



Universiteit
Leiden
The Netherlands

Spectral localisers and aperiodic topological phases in noncommutative geometry

Li, Y.

Citation

Li, Y. (2026, February 26). *Spectral localisers and aperiodic topological phases in noncommutative geometry*. Retrieved from <https://hdl.handle.net/1887/4293907>

Version: Publisher's Version

License: [Licence agreement concerning inclusion of doctoral thesis in the Institutional Repository of the University of Leiden](#)

Downloaded from: <https://hdl.handle.net/1887/4293907>

Note: To cite this publication please use the final published version (if applicable).

Part I.

**Spectral truncation of
index pairings**

2. Bivariant K-theories

Kasparov developed his theory as a tool in differential topology, and indeed some of the most powerful theorems in the topological theory of manifolds (pertaining particularly to the Novikov conjecture) rely very heavily on Kasparov's machinery. Thus, thanks to Kasparov's discoveries, functional analysis has repaid to topology the debt incurred by Brown, Douglas and Fillmore!

Nigel Higson and John Roe
[HR00, Preface]

2.1. Introduction

The first part of this thesis is devoted to some novel approach to compute index pairings. It is most convenient to do this within the abstract framework of bivariant K-theories.

K-theory was introduced by Grothendieck in his work on the Riemann–Roch theorem in algebraic geometry. Soon after that, Atiyah and Hirzebruch [AH61] initiated the study of K-theory as a generalised cohomology theory of topological spaces, enriching K-theory with their higher cases K^n . By their very definition, the K-theory of a compact Hausdorff space X classifies vector bundles thereover up to stable isomorphism. The celebrated Swan–Serre theorem states that every (finite-dimensional, locally trivial) vector bundle corresponds to a finitely generated, projective (fgp for short) $C(X)$ -module. Thus the study of K-theory of X is equivalent to study the properties of fgp $C(X)$ -modules.

Noncommutative geometry replaces topological spaces, or their corresponding commutative C^* -algebra $C_0(X)$, by noncommutative C^* -algebras. The above definition of fgp modules makes sense for them. This yields a homology theory of C^* -algebras. Unlike the commutative case, for which vast possibility of (co)homology theories can be defined, K-theory is a universal homology theory for a suitably large category of C^* -algebras in the following sense. Let $\mathfrak{N}\mathfrak{C}^*$ be

the full subcategory of all separable C^* -algebras, whose objects are C^* -algebras that are KK-equivalent to commutative C^* -algebras (i.e. the bootstrap category). Then the functor K_0 is uniquely characterised as the functor $F: \mathfrak{N}\mathfrak{C}^* \rightarrow \mathfrak{A}\mathfrak{b}$ that is homotopy invariant, C^* -stable, exact, continuous and satisfies

$$F(\mathbb{C}) = \mathbb{Z}, \quad F(C_0(\mathbb{R})) = 0,$$

cf. [Bla98, Theorem 22.3.6].

Higher K-theory groups K_n for $n \in \mathbb{N}$ may be defined as $K_n(A) := K_0(S^n A)$ ¹ due to the existence of the cone extension

$$SA \twoheadrightarrow CA \twoheadrightarrow A$$

together with the contractability of CA . Similar properties as in the preceding paragraph also uniquely characterises K_1 : it is the unique homotopy invariant, C^* -stable, exact, continuous functor satisfying

$$F(\mathbb{C}) = 0. \quad F(C_0(\mathbb{R})) = \mathbb{Z}.$$

It is well-known that K-theory has Bott periodicity, that is, $K_n(A) \simeq K_{n+2}(A)$ for every C^* -algebra A and every natural number $n \in \mathbb{N}$. So there are no other topological K-theory groups K_n other than K_0 or K_1 . In particular, one defines $K_n(A) := K_{n \bmod 2}$ for a negative integer n .

Since K-theory is a homology theory for C^* -algebras, it is natural to seek a cohomology theory of C^* -algebras that is dual to it. For topological spaces, the dual theory, called *K-homology*, exists by abstract nonsense. Atiyah [Ati70] suggested to use elliptic operators on manifolds as representatives (also called *K-cycles*) of K-homology classes, but failed to find the correct equivalence relation.

Kasparov [Kas75] was the first to provide a concrete model of K-homology using elliptic operators as suggested by Atiyah, referred to as *analytic K-homology*. In fact, Kasparov has already defined it for C^* -algebras: for each separable C^* -algebra A and each integer $n \in \mathbb{Z}$, Kasparov defined an abelian group $K^n(A)$, the n -th (analytic) *K-homology* group, such that there is a $\mathbb{Z}$ -bilinear map, the *index pairing*

$$K_i(A) \times K^i(A) \rightarrow \mathbb{Z},$$

that is functorial for $*$ -homomorphisms and tensor products.

¹The usual picture with unitaries can be translated to it by identifying $C_0(\mathbb{R})$ with $C_0(\mathbb{T} \setminus \{0\})$, and a unitary in A^+ with the image of a $*$ -homomorphism $C(\mathbb{T}) \rightarrow A^+$, cf. Section 4.4.

In another breakthrough work [Kas80b], Kasparov introduced a *bivariant* theory KK that unifies both K -theory and K -homology, in the sense that

$$KK(\mathbb{C}, B \otimes \mathcal{C}l_i) \simeq K_i(B), \quad KK(A \otimes \mathcal{C}l_i, \mathbb{C}) \simeq K^i(A)$$

for every pair of separable C^* -algebras A and B , and such that the index pairing can be recovered as the *Kasparov product*

$$KK(\mathbb{C}, A \otimes \mathcal{C}l_i) \times KK(A \otimes \mathcal{C}l_i, \mathbb{C}) \rightarrow KK(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}.$$

The power of Kasparov theory is soon exhibited by verifying the Novikov conjecture for several groups ([Kas88]).

Existence of the Kasparov product, in particular, yields a category $\mathfrak{K}\mathfrak{K}$, called the *Kasparov category*. The Kasparov category is a universal category of separable C^* -algebras subject to certain properties (cf. [Hig87, Cun87]), as will be discussed below. This provides a new insight to look for other bivariant homology theories of C^* -algebras, e.g. the E -theory of Connes and Higson (cf. [CH90]) and the local cyclic homology of Puschnigg (cf. [Pus96]).

Let $\mathfrak{C}^*\mathfrak{S}\mathfrak{ep}$ be the category of separable C^* -algebras with $*$ -homomorphisms as arrows. We shall describe a pair of functors $\mathfrak{C}^*\mathfrak{S}\mathfrak{ep} \rightarrow \mathfrak{K}\mathfrak{K}$ and $\mathfrak{C}^*\mathfrak{S}\mathfrak{ep} \rightarrow \mathfrak{C}$ whose ranges are additive categories, and are universal for prescribed properties. The arrow sets of the target additive categories can be identified with the corresponding Kasparov or E -theory groups.

Definition 2.1 (cf. [Mey08]). Let $\mathfrak{C}$ be an additive category, and $F: \mathfrak{C}^*\mathfrak{S}\mathfrak{ep} \rightarrow \mathfrak{C}$ an additive functor. Then F is:

- *C^* -stable*, if any rank-one embedding $e: \mathbb{C} \hookrightarrow \mathbb{K}$ induces an isomorphism

$$F(\text{id} \otimes e): F(A) \xrightarrow{\sim} F(A \otimes \mathbb{K});$$

- *exact*, if for any extension of separable C^* -algebras

$$I \xrightarrow{i} E \xrightarrow{p} \gg Q,$$

then

$$F(I) \xrightarrow{F(i)} F(E) \xrightarrow{F(p)} F(Q)$$

is exact at $F(E)$;

- *semi-split exact*, if for any extension of separable C^* -algebras with a completely positive split section $s: Q \rightarrow E$:

$$I \xrightarrow{i} E \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} \!\!\!\! \gg Q,$$

then

$$F(I) \xrightarrow{F(i)} F(E) \xrightarrow{F(p)} F(Q)$$

is exact at $F(E)$;

- *split exact*, if for any extension of separable C^* -algebras which splits by a $*$ -homomorphism $s: Q \rightarrow E$:

$$I \xrightarrow{i} E \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} \!\!\!\! \gg Q,$$

the map $F(i) + F(s): F(I) \oplus F(Q) \rightarrow F(E)$ is an isomorphism;

- *homotopy invariant*, if F maps a homotopy equivalence to an isomorphism.
- *has Bott periodicity*, if $F(A)$ is naturally isomorphic to $F(S^2 A)$ for any separable C^* -algebra A .

Definition 2.2. Let P be a subset of properties listed in Definition 2.1. We say an additive functor $F: \mathfrak{C}^* \mathfrak{S}ep \rightarrow \mathfrak{C}$ into an additive category $\mathfrak{C}$ is *universal* for the properties in P , if the following holds:

- F satisfies the properties in P ;
- for any additive functor $F': \mathfrak{C}^* \mathfrak{S}ep \rightarrow \mathfrak{C}'$ which satisfies the properties in P , there exists a unique functor $u: \mathfrak{C} \rightarrow \mathfrak{C}'$ such that $F' = u \circ F$.

Theorem 2.3 ([Hig87, Hig90]). *The following universal functors exist:*

- *The universal split exact and C^* -stable functor $KK: \mathfrak{C}^* \mathfrak{S}ep \rightarrow \mathfrak{K}\mathfrak{K}$;*
- *The universal C^* -stable, exact and homotopy invariant functor $E: \mathfrak{C}^* \mathfrak{S}ep \rightarrow \mathfrak{E}$.*

We take Theorem 2.3 as *definitions* of KK -theory and E -theory: for every pair of separable C^* -algebras A and B , define the abelian group $KK(A, B)$ as the set of arrows from A to B in $\mathfrak{K}\mathfrak{K}$; and the abelian group $E(A, B)$ as the set of arrows from A to B in $\mathfrak{E}$. The *Kasparov product* and the *E-theory product* are compositions of arrows in $\mathfrak{K}\mathfrak{K}$ and $\mathfrak{E}$. By definition, the universal functors KK and E send a

*-homomorphism $f: A \rightarrow B$ to an element in $KK(A, B)$ and an element in $E(A, B)$, respectively.

We note that although the functors $KK: \mathcal{C}^*\mathfrak{Sep} \rightarrow \mathfrak{K}\mathfrak{K}$ and $E: \mathcal{C}^*\mathfrak{Sep} \rightarrow \mathfrak{E}$ are only assumed to be universal for very few properties, the combination of them may imply quite strong results.

Theorem 2.4 ([Kas80b, Cun84, Hig88]). *Let $F: \mathcal{C}^*\mathfrak{Sep} \rightarrow \mathfrak{E}$ be an split exact and C^* -stable additive functor, then F is also semi-split exact, homotopy invariant and has Bott periodicity.*

Therefore, the functor $KK: \mathcal{C}^*\mathfrak{Sep} \rightarrow \mathfrak{K}\mathfrak{K}$ has Bott periodicity, i.e. S^2A is isomorphic to A in $\mathfrak{K}\mathfrak{K}$. Since a functor always preserves isomorphisms, it follows that E -theory has Bott periodicity, too.

Since every exact functor is also split exact, it follows from the universal property of $\mathfrak{K}\mathfrak{K}$ that there is a canonical functor

$$\natural: \mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{E},$$

such that the universal functor $\mathcal{C}^*\mathfrak{Sep} \rightarrow \mathfrak{E}$ factors through it. Such a functor induces a natural group homomorphism

$$\natural(A, B): KK(A, B) \rightarrow E(A, B)$$

for every pair of separable C^* -algebras A and B , and indeed a natural transformation $KK(-, -) \Rightarrow E(-, -)$. A concrete realisation of this canonical functor will be discussed in Chapter 3 and will be utilised in Chapter 4.

