Spectral localisers and aperiodic topological phases in noncommutative geometry
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1. Introduction

> Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezzi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.
>
> Galileo Galilei
> *Il Saggiatore*

Galileo believed that law of the universe is written in the language of mathematics. This is genuinely reflected by the philosophy of mathematical physics, which models physical systems with mathematics. A computation in mathematical physics is thus, the procedure of obtaining some numerical quantity from logical deduction, which ought to agree with the physically observed one in an experiment. Immediately, we are laden with the following challenges:

Modelling — A physical system always contains a huge number of particles. These particles carry large amount of information. It is difficult to extract useful physical quantities from them.

Computation — Physical experiments are always restricted by an energy threshold or space scale. Therefore one cannot recognise the complete information of the system.

The present dissertation consists of a few results and discussions surrounding these questions. The first part of the dissertation is devoted to a bivariant K-theoretic interpretation of a numerical method, the *spectral localiser*, introduced originally by Loring and Schulz-Baldes. The work presented there provides a novel interpretation of the odd spectral localiser, and extends its definition to the setting of Hilbert C*-modules. This is a joint work with my supervisor Bram Mesland [LM25], which has been submitted for publication.

The second part of the dissertation studies different C*-algebraic modelling techniques of topological materials, in particular, on aperiodic lattices. We study and compare the the dynamical (groupoid) and the coarse-geometric models of aperiodic materials. The regular representation of the groupoid model serves as a comparison map between both models, and yields a creterion for *robustness* of topological phases. This work has become a separate article [Li25] and submitted for publication. The one-dimensional groupoid model attains a special interest, due to its alternative descriptions as topological graphs or semigroup crossed products. These alternative descriptions allow for better understanding of the topological information of a one-dimensional aperiodic material. The last chapter of the thesis is an ongoing joint work [LT25a] with Guo Chuan Thiang, in which we justify the robustness of topological phases from a *symmetry-breaking* perspective.

1.1. Spectral truncation of index pairings

The first part of the thesis is about a bivariant K-theoretic interpretation of the *spectral localiser* introduced by Loring and Schulz-Baldes. This is a purely mathematical result, whereas the motivation comes strongly from physics, cf. Section 1.2 and the paragraph below.

Index pairings

In the recent decade, K-theory of C*-algebras or Banach algebras has been widely applied in physics, in particular, the theory of topological materials. The basic idea is that, the Hamiltonian of the system represents a K-theory class of a suitably chosen C*-algebra (called the observable C*-algebra, cf. Section 1.2). Quantities of topological materials, like the Chern numbers and winding numbers, can be read off by mapping this K-theory class to a number or the K-theory group of another C*-algebra that is better understood. This is usually done by pairing the K-theory class with a class in *bivariant* K-*theory*.

An index pairing is, by definition, the pairing between a K-theory class and a KK-theory class, via the celebrated Kasparov product

$$\underbrace{\mathrm{KK}_i(\mathbb{C},A)}_{\simeq \mathrm{K}_i(A)} \times \mathrm{KK}_j(A,B) \to \underbrace{\mathrm{KK}_{i+j}(\mathbb{C},B)}_{\simeq \mathrm{K}_{i+j}(B)}.$$

If $i=j=1$ and $B=\mathbb{C}$, then such an index pairing can be realised as a Fredholm index as follows. Let A be a unital C*-algebra, $(\mathscr{A},\mathcal{H},D)$ be an odd spectral triple over A and $v\in A$ be a unitary. Then v defines an odd K-theory class $[v]\in \mathrm{K}_1(A)$

and $(\mathscr{A},\mathcal{H},D)$ defines an odd K-homology class $[D]\in K^1(A)$. Their Kasparov product $[v]\times[D]\in K_0(\mathbb{C})\simeq\mathbb{Z}$ is represented by the Fredholm index of the following operator:

$$PvP+(1-P)\in\mathbb{B}(\mathcal{H})$$

where $P:=\chi_{>0}(D)$ is the positive spectral projection of D. Therefore, in order to compute the index of the Fredholm operator $PvP+1-P$, it is necessary to understand the whole spectrum of P (or D) and v, which are operators on the *infinite-dimensional* Hilbert space $\mathcal{H}$. Numerical computations with such objects require an approximation of $\mathcal{H}$ and operators therefore by finite-dimensional Hilbert spaces and finite matrices. However, every Fredholm operator on a finite-dimensional Hilbert space necessarily has zero index. It is, therefore, not even clear what finite-dimensional approximation really means in this context.

