le 0. \end{cases} \tag{5.58}$$

In particular,

$$\sum_{m\in\mathbb{Z}} \Big( \int_0^\infty e^{-b\tau} e^{-\gamma|m+c_*\tau|} \mathrm{d}\tau \Big)^2 \le \tilde{C} \Big( \sum_{m=0}^\infty e^{-2m} + e^{-2bm} \Big) < \infty.$$

We have thus shown that $\overline{R}^\infty(\cdot, \cdot, p)$ lies in $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$. An analogous computation based on (5.56) shows that $\overline{R}(\cdot + n, \cdot + n, t_n)$—and consequently its translate $\overline{R}(\cdot, \cdot, t_n)$—is also contained in $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$. □

We are then ready to prove Proposition 5.4.7.

*Proof of Proposition 5.4.7.* We apply the same estimates as in Lemma 5.4.9 to the difference

$$\begin{aligned}
\mathcal{R}_n :&= \sum_{m\in\mathbb{Z}} \|e^{a\cdot} \big(\overline{R}(\cdot + n, m + n, t_n) - \overline{R}^\infty(\cdot, m, p)\big)\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \\
&= \sum_{m\in\mathbb{Z}} \Big\| \int_{t_n}^\infty e^{a\cdot} U_{c_*}(p/c_*, p/c_* - \tau) T^{1,0}(p - c_*\tau, c_*)[\delta_m] \, \mathrm{d}\tau \Big\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^2)}.
\end{aligned}$$

Applying the stability bound (5.44) gives

$$\begin{aligned}\mathcal{R}_n &\le K^2 \sum_{m\in\mathbb{Z}} \Big( \int_{t_n}^{\infty} e^{-b\tau} \| e^{a(\cdot + c_*\tau)} T^{1,0}(p - c_*\tau, c_*)[\delta_m] \|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \, \mathrm{d}\tau \Big)^2 \\ &\le K^2 \tilde{C}^2 \sum_{m\in\mathbb{Z}} \Big( \int_{t_n}^{\infty} e^{-b\tau} e^{-\gamma|m+c_*\tau|} \mathrm{d}\tau \Big)^2.\end{aligned}$$

Inspecting (5.58), we now arrive at

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} \Big( \int_{t}^{\infty} e^{-b\tau} e^{-\gamma|m+c_*\tau|} \mathrm{d}\tau \Big)^2 \\ &\qquad \le \tilde{C} \Big( e^{-2bt} \sum_{m\ge -t} e^{-2(t+m)} + \sum_{m\le -t} e^{2bm} + e^{-2bt} \sum_{m\le -t} e^{2(t+m)} \Big) \\ &\qquad \le 3\tilde{C} \frac{e^{2b}}{e^{2b}-1} e^{-2bt},\end{aligned}$$

which shows (5.57). □

### 5.4.3 Asymptotic tail

Following the convergence of $\overline{R}$ to $\overline{R}^{\infty}$, we now turn to Proposition 5.4.1, concerning the convergence of the shifted tail to $\overline{\eta}^{\infty}$. We will see that $\overline{\eta}^{\infty}$ can be represented by the random series

$$\overline{\eta}^{\infty}(x) = \sum_{m\in\mathbb{Z}} \zeta(m - \lceil x \rceil) \overline{R}^{\infty}(\lceil x \rceil, m, \lceil x \rceil - x), \quad x \in \mathbb{R}, \tag{5.59}$$

where $\zeta(j)$, $j \in \mathbb{Z}$ is an i.i.d. sequence with the same distribution as the random coefficients $\kappa(j)$. Regarding the continuity of this representation (Remark 5.4.2), we note that

$$\overline{R}^{\infty}(j, m, 1) = \overline{R}^{\infty}(j-1, m-1, 0), \quad j, m \in \mathbb{Z},$$

which shows that the specific representation (5.59) satisfies

$$\lim_{x\downarrow j} \overline{\eta}^{\infty}(x) = \overline{\eta}^{\infty}(j), \quad j \in \mathbb{Z}.$$

In between lattice points $x \notin \mathbb{Z}$, the continuity of $\overline{\eta}^{\infty}$ is determined by that of $\overline{R}^{\infty}(j, m, p)$ in the $p$-variable, for which we do not pursue a proof here. The covariance of $\overline{\eta}^{\infty} = (\overline{\eta}_r^{\infty}, \overline{\eta}_p^{\infty})^{\top}$ is given by

$$\mathbb{E}\big(\overline{\eta}_r^{\infty}(x) \overline{\eta}_r^{\infty}(y)\big) = \Big\langle \overline{R}_r^{\infty}(\lfloor x \rfloor, \cdot, x - \lfloor x \rfloor), \overline{R}_r^{\infty}(\lfloor y \rfloor, \cdot, y - \lfloor y \rfloor) \Big\rangle_{\ell^2(\mathbb{Z};\mathbb{R})},$$

with an analogous expression for $\overline{\eta}_p^{\infty}$. See Figure 5.7 for a numerical evaluation. We are then ready to prove the main result of this section.

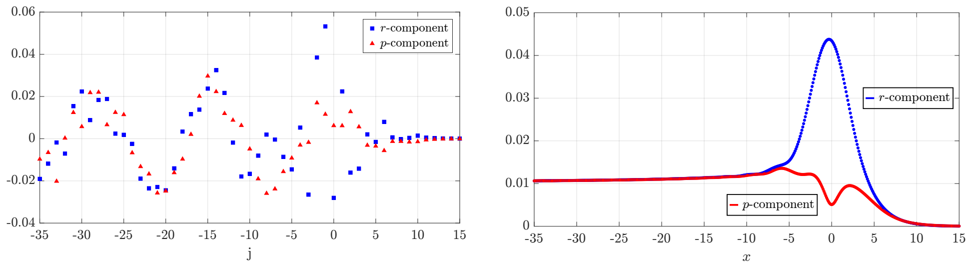


**Figure 5.7:** Realization (left) and standard deviation (right) of the limiting tail $\overline{\eta}^{\infty}$, based on $\overline{\eta} = \eta_1^{\mathrm{h}}$. ( $c_* = 1.015$ and $\zeta(i) \in \mathrm{Unif}([-\sqrt{3}, \sqrt{3}])$.)

*Proof of Proposition 5.4.1.* Once again, we pick $p \in [0, 1)$ and write $t_n = (n+p)/c_*$, so that at $x = j - p$, the representation (5.59) reads

$$\overline{\eta}^{\infty}(j-p) = \sum_{m \in \mathbb{Z}} \zeta(m-j)\overline{R}^{\infty}(j, m, p),$$

where $\zeta(j)$, $j \in \mathbb{Z}$ is an i.i.d. sequence with the same distribution as the random coefficients $\kappa(j)$. We introduce

$$Y_n(j) = \sum_{m \in \mathbb{Z}} \zeta(m-j)\overline{R}(j+n, m+n, t_n),$$

which, for each $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, has the same law as

$$\overline{\eta}(j+n, t_n) = \sum_{m \in \mathbb{Z}} \kappa(m+n)\overline{R}(j+n, m+n, t_n).$$

Indeed, the i.i.d. sequences $\kappa$ and $\zeta$ follow the same distribution. We furthermore have

$$Y_n(j) - \overline{\eta}^{\infty}(j-p) = \sum_{m \in \mathbb{Z}} \zeta(m-j)\Big(\overline{R}(j+n, m+n, t_n) - \overline{R}^{\infty}(j, m, p)\Big)$$

and as a consequence of Proposition 5.4.7:

$$\mathbb{E}[(Y_n(j) - \overline{\eta}^{\infty}(j-p))^2] = \sum_{m \in \mathbb{Z}} \Big(\overline{R}(j+n, m+n, t_n) - \overline{R}^{\infty}(j, m, p)\Big)^2 \to 0$$

as $n \to \infty$. In particular, $Y_n(j) \xrightarrow{d} \overline{\eta}^{\infty}(j-p)$. It follows that also

$$\overline{\eta}(j+n, t_n) \xrightarrow{d} \overline{\eta}^{\infty}(j-p). \qquad \square$$

## 5.5 Amplitude Attenuation

We finally return to the main goal of this chapter: capturing the amplitude attenuation that affects the propagation of solitary waves in (5.1). We build on the results of Section 5.4, where we analyzed linear tail approximations $\overline{\eta}$ based on a general time-dependent linearization $\mathcal{JA}_{c_* t, c_*}$; see (5.42) and Hypothesis 2. The representation of $\overline{\eta}(t)$ as a random series (Proposition 5.4.3) allows us to compute expectations of the second-order corrections $\overline{\gamma}_2(t)$ and $\overline{c}_2(t)$, defined via (5.41). Although the random coefficients $\kappa$—and by extension the linear tail $\overline{\eta}(t)$—are mean-zero, the quadratic terms in (5.41) lead to a non-zero expectation of $\dot{\overline{\gamma}}_2(t)$ and $\dot{\overline{c}}_2(t)$.