For the purpose of explicit computation, one also needs concrete models for these bivariant K -theories. By a *picture* of a bivariant K -theory $F \in \{KK, E\}$, we shall mean the following data (BK1)–(BK4):

- (BK1) for each pair of separable C^* -algebras A and B , a proper class $\mathbb{F}(A, B)$ of elements called “ F -theory cycles” (of the bivariant group $F(A, B)$);
- (BK2) an equivalence relation “ $\sim$ ” on $\mathbb{F}(A, B)$;
- (BK3) an abelian group structure on the set of equivalence classes $F(A, B) := \mathbb{F}(A, B)/\sim$, giving the “ F -theory” group;
- (BK4) for each triple of separable C^* -algebras A , B and C , a (natural) group homomorphism

$$F(A, B) \otimes F(B, C) \rightarrow F(A, C),$$

called the “ F -theory product”.

We shall describe the Kasparov's Fredholm picture of KK-theory in Section 2.2, and Connes–Higson's asymptotic morphism picture of E-theory in Section 2.3. Based on another closely related picture, the *unbounded* picture of KK-theory, we shall describe the canonical comparison functor $\mathfrak{k}$ in Chapter 3.

KK-theory and E-theory not only share similar universal characterisations. By a result of Thomsen [Tho99] as well as Effros–Choi lifting theorem, it follows that $\mathfrak{k}(A, B): \text{KK}(A, B) \rightarrow \text{E}(A, B)$ is a natural isomorphism of abelian groups if A is a nuclear C^* -algebra. This applies, in particular, to the cases $A = \mathbb{C}$ and yields maps between K-theory groups called *index pairings*. The coincidence of KK-theoretic and E-theoretic index pairings is provided in Section 2.4.

2.2. Kasparov theory

The first picture of KK-theory was introduced by Kasparov himself in [Kas80b], which generalised his model of K-homology in [Kas75]. The cycles of Kasparov's picture of KK-theory are called *Kasparov modules*, which can be thought of as zeroth-order *pseudo-differential* operators on Hilbert C^* -modules, satisfying certain “elliptic” conditions. There are several different ways to yield an equivalence relation on Kasparov modules, whereas for a large class of C^* -algebras, all of them give the same KK-theory groups (cf. [Bla98, Section 17.3]).

2.2.1. $\mathbb{Z}/2$ -graded C^* -algebras and Hilbert modules

Although KK-theory treats (separable) C^* -algebras as its objects, Kasparov has realised in [Kas80b] that it is most convenient to work with $\mathbb{Z}/2$ -graded C^* -algebras. We recall some definitions.

$\mathbb{Z}/2$ -graded C^* -algebras A $\mathbb{Z}/2$ -graded C^* -algebra is a C^* -algebra A together with an involutive $*$ -automorphism $\gamma: A \rightarrow A$. A $\mathbb{Z}/2$ -graded C^* -algebra (A, γ) is called *inner graded* if γ is an inner automorphism, i.e. if there exists $\Gamma \in \mathcal{M}(A)$ such that $\gamma(a) = \Gamma a \Gamma^*$.

An element $a \in A$ is *odd* (or has degree 1) if $\gamma(a) = -a$, and *even* (or has degree 0) if $\gamma(a) = a$. We write $\deg(a) \in \mathbb{Z}/2$ for the degree of a . If a is either odd or even, then we say a is *homogeneous*.

Let a, b be homogeneous elements in a $\mathbb{Z}/2$ -graded C^* -algebra (A, γ) . Their *graded commutator* is given by

$$[a, b] := ab - (-1)^{\deg(a)\deg(b)}ba.$$

It extends to all element of A by linearity.

Let (A_1, γ_1) and (A_2, γ_2) be $\mathbb{Z}/2$ -graded C^* -algebras. A $\mathbb{Z}/2$ -graded $*$ -homomorphism $A_1 \rightarrow A_2$ is a $*$ -homomorphism $\varphi: A_1 \rightarrow A_2$ that intertwines the gradings γ_1 and γ_2 , i.e. the following diagram commutes:

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi} & A_2 \\ \gamma_1 \downarrow & & \downarrow \gamma_2 \\ A_1 & \xrightarrow{\varphi} & A_2. \end{array}$$

$\mathbb{Z}/2$ -graded tensor product Let $A_1 \odot A_2$ be the algebraic tensor product space of A_1 and A_2 . There are two ways to define the multiplication, involution and grading structures thereon, so as to complete it into a $\mathbb{Z}/2$ -graded $*$ -algebra:

- the *ungraded algebraic tensor product* $A_1 \odot A_2$ has the the product, involution and degree maps given by

$$\begin{aligned} (a_1 \odot a_2) \cdot (b_1 \odot b_2) &:= a_1 b_1 \odot a_2 b_2, \\ (a_1 \odot a_2)^* &:= a_1^* \odot a_2^*, \\ \deg(a_1 \odot a_2) &:= \deg(a_1) + \deg(a_2); \end{aligned}$$

- the *graded algebraic tensor product* $A_1 \hat{\odot} A_2$ has

$$\begin{aligned} (a_1 \odot a_2) \cdot (b_1 \odot b_2) &:= (-1)^{\deg(a_2) \cdot \deg(b_1)} a_1 b_1 \odot a_2 b_2, \\ (a_1 \odot a_2)^* &:= (-1)^{\deg(a_1) \cdot \deg(a_2)} a_1^* \odot a_2^*, \\ \deg(a_1 \odot a_2) &:= \deg(a_1) + \deg(a_2), \end{aligned}$$

where $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. Complete $A_1 \odot A_2$ or $A_1 \hat{\odot} A_2$ under suitable C^* -norms, then we obtain the ungraded or graded tensor product C^* -algebras $A_1 \otimes A_2$ or $A_1 \hat{\otimes} A_2$. They are, in general, not isomorphic to one another as $\mathbb{Z}/2$ -graded C^* -algebras. However, if (A_2, γ_2) is inner graded by $\gamma_2(a_2) = \Gamma a \Gamma^*$, then

$$A_1 \hat{\otimes} A_2 \rightarrow A_1 \otimes A_2, \quad a_1 \hat{\otimes} a_2 \mapsto a_1 \Gamma^{\deg(a_2)} \otimes b$$

is an isomorphism of $\mathbb{Z}/2$ -graded C^* -algebras.

$\mathbb{Z}/2$ -graded Hilbert C^* -modules Let (A, γ) be a $\mathbb{Z}/2$ -graded C^* -algebra. A $\mathbb{Z}/2$ -graded Hilbert A -module is a Hilbert A -module $\mathcal{E}$ together with a involutive

linear map $\Gamma: \mathcal{E} \rightarrow \mathcal{E}$, such that for all $x, x_1, x_2 \in \mathcal{E}$ and $a \in A$:

$$\Gamma(xa) = \Gamma(x)\gamma(a), \quad \langle \Gamma(x_1), \Gamma(x_2) \rangle = \gamma(\langle x_1, x_2 \rangle).$$

Let $(\mathcal{E}, \Gamma)$ be a $\mathbb{Z}/2$ -graded Hilbert A -module. Then $\mathbb{B}_A(\mathcal{E})$ carries an inner $\mathbb{Z}/2$ -grading $\text{Ad}_\Gamma: T \mapsto \Gamma T \Gamma^{-1}$. Call it the $\mathbb{Z}/2$ -grading on $\mathbb{B}_A(\mathcal{E})$ induced by $\mathcal{E}$.

Let (A, γ) be a $\mathbb{Z}/2$ -graded C^* -algebra and $(\mathcal{E}, \Gamma)$ be a $\mathbb{Z}/2$ -graded Hilbert A -module. We write A^{op} for the $\mathbb{Z}/2$ -graded C^* -algebra $(A, -\gamma)$, and $\mathcal{E}^{\text{op}}$ for the $\mathbb{Z}/2$ -graded Hilbert A -module $(\mathcal{E}, -\Gamma)$.

Example 2.5 ($\mathbb{Z}/2$ -graded Hilbert modules over ungraded C^* -algebras). Let A be a trivially graded C^* -algebra and $\mathcal{E}$ be a $\mathbb{Z}/2$ -graded Hilbert A -module. Then the grading operator Γ satisfies

$$\Gamma(xa) = \Gamma(x)a, \quad \langle \Gamma(x_1), \Gamma(x_2) \rangle = \langle x_1, x_2 \rangle$$

for all $x, x_1, x_2 \in \mathcal{E}$ and $a \in A$. Thus Γ is an A -linear, self-adjoint unitary operator on $\mathcal{E}$. Define

$$P_\pm := \frac{1}{2}(\mathbb{1}_{\mathcal{E}} \pm \Gamma),$$

then both $P_\pm$ are orthogonal projections in $\mathbb{B}_A(\mathcal{E})$, and $\mathcal{E}$ decomposes as ± 1 -eigenmodules of the grading operator Γ :

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-, \quad \mathcal{E}_+ := P_+ \mathcal{E}, \quad \mathcal{E}_- := P_- \mathcal{E},$$

where both $\mathcal{E}_\pm$ are Hilbert A -modules. We note that this happens only if A is trivially graded, as otherwise $\mathcal{E}_\pm$ are not closed under A .

Clifford algebras An important class of $\mathbb{Z}/2$ -graded C^* -algebras are Clifford algebras. We describe some of their structural properties here.

Let $\mathbb{C}\ell_n$ be the finite-dimensional $\mathbb{Z}/2$ -graded C^* -algebra generated by n self-adjoint anti-commuting unitaries $e_1, \dots, e_n$. Thus, $\mathbb{C}\ell_n$ is generated as a vector space by $e_{i_1} \dots e_{i_k}$, where k ranges over $\{0, \dots, n\}$ and $1 \leq i_1 < \dots < i_k \leq n$. The grading on $\mathbb{C}\ell_n$ is given by $\deg(e_{i_1} \dots e_{i_k}) := (-1)^k$. Obviously, there are isomorphisms of $\mathbb{Z}/2$ -graded C^* -algebras for all m and n :

$$\mathbb{C}\ell_m \hat{\otimes} \mathbb{C}\ell_n \rightarrow \mathbb{C}\ell_{m+n}, \quad e_{i_1} \dots e_{i_k} \hat{\otimes} e_{j_1} \dots e_{j_\ell} \mapsto e_{i_1} \dots e_{i_k} e_{m+j_1} \dots e_{m+j_\ell}.$$

The Clifford algebra $\mathbb{C}\ell_1$ is, as a $\mathbb{Z}/2$ -graded C^* -algebra, isomorphic to $\mathbb{C} \oplus \mathbb{C}$ via $1 \mapsto (1, 1)$ and $e_1 \mapsto (1, -1)$. The grading on $\mathbb{C} \oplus \mathbb{C}$ is the flip automorphism

$(z_1, z_2) \mapsto (z_2, z_1)$. The Clifford algebra $\mathcal{C}\ell_2$ is isomorphic to $M_2(\mathbb{C})$ with grading Ad_{Γ_3} , where

$$\Gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2.2.2. Kasparov's picture of KK_0

Let A and B be $\mathbb{Z}/2$ -graded C^* -algebras.

Definition 2.6. A $\mathbb{Z}/2$ -graded C^* - A - B -correspondence is given by a pair $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a $\mathbb{Z}/2$ -graded Hilbert B -module $\mathcal{E}$ and $\varphi: A \rightarrow \mathbb{B}_B(\mathcal{E})$ is a grading-preserving $*$ -homomorphism.

A $\mathbb{Z}/2$ -graded C^* - A - B -correspondence $(\mathcal{E}, \varphi)$ is called:

- *proper*, if $\varphi(A) \subseteq \mathbb{K}_B(\mathcal{E})$;
- *essential*, if $\varphi(A)\mathcal{E}$ is dense in $\mathcal{E}$.
- *Morita equivalence*, if φ is an isomorphism onto $\mathbb{K}_B(\mathcal{E})$ and $\mathcal{E}$ is a full² Hilbert B -module.