The spectral localiser of Loring and Schulz-Baldes

A breakthrough in the numerical computation of index pairings has been made by Loring and Schulz-Baldes in a series of articles [LSB17, LSB20, LSB19, VSSB19]. Given an odd unital spectral triple $(\mathscr{A},\mathcal{H},D)$ and a unitary (or invertible) $v\in\mathscr{A}$, Loring and Schulz-Baldes define the following unbounded, self-adjoint operator

$$L_\kappa:=\begin{pmatrix}\kappa D & v\\ v^* & -\kappa D\end{pmatrix}$$

and $L_{\kappa,\lambda}$ as its truncation onto the subspace

$$\mathcal{H}_\lambda:=\chi_\lambda(D\oplus D)(\mathcal{H}\oplus\mathcal{H}),$$

where χ_λ is the characteristic function of the interval $[-\lambda,\lambda]$. The Hilbert space $\mathcal{H}_\lambda$ has finite dimension because D has discrete spectrum, thus $L_{\kappa,\lambda}$ is a *finite* matrix. The main theorem of [LSB17, LSB19] says that

> For sufficiently *small* tuning parameter $\kappa>0$ and sufficiently *large* spectral threshold $\lambda>0$, the odd–odd index pairing coincides with the half-signature of the finite-dimensional matrix $L_{\kappa,\lambda}$.

A similar construction for the case $B=\mathbb{C}$ and $i=j=0$ was provided in [LSB20, VSSB19].

The construction of the spectral localiser is somewhat mysterious, as it is a mixture of an *unbounded* operator D (coming from a spectral triple $(\mathscr{A},\mathcal{H},D)$) as well as a *bounded* unitary operator $v\in A$ (or rather an invertible operator $H\in A$ as used originally by Loring and Schulz-Baldes). The formula of L_κ itself resembles an unbounded Kasparov product, whereas the occurence of the bounded term v indicates that it cannot be (a bounded perturbation can always be neglected in the realm of unbounded Kasparov product). The pre-existing proofs from the literature rely on different methods: the fuzzy sphere (cf. [LSB17, LSB20]); or the spectral flow (cf. [LSB19, VSSB19]).

The main result of the first part of the dissertation provides the following interpretation of the spectral localiser: it arises as a pairing between the K-theory class $[v] \in \mathrm{K}_1(A)$ and an *asymptotic morphism* $(\Phi_t^D)_{t\in[1,\infty)}$ constructed from the spectral triple $(\mathscr{A}, \mathcal{H}, D)$. The pairing is represented first by a formal difference $[e_t] - [f_t]$ of *quasi-projections* in $\mathbb{M}_2(\mathbb{K}(\mathcal{H})^+)$. To get the spectral localiser, a further step of *spectral truncation* is needed, which allows for reducing the K-theory class of $[e_t] - [f_t]$, up to homotopy of quasi-projections, to a formal difference of *truncated operators* $[p^e_{t,\lambda}] - [p^f_{t,\lambda}]$. The spectral localiser $L_{t^{-1},\lambda}$ is, up to congruence, given by the operator $2p^e_{t,\lambda} - \mathbf{1}$. This leads to fact that the spectral localiser can be understood as a *quasi-projection* representative of the index pairing between $[v] \in \mathrm{K}_1(A)$ and $[D] \in \mathrm{KK}_1(A, \mathbb{C})$.