In Proposition 5.5.2 below, we give an explicit representation for the amplitude attenuation via the bilinear maps $\Gamma^{1,1}, \Gamma^{0,2}, C^{1,1}$ and $C^{0,2}$ introduced in (5.26)–(5.27), and the response function $\overline{R}$ introduced in (5.46). Recall that $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$ denotes the exponentially weighted spaces

$$\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2) := \left\{ f : \mathbb{Z}^2 \to \mathbb{R}^2 \;\middle|\; \|f\|_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)} < \infty \right\},$$

with norm

$$\|f\|^2_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)} = \sum_{m\in\mathbb{Z}} \|e^{a\cdot} f(\cdot, m)\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^2)}.$$

For $\xi \in \mathbb{R}$, $c \in (c_-, c_+)$, we define the linear map $M^{1,1}(\xi, c) : \ell^2_a(\mathbb{Z}^2;\mathbb{R}^2) \to \mathbb{R}^2$ via

$$M^{1,1}(\xi, c)[\alpha] = \sum_{m\in\mathbb{Z}} \begin{bmatrix} \Gamma^{1,1}(\xi, c) \\ C^{1,1}(\xi, c) \end{bmatrix} [\delta_m, \alpha(\cdot, m)], \tag{5.60}$$

and the bilinear map $M^{0,2}(\xi, c) : \ell^2_a(\mathbb{Z}^2;\mathbb{R}^2) \times \ell^2_a(\mathbb{Z}^2;\mathbb{R}^2) \to \mathbb{R}^2$ by

$$M^{0,2}(\xi, c)[\alpha, \beta] = \sum_{m\in\mathbb{Z}} \begin{bmatrix} \Gamma^{0,2}(\xi, c) \\ C^{0,2}(\xi, c) \end{bmatrix} [\alpha_r(\cdot, m), \beta_r(\cdot, m)]. \tag{5.61}$$

Both $M^{1,1}$ and $M^{0,2}$ are bounded (see Section 5.9) and we recall that $\overline{R}(\cdot, \cdot, t)$ is contained in $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$ for each $t \geq 0$ on account of Lemma 5.4.9. Finally, we define $M^{1,0}_{I}(\xi, c), \ldots, M^{1,0}_{IV}(\xi, c)$ as the correlations

$$M^{1,0}_{I}(\xi, c) = -\frac{\alpha_0^{-2}(c)\alpha_1(c)}{2c} \mathbb{E} \sum_{j\in\mathbb{Z}} \kappa(j)[r_c^2(j-\xi) - r_c^2(j)]\partial_\xi \begin{bmatrix} \Gamma^{1,0}(\xi, c) \\ C^{1,0}(\xi, c) \end{bmatrix} [\kappa], \tag{5.62}$$

$$M^{1,0}_{II}(\xi, c) = \frac{1}{2c\alpha_0(c)} \mathbb{E} \sum_{j\in\mathbb{Z}} \kappa(j) \int_0^\xi \partial_c r_c^2(j-s) \mathrm{d}s \; \partial_\xi \begin{bmatrix} \Gamma^{1,0}(\xi, c) \\ C^{1,0}(\xi, c) \end{bmatrix} [\kappa], \tag{5.63}$$

$$M^{1,0}_{III}(\xi, c) = -\frac{\alpha_0^{-1}(c)}{2c^2} \mathbb{E} \sum_{j\in\mathbb{Z}} \kappa(j) \Big( \int_0^\xi r_c^2(j-s)\; \mathrm{d}s - \xi r_c^2(j) \Big) \; \partial_\xi \begin{bmatrix} \Gamma^{1,0}(\xi, c) \\ C^{1,0}(\xi, c) \end{bmatrix} [\kappa], \tag{5.64}$$

and

$$M_{IV}^{1,0}(\xi,c) = -\frac{\alpha_0^{-1}(c)}{2c}\mathbb{E}\sum_{j\in\mathbb{Z}}\kappa(j)[r_c^2(j-\xi)-r_c^2(j)]\partial_c\begin{bmatrix}\Gamma^{1,0}(\xi,c)\\ C^{1,0}(\xi,c)\end{bmatrix}[\kappa]. \tag{5.65}$$

We refer to Section 5.10 for a fully deterministic representation. As a preparation, we make the following observation regarding these correlations.

**Lemma 5.5.1** (See Section 5.10). *Assume Hypothesis 1 and Hypothesis 2. For each $\xi \in \mathbb{R}, c \in (c_-, c_+)$ and $i \in \{I,\dots,IV\}$, the correlation $M_i^{1,0}(\xi,c)$ can be decomposed as*

$$M_i^{1,0}(\xi,c) = M_{i,\mathrm{per}}^{1,0}(\xi,c) + M_{i,\mathrm{trans}}^{1,0}(\xi,c),$$

*with $M_{i,\mathrm{per}}^{1,0}(\xi,c)$ periodic in $\xi$:*

$$M_{i,\mathrm{per}}^{1,0}(\xi,c) = M_{i,\mathrm{per}}^{1,0}(\xi+d,c), \quad d\in\mathbb{Z},$$

*and $M_{i,\mathrm{trans}}^{1,0}(\xi,c)$ exponentially decaying in $|\xi|$: there exist constants $C,q>0$ such that*

$$|M_{i,\mathrm{trans}}^{1,0}(\xi,c)| \le Ce^{-q|\xi|}.$$

Upon introducing

$$\begin{aligned} M^{1,0}(\xi,c) &= M_{\mathrm{per}}^{1,0}(\xi,c) + M_{\mathrm{trans}}^{1,0}(\xi,c)\\ &= M_I^{1,0}(\xi,c) + M_{II}^{1,0}(\xi,c) + M_{III}^{1,0}(\xi,c) + M_{IV}^{1,0}(\xi,c), \end{aligned}$$

we then obtain the following.

**Proposition 5.5.2.** *Assume Hypothesis 1 and Hypothesis 2. For each $t\ge 0$, we have*

$$\mathbb{E}\begin{bmatrix}\dot{\overline{\gamma}}_2(t)\\ \dot{\overline{c}}_2(t)\end{bmatrix} = M^{1,0}(c_*t,c_*) + M^{0,2}(c_*t,c_*)[\overline{R}(\cdot,\cdot,t),\overline{R}(\cdot,\cdot,t)] + M^{1,1}(c_*t,c_*)[\overline{R}(\cdot,\cdot,t)]. \tag{5.66}$$

*Proof.* By Hypothesis 1, the random coefficients satisfy

$$\mathbb{E}[\kappa(j)] = 0 \quad \text{and} \quad \mathbb{E}[\kappa(i)\kappa(j)] = \delta_{ij}, \quad i,j\in\mathbb{Z}.$$

We proceed by computing the expectation of (5.41) using the representation

$$\overline{\eta}(j,t) = \sum_{m\in\mathbb{Z}}\kappa(m)\overline{R}(j,m,t).$$

Via the definitions (5.62)–(5.65), the gradient terms in (5.41) satisfy

$$\begin{aligned}
\mathbb{E}\ \gamma_{1,\mathit{I}}(t)\partial_\xi \begin{bmatrix}\Gamma^{1,0}(c_*t,c_*)\\ C^{1,0}(c_*t,c_*)\end{bmatrix}[\kappa] &= M^{1,0}_{\mathit{I}}(c_*t,c_*),\\
\mathbb{E}\ \gamma_{1,\mathit{II}}(t)\partial_\xi \begin{bmatrix}\Gamma^{1,0}(c_*t,c_*)\\ C^{1,0}(c_*t,c_*)\end{bmatrix}[\kappa] &= M^{1,0}_{\mathit{II}}(c_*t,c_*),\\
\mathbb{E}\ \int_0^t c_1(s)\ \mathrm{d}s\partial_\xi \begin{bmatrix}\Gamma^{1,0}(c_*t,c_*)\\ C^{1,0}(c_*t,c_*)\end{bmatrix}[\kappa] &= M^{1,0}_{\mathit{III}}(c_*t,c_*),
\end{aligned}$$

and

$$\mathbb{E}\ c_1(t)\partial_c \begin{bmatrix}\Gamma^{1,0}(c_*t,c_*)\\ C^{1,0}(c_*t,c_*)\end{bmatrix}[\kappa] = M^{1,0}_{\mathit{IV}}(c_*t,c_*).$$

Here, we recall that the components in $\gamma_{1,\mathit{I}}(t)$ and $\gamma_{1,\mathit{II}}(t)$ sum to $\gamma_1(t)$, and are defined in (5.34) and (5.35), respectively. We proceed with the $\mathcal{O}(\overline{\eta}_r^2)$ term in (5.41). The bilinearity of $\Gamma^{0,2}(c_*t,c_*)$ and $C^{0,2}(c_*t,c_*)$ implies

$$\begin{aligned}
\mathbb{E}&\begin{bmatrix}\Gamma^{0,2}(c_*t,c_*)\\ C^{0,2}(c_*t,c_*)\end{bmatrix}[\overline{\eta}_r(t),\overline{\eta}_r(t)]\\
&= \mathbb{E}\begin{bmatrix}\Gamma^{0,2}(c_*t,c_*)\\ C^{0,2}(c_*t,c_*)\end{bmatrix}\Big[\sum_{m\in\mathbb{Z}}\kappa(m)\overline{R}_r(\cdot,m,t),\sum_{m'\in\mathbb{Z}}\kappa(m')\overline{R}_r(\cdot,m',t)\Big]\\
&= \sum_{m\in\mathbb{Z}}\sum_{m'\in\mathbb{Z}}\mathbb{E}[\kappa(m)\kappa(m')]\begin{bmatrix}\Gamma^{0,2}(c_*t,c_*)\\ C^{0,2}(c_*t,c_*)\end{bmatrix}[\overline{R}_r(\cdot,m,t),\overline{R}_r(\cdot,m',t)]\\
&= M^{0,2}(c_*t,c_*)[\overline{R}(\cdot,\cdot,t),\overline{R}(\cdot,\cdot,t)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\begin{bmatrix}\Gamma^{1,1}(c_*t,c_*)\\ C^{1,1}(c_*t,c_*)\end{bmatrix}[\kappa,\overline{\eta}(t)] &= \mathbb{E}\begin{bmatrix}\Gamma^{1,1}(c_*t,c_*)\\ C^{1,1}(c_*t,c_*)\end{bmatrix}\Big[\sum_{m\in\mathbb{Z}}\kappa(m)\delta(\cdot-m),\sum_{m'\in\mathbb{Z}}\kappa(m')\overline{R}(\cdot,m',t)\Big]\\
&= \sum_{m\in\mathbb{Z}}\sum_{m'\in\mathbb{Z}}\mathbb{E}[\kappa(m)\kappa(m')]\begin{bmatrix}\Gamma^{1,1}(c_*t,c_*)\\ C^{1,1}(c_*t,c_*)\end{bmatrix}\Big[\delta_m,\overline{R}(\cdot,m',t)\Big]\\
&= M^{1,1}(c_*t,c_*)[\overline{R}(\cdot,\cdot,t)]. \qquad \square
\end{aligned}$$

As a consequence of Proposition 5.4.7—the convergence of $\overline{R}$ to $\overline{R}^\infty$ (defined in (5.54)) in the weighted space $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$—implies that the amplitude attenuation $\mathbb{E}[\dot{\overline{c}}_2(t)]$ also converges in a periodic sense.

