Spectral localisers and aperiodic topological phases in noncommutative geometry

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Summary

The present dissertation studies several computational and modelling problems related to *topological phases of matter* within the framework of *noncommutative geometry*. In the following, we shall provide a brief invitation to the topics covered in the dissertation.

Noncommutative geometry

Geometry is the branch of mathematics concerned with properties of spaces, like shape, size and volume. Rather than drawing it on a piece of paper, it is more convenient to describe a space *algebraically*. The unit circle in the plane may be described as the set of solutions to the equation $x^2 + y^2 = 1$. Equivalently, it may be described by the collection of all polynomial functions defined on it. This corresponds to a *commutative algebra* $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$. Being commutative means that for any such polynomial f and g, their pointwise multiplication satisfies $fg = gf$.

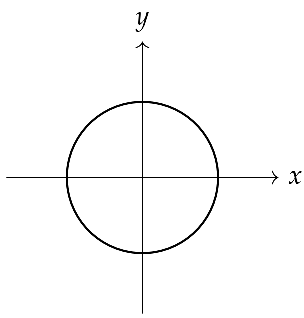

Figure 9.1.: The unit circle $x^2 + y^2 = 1$.

By contrast, *noncommutative algebras* are algebras in which the product of two elements need not commute: given elements $a, b \in A$ in a noncommutative algebra A, then in general $ab \neq ba$. Such algebras arise naturally in quantum mechanics, dynamical systems, and the study of singular spaces. Although noncommutative algebras do not correspond to "classical" geometric spaces in the usual sense, they exhibit properties to analogous to algebras of functions. *Noncommutative geometry* studies such algebras as if they encode the geometry of "virtually existing" spaces.

C*-algebras and K-theory

The celebrated Gelfand–Naimark theorem estalishes a one-to-one correspondence between spaces and a particular classes of algebras. More precisely, the algebra of continuous functions on a compact Hausdorff space X is a commutative, unital C*-algebra $\mathrm{C}(X)$. The philosophy of noncommutative geometry thus suggests to study *noncommutative* C*-algebras as if they are *noncommutative* spaces.

A central theme of mathematics is the classification of objects via invariants capturing their characteristic properties. *Algebraic topology* provide such tools for spaces, allowing them to be distinguished by *topological invariants*, which remain unchanged under continuous deformations. Examples of topological invariants are the Euler charateristic, the genus, or more generally *homology groups*. For instance, a sphere and a torus can be distinguished by their genus (roughly, number of holes): the sphere has genus zero, whereas the torus has genus one.

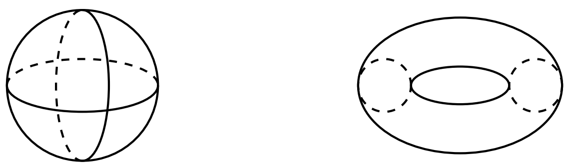

Figure 9.2.: The sphere (genus 0) and the torus (genus 1)

For noncommutative spaces described by C*-algebras, the role of algebraic topology is played by K-*theory*. To a C*-algebra A, one associates two abelian groups, $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$, called the K-theory groups of A. These groups capture essential structural information about A, and allow for distinguishing C*-algebras that are not *isomorphic*. In favourable situations, K-theory yields a *complete* classification within a given class of C*-algebras, meaning that two such C*-algebras are isomorphic if and only if they have the same K-theory.

The standpoint of the dissertation is to associate C*-algebras to *topological* *materials*, in such a way that their K-theory accommodate physically observable quantities, i.e. accessible through experiemental measurements. This provides a robust framework for modelling and analysing physical systems with non-trivial topological structures.

Topological phases of matter

A *phase* of matter refers to a region of matter that is homogeneous with respect to its physical or chemical properties. A *phase transition* describes the process by which a system changes from one phase into another.

According to Landau theory, phases of matter are classified by their symmetries, or more precisely, *order parameters*. For example, the ice and water are distinguished as the ice crystal has a hexagonal crystalline structure, whereas water has a more more disordered liquid structure.

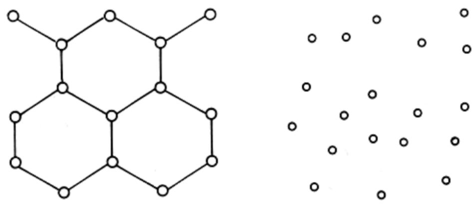

Figure 9.3.: Different phases of H_2O.

However, certain exotic phases of matter (referred to as *topological matter* or *topological materials*) cannot be characterised in this way, which are indeed distinguished by their *topological invariants*. These invariants have direct physical significance, as they correspond to *macroscopic* observables in physical experiments. The discovery of such topological effects has been honoured with several Nobel Prizes in Physics, including those for the integer quantum Hall effect (1985), the fractional quantum Hall effect (1998), and most recently the Josephson effect (2025).

The great contribution by Bellissard in the 1980s was a formulation of such topological invariants within the framework of C*-algebras and K-theory. The observable quantities of a topological material generate a noncommutative C*-algebra, referred to as the *observable* C*-*algebra*, whose K-theory encodes the system's topological invariants.

Contributions of the work

We have seen that the algebraic topology of noncommutative spaces is not merely a mathematical abstraction, but manifests concretely in experimentally measurable physical properties. In practice, however, we are often laden with extra challenges.

(1) **Computation**: Physical measurements are limited by finite energy resolution and spatial scale, so that only finite spectral data are available in numerical simulations. This poses significant challenges for the computation of topological invariants.

(2) **Modelling**: Realistic physical systems consist of a large number of particles, carrying vast amounts of information. Extracting relevant topological features from such systems requires effective and robust modelling approaches.

The dissertation offers new tools and insights to address these challenges. The first part develops an abstract mathematical framework for computing topological invariants of quantum systems using finite spectral data, and provides a conceptual foundation for the *spectral localiser* introduced by Loring and Schulz-Baldes. To this end, we employ E-*theory*, a bivariant K-theory introduced by Connes and Higson. This framework also allows for a generalisation of the spectral localiser to the Hilbert module setting, thereby enabling finite-rank computations of the topological invariants associated with families of operators.

The second part of the dissertation further develops C*-algebraic models of aperiodic topological materials, and explicitly describes the comparison maps between them. From a physical perspective, this yields a characterisation of the *robustness* of topological phases and their invariants, leading to the distinction between *strong* and *weak* topological phases, determined by their stability under perturbations. We also provide alternative dynamical descriptions of the C*-algebraic model of one-dimensional aperiodic topological materials, and interpret the Cuntz–Pimsner model of Bourne and Mesland as a topological graph C*-algebra.

We introduce an inductive limit C*-algebra, called the *symmetry-breaking Roe C*-algebra*, as a C*-algebraic model that encodes the symmetry-breaking processes of lattices. The K-theory of this C*-algebra accommodates both strong and weak topological phases, and distinguishes them via their divisibility. This allows us to interpret strong topological phases as those that are invariant under symmetry-breaking processes, corresponding physically to the independence from the specific microscopic partitioning of the material into unit cells.