E-mergence of the spectral localiser

Let A and B be C*-algebras. An *asymptotic morphism* is a family of linear maps $\Phi_t \colon A \to B$, parametrised by $t \in [1, \infty)$, that "asymptotically" becomes a *-homomorphism (cf. Definition 2.26). Asymptotic morphisms were introduced by Connes and Higson in [CH90] to concretely realise Higson's abstractly defined E-theory category [Hig90]. This is a another bivariant extension of K-theory, characterised by different universal properties from Kasparov's KK-theory. The universal properties of both theories guarantee the existence of a canonical comparison functor $\natural \colon \mathfrak{KK} \to \mathfrak{E}$, which yields for each pair of separable C*-algebra A and B, a natural group homomorphism $\natural(A,B) \colon \mathrm{KK}(A,B) \to \mathrm{E}(A,B)$. For many C*-algebras A, e.g. if A is nuclear, then $\natural(A,B)$ is a natural isomorphism. The naturality here means that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{K}_1(A) & \xrightarrow{\mathrm{KK}_1(A,B)} & \mathrm{K}_0(B) \\ \big\downarrow \mathrm{ind} & \big\downarrow \natural(A,B) & \big\downarrow \pm 1 \\ \mathrm{K}_0(\mathrm{S}A) & \xrightarrow[\mathrm{E}_1(A,B)]{} & \mathrm{K}_0(B). \end{array}$$

That is, the KK-theoretic index pairing induced the same K-theory map $\mathrm{K}_i(A) \to \mathrm{K}_{i+j}(B)$ (up to a sign) as the E-theoretic index pairing

$$\underbrace{\mathrm{E}_i(\mathbb{C}, A)}_{\simeq \mathrm{K}_i(A)} \times \mathrm{E}_j(A, B) \to \underbrace{\mathrm{E}_{i+j}(\mathbb{C}, B)}_{\simeq \mathrm{K}_{i+j}(B)} . \tag{E}$$

Therefore, a first step of such a computation is to concretely realise the comparison functor $\natural\colon \mathfrak{KK} \to \mathfrak{E}$. To this end, we utilise the *unbounded* picture of Kasparov theory, representing KK-classes by unbounded Kasparov modules. The following result has been alluded to in the literature, whereas we establish it rigorously here:

Theorem A (Theorem 3.26)**.** *Let $(\mathscr{A}, \mathcal{E}, D)$ be an odd, essential unbounded Kasparov A-B–module. Then the asymptotic morphism*

$$\Phi_t^D \colon \mathrm{C}_0(\mathbb{R}) \otimes A \longrightarrow \mathbb{K}_B(\mathcal{E}), \quad f \otimes a \mapsto f(t^{-1}D)a$$

represents the class of the image of $[D] \in \mathrm{KK}_1(A, B)$ in $\mathrm{E}_1(A, B)$ under the universal comparison functor $\natural\colon \mathfrak{KK} \to \mathfrak{E}$. In particular, Φ_t^D induces the same map $\mathrm{K}_0(\mathrm{S}A) \to \mathrm{K}_0(\mathbb{K}_B(\mathcal{E}))$ as $[D] \in \mathrm{KK}_1(A, B)$, up to an isomorphism $\mathrm{K}_1(A) \simeq \mathrm{K}_0(\mathrm{S}A)$.

We use the asymptotic morphism $(\Phi_t^D)_{t\in[1,\infty)}$ to compute the index pairing between the odd K-theory class $[v] \in \mathrm{K}_1(A)$ and $[D] \in \mathrm{KK}_1(A, B)$. This gives a *quasi-projection* representative (cf. Theorem 4.14) $[e_t] - [f_t]$ of the index pairing $\langle [v], [D] \rangle$. The step of *spectral truncation* comes from a homotopy of *quasi-projections*: we show that the formal difference of quasi-projections $[e_t] - [f_t]$ can be homotoped to quasi-projections $[p^e_{t,\lambda}] - [p^f_{t,\lambda}]$ supported on a *spectral submodule* $\mathcal{E}_\lambda^{\downarrow}$. This requires an extra parameter λ. If $B = \mathbb{C}$ (so $\mathcal{E} = \mathcal{H}$ is a Hilbert space), then these quasi-projections are supported on a *finite-dimensional* Hilbert space. The index pairing $\langle [v], [D] \rangle$ is thus given by the difference of ranks of two projections, which come from these finite-dimensional quasi-projections $p^e_{t,\lambda}$ and $p^f_{t,\lambda}$.