**Corollary 5.5.3.** *Assume Hypothesis 1 and Hypothesis 2. For each $p\in[0,1)$, we have*

$$\lim_{n\to\infty}\mathbb{E}\begin{bmatrix}\dot{\overline{\gamma}}_2\big((n+p)/c_*\big)\\ \dot{\overline{c}}_2\big((n+p)/c_*\big)\end{bmatrix} = M^{1,0}_{\mathrm{per}}(p,c_*) + M^{0,2}(p,c_*)[\overline{R}^\infty(\cdot,\cdot,p),\overline{R}^\infty(\cdot,\cdot,p)]$$

$$+ M^{1,1}(p,c_*)[\overline{R}^{\infty}(\cdot,\cdot,p)].$$

*The convergence holds with the same exponential rates as Proposition 5.4.7 and Lemma 5.5.1, which is independent of* $\sigma$*.*

*Proof.* Using the shift invariance

$$M^{1,1}(c_*t,c_*)[\alpha] = M^{1,1}(c_*t-j,c_*)[\alpha(\cdot+j,\cdot+j)], \quad j\in\mathbb{Z},$$

and

$$M^{0,2}(c_*t,c_*)[\alpha,\beta] = M^{0,2}(c_*t-j,c_*)[\alpha(\cdot+j,\cdot+j),\beta(\cdot+j,\cdot+j)], \quad j\in\mathbb{Z},$$

we rewrite (5.66) as

$$\begin{aligned}\mathbb{E}\begin{bmatrix}\dot{\overline{\gamma}}_2(t)\\ \dot{\overline{c}}_2(t)\end{bmatrix} &= M^{0,2}\big(c_*t-\lfloor c_*t\rfloor,c_*\big)\big[\overline{R}(\cdot+\lfloor c_*t\rfloor,\cdot+\lfloor c_*t\rfloor,t),\overline{R}(\cdot+\lfloor c_*t\rfloor,\cdot+\lfloor c_*t\rfloor,t)\big]\\ &\quad + M^{1,0}(c_*t,c_*) + M^{1,1}\big(c_*t-\lfloor c_*t\rfloor,c_*\big)\big[\overline{R}(\cdot+\lfloor c_*t\rfloor,\cdot+\lfloor c_*t\rfloor,t)\big].\end{aligned}$$

Thus, for $p\in[0,1)$ and

$$t_n = (n+p)/c_*, \quad n\in\mathbb{N},$$

we have

$$\begin{aligned}\mathbb{E}\begin{bmatrix}\dot{\overline{\gamma}}_2(t_n)\\ \dot{\overline{c}}_2(t_n)\end{bmatrix} &= M^{1,0}(n+p,c_*) + M^{0,2}(p,c_*)\big[\overline{R}(\cdot+n,\cdot+n,t_n),\overline{R}(\cdot+n,\cdot+n,t_n)\big]\\ &\quad + M^{1,1}(p,c_*)\big[\overline{R}(\cdot+n,\cdot+n,t_n)\big].\end{aligned}$$

The result now follows from the convergence

$$M^{1,0}(n+p,c_*) = M^{1,0}_{\mathrm{per}}(p,c_*) + M^{1,0}_{\mathrm{trans}}(n+p,c_*) \to M^{1,0}_{\mathrm{per}}(p,c_*),$$

through Lemma 5.5.1, and the convergence of

$$\overline{R}(\cdot+n,\cdot+n,t_n)\to\overline{R}^{\infty}(\cdot,\cdot,p)$$

in $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$ (Proposition 5.4.7). Indeed, the linear operator $M^{1,1}(p,c_*)$ and bilinear map $M^{0,2}(p,c_*)$ are bounded on $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$ (Section 5.9). □

Since $\mathbb{E}[\dot{\overline{c}}_2(t)]$ converges to a $1/c_*$-periodic function, the quantity which effectively captures the decay-rate of the solitary wave amplitude is the second component of

$$\begin{aligned}\begin{bmatrix}\overline{\mathcal{Q}}_\gamma(c_*)\\ \overline{\mathcal{Q}}_c(c_*)\end{bmatrix} &:= c_*^{-1}\int_0^1 M^{1,0}_{\mathrm{per}}(p,c_*) + M^{0,2}(p,c_*)[\overline{R}^{\infty}(\cdot,\cdot,p),\overline{R}^{\infty}(\cdot,\cdot,p)]\mathrm{d}p\\ &\quad + c_*^{-1}\int_0^1 M^{1,1}(p,c_*)[\overline{R}^{\infty}(\cdot,\cdot,p)]\,\mathrm{d}p. \qquad (5.67)\end{aligned}$$

In particular, we write $\mathcal{Q}_c(c_*)$ for the version of $\overline{\mathcal{Q}}_c(c_*)$ that arises by taking $\mathcal{A}_{c_*t,c_*} = \mathcal{L}_{c_*t,c_*}$ in (5.42). Similarly, we write $\mathcal{Q}_c^{\mathrm{h}}(c_*)$ for the version of $\overline{\mathcal{Q}}_c(c_*)$ corresponding to the constant coefficient reduction $\mathcal{A}_{c_*t,c_*} = I$ in (5.42). Associated to these quantities, we can define the limiting ODEs

$$\dot{c}_{\mathrm{lim}}(\tau) = \mathcal{Q}_c(c_{\mathrm{lim}}(\tau)), \qquad \text{with} \quad c_{\mathrm{lim}}(0) = c_*, \tag{5.68}$$

and

$$\dot{c}^{\mathrm{h}}_{\mathrm{lim}}(\tau) = \mathcal{Q}^{\mathrm{h}}_c(c^{\mathrm{h}}_{\mathrm{lim}}(\tau)), \qquad \text{with} \quad c^{\mathrm{h}}_{\mathrm{lim}}(0) = c_*, \tag{5.69}$$

which capture the substantial deviations of $c$ from $c_*$ in the slow time variable $\tau = \sigma^2 t$.

Figure 5.8 shows that the dependence of the integrand in (5.67) on $p$ is very minimal. The right panel of Figure 5.9 displays the nonlinear dependence on the

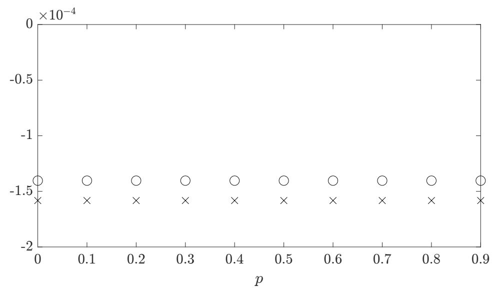


**Figure 5.8:** Asymptotic mean attenuation $\lim_{n\to\infty} \mathbb{E}[\dot{c}_2((n+p)/c_*)]$ (marked with $\circ$) and $\lim_{n\to\infty} \mathbb{E}[\dot{c}_2^{\mathrm{h}}((n+p)/c_*)]$ (marked with $\times$), as given by Corollary 5.5.3, for $p \in [0,1)$.

decay rates $\mathcal{Q}_c(c)$ and $\mathcal{Q}_c^{\mathrm{h}}(c)$ on the amplitude $c$. This sheds light on the nature of the amplitude decay through (5.68). A polynomial fit on the datapoints in Figure 5.9 suggests a relation of the form $\mathcal{Q}_c(c) \sim -(c-1)^2$, corresponding to a polynomial decay $c(t) - 1 \sim \frac{1}{\sigma^2 t}$. Figure 5.10 confirms that the limiting ODEs (5.68) and (5.69) track the amplitude decay remarkably well over long timescales.

5

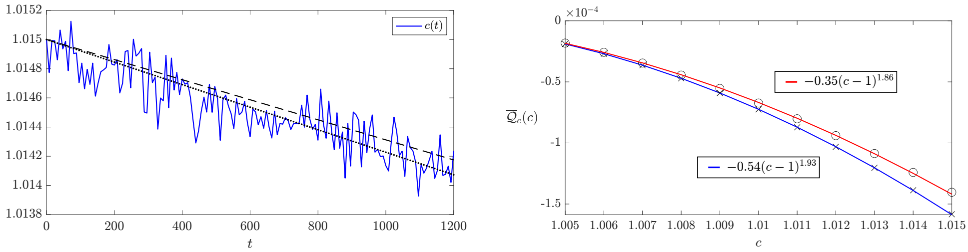


**Figure 5.9:** The left panel shows a realization of $c(t)$ and the asymptotic means $c_* + \mathcal{Q}_c(c_*)\sigma^2 t$ (dashed) and $c_* + \mathcal{Q}_c^{\mathrm{h}}(c_*)\sigma^2 t$ (dotted), via a numerical implementation of (5.67). For this realization, $\sigma = 0.07$, $c_* = 1.015$ and $\kappa(i) \in \mathrm{Unif}([-\sqrt{3}, \sqrt{3}])$. The right panel displays numerical evaluations of $\mathcal{Q}_c(c)$ (marked with $\circ$) and $\mathcal{Q}_c^{\mathrm{h}}(c)$ (marked with $\times$), for various values of $c$; see Section 5.11. The solid lines represent fitted power laws.