In what follows, we shall often oppress φ . Thus a $\mathbb{Z}/2$ -graded C^* - A - B -correspondence will be denoted by its underlying $\mathbb{Z}/2$ -graded Hilbert B -module $\mathcal{E}$, and A is identified with the C^* -subalgebra $\varphi(A)$ of $\mathbb{B}_B(\mathcal{E})$.

Definition 2.7. Two $\mathbb{Z}/2$ -graded C^* - A - B -correspondences $\mathcal{E}_0$ and $\mathcal{E}_1$ are *unitarily equivalent*, if there exists a unitary equivalence of Hilbert B -modules $U: \mathcal{E}_0 \rightarrow \mathcal{E}_1$ that intertwines A . That is, the following diagram commutes for each $a \in A$:

$$\begin{array}{ccc} \mathcal{E}_0 & \xrightarrow{a} & \mathcal{E}_0 \\ U \downarrow & & \downarrow U \\ \mathcal{E}_1 & \xrightarrow{a} & \mathcal{E}_1. \end{array}$$

Definition 2.8. A *Kasparov A - B -module* is a pair $(\mathcal{E}, F)$, where

- $\mathcal{E}$ is a $\mathbb{Z}/2$ -graded C^* - A - B -correspondence;
- $F \in \mathbb{B}_B(\mathcal{E})$ is an odd element for the grading on $\mathbb{B}_B(\mathcal{E})$ induced by $\mathcal{E}$, and satisfies

$$[F, a] \in \mathbb{K}_B(\mathcal{E}), \quad a(F^2 - 1) \in \mathbb{K}_B(\mathcal{E}), \quad a(F - F^*) \in \mathbb{K}_B(\mathcal{E}) \quad (2.9)$$

²A Hilbert B -module $\mathcal{E}$ is full if $\langle \mathcal{E}, \mathcal{E} \rangle$ is dense in B .

for all $a \in A$.

Write $\mathbb{K}\mathbb{K}(A, B)$ for the collection of all Kasparov A - B -modules.

A special class of Kasparov modules are *proper C^* -correspondences*.

Lemma 2.10. *Let $\mathcal{E}$ be a $\mathbb{Z}/2$ -graded C^* - A - B -correspondence. Then $(\mathcal{E}, 0)$ is a Kasparov A - B -module iff $\mathcal{E}$ is proper. In particular, every $\mathbb{Z}/2$ -graded $*$ -homomorphism $\varphi: A \rightarrow B$ gives a Kasparov A - B -module $(B, 0)$.*

Proof. If $F = 0$, then the conditions in (2.9) is equivalent to saying that $a \in \mathbb{K}_B(\mathcal{E})$ for all $a \in A$. Every $\mathbb{Z}/2$ -graded $*$ -homomorphism $\varphi: A \rightarrow B$ gives a C^* - A - B -correspondence (B, φ) , where B is viewed as a $\mathbb{Z}/2$ -graded Hilbert module over itself. Since $B \simeq \mathbb{K}_B(B)$, such a C^* -correspondence is proper. $\square$

Kasparov modules are cycles of $\mathbb{K}\mathbb{K}$ -theory. In his original article [Kas80b], Kasparov introduced several different equivalence relations on $\mathbb{K}\mathbb{K}(A, B)$. Nevertheless, the slogan that we have in mind is that for sufficiently nice C^* -algebras, all these equivalence relations coincide. The most commonly used one is called *homotopy* of Kasparov modules, which we describe below.

Definition 2.11. Let $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$ be Kasparov A - B -modules.

- (1) $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$ are *unitarily equivalent*, denoted by $(\mathcal{E}_0, F_0) \simeq (\mathcal{E}_1, F_1)$, if there exists a unitary equivalence of C^* -correspondences $U: \mathcal{E}_0 \rightarrow \mathcal{E}_1$ that intertwines F_i , that is, for each $a \in A$, the following diagrams commute:

$$\begin{array}{ccc} \mathcal{E}_0 & \xrightarrow{a} & \mathcal{E}_0 \\ U \downarrow & & \downarrow U \\ \mathcal{E}_1 & \xrightarrow{a} & \mathcal{E}_1 \end{array} \quad \begin{array}{ccc} \mathcal{E}_0 & \xrightarrow{F_0} & \mathcal{E}_0 \\ U \downarrow & & \downarrow U \\ \mathcal{E}_1 & \xrightarrow{F_1} & \mathcal{E}_1 \end{array}$$

- (2) $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$ are *homotopic*, denoted by $(\mathcal{E}_0, F_0) \sim (\mathcal{E}_1, F_1)$, if there exists a Kasparov A - $C([0, 1], B)$ -module $(\mathcal{E}, F)$, such that

$$\begin{aligned} (\mathcal{E}_0, F_0) &\simeq (\mathcal{E} \otimes_{\text{ev}_0} B, F \otimes_{\text{ev}_0} 1), \\ (\mathcal{E}_1, F_1) &\simeq (\mathcal{E} \otimes_{\text{ev}_1} B, F \otimes_{\text{ev}_1} 1). \end{aligned}$$

- (3) One checks easily that $\sim$ is an equivalence relation on $\mathbb{K}\mathbb{K}(A, B)$. Define

$$\mathbb{K}\mathbb{K}(A, B) := \mathbb{K}\mathbb{K}(A, B) / \sim.$$

We impose an abelian group structure on $\mathrm{KK}(A, B)$: the addition is implemented by the direct sum of Kasparov modules,

$$(\mathcal{E}_1, F_1) \oplus (\mathcal{E}_2, F_2) := (\mathcal{E}_1 \oplus \mathcal{E}_2, F_1 \oplus F_2),$$

and the zero element in $\mathrm{KK}(A, B)$ is the class of the zero module $(0, 0)$. The inverse of the class of $(\mathcal{E}, F)$ is represented by $(\mathcal{E}^{\mathrm{op}}, -F)$.

Theorem 2.12 ([Hig87]). *Let $\mathfrak{K}\mathfrak{K}$ be the category whose objects are separable C^* -algebras, and an arrow from A to B is an element in $\mathrm{KK}(A, B)$. Then $\mathfrak{K}\mathfrak{K}$ is an additive category, and the canonical functor $\mathcal{C}^*\mathfrak{S}\mathfrak{e}\mathfrak{p} \rightarrow \mathfrak{K}\mathfrak{K}$ is the universal C^* -stable, split exact functor.*

The universal functor $\mathrm{KK}: \mathcal{C}^*\mathfrak{S}\mathfrak{e}\mathfrak{p} \rightarrow \mathfrak{K}\mathfrak{K}$ maps an arrow in $\mathcal{C}^*\mathfrak{S}\mathfrak{e}\mathfrak{p}$, i.e. a $*$ -homomorphism $\varphi: A \rightarrow B$, to the class of the Kasparov A - B -module $(B, 0)$ as in Lemma 2.10.

Kasparov product The composition product in the category $\mathfrak{K}\mathfrak{K}$ is called the *Kasparov product*. This is a $\mathbb{Z}$ -bilinear group homomorphism

$$\mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \rightarrow \mathrm{KK}(A, C) \quad (\mathrm{KK})$$

for every triple of separable C^* -algebras A, B and C . The existence and uniqueness of the Kasparov product has been proved by Kasparov in [Kas80b], based on a few fairly technical lemmas. Later, Connes and Skandalis [CS84, Appendix A] provided an axiomatic characterisation, which allows for a guess-and-check of the Kasparov product in concrete settings:

Theorem 2.13. *Let $(\mathcal{E}_1, F_1) \in \mathrm{KK}(A, B)$ and $(\mathcal{E}_2, F_2) \in \mathrm{KK}(B, C)$. Then their Kasparov product is represented by any element of the form $(\mathcal{E}, F) \in \mathrm{KK}(A, C)$, where $\mathcal{E} = \mathcal{E}_1 \otimes_B \mathcal{E}_2$ and F satisfies the following connection (CoSk1) and positivity (CoSk2) conditions:*

(CoSk1) *For all $x \in \mathcal{E}_1$: the operator*

$$T: \mathcal{E}_2 \rightarrow \mathcal{E}_1 \otimes_B \mathcal{E}_2, \quad Ty := F(x \otimes y) - \Gamma_1(x) \otimes F_2 y$$

is compact, where Γ_1 is the grading on $\mathcal{E}_1$.

(CoSk2) *There exists $0 < \kappa < 2$, such that for all $a \in A$, the image of the operator*

$$a^*[F_1 \otimes 1, F]a + \kappa a^* a$$

is positive in the Calkin algebra $\mathbb{B}_B(\mathcal{E})/\mathbb{K}_B(\mathcal{E})$.

Theorem 2.13 provides a gadget to compute the Kasparov product (KK) with a guess-and-check process: one first “guesses” an operator $F \in \mathbb{B}_C(\mathcal{E})$, then “checks” that $(\mathcal{E}, F)$ is a Kasparov A - C -module that satisfies Connes–Skandalis’ conditions.

The ring $\mathrm{KK}(A, A)$ Let $A = B = C$, then the Kasparov product

$$\mathrm{KK}(A, A) \times \mathrm{KK}(A, A) \rightarrow \mathrm{KK}(A, A)$$

yields a *ring* structure on $\mathrm{KK}(A, A)$. Denote by $\mathbf{1}_A$ the unit of $\mathrm{KK}(A, A)$.

Lemma 2.14. *The unit $\mathbf{1}_A \in \mathrm{KK}(A, A)$ is represented by $(A, 0)$.*

Proof. Let A be a $\mathbb{Z}/2$ -graded C^* -algebra. Then it is a $\mathbb{Z}/2$ -graded Hilbert A -module, and $\mathbb{K}_A(A) = A$. Thus $(A, \mathrm{id}: A \rightarrow A)$ is a proper $\mathbb{Z}/2$ -graded C^* - A - A -correspondence, and $(A, 0)$ is a Kasparov A - A -module by Lemma 2.10.

Let $(\mathcal{E}, F)$ be any Kasparov A - A -module. Using the conditions of Connes and Skandalis Theorem 2.13, one readily checks that the Kasparov product of either

$$[\mathcal{E}, F] \times [A, 0]$$

or

$$[A, 0] \times [\mathcal{E}, F]$$

equals $[\mathcal{E}, F]$. This proves the claim. $\square$

Two $\mathbb{Z}/2$ -graded C^* -algebras A and B are called *KK-equivalent*, if they are isomorphic as objects in $\mathfrak{KK}$. Concretely, this means that there exists elements $\alpha \in \mathrm{KK}(A, B)$ and $\beta \in \mathrm{KK}(B, A)$ such that their Kasparov products satisfy

$$\alpha \times \beta = \mathbf{1}_A, \quad \beta \times \alpha = \mathbf{1}_B.$$

General Kasparov product The Kasparov category $\mathfrak{KK}$ is a universal additive category, i.e. codomain of the universal split exact and homotopy invariant functor. It turns out that $\mathfrak{KK}$ carries more structure. A relevant structure for the discussion below is the symmetric monoidal structure of $\mathfrak{KK}$ implemented by the *exterior Kasparov product*

$$\boxtimes: \mathrm{KK}(A, B) \times \mathrm{KK}(C, D) \rightarrow \mathrm{KK}(A \hat{\otimes} C, B \hat{\otimes} D) \quad (2.15)$$

for a quadruple of $\mathbb{Z}/2$ -graded separable C^* -algebras A, B, C and D , which is associative, and functorial for $*$ -homomorphisms. This amounts to saying that the

universal functor $\mathrm{KK}: \mathfrak{C}^*\mathfrak{Sep} \rightarrow \mathfrak{KK}$ is symmetric monoidal, where the symmetric monoidal structure on $\mathfrak{C}^*\mathfrak{Sep}$ is implemented by the (spatial) tensor products of separable C^* -algebras. We refer to the work of Joachim and Stolz [JS09] for a more precise description of this structure.