The odd spectral localiser of Loring and Schulz-Baldes comes from a special choice of the functions $c(x)$ and $s(x)$ (cf. Theorem 4.21) that is used to construct the K-theory class of $\mathrm{K}_0(\mathrm{S}A)$ under the suspension isomorphism $\mathrm{K}_1(A) \simeq \mathrm{K}_0(\mathrm{S}A)$. Thus we justify that the spectral localiser can be viewed as an E-theoretic index pairing:

Theorem B (Theorem 4.27)**.** *For every $t, \lambda > 0$, there is an equality*

$$\frac{1}{2}\operatorname{sig}(L_{t^{-1},\lambda}) = \operatorname{rank}(\kappa_0(p^e_{t,\lambda})) - \operatorname{rank}(\kappa_0(p^f_{t,\lambda})).$$

As a consequence, let $\varepsilon, \delta > 0$ satisfy $\varepsilon + \delta < \frac{1}{400}$. For any choice t, λ satisfying $t > 4\varepsilon^{-1}\|[D, v]\|$ and $\lambda > t\delta^{-1}$, the half-signature $\frac{1}{2}\operatorname{sig}(L_{t^{-1},\lambda})$ coincides with the index pairing $\langle [v], [D]\rangle$.

Similar ideas can be applied to compute numerical index pairings, which are compositions of maps of the form

$$\mathrm{K}_1(A) \xrightarrow{\times[D]} \mathrm{K}_0(B) \xrightarrow{\tau_*} \mathbb{R},$$

where $\tau\colon B \longrightarrow \mathbb{C}$ is a finite, faithful trace on a unital C*-algebra B. In this case, to accommodate the truncation step, one must pass to a larger C*-algebra generated by all $\widehat{\tau}$-finite projections, where $\widehat{\tau}$ is the transferred trace on $\mathbb{K}_B(\mathcal{E})$. Then we recover the main result of [SBS21] in this special setting:

Theorem C (Theorem 4.40)**.** *For every $t, \lambda > 0$, $\operatorname{sig}_{\widehat{\tau}}(L^{\tau}_{t^{-1},\lambda})$, $\operatorname{rank}_{\widehat{\tau}}(p^e_{t,\lambda})$ and $\operatorname{rank}_{\widehat{\tau}}(p^f_{t,\lambda})$ are well-defined and satisfy the equality*

$$\frac{1}{2}\operatorname{sig}_{\widehat{\tau}}(L^{\tau}_{t^{-1},\lambda}) = \operatorname{rank}_{\widehat{\tau}}(p^e_{t,\lambda}) - \operatorname{rank}_{\widehat{\tau}}(p^f_{t,\lambda}).$$

As a consequence, let $\varepsilon, \delta > 0$ satisfy $\varepsilon + \delta < \frac{1}{400}$. For any choice t, λ satisfying $t > 4\varepsilon^{-1}\|[D, v]\|$ and $\lambda > t\delta^{-1}$, the half–$\widehat{\tau}$-signature $\frac{1}{2}\operatorname{sig}_{\widehat{\tau}}(L_{t^{-1},\lambda})$ coincides with the numerical index pairing $\tau_*(\langle [v], [D]\rangle)$.

Related works

Finite-dimensional representatives of K-homology classes have been considered by Willett in [Wil21, Wil20]. It was proven in [Wil21, Section 7] that there exists an asymptotic morphism $(\alpha^A_t\colon \mathrm{S}^2A \longrightarrow A \otimes \mathbb{K})_{t\in[1,\infty)}$ such that for each $t \in [1, \infty)$ and $a \in \mathrm{S}^2A$, then $\alpha^A_t(a)$ belongs to a matrix algebra $\mathbb{M}_n(A)$. In [Wil20], a finite-dimensional representative of the Dirac (inverse Bott) element is constructed based on the *localisation algebra* of Yu [Yu97]. The explicit computation in [Wil20] has been the inspiration for computing the odd index pairing via asymptotic morphisms and quasi-projections.

The localisation algebra provides a model for Kasparov theory [DWW18], with the advantage that the universal comparison functor $\natural\colon \mathfrak{KK} \to \mathfrak{E}$ can be described in an explicit way, cf. [DWW18, Section 5]. We expect that there is a close relation between this model and the constructions in the present dissertation.