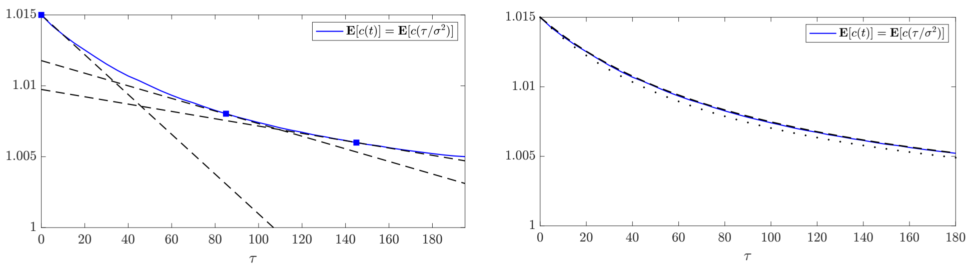


**Figure 5.10:** Figure 5.2 revisited. The solid curves represent the sample mean $\mathbf{E}[c(t)]$, with $t = \tau/\sigma^2$, computed over 100 realizations, for $\sigma = 0.1$, $c_* = 1.015$, and $\kappa(i)$ drawn from a uniform distribution on $[-\sqrt{3}, \sqrt{3}]$. The dashed lines in the left panel have slope $\mathcal{Q}_c(c)$ for $c \in \{1.015, 1.008, 1.006\}$, and are vertically shifted to intersect the $\mathbf{E}[c(t)]$ curve at the point where $\mathbf{E}[c(t)] = c$. The dashed and dotted curves in the right panel represent $c_{\lim}(\tau)$ and $c_{\lim}^{\mathrm{h}}(\tau)$, respectively, with $c_* = 1.015$. Observe that the dashed curve coincides with the solid curve.

## 5.6 Outlook



While the second-order expansion terms identified in Section 5.3 reveal the $\mathcal{O}(\sigma^2)$ attenuation affecting the propagation of solitary waves in (5.1), our future goal is to rigorously justify its effect on the exact amplitude $c(t)$. This requires, presumably among other things, asserting that the solitary waves remain coherent as they slowly decay in amplitude. That is, $\eta(t)$ must stay small in some (weighted) norm, at least for times proportional to $\sigma^{-2}$. The main difficulty towards establishing this via the linear stability tool Proposition 5.1.1 is to control the nonlinearity $\eta_r^2$ present in (5.17). Classically, this is controlled by estimating the weighted norm

$$\|e^{a(\cdot-\xi(t))}\eta_r^2(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})} \leq \|\eta_r(t)\|_{\ell^\infty(\mathbb{Z};\mathbb{R})}\|e^{a(\cdot-\xi(t))}\eta_r(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})}$$

$$\leq \|\eta_r(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})} \|e^{a(\cdot - \xi(t))}\eta_r(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})},$$

where the unweighted norm $\|\eta_r(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})}$ can in turn be understood via energy methods. However, since the tail $\eta_r(t)$ expands behind the soliton, its squared $\ell^2(\mathbb{Z};\mathbb{R})$-norm grows linearly with time. Indeed, the amplitude attenuation goes hand-in-hand with the formation of a tail. We anticipate that future work will focus on controlling the supremum norm $\|\eta_r(t)\|_{\ell^\infty(\mathbb{Z};\mathbb{R})}$. The right panel of Figure 5.7 suggests that the linear tail $\overline{\eta}(t) = \eta_1^{\mathrm{h}}(t)$ satisfies uniform bounds in space and time. The challenge will be to develop a pointwise analysis for the nonlinear tail $\eta(t)$.

We also briefly discussed the near-sonic behavior of the amplitude attenuation, i.e. the nature of the decay as $c \downarrow 1$. Future work can analyze the derivatives of $\mathcal{Q}_c(c)$ (Section 5.5) at $c = 1$ in more detail, to identify the lowest non-zero coefficient in an expansion

$$\mathcal{Q}_c(c) = k_1(c-1) + k_2(c-1)^2 + \dots \ .$$

This could confirm if the decay is of the form

$$c(t) - 1 \sim \frac{1}{\sigma^2 t}.$$

## 5.7 Statistics of leading-order phase and amplitude

Below, we prove elementary statistical properties of the leading-order phase and amplitude contributions $(\gamma_1(t), c_1(t))$ in (5.29) and (5.30).

*Proof of Proposition 5.3.1.* From the explicit solutions (5.33), (5.34), (5.35) and the assumption

$$\mathbb{E}[\kappa(i)] = 0, \quad \mathbb{E}[\kappa^2(i)] = 1, \quad i \in \mathbb{Z}$$

we immediately obtain

$$\mathbb{E}[c_1(t)] = c_*, \quad \mathbb{E}[\gamma_1(t)] = 0, \quad t \geq 0.$$

Turning to the covariance, we compute

$$\begin{aligned}&\mathbb{E}[c_1(t)c_1(s)]\\ &\quad = \frac{\alpha_0^{-2}(c_*)}{4c_*^2} \sum_{j\in\mathbb{Z}} \sum_{j'\in\mathbb{Z}} \mathbb{E}[\kappa(j)\kappa(j')](r_{c_*}^2(j - c_* t) - r_{c_*}^2(j))(r_{c_*}^2(j' - c_* s) - r_{c_*}^2(j')).\end{aligned}$$

Since the $\kappa(j)$'s are independent, this reduces to

$$\mathbb{E}[c_1(t)c_1(s)] = \frac{\alpha_0^{-2}(c_*)}{4c_*^2} \sum_{j\in\mathbb{Z}} (r_{c_*}^2(j) - r_{c_*}^2(j - c_* t))(r_{c_*}^2(j) - r_{c_*}^2(j - c_* s))$$

and the variance bound (5.36) then follows directly. Moving on to the variance of $\gamma_1(t) = \gamma_{1,I}(t) + \gamma_{1,II}(t)$, we obtain in an analogous way from (5.34) that

$$\mathbb{E}[\gamma_{1,I}^2(t)] \leq \frac{\alpha_0^{-4}(c_*)\alpha_1^2(c_*)}{c_*^2} \|r_{c_*}\|_{\ell^4(\mathbb{Z};\mathbb{R})}^4.$$

Furthermore, we obtain from (5.35) that

$$\begin{aligned}
\mathbb{E}[\gamma_{1,II}^2(t)] &= \frac{\alpha_0^{-2}(c_*)}{4} \sum_{j\in\mathbb{Z}} \Big( \int_0^t \partial_c r_{c_*}^2(j - c_* s)\mathrm{d}s \Big)^2 \\
&= \frac{\alpha_0^{-2}(c_*)}{4} \sum_{j\in\mathbb{Z}} \Big( \int_{j-c_*t}^j \partial_c r_{c_*}^2(y)\mathrm{d}y \Big)^2 \\
&\leq C\frac{\alpha_0^{-2}(c_*)}{4} \sum_{j\in\mathbb{Z}} \Big( \int_{j-c_*t}^j e^{-\beta|y|}\mathrm{d}y \Big)^2.
\end{aligned}$$

Here, we used that $\partial_c r_{c_*}^2$ is exponentially localized [41, Proposition 5.5]:

$$\partial_c r_{c_*}^2(x) \leq Ce^{-\beta|x|}$$

for some $C, \beta > 0$. Note that

$$\int_{j-c_*t}^j e^{-\beta|y|}\,\mathrm{d}y = \begin{cases} \beta^{-1}\left(e^{\beta j} - e^{\beta(j-c_*t)}\right), & \text{if } j \leq 0, \\ \beta^{-1}\left(2 - e^{\beta(j-c_*t)} - e^{-\beta j}\right), & \text{if } 0 < j \leq c_*t, \\ \beta^{-1}\left(e^{-\beta(j-c_*t)} - e^{-\beta j}\right), & \text{if } j > c_*t. \end{cases}$$

Hence,

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \Big( \int_{j-c_*t}^j e^{-\beta|y|}\mathrm{d}y \Big)^2 &\leq 2\beta^{-2}\sum_{j\leq 0} e^{2\beta j} + 4c_*t + 2\beta^{-2}\sum_{j>ct} e^{-2\beta j} \\
&\leq \frac{4\beta^{-2}}{1-e^{-2\beta}} + 4c_*t,
\end{aligned}$$

and

$$\mathbb{E}[\gamma_{1,II}^2(t)] \leq C\frac{\alpha_0^{-2}(c_*)}{4}\Big(\frac{4\beta^{-2}}{1-e^{-2\beta}} + 4c_*t\Big).$$

The result now follows by estimating

$$\mathbb{E}[\gamma_1^2(t)] = \mathbb{E}[\gamma_{1,I}^2(t)] + 2\mathbb{E}[\gamma_{1,I}(t)\gamma_{1,II}(t)] + \mathbb{E}[\gamma_{1,II}^2(t)] \leq 2\mathbb{E}[\gamma_{1,I}^2(t)] + \mathbb{E}[\gamma_{1,II}^2(t)]. \quad \square$$

## 5.8 Kernel representations

Here, we derive the explicit kernel formula (5.53) and prove Lemma 5.4.4 regarding the Green's function associated to general time/shift invariant linear operators.