Combining the exterior Kasparov product with the (interior) Kasparov product, we have a general product structure on $\mathfrak{KK}$ of the form

$$\mathrm{KK}(A, B \hat{\otimes} C) \times \mathrm{KK}(C \hat{\otimes} D, E) \rightarrow \mathrm{KK}(A \hat{\otimes} D, B \hat{\otimes} E). \quad (2.16)$$

defined as

$$(x, y) \mapsto (x \boxtimes \mathbf{1}_D) \times (\mathbf{1}_B \boxtimes y),$$

where $\mathbf{1}_D := [D, 0] \in \mathrm{KK}(D, D)$ (resp. $\mathbf{1}_B := [B, 0] \in \mathrm{KK}(B, B)$) is the unit of the ring $\mathrm{KK}(D, D)$ (resp. $\mathrm{KK}(B, B)$) as in Lemma 2.14.

The symmetric monoidal structure of $\mathfrak{KK}$ immediately leads to the following fact³:

Lemma 2.17. *Let A_1, A_2, B_1 and B_2 be $\mathbb{Z}/2$ -graded C^* -algebras such that A_1 is KK-equivalent to A_2 and B_1 is KK-equivalent to B_2 , then $A_1 \hat{\otimes} B_1$ is KK-equivalent to $A_2 \hat{\otimes} B_2$.*

Proof. Let $\alpha_1 \in \mathrm{KK}(A_1, A_2)$ be a KK-equivalence with inverse $\alpha_2 \in \mathrm{KK}(A_2, A_1)$; and $\beta_1 \in \mathrm{KK}(B_1, B_2)$ be a KK-equivalence with inverse $\beta_2 \in \mathrm{KK}(B_2, B_1)$. Set

$$\gamma_1 := \alpha_1 \boxtimes \beta_1 \in \mathrm{KK}(A_1 \hat{\otimes} B_1, A_2 \hat{\otimes} B_2), \quad \gamma_2 := \alpha_2 \boxtimes \beta_2 \in \mathrm{KK}(A_2 \hat{\otimes} B_2, A_1 \hat{\otimes} B_1).$$

Then their Kasparov products satisfy

$$\gamma_1 \times \gamma_2 = (\alpha_1 \boxtimes \beta_1) \times (\alpha_2 \boxtimes \beta_2) = (\alpha_1 \times \alpha_2) \boxtimes (\beta_1 \times \beta_2) = \mathbf{1}_{A_1} \boxtimes \mathbf{1}_{B_1} = \mathbf{1}_{A_1 \hat{\otimes} B_1}$$

and similarly $\gamma_2 \times \gamma_1 = \mathbf{1}_{A_2 \hat{\otimes} B_2}$. $\square$

Bott periodicity The Bott periodicity in KK-theory states that A is KK-equivalent to $S^2 A$ for every separable C^* -algebra A . This is a special case of the following theorem.

Theorem 2.18. *There are KK-equivalences between:*

- $C_0(\mathbb{R})$ and $\mathcal{C}\ell_1$;

³Here only the monoidal structure is needed, as no factors are swapped; and the equality holds strictly due to functoriality.

- $\mathbb{C}$, $\mathbb{C}\ell_2$ and $C_0(\mathbb{R}^2)$.

Proof. The KK-equivalence between $C_0(\mathbb{R})$ and $\mathbb{C}\ell_1$ is a special case of [Kas80b, Section 5, Theorem 7] for $n = 1$. In particular, since

$$\mathbb{C}\ell_2 \simeq \mathbb{C}\ell_1 \hat{\otimes} \mathbb{C}\ell_1, \quad C_0(\mathbb{R}^2) \simeq C_0(\mathbb{R}) \hat{\otimes} C_0(\mathbb{R}),$$

it follows from Lemma 2.17 that $C_0(\mathbb{R}^2)$ is KK-equivalent to $\mathbb{C}\ell_2$. The isomorphism of $\mathbb{Z}/2$ -graded C^* -algebras $\mathbb{C}\ell_2 \simeq (M_2(\mathbb{C}), \text{Ad}_{\Gamma_3})$ yields an isomorphism

$$\mathbb{C}\ell_2 \xrightarrow{\sim} K(\mathbb{C} \oplus \mathbb{C}),$$

where $\mathbb{C} \oplus \mathbb{C}$ is viewed as a $\mathbb{Z}/2$ -graded Hilbert space with grading operator $1 \oplus -1$. This yields a Morita equivalence between the $\mathbb{Z}/2$ -graded C^* -algebras $\mathbb{C}\ell_2$ and $\mathbb{C}$, hence a KK-equivalence between them. $\square$

Corollary 2.19. *The followings are all isomorphic to $\mathbb{Z}$ as abelian groups:*

$$KK(C_0(\mathbb{R}), \mathbb{C}\ell_1), \quad KK(\mathbb{C}, C_0(\mathbb{R}^2)), \quad KK(\mathbb{C}, \mathbb{C}\ell_2). \quad (2.20)$$

Proof. It follows from the KK-equivalence between $C_0(\mathbb{R})$ and $\mathbb{C}\ell_1$ that

$$KK(C_0(\mathbb{R}), \mathbb{C}\ell_1) \simeq KK(C_0(\mathbb{R}), C_0(\mathbb{R})).$$

The C^* -algebra $C_0(\mathbb{R})$ is commutative, hence belongs to the UCT class of Rosenberg and Schochet [RS87]. Moreover, it has torsion-free K-theory groups. Hence

$$\begin{aligned} KK(C_0(\mathbb{R}), C_0(\mathbb{R})) &\simeq \text{Hom}(K_0(C_0(\mathbb{R})), K_0(C_0(\mathbb{R}))) \oplus \text{Hom}(K_1(C_0(\mathbb{R})), K_1(C_0(\mathbb{R}))) \\ &\simeq \mathbb{Z}. \end{aligned}$$

The cases of $KK(C_0(\mathbb{R}^2), \mathbb{C})$ and $KK(C_0(\mathbb{R}^2), \mathbb{C}\ell_2)$ can be reduced to $KK(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}$ by their KK-equivalences. $\square$

On the sign of Bott periodicity We remark that the generator of any of the abelian groups in (2.20) cannot be chosen in a completely canonical way. That means, if $\alpha \in KK(C_0(\mathbb{R}), \mathbb{C}\ell_1)$ and $\beta \in KK(\mathbb{C}\ell_1, C_0(\mathbb{R}))$ are KK-equivalences such that

$$\alpha \times \beta = 1_{C_0(\mathbb{R})}, \quad \beta \times \alpha = 1_{\mathbb{C}\ell_1},$$

then replacing α and β by their inverses $-\alpha$ and $-\beta$ yields another pair of KK-equivalences.

One might, instead, attempt to seek canonical choices of the generators from other data. For KK-equivalences regarding $\mathbb{C}l_1$ or $\mathbb{C}l_2$, this is also not possible due to the fact that there is an automorphism of $\mathbb{Z}/2$ -graded C^* -algebras

$$\alpha: \mathbb{C}l_1 \rightarrow \mathbb{C}l_1, \quad e \mapsto -e$$

where e is *any* choice of the odd, self-adjoint generator of $\mathbb{C}l_1$. This, in particular, says that there is no canonical choice of the odd, self-adjoint generator of $\mathbb{C}l_1$. Taking the Kasparov product with the KK-class $[\alpha] \in \text{KK}(\mathbb{C}l_1, \mathbb{C}l_1)$ gives an automorphism of $\text{KK}(C_0(\mathbb{R}), \mathbb{C}l_1)$ that maps a generator to its inverse. Thus the generator of $\text{KK}(C_0(\mathbb{R}), \mathbb{C}l_1)$ cannot be canonical either. A similar statement holds for the KK-equivalence between $C_0(\mathbb{R}^2)$ and $\mathbb{C}l_2$, too, as $\mathbb{C}l_2 \simeq \mathbb{C}l_1 \hat{\otimes} \mathbb{C}l_1$ carries $\text{id} \hat{\otimes} \alpha$, and $C_0(\mathbb{R}^2)$ carries the antipodal map, either induces an automorphism of the K_0 -group that maps a generator to its inverse.

For the KK-equivalence between $C_0(\mathbb{R}^2)$ and $\mathbb{C}$, one may hope to seek a canonical generator using the ordered structure on K_0 -groups of *ungraded* C^* -algebras. More precisely, if A is a stably finite and unital C^* -algebra⁴, then it is well-known (cf. [Bla98, Proposition 6.3.3]) that $K_0(A)$ is an ordered abelian group. Since $C_0(\mathbb{R}^2)$ is not unital, an order structure has to come from a unitisation of $C_0(\mathbb{R}^2)$, whereas every projection in $\mathbb{M}_n(C_0(\mathbb{R}^2))$ represents the zero class in $K_0(C_0(\mathbb{R}^2))$. This means that for any unitisation of $C_0(\mathbb{R}^2)$, restricting its order structure to $C_0(\mathbb{R}^2)$ yields a trivial partial order. So there is no canonical way to distinguish the two generators of $K_0(C_0(\mathbb{R}^2))$, either.

The lengthy discussion presented above aims at clarifying the possible occurrence of a sign from the *odd* index pairing, that is, a (general) Kasparov product of either of the forms

$$\begin{aligned} \text{KK}(\mathbb{C}, A) \times \text{KK}(A, B \hat{\otimes} \mathbb{C}l_1) &\rightarrow \text{KK}(\mathbb{C}, B \hat{\otimes} \mathbb{C}l_1), \\ \text{KK}(\mathbb{C}, A \hat{\otimes} \mathbb{C}l_1) \times \text{KK}(A, B \hat{\otimes} \mathbb{C}l_1) &\rightarrow \text{KK}(\mathbb{C}, B \hat{\otimes} \mathbb{C}l_2). \end{aligned}$$

Due to the arbitrary choice of the KK-equivalences $\text{KK}(\mathbb{C}l_1, C_0(\mathbb{R}))$ and $\text{KK}(\mathbb{C}l_2, \mathbb{C})$, the isomorphisms $\text{KK}(\mathbb{C}, B \hat{\otimes} \mathbb{C}l_1) \simeq K_1(B)$ and $\text{KK}(\mathbb{C}, B \hat{\otimes} \mathbb{C}l_2) \simeq K_0(B)$ cannot be completely canonical. This means, in particular, that there is an artificial “choice” of the sign in odd index pairings. The emergence of such a sign is usually

⁴A unital C^* -algebra A is stably finite iff every isometry in $\mathbb{M}_n(A)$ is a unitary.

neglected due to certain *conventions* arising in the following incomplete list of objects:

- winding number (clockwise or anti-clockwise);
- boundary map in K-theory (with plus or minus sign, cf. [Ell14]);
- Fredholm index (either $\dim \ker - \dim \operatorname{coker}$ or $\dim \operatorname{coker} - \dim \ker$);
- spectral flow (positive-to-negative or negative-to-positive);
- signature of the *spectral localiser* (with plus or minus sign, cf. Chapter 4 and Remark 4.41);
- ...

Each of these conventions gives a particular choice of a sign. They are not always compatible with each other. For example, the Noether–Gohberg–Krein index theorem identifies the index of a Toeplitz operator on the Hardy space with *minus* the winding number of its symbol.