Connes and van Suijlekom have studied spectral truncation in [CvS21], in order to provide a framework of coarse graining approximation of geometric space at a finite resolution. This yields a spectral triple over an *operator system* (as opposed to a C*-algebra). More recently, van Suijlekom has introduced a version of K-theory for operator systems in [vS24a, vS24b], and shown that the spectral localiser yields a well-defined index map for it. It is possible that results in the present article might shed some light on how K-*homology*, or more generally, *bivariant* K-theory, should be defined for operator systems.

Recently, Kaad [Kaa25] has provided a KK-theoretic perspective of the even spectral localiser. In particular, Kaad developed a continuous version of spectral truncation for Hilbert C*-modules. The relation between this KK-theoretic approach and the E-theoretic approach developed in this dissertation is left for future work.

1.2. Topological phases of aperiodic matter

The second part of the dissertation focuses on modelling topological phases supported on an aperiodic media, and topological invariant thereof. The meaning of the word *phase* depends on the context, but a general feature is summarised by Thiang [Thi17] as follows:

> There are typically some parameters specifying a physical *state* (e.g. temperature, pressure, . . .) and the parameter space (*phase/moduli space*) divides up into various connected pieces (phases) separated by *phase transitions*.

Examples of phases of matter are crystals, liquid and gas. An example of a phase transition from crystal to liquid is melting of ice, which falls into the paradigm of a Landau-like phase transition: the initial phase (ice) and final phase (water) are distinguished by the degree of structural order of the H_2O molecules constituting the matter. The ice crystal has a beautiful hexagonal crystalline structure, and is transtioned into the more disordered liquid structure of water.

Topological phases are quite different from crystals, liquid or gas, and a *topological phase transition* does not fall in Landau's paradigm. What distinguishes the initial and final phases is a certain *topological invariant* associated to the matter. Here *topological* refers to a suitable sense of *robustness*: the topological phases (or rather their associated topological invariants) are immune to impurity of the samples.

In the recent decade, there has been a trend of studying topological phases of matter with tools from operator algebras and K-theory, thanks to the vast development of insights and techniques in these fields. The paradigm starts with choosing a suitable C*-algebra, referred to as the *observable* C*-algebra*, then proceeds with studying its relevant K-theory and index theory. We focus on the following two classes of candicate observable C*-algebras:

- *dynamical* model describes a crossed product C*-algebra, covariant for a prescribed action of the symmetry group or groupoid. This gives a group(oid) C*-algebra.
- *coarse-geometric* model describes a C*-algebra which is stable under all *short-range*, *locally–finite-rank* perturbations. This leads to the Roe C*-algebra.

The dynamical model: crossed product and groupoid C*-algebras

The emergence of noncommutative C*-algebras in modelling materials is due to the following fact: all materials contain a numerable amount of particles, hence the occurence of disorder and impurity destroys the translation invariance of the Hamiltonian, forcing the Brillouin zone to be a *noncommutative space*. Furthermore, as pioneered by Bellissard [Bel86], the *homogeneity* of such a system indicates that all particles of the system should be treated on an equal footing:

> Even though in practice a sample is given from the beginning and will not change at low temperature during the experiment, the physical energy operator is not the operator H_ω alone corresponding to the given configuration of the disorder. Otherwise it would imply the choice of an origin of coordinates inside the sample, preventing the use of its homogeneity : there is no physical reason "a priori" to prefer one origin instead of one another! Therefore the observable algebra must contain not only H_ω but also the whole family of its translates. Since in general two elements of this family do not commute the observable algebra will be non-commutative in a fundamental way.

This motivates Bellissard to consider a certain class of *noncommutative* C*-algebras as the possible candidate of the observable C*-algebra: *crossed products*. These are C*-algebras that are built from a topological dynamical system $(\Omega \curvearrowleft \mathbb{Z}^d)$ or $(\Omega \curvearrowleft \mathbb{R}^d)$, where Ω is the *moduli space* of disorder configurations: Bellissard called it the *hull* of disordered Hamiltonians; and $\mathbb{Z}^d$ or $\mathbb{R}^d$ acts by translation. This new recipe provides an exceptionally nice interpretation of the quantum Hall effect (cf. [BvESB94]) and the bulk–boundary correspondence of topological insulators (cf. [KRSB02]).