**Lemma 5.8.1.** *Let $u_0, v_0 \in \ell^2(\mathbb{Z};\mathbb{R})$. The linear system*

$$\begin{bmatrix} \dot{u}(t,j) \\ \dot{v}(t,j) \end{bmatrix} = \mathcal{J} \begin{bmatrix} u(t,j) \\ v(t,j) \end{bmatrix}, \quad \textit{with} \quad \begin{bmatrix} u(0,j) \\ v(0,j) \end{bmatrix} = \begin{bmatrix} u_0(j) \\ v_0(j) \end{bmatrix}$$

*is solved by*

$$\begin{bmatrix} u(t,j) \\ v(t,j) \end{bmatrix} = \sum_{k\in\mathbb{Z}} \Phi(j-k,t) \begin{bmatrix} u_0(k) \\ v_0(k) \end{bmatrix}, \quad j \in \mathbb{Z},\ t \in \mathbb{R}.$$

*Proof.* We note that the Bessel functions satisfy $J_n(0) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ and meet the recurrence relation

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x), \tag{5.70}$$

see for instance [101, §2.12]. Consider initial conditions given by the basis vectors $(u_0,\ v_0)^\top = (\delta_0,\ 0)^\top$ and $(u_0,\ v_0)^\top = (0,\ \delta_0)^\top$. These give rise to the fundamental solutions

$$\phi(j,t) = \begin{bmatrix} J_{2j}(2t) \\ -J_{2j-1}(2t) \end{bmatrix} \quad \text{and} \quad \psi(j,t) = \begin{bmatrix} -J_{2j+1}(2t) \\ J_{2j}(2t) \end{bmatrix}$$

for the discrete wave operator $\mathcal{J}$. Indeed, through the recurrence relation (5.70) one verifies that

$$\dot{\phi}(j,t) = \mathcal{J}\phi(j,t), \quad \text{with} \quad \phi(j,0) = \begin{bmatrix} \delta_0(j) \\ 0 \end{bmatrix}$$

and

$$\dot{\psi}(j,t) = \mathcal{J}\psi(j,t), \quad \text{with} \quad \psi(j,0) = \begin{bmatrix} 0 \\ \delta_0(j) \end{bmatrix}.$$

For arbitrary initial conditions $(u_0,\ v_0)^\top \in \ell^2(\mathbb{Z};\mathbb{R}^2)$, we obtain the solution (5.52) through a convolution with the fundamental solutions. □

We then move on to the proof of Lemma 5.4.4. We refer to [8, Theorem 4.2] for a constructive proof in the context of the discretized conservation laws.

*Proof of Lemma 5.4.4.* In terms of the shift-operation $S\eta = \eta(\cdot + 1)$, (5.43) reads

$$S^d \mathcal{A}_{c_* t, c_*} = \mathcal{A}_{c_* t - d, c_*} S^d, \quad d \in \mathbb{Z}.$$

We claim that this translates to the following property of the associated evolution family:

$$S^d U_{c_*}(t,s) = U_{c_*}(t-d/c_*, s-d/c_*)S^d, \quad d\in\mathbb{Z}. \tag{5.71}$$

We prove the case $d=1$ by first differentiating the left-hand side:

$$\partial_t(SU_{c_*}(t,s)) = S\mathcal{A}_{c_*t,c_*}U_{c_*}(t,s) = \mathcal{A}_{c_*t-1,c_*}SU_{c_*}(t,s).$$

On the other hand:

$$\partial_t(U_{c_*}(t-1/c_*, s-1/c_*)S) = \mathcal{A}_{c_*t-1,c_*}U_{c_*}(t-1/c_*, s-1/c_*)S.$$

Hence, both terms satisfy the (operator-valued) ODE

$$\dot{x}(t) = \mathcal{A}_{c_*t-1,c_*}x(t) \quad \text{with} \quad x(s) = S.$$

Next, we observe that for each $t\geq s$, the bounded operator $U_{c_*}(t,s)$ admits a kernel representation

$$U_{c_*}(t,s)\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(j) = \sum_k \Phi_{c_*}(t,s;j,k)\begin{bmatrix}\alpha(k)\\ \beta(k)\end{bmatrix}.$$

Identity (5.71) has immediate consequences for the structure of the kernel $\Phi_{c_*}$:

$$\begin{aligned}\sum_k \Phi_{c_*}(t,s;j+d,k)\begin{bmatrix}\alpha(k)\\ \beta(k)\end{bmatrix} &= \left(U_{c_*}(t-d/c_*, s-d/c_*)\begin{bmatrix}\alpha(\cdot+d)\\ \beta(\cdot+d)\end{bmatrix}\right)(j)\\ &= \sum_k \Phi_{c_*}(t-d/c_*, s-d/c_*;j,k)\begin{bmatrix}\alpha(k+d)\\ \beta(k+d)\end{bmatrix}\\ &= \sum_{k'} \Phi_{c_*}(t-d/c_*, s-d/c_*;j,k'-d)\begin{bmatrix}\alpha(k')\\ \beta(k')\end{bmatrix}\end{aligned}$$

and consequently

$$\Phi_{c_*}(t,s;j,k) = \Phi_{c_*}(t-d/c_*, s-d/c_*;j-d,k-d), \quad d\in\mathbb{Z}. \tag{5.72}$$

Through a basis transformation of the arguments $(t,s;j,k)$, we obtain a kernel $\tilde{\mathcal{G}}_{c_*}$ which satisfies

$$\Phi_{c_*}(t,s;j,k) = \tilde{\mathcal{G}}_{c_*}(t-s, j-c_*t, k-c_*s, k).$$

In view of (5.72), $\tilde{\mathcal{G}}_{c_*}$ is constant in its last variable. It follows that the kernel only depends on three effective variables:

$$\Phi_{c_*}(t,s;j,k) = \mathcal{G}_{c_*}(t-s, j-c_*t, k-c_*s). \qquad \square$$

## 5.9 Continuity of (bi)linear maps

Here, we prove that the linear map $M^{1,1}(\xi,c):\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)\to\mathbb{R}^2$ defined via

$$M^{1,1}(\xi,c)[\alpha]=\sum_{m\in\mathbb{Z}}\begin{bmatrix}\Gamma^{1,1}(\xi,c)\\ C^{1,1}(\xi,c)\end{bmatrix}[\delta_m,\alpha(\cdot,m)],$$

and the bilinear map $M^{0,2}(\xi,c):\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)\times\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)\to\mathbb{R}^2$ defined by

$$M^{0,2}(\xi,c)[\alpha,\beta]=\sum_{m\in\mathbb{Z}}\begin{bmatrix}\Gamma^{0,2}(\xi,c)\\ C^{0,2}(\xi,c)\end{bmatrix}[\alpha_r(\cdot,m),\beta_r(\cdot,m)],$$

used in Section 5.5, are continuous on exponentially weighted spaces

$$\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2):=\left\{f:\mathbb{Z}^2\to\mathbb{R}^2\;\middle|\;\|f\|_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)}<\infty\right\},$$

with norm

$$\|f\|^2_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)}=\sum_{m\in\mathbb{Z}}\|e^{a\cdot}f(\cdot,m)\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^2)}.$$

We recall that the weight $a>0$ was introduced in Proposition 5.1.1, and that $\Gamma^{1,1},\Gamma^{0,2},C^{1,1}$ and $C^{0,2}$ are expansion terms of the modulation system, as introduced in Section 5.2.

**Lemma 5.9.1.** *For each $p\in[0,1)$ and $c\in(c_-,c_+)$, the linear operator $M^{1,1}(p,c)$ is bounded from $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$ to $\mathbb{R}^2$.*

*Proof.* Let $p\in[0,1),c\in(c_-,c_+)$, and $\alpha=(\alpha_r,\ \alpha_p)^\top\in\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$. From (5.60), we get

$$\left|M^{1,1}(p,c)[\alpha]\right|\le\sum_{m\in\mathbb{Z}}\left|\begin{bmatrix}\Gamma^{1,1}(\xi,c)\\ C^{1,1}(\xi,c)\end{bmatrix}[\delta_m,\alpha(\cdot,m)]\right|\le S_1+S_2,$$

with

$$S_1=\sum_{m\in\mathbb{Z}}\left|\begin{bmatrix}\Gamma^{0,2}(p,c)\\ C^{0,2}(p,c)\end{bmatrix}[\delta_m,\alpha_r(\cdot,m)]\right|,$$

and

$$S_2=\sum_{m\in\mathbb{Z}}\left|A^{-1}(c)B(p,c,\alpha(\cdot,m))\begin{bmatrix}\Gamma^{1,0}(p,c)\\ C^{1,0}(p,c)\end{bmatrix}[\delta_m]\right|.$$

We estimate

$$\left|\begin{bmatrix}\Gamma^{0,2}(p,c)\\ C^{0,2}(p,c)\end{bmatrix}[\delta_m,\alpha_r(\cdot,m)]\right|\le C\Big(|\partial_\xi r_c(m-p)\alpha_r(m,m)|+|\partial_c r_c(m-p)\alpha_r(m,m)|\Big),$$

so that

$$\begin{aligned} S_1 &\le C \sum_{m\in\mathbb{Z}} \big(|\partial_\xi r_c(m-p) + |\partial_c r_c(m-p)|\big)|\alpha_r(m,m)| \\ &\le C\Big(\sum_{m\in\mathbb{Z}} e^{2am}|\alpha_r(m,m)|^2\Big)^{1/2}\Big(\sum_{m\in\mathbb{Z}} e^{-2am}\big(|\partial_\xi r_c(m-p) + |\partial_c r_c(m-p)|\big)^2\Big)^{1/2} \\ &\le \tilde{C}\Big(\sum_{m\in\mathbb{Z}} \|e^{a\cdot}\alpha_r(\cdot,m)\|^2_{\ell^2(\mathbb{Z};\mathbb{R})}\Big)^{1/2} \le \tilde{C}\|\alpha\|_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)}. \end{aligned}$$

We proceed with

$$S_2 \le \Big(\sum_{m\in\mathbb{Z}} \Big|A^{-1}(c)B(p,c,\alpha(\cdot,m))\Big|^2\Big)^{1/2}\Big(\sum_{m\in\mathbb{Z}} \Big|\begin{bmatrix}\Gamma^{1,0}(p,c)\\ C^{1,0}(p,c)\end{bmatrix}[\delta_m]\Big|^2\Big)^{1/2},$$

where

$$\sum_{m\in\mathbb{Z}} \Big|\begin{bmatrix}\Gamma^{1,0}(p,c)\\ C^{1,0}(p,c)\end{bmatrix}[\delta_m]\Big|^2 \le C\sum_{m\in\mathbb{Z}}\Big(|\partial_\xi r_c^2(m-p)| + |\partial_c r_c^2(m-p)|\Big)^2 < \infty.$$

The result then follows by estimating

$$\Big|A^{-1}(c)B(p,c,\alpha(\cdot,m))\Big| \le CW\|e^{a\cdot}(\alpha(\cdot,m))\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)},$$

with

$$\begin{aligned} W := {}& \|e^{-a\cdot}\mathcal{J}^{-1}\partial^2_{\xi\xi}\phi_c(\cdot-p)\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} + 2\|e^{-a\cdot}\mathcal{J}^{-1}\partial^2_{\xi c}\phi_c(\cdot-p)\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)} \\ &+ \|e^{-a\cdot}\mathcal{J}^{-1}\partial^2_{cc}\phi_c(\cdot-p)\|_{\ell^2(\mathbb{Z};\mathbb{R}^2)}. \end{aligned}$$

In particular

$$\sum_{m\in\mathbb{Z}} \Big|A^{-1}(c)B(p,c,\alpha(\cdot,m))\Big|^2 \le C^2W^2\sum_{m\in\mathbb{Z}}\|e^{a\cdot}(\alpha(\cdot,m))\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^2)} = C^2W^2\|\alpha\|^2_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)}.$$

□

We proceed with the bilinear map $M^{0,2}$.