2.2.3. The graded, ungraded and extension pictures of KK_1

Similar to K_1 , the higher Kasparov group $\mathrm{KK}_1(A, B)$ can be defined as $\mathrm{KK}(A, SB)$ due to the split exactness of the functor $\mathrm{KK}: \mathcal{C}^*\mathfrak{Sep} \rightarrow \mathfrak{Ab}$. Alternatively, one may use the Clifford suspension to define

$$\mathrm{KK}_1(A, B) := \mathrm{KK}(A, B \otimes \mathbb{C}\ell_1).$$

This definition coincides with $\mathrm{KK}(A, SB)$ due to Bott periodicity (cf. Theorem 2.18). It also follows from Bott periodicity that $\mathrm{KK}_n(A, B) := \mathrm{KK}(A, B \otimes \mathbb{C}\ell_n)$ is isomorphic to $\mathrm{KK}_{n-2}(A, B)$ for $n \geq 2$. Thus there are no more KK-theory groups for C^* -algebras other than KK_0 and KK_1 .

We highlight the following well-known result:

Lemma 2.21. *Let A and B be ungraded separable C^* -algebras. Every class in $\mathrm{KK}(A, B \otimes \mathbb{C}\ell_1)$ can be represented by a Kasparov A - $B \otimes \mathbb{C}\ell_1$ -module of the form*

$$(\mathcal{E} \otimes \mathbb{C}\ell_1, F \otimes e)$$

where $\mathcal{E}$ is an ungraded C^* - A - B -correspondence, $F \in \mathbb{B}_B(\mathcal{E})$, and e is a choice of an odd generator of $\mathbb{C}\ell_1$. Two Kasparov A - $B \otimes \mathbb{C}\ell_1$ -modules of the form $(\mathcal{E}_1 \otimes \mathbb{C}\ell_1, F_1 \otimes e)$ and $(\mathcal{E}_2 \otimes \mathbb{C}\ell_1, F_2 \otimes e)$ are homotopic, iff there exists a Kasparov A - $C([0, 1], B) \otimes \mathbb{C}\ell_1$ -module

$(\mathcal{E}, F \otimes e)$, such that

$$\begin{aligned}(\mathcal{E}_0, F_0) &\simeq (\mathcal{E} \otimes_{\text{ev}_0} B, F \otimes_{\text{ev}_0} 1), \\(\mathcal{E}_1, F_1) &\simeq (\mathcal{E} \otimes_{\text{ev}_1} B, F \otimes_{\text{ev}_1} 1).\end{aligned}$$

As a consequence, $\text{KK}_1(A, B)$ may be defined via *ungraded* or *odd* Kasparov modules:

Definition 2.22. Let A and B be ungraded C^* -algebras. An *odd Kasparov A - B -module* is a pair $(\mathcal{E}, F)$, where

- $\mathcal{E}$ is an ungraded C^* - A - B -correspondence;
- $F \in \mathbb{B}_B(\mathcal{E})$ satisfies

$$[F, a] \in \mathbb{K}_B(\mathcal{E}), \quad a(F^2 - 1) \in \mathbb{K}_B(\mathcal{E}), \quad a(F - F^*) \in \mathbb{K}_B(\mathcal{E}) \quad (2.23)$$

for all $a \in A$.

Write $\text{KK}_1(A, B)$ for the collection of all *odd* Kasparov (A, B) -modules.

It follows immediately from the previous lemma that there is an isomorphism of abelian groups:

$$\text{KK}_1(A, B)/\sim \quad \text{and} \quad \text{KK}(A, B \otimes \mathbb{C}\ell_1).$$

Now we describe the relation between $\text{KK}_1(A, B)$ with invertible extensions. Let M be a C^* -algebra and $J \triangleleft M$ be an ideal. By universal property, there is a unique $*$ -homomorphism $\beta: A \rightarrow \mathcal{Q}(J)$ making the following diagram commutes:

$$\begin{array}{ccccc} J & \hookrightarrow & M & \twoheadrightarrow & A \\ \parallel & & \downarrow & & \downarrow \beta \\ J & \hookrightarrow & \mathcal{M}(J) & \xrightarrow{q} & \mathcal{Q}(J) \end{array}$$

The $*$ -homomorphism $\beta: A \rightarrow \mathcal{Q}(J)$ is called the *Busby invariant* of the extension $J \hookrightarrow M \twoheadrightarrow A$.

Let $\mathcal{H}_B$ be the standard countably generated Hilbert B -module, and $J := \mathbb{K}_B(\mathcal{H}_B) \simeq \mathbb{K} \otimes B$. Then $\mathcal{M}(J) = \mathcal{M}(B \otimes \mathbb{K})$. There is a direct sum operation on extensions of the form

$$\mathbb{K}_B(\mathcal{H}_B) \hookrightarrow M \twoheadrightarrow A, \quad (2.24)$$

defined in terms of their corresponding Busby invariants:

$$\beta_1 \oplus \beta_2: A \rightarrow \mathbb{M}_2(\mathcal{Q}(\mathbb{K} \otimes B)) \simeq \mathcal{Q}(\mathbb{K} \otimes B).$$

An extension of the form (2.24) is *invertible* iff its Busby invariant $\beta: A \rightarrow \mathcal{Q}(\mathbb{K} \otimes B)$ lifts to a completely positive contraction $s: A \rightarrow \mathcal{M}(\mathbb{K} \otimes B)$. By Kasparov–Stinespring dilation theorem [Kas80a, Theorem 3], such a completely positive contraction dilates to a $*$ -homomorphism $\bar{s}: A \rightarrow \mathcal{M}(\mathbb{K} \otimes B)$: there exists an orthogonal projection $P \in \mathcal{M}(\mathbb{K} \otimes B)$ such that $s(a) = P\bar{s}(a)P$.

The main result of [Kas80b, Section 7] can be rephrased as follows: given an *invertible* extension of the form (2.24) with Busby invariant $\beta: A \rightarrow \mathcal{Q}(\mathbb{K} \otimes B)$ and completely positive contraction $s: A \rightarrow \mathcal{M}(\mathbb{K} \otimes B)$, dilate s to a $*$ -homomorphism $\bar{s}: A \rightarrow \mathcal{M}(\mathbb{K} \otimes B)$ such that $s(a) = P\bar{s}(a)P$, then there is an odd Kasparov A - B -module

$$(\mathcal{H}_B, 2P - 1).$$

Conversely, given an odd Kasparov A - B -module of the form $(\mathcal{H}_B, F)$, it yields the Busby invariant

$$\beta: A \rightarrow \mathcal{Q}(\mathbb{K} \otimes B), \quad \beta(a) := q \left(\frac{F+1}{2} a \frac{F+1}{2} \right).$$

Upon choosing suitable equivalence relations on the set of invertible extensions, Kasparov showed that there is an isomorphism (cf. [Kas80b, Section 7, Lemma 2 and Theorem 1]):

$$\text{Ext}(A, B)^{-1} \simeq \text{KK}_1(A, B) \tag{2.25}$$

between the group of invertible extensions of A by $\mathbb{K} \otimes B$, and the Kasparov group $\text{KK}_1(A, B)$.

2.3. Connes–Higson E-theory

By Theorem 2.4, for every separable C^* -algebra A and *semi-split* extension

$$I \longrightarrow E \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow[s]{\quad} \end{array} Q$$

of separable C^* -algebras, there is a long exact sequence

$$\begin{aligned} \cdots \rightarrow \mathrm{KK}_1(A, I) \rightarrow \mathrm{KK}_1(A, E) \rightarrow \mathrm{KK}_1(A, Q) \\ \rightarrow \mathrm{KK}_0(A, I) \rightarrow \mathrm{KK}_0(A, E) \rightarrow \mathrm{KK}_0(A, Q) \rightarrow \cdots \end{aligned}$$

of abelian groups. One also says that KK-theory has *excision* for all semi-split extensions. In particular, if E is K-nuclear in the sense of [Kas80b, Section 7], then any extension $I \hookrightarrow E \twoheadrightarrow Q$ admits a completely positive section $s: Q \rightarrow E$. Whilst the class of K-nuclear C*-algebras is rather large, Skandalis [Ska88] has shown that there exist C*-algebras that are nevertheless not K-nuclear. As a consequence, excision in KK-theory fails for them.

As a replacement, Higson constructed in [Hig90] a category, the E-theory category, that possesses a excision property for *all* extensions of C*-algebras, based on purely categorical methods. Later, Connes and Higson [CH90, CH89] provided a concrete model for E-theory based on *asymptotic morphisms*. In a similar fashion, E-theory can be defined for C*-algebras equipped with continuous actions of a second countable, locally compact group (cf. [HK01, GHT00]), and for $\mathbb{Z}/2$ -graded C*-algebras (cf. [HKT98, HG04]).

2.3.1. Asymptotic morphisms and K-theory

Definition 2.26. Let A and B be C*-algebras. An *asymptotic morphism* $\Phi: A \dashrightarrow B$ is a family of maps $(\Phi_t: A \rightarrow B)_{t \in [1, \infty)}$ such that:

- (1) $t \mapsto \Phi_t(a)$ is continuous for all $a \in A$;
- (2) for all $a, a' \in A$ and $\lambda \in \mathbb{C}$,

$$\Phi_t(a^*) - \Phi_t(a)^*, \quad \Phi_t(aa') - \Phi_t(a)\Phi_t(a'), \quad \Phi_t(\lambda a) - \lambda\Phi_t(a)$$

all tend to 0 as $t \rightarrow \infty$.

Two asymptotic morphisms $(\Phi_t: A \rightarrow B)_t$ and $(\Psi_t: A \rightarrow B)_t$ are *asymptotically equivalent*, if

$$\lim_{t \rightarrow \infty} \Phi_t(a) - \Psi_t(a) = 0, \quad \text{for all } a \in A.$$

Operations on asymptotic morphisms Let $(\Phi_t: A \rightarrow B)_{t \in [1, \infty)}$ be an asymptotic morphism.

- (1) The unitisation of $(\Phi_t)_t$ is the asymptotic morphism $(\Phi_t^+: A^+ \rightarrow B^+)$,

$$\Phi_t^+: A^+ \rightarrow B^+, \quad \Phi_t^+(a + \lambda) := \Phi_t(a) + \lambda.$$

- (2) For each natural number n , define the asymptotic morphism $(\Phi_t^n: \mathbb{M}_n(A) \rightarrow \mathbb{M}_n(B))$,

$$\Phi_t^n: \mathbb{M}_n(A) \rightarrow \mathbb{M}_n(B), \quad \Phi_t^n((a_{ij})_{i,j=1}^n) := (\Phi_t(a_{ij}))_{i,j=1}^n.$$

- (3) Extension to matrix algebras as in (2) may be regarded as a special case of the following tensor product construction. Let $(\Phi_t: A \rightarrow B)_t$ and $(\Psi_t: C \rightarrow D)_t$ be two asymptotic morphisms. Then there is an asymptotic morphism $((\Phi \otimes \Psi)_t: A \otimes_{\max} C \rightarrow B \otimes_{\max} D)$,

$$(\Phi \otimes \Psi)_t: A \otimes_{\max} C \rightarrow B \otimes_{\max} D, \quad (\Phi \otimes \Psi)_t(a \otimes c) := \Phi_t(a) \otimes \Psi_t(c),$$

where $\otimes_{\max}$ denotes the *maximal* tensor product of C^* -algebras.

An asymptotic morphism $\Phi: A \dashrightarrow B$ induces a map $\Phi_*: K_i(A) \rightarrow K_i(B)$ in K-theory for $i \in \{0, 1\}$. The case that we will be mostly interested in is $i = 0$. To define this induced map, we make the following simple observation.