**Lemma 5.9.2.** *For each $p \in [0,1)$ and $c \in (c_-, c_+)$, the bilinear form $M^{0,2}(p,c)$ is bounded from $\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2) \times \ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$ to $\mathbb{R}^2$. In particular, there exists a constant $C > 0$ such that*

$$|M^{0,2}(p,c)[\alpha,\beta]| \le C\|\alpha\|_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)}\|\beta\|_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)},$$

*for all $\alpha, \beta \in \ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)$.*

*Proof.* Let $p \in [0,1)$ and $c \in (c_-, c_+)$. Let furthermore

$$\alpha = (\alpha_r,\ \alpha_p)^\top, \beta = (\beta_r,\ \beta_p)^\top \in \ell^2_a(\mathbb{Z}^2; \mathbb{R}^2).$$

From (5.61), we estimate

$$|M^{0,2}(p,c)[\alpha,\beta]| \leq \sum_{m\in\mathbb{Z}} \Big( \Big| \begin{bmatrix} \Gamma^{0,2}(p,c) \\ C^{0,2}(p,c) \end{bmatrix} [\alpha_r(\cdot,m), \beta_r(\cdot,m)] \Big| \Big).$$

For all $m \in \mathbb{Z}$, we have

$$\begin{aligned} \Big| \begin{bmatrix} \Gamma^{0,2}(p,c) \\ C^{0,2}(p,c) \end{bmatrix} [\alpha_r(\cdot,m), \beta_r(\cdot,m)] \Big| &\leq C |\langle \partial_\xi r_c(\cdot - p), \alpha_r(\cdot,m)\beta_r(\cdot,m)\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}| \\ &\quad + C |\langle \partial_c r_c(\cdot - p), \alpha_r(\cdot,m)\beta_r(\cdot,m)\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}| \\ &\leq C \sup_{j\in\mathbb{Z}} e^{-2aj} \big( |\partial_\xi r_c(j-p)| + |\partial_c r_c(j-p)| \big) \\ &\quad \times |\langle e^{a\cdot}\alpha_r(\cdot,m), e^{a\cdot}\beta_r(\cdot,m)\rangle_{\ell^2(\mathbb{Z};\mathbb{R})}| \\ &\leq \tilde{C} \|e^{a\cdot}\alpha_r(\cdot,m)\|_{\ell^2(\mathbb{Z};\mathbb{R})} \|e^{a\cdot}\beta_r(\cdot,m)\|_{\ell^2(\mathbb{Z};\mathbb{R})}, \end{aligned}$$

so that

$$\begin{aligned} |M^{0,2}(p,c)[\alpha,\beta]| &\leq \tilde{C} \sum_{m\in\mathbb{Z}} \|e^{a\cdot}\alpha_r(\cdot,m)\|_{\ell^2(\mathbb{Z};\mathbb{R})} \|e^{a\cdot}\beta_r(\cdot,m)\|_{\ell^2(\mathbb{Z};\mathbb{R})} \\ &\leq \tilde{C} \Big( \sum_{m\in\mathbb{Z}} \|e^{a\cdot}\alpha_r(\cdot,m)\|^2_{\ell^2(\mathbb{Z};\mathbb{R})} \Big)^{1/2} \Big( \sum_{m\in\mathbb{Z}} \|e^{a\cdot}\beta_r(\cdot,m)\|^2_{\ell^2(\mathbb{Z};\mathbb{R})} \Big)^{1/2} \\ &\leq \tilde{C} \|\alpha\|_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)} \|\beta\|_{\ell^2_a(\mathbb{Z}^2;\mathbb{R}^2)}. \end{aligned}$$

□

## 5.10 Correlations

Here, we prove Lemma 5.5.1 regarding the correlations $M^{1,0}_{I}, \ldots, M^{1,0}_{IV}$ introduced in (5.62)–(5.65). By Hypothesis 1, we have

$$\mathbb{E}[\kappa(j)] = 0 \quad \text{and} \quad \mathbb{E}[\kappa(i)\kappa(j)] = \delta_{ij}, \quad i,j \in \mathbb{Z}.$$

This allows us to explicitly compute

$$\begin{aligned} M^{1,0}_1(\xi,c) &= \frac{\alpha_0^{-2}(c)\alpha_1(c)}{4c} \sum_{j\in\mathbb{Z}} [r_c^2(j-\xi) - r_c^2(j)] A^{-1}(c) \begin{bmatrix} (\partial^2_{\xi\xi} r_c^2)(j-\xi) \\ (\partial^2_{\xi c} r_c^2)(j-\xi) \end{bmatrix}, \\ M^{1,0}_2(\xi,c) &= -\frac{\alpha_0^{-1}(c)}{4c} \sum_{j\in\mathbb{Z}} \int_0^\xi \partial_c r_c^2(j-s) \mathrm{d}s A^{-1}(c) \begin{bmatrix} (\partial^2_{\xi\xi} r_c^2)(j-\xi) \\ (\partial^2_{\xi c} r_c^2)(j-\xi) \end{bmatrix}, \end{aligned}$$

$$M_{\mathit{III}}^{1,0}(\xi,c) = -\frac{\alpha_0^{-1}(c)}{4c^2}\sum_{j\in\mathbb{Z}}\Big(\int_0^{\xi} r_c^2(j-s)\,\mathrm{d}s - \xi r_c^2(j)\Big)A^{-1}(c)\begin{bmatrix}(\partial^2_{\xi\xi}r_c^2)(j-\xi)\\(\partial^2_{\xi c}r_c^2)(j-\xi)\end{bmatrix},$$

together with

$$\begin{aligned}M_{\mathit{IV}}^{1,0}(\xi,c) &= \frac{\alpha_0^{-1}(c)}{4c}\sum_{j\in\mathbb{Z}}[r_c^2(j-\xi)-r_c^2(j)]\partial_c A^{-1}(c)\begin{bmatrix}(\partial_{\xi}r_c^2)(j-\xi)\\(\partial_{c}r_c^2)(j-\xi)\end{bmatrix}\\&\quad+\frac{\alpha_0^{-1}(c)}{4c}\sum_{j\in\mathbb{Z}}[r_c^2(j-\xi)-r_c^2(j)]A^{-1}(c)\begin{bmatrix}(\partial^2_{c\xi}r_c^2)(j-\xi)\\(\partial^2_{cc}r_c^2)(j-\xi)\end{bmatrix}.\end{aligned}$$

*Proof of Lemma 5.5.1.* We prove that

$$M_{\mathit{III}}^{1,0}(\xi,c) = M_{\mathit{III},\mathrm{per}}^{1,0}(\xi,c) + M_{\mathit{III},\mathrm{trans}}^{1,0}(\xi,c),$$

with $M_{\mathit{III},\mathrm{per}}^{1,0}(\xi,c)$ 1-periodic in $\xi$ and $M_{\mathit{III},\mathrm{trans}}^{1,0}(\xi,c)$ exponentially decaying in $|\xi|$. The proof for the remaining mappings is then analogous. Decomposing

$$\mathbb{R}\ni\xi = n+p,\quad n\in\mathbb{Z},\ p\in[0,1),$$

we observe that

$$\begin{aligned}M_{\mathit{III}}^{1,0}(\xi,c) &= -\frac{\alpha_0^{-1}(c)}{4c^2}\sum_{j\in\mathbb{Z}}\int_0^{n+p} r_c^2(j+n+p-s-p)\,\mathrm{d}sA^{-1}(c)\begin{bmatrix}(\partial^2_{\xi\xi}r_c^2)(j-p)\\(\partial^2_{\xi c}r_c^2)(j-p)\end{bmatrix}\\&\quad+\frac{\alpha_0^{-1}(c)}{4c^2}\sum_{j\in\mathbb{Z}}(j+n)r_c^2(j+n)A^{-1}(c)\begin{bmatrix}(\partial^2_{\xi\xi}r_c^2)(j-p)\\(\partial^2_{\xi c}r_c^2)(j-p)\end{bmatrix}\\&= -\frac{\alpha_0^{-1}(c)}{4c^2}\sum_{j\in\mathbb{Z}}\int_0^{n+p} r_c^2(j-p+\tau)\,\mathrm{d}\tau A^{-1}(c)\begin{bmatrix}(\partial^2_{\xi\xi}r_c^2)(j-p)\\(\partial^2_{\xi c}r_c^2)(j-p)\end{bmatrix}\\&\quad+\frac{\alpha_0^{-1}(c)}{4c^2}\sum_{j\in\mathbb{Z}}(j+n)r_c^2(j+n)A^{-1}(c)\begin{bmatrix}(\partial^2_{\xi\xi}r_c^2)(j-p)\\(\partial^2_{\xi c}r_c^2)(j-p)\end{bmatrix}.\end{aligned}$$

Hence, we identify

$$M_{\mathit{III},\mathrm{per}}^{1,0}(\xi,c) = -\frac{\alpha_0^{-1}(c)}{4c^2}\sum_{j\in\mathbb{Z}}\int_0^{\infty} r_c^2(j-p+\tau)\,\mathrm{d}\tau A^{-1}(c)\begin{bmatrix}(\partial^2_{\xi\xi}r_c^2)(j-p)\\(\partial^2_{\xi c}r_c^2)(j-p)\end{bmatrix},$$

and

$$\begin{aligned}M_{\mathit{III},\mathrm{trans}}^{1,0}(\xi,c) &= \frac{\alpha_0^{-1}(c)}{4c^2}\sum_{j\in\mathbb{Z}}\int_{n+p}^{\infty} r_c^2(j-p+\tau)\,\mathrm{d}\tau A^{-1}(c)\begin{bmatrix}(\partial^2_{\xi\xi}r_c^2)(j-p)\\(\partial^2_{\xi c}r_c^2)(j-p)\end{bmatrix}\\&\quad+\frac{\alpha_0^{-1}(c)}{4c^2}\sum_{j\in\mathbb{Z}}(j+n)r_c^2(j+n)A^{-1}(c)\begin{bmatrix}(\partial^2_{\xi\xi}r_c^2)(j-p)\\(\partial^2_{\xi c}r_c^2)(j-p)\end{bmatrix},\end{aligned}$$

which sum to $M_{I\!I\!I}^{1,0}(\xi, c)$. The component $M_{I\!I\!I,\mathrm{per}}^{1,0}(\xi, c)$ is clearly periodic with period 1. It remains to be shown that $M_{I\!I\!I,\mathrm{trans}}^{1,0}(\xi, c)$ is exponentially decaying in $|\xi| = |n + p|$. The exponential decay of the wave profiles and their derivatives [41, Proposition 5.5] provides

$$\sum_{j\in\mathbb{Z}}(j+n)r_c^2(j+n)\left|A^{-1}(c)\begin{bmatrix}(\partial_{\xi\xi}^2 r_c^2)(j-p)\\(\partial_{\xi c}^2 r_c^2)(j-p)\end{bmatrix}\right| \le C\sum_{j\in\mathbb{Z}} e^{-\beta|j+n|}e^{-\beta|j-p|} \le \tilde{C}e^{-\tilde{\beta}|n|},$$

for some $C, \tilde{C}, \beta > 0$. For the integral component, we similarly find

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\int_0^{n+p} r_c^2(j-p+\tau)\,\mathrm{d}\tau\left|A^{-1}(c)\begin{bmatrix}(\partial_{\xi\xi}^2 r_c^2)(j-p)\\(\partial_{\xi c}^2 r_c^2)(j-p)\end{bmatrix}\right| &\le \sum_{j\in\mathbb{Z}}\int_0^{n+p} e^{-\beta|j-p+\tau|}e^{-\beta|j-p|}\,\mathrm{d}\tau\\ &\le \tilde{C}e^{-\tilde{\beta}|n|},\end{aligned}$$

upon increasing $\tilde{C}$ if necessary. □

## 5.11 Numerical Schemes

Our direct simulations of the FPUT system (5.1) were carried out using a fourth-order Runge–Kutta scheme. We extracted the modulation parameters from the numerical solution by enforcing the orthogonality conditions (5.13) through a nonlinear solver. Specifically, given the numerical solution $u(t) = (r(t), p(t))^\top$ to (5.1) at time $t$, we determined parameters $c_{\mathrm{fit}}(t)$ and $\xi_{\mathrm{fit}}(t)$ from the equations

$$\Omega\Big(\partial_\xi\phi_{c_{\mathrm{fit}}(t)}(\cdot-\xi_{\mathrm{fit}}(t)),\, u-\phi_{c_{\mathrm{fit}}(t)}(\cdot-\xi_{\mathrm{fit}}(t))\Big) = 0,$$

and

$$\Omega\Big(\partial_c\phi_{c_{\mathrm{fit}}(t)}(\cdot-\xi_{\mathrm{fit}}(t)),\, u-\phi_{c_{\mathrm{fit}}(t)}(\cdot-\xi_{\mathrm{fit}}(t))\Big) = 0.$$

We computed the wave profiles and their derivatives by solving the traveling wave equation (5.5) following the approach outlined in [37, Section 3.1]. The phase parameter $\gamma_{\mathrm{fit}}$ was then obtained via the numerical integration

$$\gamma_{\mathrm{fit}}(t) \;=\; \xi_{\mathrm{fit}}(t) - \int_0^t c_{\mathrm{fit}}(s)\,\mathrm{d}s.$$

For comparison, the theoretically equivalent parameters $c_{\mathrm{mod}}(t)$ and $\gamma_{\mathrm{mod}}(t)$ were computed by numerically integrating (5.15)–(5.17), again with a fourth-order Runge–Kutta method. Figure 5.11 demonstrates the close agreement between the "fitted" and "modulation" parameters, with discrepancies limited to negligible truncation errors. Throughout the numerical figures in this chapter, we have used $c(t) = c_{\mathrm{fit}}(t)$ and $\gamma(t) = \gamma_{\mathrm{fit}}(t)$.

For the numerical evaluation of the asymptotic tail $\overline{\eta}^\infty$ in Figure 5.7, based

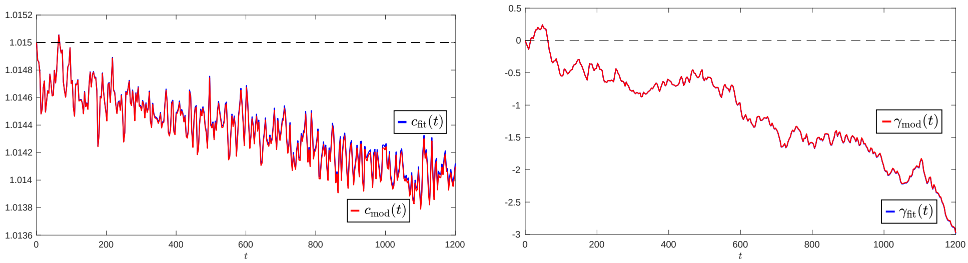


**Figure 5.11:** Comparison of $c_{\text{fit}}(t)$ with the theoretically equivalent parameter $c_{\text{mod}}(t)$ (left) and comparison of $\gamma_{\text{fit}}(t)$ with $c_{\text{mod}}(t)$ (right).

on $\overline{\eta} = \eta_1^{\text{h}}$, we computed the corresponding asymptotic response function $\overline{R}^{\infty}$ directly from a discretization of (5.54) with the explicit kernel representation from Lemma 5.8.1. Additionally, we computed the response function $\overline{R}^{\infty}$ based on $\overline{\eta} = \eta_1$ using the true soliton linearization $\mathcal{A}_{c_* t, c_*} = \mathcal{L}_{c_* t, c_*}$, performing a numerical integration of (5.46) until satisfactory convergence was achieved. We then used the resulting $\overline{R}^{\infty}$ functions to determine the asymptotic attenuation rate $\mathcal{Q}_c(c_*)$ according to Corollary 5.5.3 and (5.67), in which the periodic quantity $M^{1,0}_{\text{per}}(p, c_*)$ was evaluated by computing $M^{1,0}(p+n, c_*)$ for sufficiently large $n$.

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# Samenvatting

Dit proefschrift bestudeert het gedrag van golven onder invloed van verstoringen uit hun omgeving. Hoewel we ons richten op de *wiskundige* beschrijving van golven en hun theoretische gedrag, vindt deze beschrijving wel degelijk haar oorsprong in de realiteit. Men kan hierbij denken aan golven op het oppervlak van laagstaand water: golven die bijvoorbeeld ontstaan door een passerende boot op een gracht, en die ook nadat de boot verdwenen is nog lange tijd hun vorm behouden—tot steeds verder vermengen met de omgevende golven wanneer ze oneffenheden tegenkomen. De kracht van een wiskundige benadering is dat de inzichten niet beperkt blijven tot één specifieke situatie. Ze kunnen bovendien helder maken wat in experimenten lastig te onderscheiden is.

In dit proefschrift onderzoeken we hoe golven in dispersieve media, zoals water, in hoogte veranderen door realistische imperfecties. We modelleren die imperfecties als willekeurige verstoringen (ruis) en focussen hierbij op een bekend model uit de mathematische fysica: de Korteweg–de Vries vergelijking. Deze vergelijking is al sinds de 19e eeuw bekend vanwege de golfverschijnselen die zij beschrijft, en vormt nog altijd een basis voor moderne analyse van golven. Een bijzonder kenmerk van deze vergelijking is dat zij oplossingen heeft die golven voorspellen van verschillende hoogtes (amplitudes), waarbij hogere golven sneller reizen. Dit koppelt amplitude en snelheid op een directe manier—een interessant uitgangspunt voor het bestuderen van golfverstoring in een imperfecte omgeving.

Hoofdstuk 2 richt zich op de KdV-vergelijking met zogeheten multiplicatieve ruis: een willekeurige verstoring die direct invloed heeft op de golf. Hierdoor ondergaan de golven in het KdV-systeem grote veranderingen in hoogte en snelheid. De sleutel hierin is de rol van de energie van het systeem, die door de ruis voortdurend op en neer schommelt.

We introduceren in dit hoofdstuk een techniek om de positie en amplitude van zulke golven te traceren en te voorspellen. Deze “effectieve” positie en amplitude zijn kansprocessen: de ruis veroorzaakt immers willekeurige fluctuaties. Het idee is dat we steeds het golfprofiel zoeken dat het best past op de verstoorde golf. Perfect past het nooit; de verstoring vormt een “staart” achter de golf, vergelijkbaar met het kielzog van een boot. Onze techniek levert wiskundige vergelijkingen op die het verloop van amplitude, positie en staart beschrijven. Exact oplossen is onhaalbaar, zoals vaak bij niet-lineaire systemen, maar via computersimulaties en systematische benaderingen krijgen we inzicht in het voorspelde gedrag. Zo laten we zien dat een veelvoorkomende vorm van ruis gemiddeld een langzame gemiddelde stijging van de amplitude veroorzaakt. Onze analyse voorspelt deze groei nauwkeurig, en deze voorspelling komt overeen met de simulaties.