Definition and Lemma 2.27 ([GHT00, Definition 1.1]). *Let B be a C^* -algebra. Define the asymptotic algebra of B as*

$$\mathfrak{A}B := C_b([1, \infty), B) / C_0([1, \infty), B).$$

Then there is a bijection between asymptotic equivalence classes of asymptotic morphisms $A \dashrightarrow B$ and $*$ -homomorphisms $A \rightarrow \mathfrak{A}B$.

Proof. Any asymptotic morphism $(\Phi_t: A \rightarrow B)_{t \in [1, \infty)}$ defines a $*$ -homomorphism

$$\tilde{\Phi}: A \rightarrow C_b([1, \infty), B), \quad a \mapsto (t \mapsto \Phi_t(a)).$$

Composition with the quotient map $q: C_b([1, \infty), B) \rightarrow \mathfrak{A}B$ yields a $*$ -homomorphism $A \rightarrow \mathfrak{A}B$. Two asymptotic morphisms (Φ_t) and (Φ'_t) define the same $*$ -homomorphism iff $t \mapsto \Phi_t(a) - \Phi'_t(a)$ belongs to $C_0([1, \infty), B)$ for all $a \in A$. This happens iff Φ and Φ' are asymptotically equivalent.

Conversely, let $\tilde{\Phi}: A \rightarrow \mathfrak{A}B$ be a $*$ -homomorphism. Choose any set-theoretic section $s: \mathfrak{A}B \rightarrow C_b([1, \infty), B)$, which exists by surjectivity of q . Define a family of linear maps $(\Phi_t: A \rightarrow B)_{t \in [1, \infty)}$ as

$$\Phi_t: A \xrightarrow{\tilde{\Phi}} \mathfrak{A}B \xrightarrow{s} C_b([1, \infty), B) \xrightarrow{\text{ev}_t} B.$$

We claim that $(\Phi_t)_{t \in [1, \infty)}$ is an asymptotic morphism. Let $a, a' \in A$ and $\lambda \in \mathbb{C}$. Since $\tilde{\Phi}$ is a $*$ -homomorphism, we have

$$s(\Phi(a^*)) - s(\Phi(a)^*), \quad s(\Phi(aa')) - s(\Phi(a)\Phi(a')), \quad s(\Phi(\lambda a)) - s(\lambda\Phi(a))$$

belongs to $C_0([1, \infty), B)$. Thus

$$\Phi_t(a^*) - \Phi_t(a)^*, \quad \Phi_t(aa') - \Phi_t(a)\Phi_t(a'), \quad \Phi_t(\lambda a) - \lambda\Phi_t(a)$$

all tends to 0 as $t \rightarrow \infty$. If s and s' are two different sections that give two different asymptotic morphisms (Φ_t) and (Φ'_t) , then $s \circ \Phi(a) - s' \circ \Phi(a) \in C_0([1, \infty), B)$ for all $a \in A$. Thus (Φ_t) and (Φ'_t) are asymptotically equivalent. $\square$

Lemma 2.28. *Let $(\Phi_t : A \rightarrow B)_{t \in [1, \infty)}$ be an asymptotic morphism. Then there is a well-defined group homomorphism*

$$\Phi_* : K_0(A) \xrightarrow{\tilde{\Phi}_*} K_0(\mathfrak{A}B) \xrightarrow{\sim} K_0(C_b([1, \infty), B)) \xrightarrow{\text{ev}_1} K_0(B).$$

Proof. To see the well-definedness, we only need to address the isomorphism $K_0(\mathfrak{A}B) \simeq K_0(C_b([1, \infty), B))$. This follows from the contractibility of $C_0([1, \infty), B)$ and the long exact sequence associated to the extension

$$C_0([1, \infty), B) \hookrightarrow C_b([1, \infty), B) \twoheadrightarrow \mathfrak{A}B. \quad \square$$

2.3.2. Asymptotic category and E-theory

Asymptotic morphisms can be composed, but not in a naïve way. Namely, the naïve guess $(\Psi_t \circ \Phi_t)$ for asymptotic morphisms $(\Phi_t : A \rightarrow B)_t$ and $(\Psi_t : B \rightarrow C)_t$ is in general not an asymptotic morphism. To overcome this, Connes and Higson showed that [CH90, Proposition 4, CH89, Section 2] that if A is separable, then there always exists a *reparametrisation* $s : [1, \infty) \rightarrow [1, \infty)$ of $(\Phi : A \rightarrow B)_t$, such that

$$(\Psi_t \circ \Phi_{s(t)})_t$$

is an asymptotic morphism. This yields a well-defined composition product in the *homotopy category* of asymptotic morphisms.

Below, we shall follow a different approach introduced in [GHT00, Section 2]. In this approach, the separability of A is not assumed at the first place, whereas there is still a well-defined composition product in the homotopy category of asymptotic

morphisms. The separability becomes crucial to descend the composition product to a smaller class of asymptotic morphisms, which corresponds to 1-homotopies thereof.

Definition and Lemma 2.29. *There is an endofunctor $\mathfrak{A}$ on the category of C^* -algebras, which on objects sends B to $\mathfrak{A}B$; and on arrows sends a $*$ -homomorphism $\varphi: A \rightarrow B$ to the $*$ -homomorphism*

$$\mathfrak{A}\varphi: \mathfrak{A}A \xrightarrow{s} C_b([1, \infty), A) \xrightarrow{\varphi_*} C_b([1, \infty), B) \xrightarrow{q} \mathfrak{A}B,$$

where $s: \mathfrak{A}B \rightarrow C_b([1, \infty), B)$ is any set-theoretic section of q .

Proof. Clearly, the post-composition

$$\varphi_*: C_b([1, \infty), A) \rightarrow C_b([1, \infty), B), \quad f \mapsto \varphi \circ f$$

maps $C_0([1, \infty), A)$ into $C_0([1, \infty), B)$. Thus it descends to a $*$ -homomorphism between the quotients $\mathfrak{A}A \rightarrow \mathfrak{A}B$ in the commutative diagram:

$$\begin{array}{ccccc} C_0([1, \infty), A) & \twoheadrightarrow & C_b([1, \infty), A) & \xrightarrow{q} & \mathfrak{A}A \\ \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \text{dashed} \\ C_0([1, \infty), B) & \twoheadrightarrow & C_b([1, \infty), B) & \xrightarrow{q} & \mathfrak{A}B. \end{array}$$

The dashed arrow is realised by the claimed formula. □

Let A be a C^* -algebra and $n \in \mathbb{N}$. Write

$$\mathfrak{A}^n A := \underbrace{\mathfrak{A} \circ \mathfrak{A} \circ \cdots \circ \mathfrak{A}}_{n\text{-fold composition}} A.$$

In particular, $\mathfrak{A}^0 = \text{id}$ and $\mathfrak{A}^0 A = A$.

Definition 2.30. Two $*$ -homomorphisms $\Phi_0, \Phi_1: A \rightarrow \mathfrak{A}^n B$ are n -homotopic, if there exists a $*$ -homomorphism

$$\Phi: A \rightarrow \mathfrak{A}^n(C[0, 1], B),$$

such that $\Phi_i = \mathfrak{A}^n(\text{ev}_i) \circ \Phi$ for $i = 0, 1$.

Example 2.31. A 0-homotopy is the same thing as a homotopy of $*$ -homomorphisms $A \rightarrow B$. A 1-homotopy is the same thing as a homotopy of asymptotic morphisms in the sense of [CH90, CH89]: two asymptotic morphisms $A \dashrightarrow B$, given by $(\Phi_t^0: A \rightarrow B)_{t \in [1, \infty)}$ and $(\Phi_t^1: A \rightarrow B)$, are *asymptotically homotopic*, if there exists an asymptotic morphism $A \dashrightarrow C[0, 1] \otimes B$ given by $(\Phi_t: A \rightarrow C([0, 1], B))_{t \in [1, \infty)}$, such that $\text{ev}_i \circ \Phi_t = \Phi_t^i$ for $i \in \{0, 1\}$ and $t \in [1, \infty)$.

The following lemma shows that if an asymptotic morphism is given by $*$ -homomorphisms Φ_t for all $t \in [1, \infty)$, then Φ_t is asymptotically homotopic to a constant asymptotic morphism.

Lemma 2.32. *If $(\Phi_t)_{t \in [1, +\infty)}$ is an asymptotic morphism such that each Φ_t is a $*$ -homomorphism. Then (Φ_t) is asymptotically homotopic to the constant family $(\Phi_1)_{t \in [1, \infty)}$.*

Proof. For each $t \in [1, +\infty)$, define $\Psi_t: A \rightarrow C([0, 1], B)$ by

$$\Psi_t(a)(s) := \Phi_{s(t-1)+1}(a).$$

Then $(\Psi_t)_{t \in [1, +\infty)}$ is an asymptotic morphism, satisfying $\text{ev}_0 \circ \Psi_t = \Phi_1$ and $\text{ev}_1 \circ \Psi_t = \Phi_t$. $\square$

Definition and Lemma 2.33. *Being n -homotopic is an equivalence relation on the set of $*$ -homomorphisms $A \rightarrow \mathfrak{A}^n B$. Denote by $\llbracket A, B \rrbracket_n$ the set of equivalence classes of n -homotopic $*$ -homomorphisms $A \rightarrow \mathfrak{A}^n B$.*

Proof. This is [GHT00, Proposition 2.3]. $\square$

Every $*$ -homomorphism $\varphi: A \rightarrow B$ can be regarded as a constant family $(\varphi: A \rightarrow B)_{t \in [1, \infty)}$. Equivalently, every $*$ -homomorphism φ gives another $*$ -homomorphism $\tilde{\varphi}: A \rightarrow \mathfrak{A}B$ given by $\tilde{\varphi}(a)(t) := \varphi(a)$ for all $t \in [1, \infty)$. Equivalently, there is a natural transformation of endofunctors on the category of C^* -algebras,

$$\alpha: \text{id} \Rightarrow \mathfrak{A}, \quad \alpha_A(\varphi) := \tilde{\varphi}.$$

Then α induces an inclusion of sets

$$\llbracket A, B \rrbracket_n \rightarrow \llbracket A, B \rrbracket_{n+1}$$

by sending the n -homotopy equivalence class of $\Phi: A \rightarrow \mathfrak{A}^n B$ to the $(n+1)$ -homotopy equivalence class of the $*$ -homomorphism

$$A \xrightarrow{\Phi} \mathfrak{A}^n B \xrightarrow{\mathfrak{A}^n(\alpha_B)} \mathfrak{A}^{n+1} B.$$

Lemma 2.34 ([GHT00, Definition 2.7, Proposition 2.12]). *Define*

$$\llbracket A, B \rrbracket := \varinjlim \llbracket A, B \rrbracket_n$$

as the direct limit under the inclusions $\llbracket A, B \rrbracket_n \rightarrow \llbracket A, B \rrbracket_{n+1}$ above. Then there is a well-defined composition map

$$\llbracket A, B \rrbracket \times \llbracket B, C \rrbracket \rightarrow \llbracket A, C \rrbracket$$

defined as follows: if $\Phi: A \rightarrow \mathfrak{A}^j B$ and $\Psi: B \rightarrow \mathfrak{A}^k C$ are $*$ -homomorphisms, then their composition is represented by the $*$ -homomorphism

$$A \xrightarrow{\Phi} \mathfrak{A}^j B \xrightarrow{\mathfrak{A}^j(\Psi)} \mathfrak{A}^{j+k} C.$$

Definition 2.35 ([GHT00, Definition 2.13]). *The homotopy category of asymptotic morphisms is the category $\mathfrak{A}\mathfrak{s}\mathfrak{h}\mathfrak{m}$ whose objects are C^* -algebras, and an arrow from A to B is an element in $\llbracket A, B \rrbracket$. Composition of arrows is given as in Lemma 2.34.*

Asymptotic morphisms become composable in $\mathfrak{A}\mathfrak{s}\mathfrak{h}\mathfrak{m}$, after paying the price of passing to higher homotopies. For separable C^* -algebras, such higher homotopies can always be reduced to 1-homotopies, i.e. asymptotic homotopies. The following theorem recovers the original composition product of asymptotic morphisms for separable C^* -algebras:

Theorem 2.36 ([GHT00, Theorem 2.16]). *For any separable C^* -algebra A , the natural map*

$$\llbracket A, B \rrbracket_1 \rightarrow \llbracket A, B \rrbracket$$

is a bijection. That is, every arrow in $\mathfrak{A}\mathfrak{s}\mathfrak{h}\mathfrak{m}$ with source a separable C^* -algebra can be represented by the 1-homotopy equivalence class of an asymptotic morphism.

E-theory can be viewed, in a suitable sense, as the stable homotopy category of asymptotic morphisms.

Theorem 2.37 ([CH90, Corollaire 9; GHT00, Definition 6.8]). *Let $\mathfrak{E}$ be the category whose objects are separable C^* -algebras, and an arrow from A to B is an element in*

$$E(A, B) := \llbracket C_0(0, 1) \otimes A \otimes \mathbb{K}, C_0(0, 1) \otimes B \otimes \mathbb{K} \rrbracket.$$

Then $\mathfrak{E}$ is an additive category, and the canonical functor $\mathfrak{E}^* \mathfrak{S}\mathfrak{e}\mathfrak{p} \rightarrow \mathfrak{E}$ is the universal C^* -stable, exact and homotopy invariant functor.

Remark 2.38. We remark on the following well-known facts about E-theory, and refer to their proofs in [GHT00, HG04]:

- (1) The tensor factors $C_0(0, 1)$ and $\mathbb{K}$ provides the additive structure of E-theory: the addition is implemented by inclusion of C^* -algebras

$$\mathbb{K}(\mathcal{H}) \oplus \mathbb{K}(\mathcal{H}) \hookrightarrow \mathbb{K}(\mathcal{H} \oplus \mathcal{H}) \simeq \mathbb{K}(\mathcal{H}),$$

and the inverse of a class in $E(A, B)$ is implemented by composing the reflection automorphism (cf. [CH90, Proposition 6, GHT00, Theorem 6.7]):

$$C_0(\mathbb{R}) \rightarrow C_0(\mathbb{R}), \quad f \mapsto (x \mapsto f(-x)).$$

- (2) Equivalently, one can replace $C_0(0, 1)$ by $C_0(\mathbb{R})$ or $C_0(-1, 1)$ in Theorem 2.37 as they are all isomorphic to each other. The first replacement is convenient while working with *unbounded* self-adjoint operators (whose spectra are unbounded subsets of $\mathbb{R}$), which will be adopted in Chapters 3 and 4. The second replacement will be suitable for working with bounded transforms of unbounded operators, and has been used in [CH89].
- (3) Another equivalent formulation of $E(A, B)$ is given by

$$[[C_0(0, 1) \otimes A, C_0(0, 1) \otimes B \otimes \mathbb{K}]].$$

It can be identified with the definition in Theorem 2.37 using any rank-one corner embedding $\mathbb{C} \hookrightarrow \mathbb{K}$, cf. [GHT00, Lemma 6.25; HG04, Lemma 2.2].

- (4) There is a canonical functor

$$\mathfrak{Asym} \rightarrow \mathfrak{E}, \quad \Phi \mapsto \text{id}_{C_0(0,1)} \otimes \Phi \otimes \text{id}_{\mathbb{K}}$$

for every $\Phi: A \rightarrow \mathfrak{A}''B$. In particular, the universal functor $E: \mathfrak{C}^* \mathfrak{Sep} \rightarrow \mathfrak{E}$ factors through it. Thus every asymptotic morphism $\Phi: A \dashrightarrow B$ represents a class $[[\Phi]] \in E(A, B)$ represented by $\text{id}_{C_0(0,1)} \otimes \Phi \otimes \text{id}_{\mathbb{K}}$, such that Φ and $[[\Phi]]$ induce the same map in K-theory, cf. [GHT00, Theorem 6.9].

- (5) Similar to KK-theory, the abelian group $E_1(A, B)$ may be defined as

$$E_1(A, B) := E_1(A, SB) := [[C_0(0, 1) \otimes A \otimes \mathbb{K}, C_0(0, 1)^2 \otimes B \otimes \mathbb{K}]].$$

By composing with an “inverse Bott map”, i.e. an asymptotic morphism $C_0(0, 1)^2 \dashrightarrow \mathbb{K}$ that is invertible in $\mathfrak{Asym}$ (which is well-known to exist,

cf. [Wil21, Lemma 7.3]), then $E_1(A, B)$ can equivalently be represented as

$$\llbracket C_0(0, 1) \otimes A, B \otimes \mathbb{K} \rrbracket.$$

2.4. Index pairings

The most involved construction of both bivariant K-theories $F \in \{KK, E\}$ are the composition products

$$F(A, B) \times F(B, C) \rightarrow F(A, C). \quad (F)$$

Here being involved means two things: it is both difficult to *prove* the existence and uniqueness of such a product, and difficult to *compute* in concrete examples. For the Fredholm picture of KK-theory, the technicality comes from constructing the desired Kasparov module together with a suitable connection such that Connes–Skandalis conditions are satisfied (cf. Theorem 2.13). In E-theory, the technicality comes from a suitable reparametrisation of the asymptotic morphisms, as well as describing the composition asymptotic morphism (or its induced map in K-theory) in a computable fashion.

Nevertheless, for the case $A = \mathbb{C}$, then this computational technicality vanishes simultaneously for both KK-theory and E-theory. Using the general form of the Kasparov product Equation (2.16), we have the following induced map induce a map in K-theory:

$$\underbrace{KK(\mathbb{C}, S^i B) \times KK_j(B, C)}_{\simeq K_i(B)} \rightarrow \underbrace{KK(\mathbb{C}, S^{i+j} C)}_{\simeq K_{i+j}(C)}. \quad (KK)$$

We refer to (KK) as the *KK-theoretic index pairing*, which is *even* if $j = 0$; and *odd* if $j = 1$. The *E-theoretic index pairing* is defined likewise.

2.4.1. “Triviality” of index pairings

Assume that B is a *unital* ungraded C^* -algebra. We first claim that both KK-theoretic and E-theoretic index pairings can be computed by simple formulae. This is due to the fact that every class in $KK(\mathbb{C}, B)$ can be represented by a $\mathbb{Z}/2$ -graded *fgp* Hilbert B -module together with the *zero* operator; and every class in $E(\mathbb{C}, B)$ can be represented by a **-homomorphism* instead of an asymptotic morphism.

These results are well-known, and we collect them here for convenience.

The case of $\mathbf{KK}$ For the $\mathbf{KK}$ -theoretic index pairing, we have the following simplification:

Lemma 2.39. *Let B be a unital ungraded C^* -algebra, then every class in $\mathbf{KK}(\mathbb{C}, B)$ can be represented by a Kasparov module of the form $(\mathcal{E}_1, 0)$, where $\mathcal{E}_1$ is a $\mathbb{Z}/2$ -graded fgp Hilbert B -module.*

Proof. This is proved by Kasparov [Kas80b, Section 6, Theorem 3]. We note, in particular, that every $\mathbb{Z}/2$ -graded fgp Hilbert B -module $\mathcal{E}$ takes the form (cf. Example 2.5):

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-, \quad \mathcal{E}_+ = pB^n, \quad \mathcal{E}_- = qB^m$$

where $p \in \mathbb{M}_n(B)$ and $q \in \mathbb{M}_m(B)$ are projections. This gives the standard isomorphism

$$\mathbf{KK}(\mathbb{C}, B) \rightarrow K_0(B), \quad [pB^n \oplus qB^m, 0] \mapsto [p] - [q]. \quad \square$$

Recall that the inverse of $[\mathcal{E}, F] \in \mathbf{KK}(A, B)$ is given by $[\mathcal{E}^{\text{op}}, -F]$. It follows that

$$[pB^n \oplus qB^m, 0] = [pB^n \oplus 0, 0] - [0 \oplus qB^m, 0]. \quad (2.40)$$

Hence, in order to compute the Kasparov product of the form $(\mathbf{KK})$, it suffices to consider classes in $\mathbf{KK}(\mathbb{C}, B)$ of the form $[pB^n \oplus 0, 0]$.

Theorem 2.41. *Let $\mathcal{E}_1$ be a $\mathbb{Z}/2$ -graded fgp Hilbert B -module of the form $pB^n \oplus 0$. Let $(\mathcal{E}_2, F_2)$ be a Kasparov B - $\mathbb{C}$ -module. Then the Kasparov product of $[\mathcal{E}_1, 0] \in \mathbf{KK}(\mathbb{C}, B)$ and $[\mathcal{E}_2, F_2] \in \mathbf{KK}(B, \mathbb{C})$ is represented by the Kasparov $\mathbb{C}$ - $\mathbb{C}$ -module $(p\mathcal{E}_2^n, pF_2p)$.*

Proof. We have an isomorphism of $\mathbb{Z}/2$ -graded Hilbert B -modules

$$(pB^n \oplus 0) \otimes_B \mathcal{E}_2 \simeq p\mathcal{E}_2^n, \quad p(b_1, \dots, b_n) \otimes y \mapsto p(b_1y, \dots, b_ny),$$

where $p \in \mathbb{M}_n(B) \subseteq \mathbb{M}_n(\mathbb{B}_C(\mathcal{E}_2)) = \mathbb{B}_C(\mathcal{E}_2^n)$ is even for the grading on $\mathcal{E}_2$. Thus pF_2p is a well-defined odd element in $p\mathbb{B}_C(\mathcal{E}_2^n)p = \mathbb{B}_C(p\mathcal{E}_2^n)$. The Fredholm conditions (2.9) are automatic because $p\mathcal{E}_2^n$ is fgp. Thus $(p\mathcal{E}_2^n, pF_2p)$ is a Kasparov $\mathbb{C}$ - $\mathbb{C}$ -module.

It remains to verify the connection conditions of Connes and Skandalis Theorem 2.13. Let $x \in pB^n$, i.e. there exists $(b_1, \dots, b_n) \in B^n$ such that $x = p(b_1, \dots, b_n)$.

For convenience, we introduce the following notation:

$$\begin{aligned}\mathbf{b} &:= \text{diag}(b_1, \dots, b_n) \in \mathbb{M}_n(B) \subseteq \mathbb{B}_C(\mathcal{E}_2^n), \\ \mathbf{p} &:= \text{diag}(p, \dots, p) \in \mathbb{B}_C(\mathcal{E}_2^n), \\ \mathbf{F}_2 &:= \text{diag}(F_2, \dots, F_2) \in \mathbb{B}_C(\mathcal{E}_2^n), \\ \mathbf{y} &:= (y, \dots, y) \in \mathcal{E}_2^n.\end{aligned}$$

Then the operator

$$T: \mathcal{E}_2 \rightarrow p\mathcal{E}_2^n, \quad Ty := (pF_2p)(x \otimes y) - x \otimes F_2y$$

in the connection condition (CoSk1) is given by

$$\begin{aligned}Ty &= pF_2p \cdot p(b_1y, b_2y, \dots, b_ny) - p(b_1, b_2, \dots, b_n) \cdot F_2y \\ &= \mathbf{pF}_2\mathbf{pby} - \mathbf{pbF}_2\mathbf{y} = \mathbf{p}[\mathbf{F}_2, \mathbf{pb}]y,\end{aligned}$$

while $\mathbf{p}[\mathbf{F}_2, \mathbf{b}] \in \mathbb{K}_C(\mathcal{E}_2^n)$ because $(\mathcal{E}_2, F_2)$ is an essential Kasparov B - C -module. This implies that the operator $y \mapsto \mathbf{p}[\mathbf{F}_2, \mathbf{pb}]y$ belongs to $\mathbb{K}_C(\mathcal{E}_2, p\mathcal{E}_2^n)$.