In Hoofdstuk 3 onderzoeken we een meer fundamentele vraag: overleeft een golf de willekeurige verstoring überhaupt wel? Of breekt hij uiteindelijk, zoals een golf in de branding? En zo ja, na hoeveel tijd? Om deze vraag behapbaar te maken, bekijken we eerst een eenvoudiger situatie: in plaats van willekeurige ruis bestuderen we een vooraf vastgelegde, niet-willekeurige (deterministische) energieverandering in het medium. Dit vereenvoudigt de analyse aanzienlijk. We laten zien dat zolang toevoer of afvoer van energie langzaam genoeg gebeurt, golven langzame, maar grote amplitude veranderingen kunnen doorstaan zonder hun coherente vorm te verliezen. Dit inzicht vormt het fundament voor het bestuderen van willekeurige verstoringen.

In Hoofdstuk 4 keren we terug naar het willekeurige model van Hoofdstuk 2. Gebaseerd op de resultaten uit Hoofdstuk 3 laten we zien dat golven ook onder invloed van willekeurige ruis grote amplitudevariaties kunnen verdragen terwijl ze coherent blijven. Hiermee tonen we bovendien aan dat onze beschrijving van het effectieve amplitudeverloop uit Hoofdstuk 2 wiskundig valide is. De gebruikte technieken suggereren zelfs dat de golven nóg langer coherent blijven dan we strikt kunnen aantonen. Het kielzog strekt ver achter een verstoorde golf, maar blijft laag. Dat laatste is lastig wiskundig te bewijzen, en biedt ruimte voor vervolgonderzoek.

In Hoofdstuk 5 verlaten we het KdV-model en richten we ons op een verwant, maar discreet systeem: de Fermi–Pasta–Ulam–Tsingou (FPUT) ketting, een model van massa's die via veren met elkaar zijn verbonden. Een soort oneindige ketting van kralen verbonden met elastiek. Het model heeft concrete toepassingen in de beschrijving van kristalroosters op moleculair niveau, waar atomen kunnen worden gemodelleerd als massa's die via veren met elkaar zijn verbonden. Ook in deze discrete context ontstaan golfachtige oplossingen: wanneer de massa's op een geschikte manier uit evenwicht worden gebracht, verplaatst de resulterende uitwijking zich met constante snelheid door de ketting. De connectie met de KdV-vergelijking is sterk: de volledige dynamica van de FPUT-ketting wordt door de KdV-vergelijking benaderd.

We bestuderen wat er gebeurt als de veren in de ketting niet identiek zijn, maar kleine willekeurige verschillen hebben in veerkracht. Ook hier onderzoeken we dus het effect van realistische oneffenheden op de voortplanting van golven. De verstoring heeft een langzaam maar duidelijk effect: de golven nemen af in hoogte en verliezen energie via een staart (kielzog) die achter de golf ontstaat. We berekenen expliciet hoe sterk deze amplitude afname is; de technieken uit de eerdere hoofdstukken blijken hiervoor uitstekend inzetbaar.

Samen laten deze hoofdstukken zien hoe kleine imperfecties — zowel willekeurig als deterministisch van aard — een langzame maar grote invloed kunnen hebben op de snelheid en amplitude van golven in dispersieve systemen. De in dit proefschrift ontwikkelde technieken bieden nieuwe manieren om zulke veranderingen te voorspellen. Ze laten bovendien zien dat deze golven verrassend robuust zijn en zelfs in realistische imperfecte omgevingen lange tijd kunnen overleven.

# Summary

This dissertation studies the behavior of waves under the influence of disturbances from their environment. Although we focus on the *mathematical* description of waves and their theoretical behavior, this description does indeed have its origins in reality. One can think of waves on the surface of shallow water: waves that are created, for example, by a passing boat on a canal, and which retain their shape for a long time even after the boat has disappeared—until they gradually merge with the surrounding waves as they meet irregularities. The power of a mathematical approach is that the insights are not limited to one specific situation. They can also clarify what is difficult to distinguish in experiments.

In this thesis, we investigate how waves in dispersive media, such as water, change in height due to realistic imperfections. We model these imperfections as random disturbances (noise) and focus on a well-known model from mathematical physics: the Korteweg–de Vries equation. This equation has been known since the 19th century for the wave phenomena it describes, and still forms the basis for modern wave analysis. A special feature of this equation is that it has solutions that predict waves of different heights (amplitudes), with higher waves traveling faster. This links amplitude and speed in a direct way—an interesting starting point for studying wave disturbance in an imperfect environment.

Chapter 2 focuses on the KdV equation with so-called multiplicative noise: a random disturbance that directly affects the wave. As a result, the waves in the KdV system undergo major changes in height and speed. The key here is the role of the energy of the system, which constantly fluctuates up and down due to the noise.

In this chapter, we introduce a technique for tracking and predicting the position and amplitude of such waves. These "effective" position and amplitude are random processes: after all, the noise causes random fluctuations. The idea is that we always look for the wave profile that best fits the disturbed wave. It never fits perfectly; the disturbance forms a "tail" behind the wave, similar to the wake of a boat. Our technique yields mathematical equations that describe the course of amplitude, position, and tail. Exact solutions are unfeasible, as is often the case with nonlinear systems, but computer simulations and systematic approaches provide insight into the predicted behavior. For example, we show that a common form of noise causes a slow average increase in amplitude. Our analysis accurately predicts this growth, and this prediction corresponds with the simulations.

In Chapter 3, we examine a more fundamental question: does a wave survive

random disturbance at all? Or does it eventually break, like a wave in the surf? And if so, after how long? To make this question more manageable, we first consider a simpler situation: instead of random noise, we study a predetermined, non-random (deterministic) energy change in the medium. This simplifies the analysis considerably. We show that as long as the supply or removal of energy occurs slowly enough, waves can withstand slow but large amplitude changes without losing their coherent form. This insight forms the basis for studying random perturbations.

In Chapter 4, we return to the random model from Chapter 2. Based on the results from Chapter 3, we show that waves can tolerate large amplitude variations while remaining coherent, even under the influence of random noise. This also demonstrates that our description of the effective amplitude evolution from Chapter 2 is mathematically valid. The techniques used even suggest that the waves remain coherent for even longer than we can strictly demonstrate. The wake extends far behind a disturbed wave, but remains low. The latter is difficult to prove mathematically and offers opportunity for further research.

In Chapter 5, we leave the KdV model and focus on a related but discrete system: the Fermi–Pasta–Ulam–Tsingou (FPUT) chain, a model of masses connected by springs. A kind of infinite chain of beads connected by elastic. The model has concrete applications in the description of crystal lattices at the molecular level, where atoms can be modeled as masses connected by springs. Wave-like solutions also arise in this discrete context: when the masses are brought out of equilibrium in a suitable way, the resulting deflection travels through the chain at a constant speed. The connection with the KdV equation is strong: the entire dynamics of the FPUT chain is approximated by the KdV equation.

We study what happens when the springs in the chain are not identical, but have small random differences in spring-force. Here too, we investigate the effect of realistic irregularities on the propagation of waves. The disturbance has a slow but clear effect: the waves decrease in height and lose energy via a tail (wake) that forms behind the wave. We explicitly calculate how strong this amplitude decrease is; the techniques from the previous chapters prove to be excellent for this purpose.

Together, these chapters show how small imperfections — both random and deterministic in nature — can have a slow but significant influence on the speed and amplitude of waves in dispersive systems. The techniques developed in this thesis offer new ways to predict such changes. They also show that these waves are surprisingly robust and can survive for a long time even in realistic imperfect environments.

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# Curriculum Vitae

Rik Willem Simon Westdorp was born in Roosendaal, the Netherlands, on September 28, 1997. From 2009 to 2015, he attended R.S.G. ‘t Rijks in Bergen op Zoom, where he completed the bilingual atheneum programme. In 2015, he moved to Delft to study Applied Physics and Applied Mathematics at Delft University of Technology. He completed both degrees *cum laude* in three years.

Alongside his studies, he was a member of the student rowing association D.S.R.V. ‘Laga’ and the study association W.I.S.V. ‘Christiaan Huygens’. At the latter, he joined various committees and served as full-time chair of the association during the 2018–2019 academic year.

In 2019, he began the two-year Master’s programme in Applied Mathematics, also at Delft University of Technology. During this time, he spent a semester taking courses at the National University of Singapore. In the final year of the programme, he completed an internship at CWI/QuSoft, after which he wrote his master’s thesis, *A Stochastic Parametrically-Forced NLS Equation*, under the supervision of Dr. Manuel Gnann. Rik graduated *cum laude* in the summer of 2021.

Shortly after graduation, he started as a PhD candidate at the Mathematical Institute of Leiden University, under the supervision of Prof. Hermen Jan Hupkes. There, he continued his research on nonlinear dispersive waves in the presence of noise. Over the course of the four-year programme, he presented his work at various conferences and seminars across Europe and the USA. He also co-organised the Dynamical Systems group seminar for two years, as well as the NDNS+ PhD Days in 2022.

Rik’s thesis, *Stochastic Amplitude Modulation of Nonlinear Dispersive Waves*, comprises four projects. Three of these focus on soliton propagation in the Korteweg–de Vries equation under (stochastic) forcing. The fourth project, on attenuation of solitary waves in the Fermi–Pasta–Ulam–Tsingou lattice, resulted from a collaboration with Prof. Doug Wright, whom he visited for six weeks at Drexel University in Philadelphia.