The positivity condition (CoSk2) is trivially satisfied because the zero operator commutes with F_2 . Thus the Connes–Skandalis' conditions of Theorem 2.13 are satisfied as claimed. $\square$

Remark 2.42. It is clear from the proof of Theorem 2.41 that all the simplifications of Lemma 2.39 are needed: the projectivity of $\mathcal{E}_1$ is used for constructing the operator pF_2p , which is well-defined due to the existence of the projection $p \in \mathbb{M}_n(B)$ (so pF_2p is well-defined); the zero operator trivialises the positivity condition (CoSk2), and allows for the direct sum decomposition of (2.40). The finite generation of $\mathcal{E}_1$ yields the isomorphisms $\mathcal{E}_1 \simeq pB^n$. This is used to prove the connection condition (CoSk1).

The case of E We claim that every class in $E_i(\mathbb{C}, B)$ for $i \in \{0, 1\}$ can be represented by a $*$ -homomorphism.

Theorem 2.43. *Every class in $E(\mathbb{C}, B)$ can be represented by a $*$ -homomorphism $C_0(0, 1) \rightarrow C_0(0, 1) \otimes B \otimes \mathbb{K}$. Define $E_1(\mathbb{C}, B) := E(\mathbb{C}, SB)$. Then every class in $E_1(\mathbb{C}, B)$ can be represented by a $*$ -homomorphism $C_0(0, 1) \rightarrow B \otimes \mathbb{K}$.*

Proof. We may replace $E(\mathbb{C}, B)$ by

$$\llbracket C_0(0, 1), C_0(0, 1) \otimes B \otimes \mathbb{K} \rrbracket$$

using (3) of Remark 2.38. In the following, let $A := C_0(0, 1) \otimes B$. We claim that there is a bijection between $[C_0(0, 1), A \otimes \mathbb{K}]$ and path components of the set

$$\mathcal{U} := \{u \in (A \otimes \mathbb{K})^+ \mid uu^* = u^*u = 1, \pi_{(A \otimes \mathbb{K})^+}(u) = 1\}.$$

To see this, for every *-homomorphism $\varphi: C_0(0, 1) \rightarrow A \otimes \mathbb{K}$, extend it to the unitisation to obtain the *-homomorphism $\varphi^+: C_0(0, 1)^+ \rightarrow (A \otimes \mathbb{K})^+$. The C*-algebra $C_0(0, 1)^+ \simeq C(\mathbb{T})$ is generated by the unitary function $f(x) = e^{i2\pi x}$ on $(0, 1)$. Define

$$u_\varphi := \varphi^+(f) \in (A \otimes \mathbb{K})^+,$$

then u_φ is a unitary in $(A \otimes \mathbb{K})^+$ that satisfies $\pi_{(A \otimes \mathbb{K})^+}(u_\varphi) = 1$ because $e^{i2\pi \cdot 0} = e^{i2\pi \cdot 1} = 1$. Thus u_φ belongs to the set $\mathcal{U}$. If φ_0 and φ_1 are homotopic *-homomorphisms $C_0(0, 1) \rightarrow A \otimes \mathbb{K}$, then this homotopy also yields a homotopy between their corresponding unitaries.

Conversely, assume that u belongs to the set $\mathcal{U}$. Let $\varphi_u^+: C_0(0, 1)^+ \rightarrow (A \otimes \mathbb{K})^+$ be the unique *-homomorphism sending $f(x) = e^{i2\pi x}$ to u . Since $\pi_{(A \otimes \mathbb{K})^+}(u) = 1$, the *-homomorphism restricts to $\varphi_u: C_0(0, 1) \rightarrow A \otimes \mathbb{K}$. If u_0 and u_1 are two such elements which belong to the same path component, then there is a homotopy of unitaries $(u_t)_{t \in [0, 1]}$ in $(A \otimes \mathbb{K})^+$ connecting u_0 and u_1 , and such that $\pi_{(A \otimes \mathbb{K})^+}(u_t) = 1$ for all t . Such a homotopy gives to a homotopy between *-homomorphisms $C_0(0, 1) \rightarrow A \otimes \mathbb{K}$ determined by u_0 and u_1 .

Now let $(\Phi_t: C_0(0, 1) \rightarrow A \otimes \mathbb{K})_{t \in [1, \infty)}$ be an asymptotic morphism. By passing to their unitisations, we have a family of elements $u_t \in (A \otimes \mathbb{K})^+$, defined by $u_t := \Phi_t(f)$, satisfying $\pi_{(A \otimes \mathbb{K})^+}(u_t) = 1$ and

$$u_t^* u_t \rightarrow 1, \quad u_t u_t^* \rightarrow 1 \quad \text{as } t \rightarrow \infty.$$

In particular, this shows that there exists $t_0 > 1$ such that $u_t^* u_t$ is invertible for all $t > t_0$. This allows us to define a family of unitaries in $(A \otimes \mathbb{K})^+$:

$$v_t := u_{t+t_0} (u_{t+t_0}^* u_{t_0})^{-1/2} \quad \text{for all } t \in [1, \infty)$$

Each v_t defines a *-homomorphism $\varphi_{v_t}: C_0(0, 1) \rightarrow A \otimes \mathbb{K}$, and such that (φ_{v_t}) is an asymptotic morphism that is asymptotic equivalent to $(\Phi_t)_t$. By Lemma 2.32, such an asymptotic family is asymptotically equivalent to the *-homomorphism φ_{v_1} .

The case of $E_1(\mathbb{C}, B) := E(\mathbb{C}, SB)$ can be dealt with in a similar way, using (5) of

Remark 2.38. □

Remark 2.44. The construction in proof above, namely, representing a class in $K_0(B)$ (resp. $K_1(B)$) by a homotopy class of $*$ -homomorphisms $C_0(0, 1) \rightarrow C_0(0, 1) \otimes B \otimes \mathbb{K}$ (resp. $C_0(0, 1) \rightarrow B \otimes \mathbb{K}$) is also known as the spectral picture of K-theory, cf. [Tro00].

As a consequence, we obtain the following simplification of the E-theory product. Let $\varphi: C_0(0, 1) \rightarrow C_0(0, 1) \otimes B \otimes \mathbb{K}$ be a $*$ -homomorphism, and $\Psi: C_0(0, 1) \otimes B \otimes \mathbb{K} \dashrightarrow C_0(0, 1) \otimes C \otimes \mathbb{K}$ be an asymptotic morphism. Then their composition is given by the asymptotic morphism

$$C_0(0, 1) \xrightarrow{\varphi} C_0(0, 1) \otimes B \otimes \mathbb{K} \dashrightarrow^{\Psi} C_0(0, 1) \otimes C \otimes \mathbb{K}.$$

2.4.2. Uniqueness of the even index pairing

We end this section with a characterisation of the even index pairing using abstract nonsense given in Section 2.1. We shall explain that there is a *unique* even index pairing in both KK-theory and E-theory, which induce the *same* K-theoretic even index pairing. This is due to the existence of both categories $\mathfrak{K}\mathfrak{K}$ and $\mathfrak{E}$. Such a uniqueness does not hold in the odd case as neither $\mathfrak{K}\mathfrak{K}_1$ nor $\mathfrak{E}_1$ yields a category on its own.

An *KK-theoretic even index pairing* (**KK**) is, in terms of abstract nonsense, a functor ($i = 0, 1$)

$$\sharp_i^{\mathfrak{K}\mathfrak{K}}: \mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{Ab}$$

such that $\sharp_i^{\mathfrak{K}\mathfrak{K}}(A) = K_i(A)$ for every C*-algebra A . That is, for each element $x \in \mathfrak{K}\mathfrak{K}(A, B)$, $\sharp_i(x)$ is a group homomorphism $K_i(A) \rightarrow K_i(B)$. Replacing $\mathfrak{K}\mathfrak{K}$ with $\mathfrak{E}$ yields an E-theoretic even index pairing functor $\sharp_i^{\mathfrak{E}}: \mathfrak{E} \rightarrow \mathfrak{Ab}$ for $i = 0, 1$.

Theorem 2.45. *For $i = \{0, 1\}$, there is a unique KK-theoretic and a unique E-theoretic even index pairing functor that is compatible with $*$ -homomorphisms in the following sense: if $\varphi: A \rightarrow B$ is a $*$ -homomorphism, then $\sharp_i^{\mathfrak{K}\mathfrak{K}}([\varphi]): K_i(A) \rightarrow K_i(B)$ coincides with the induced map $K_i(\varphi): K_i(A) \rightarrow K_i(B)$ in K_i .*

Write $\sharp_i^{\mathfrak{K}\mathfrak{K}}$ and $\sharp_i^{\mathfrak{E}}$ for this unique functor, then

$$\sharp_i^{\mathfrak{K}\mathfrak{K}} = \sharp_i^{\mathfrak{E}} \circ \natural_i$$

where $\natural_i: \mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{E}$ is the universal comparison functor.

Proof. Let K_i be the functor $\mathfrak{C}^*\mathfrak{S}\mathfrak{cp} \rightarrow \mathfrak{Ab}$ sending a C*-algebra A to $K_i(A)$, and a $*$ -homomorphism $f: A \rightarrow B$ to its induced map $K_i(f): K_i(A) \rightarrow K_i(B)$ in

K-theory. For both $i \in \{0, 1\}$, the functor K_i is split exact and homotopy invariant. Thus it follows from the universal property of $\mathfrak{K}\mathfrak{K}$ that the functor K_i factors through the canonical functor $\mathbf{KK}: \mathcal{C}^*\mathfrak{Sep} \rightarrow \mathfrak{K}\mathfrak{K}$, and the factorisation is unique: there is a unique functor $\natural_i^{\mathbf{KK}}: \mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{Ab}$ that fits in the dotted arrow:

$$\begin{array}{ccc} & \mathcal{C}^*\mathfrak{Sep} & \\ \mathbf{KK} \swarrow & & \searrow K_i \\ \mathfrak{K}\mathfrak{K} & \cdots \cdots \cdots & \mathfrak{Ab}. \end{array}$$

The universal property of E-theory also gives a unique even index pairing functor $\sharp_i^E: \mathcal{C} \rightarrow \mathfrak{Ab}$. In particular, $\sharp_i^E \circ \natural$ is yet another functor $\mathfrak{K}\mathfrak{K} \rightarrow \mathfrak{Ab}$ that fits in the dotted arrow above. Thus the uniqueness forces $\sharp_i^{\mathbf{KK}} = \sharp_i^E \circ \natural$. $\square$

Notes

The original formulation of both Kasparov theory [Kas80b] and E-theory [CH90] requires that C^* -algebra A to be separable, so as to guarantee the existence of the composition product (F). This extra assumption is sometimes not desirable due to the existence of important non-separable C^* -algebras, like the Roe C^* -algebras in coarse geometry.

Removing this technical assumption is highly non-trivial and requires completely new ideas. I do not know many such pictures. For Kasparov theory, one way is to pass to a more homotopy-theoretic picture due to Bunke, Engel and Land [BEL21]. For E-theory, separability is not required for constructing the composition product (cf. construction of the category $\mathfrak{A}^{\text{sym}}$ in Section 2.3), but is required for proving exactness of the functor $E: \mathcal{C}^*\mathfrak{Sep} \rightarrow \mathcal{C}$. Fixing this problem requires a larger class of cycles for E-theory. This has been achieved by Wulff [Wul22]. Homotopy-theoretic idea has also been investigated by Bunke and Dünzinger [BD24].