Replacing $\mathbb{R}^d$ by $\mathbb{Z}^d$ corresponds to the *tight-binding approximation*, which replaces the Hilbert space $L^2(\mathbb{R}^d)$ by an infinite-dimensional subspace spanned by a finite number N of eigenfunctions of a *single-particle Hamiltonian*, together with all of its $\mathbb{Z}^d$-translated copies. Each of these eigenfunctions and their translated copies might be thought of as a particle supported on individual points of a discrete set Λ. If the Hamiltonian is periodic, then the Λ is identified with a lattice subgroup of $\mathbb{R}^d$ that commutes with the Hamiltonian.

The tight-binding approximation as above entails an implicit assumption: even if the Hamiltonian is no longer periodic, there should exist a continuous $\mathbb{Z}^d$-action on the hull Ω_Λ of the Hamiltonian. Amorphous materials like liquid crystals and glass, however do not accomodate natural $\mathbb{Z}^d$-actions on their hulls. For quasi-crystals, a $\mathbb{Z}^d$-action can be generated by applying certain approximation methods as in [AP98, SW03]. [1] Nevertheless, one has to essentially provide a (non-canonical) $\mathbb{Z}^d$-labelling of the underlying lattice, under which the Hamiltonian remains short-range (cf. [BP18]). These suggest to use more general point sets in $\mathbb{R}^d$ as their underlying geometric space and study the dynamics thereon by more general actions other than a $\mathbb{Z}^d$-translation.

Bellissard [BHZ00] suggested to use *Delone sets* as such model sets Λ, which are uniformly discrete and relatively dense point sets in $\mathbb{R}^d$. The observable C*-algebra in this setting is no longer a crossed product, but a *groupoid C*-algebra* $C^*(\mathcal{G}_\Lambda)$: The groupoid $\mathcal{G}_\Lambda$ is the restriction of the *action groupoid* $\Omega_\Lambda \rtimes \mathbb{R}^d$ to an abstract *transversal* Ω_0. Thus $\mathcal{G}_\Lambda$ is an étale groupoid that is Morita equivalent to $\Omega_\Lambda \rtimes \mathbb{R}^d$. If Λ is a lattice group, then $\mathcal{G}_\Lambda$ is isomorphic (as a topological groupoid) to the discrete abelian group $\mathbb{Z}^d$, and the corresponding obserable C*-algebra is isomorphic to the $C(\mathbb{T}^d)$, the continuous functions on the Brillouin torus. This is the usual observable C*-algebra in Bloch–Floquet theory.

An interesting special case emerges from one-dimensional Delone sets, i.e. from one-dimensional materials. In such cases, Bourne and Mesland [BM19, Section 2.4] have shown that $C^*(\mathcal{G}_\Lambda)$ is isomorphic to a Cuntz–Pimsner algebra. We may strengthen this result:

[1] Such methods are merely approximations, in the sense that their associated dynamical system is *not* topologically conjugate to the original one.

Theorem D (Theorem 7.13)**.** *The groupoid* C*-*algebra* $C^*(\mathcal{G}_\Lambda)$ *of a one-dimensional Delone set* $\Lambda \subseteq \mathbb{R}$ *is isomorphic to the* C*-*algebra of a topological graph* $\mathcal{E}_\Lambda$. *Moreover, we have* $\mathcal{G}_\Lambda \simeq \Omega_0 \rtimes \mathbb{Z}$ *as topological groupoids, and* $C^*(\mathcal{G}_\Lambda) \simeq C(\Omega_0) \rtimes \mathbb{Z}$ *as* C*-*algebras.*

The proof is based on alternative dynamical descriptions of the C*-algebra $C^*(\mathcal{G}_\Lambda)$: for a one-dimensional Delone set Λ, $\mathcal{G}_\Lambda$ can be viewed either as the Renault–Deaconu groupoid of a singly generated dynamical system on the abstract transversal Ω_0; or as a semigroup action groupoid of a wide inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G}_\Lambda)$, where $\mathrm{Bis}(\mathcal{G}_\Lambda)$ is the inverse semigroup of local bisections of $\mathcal{G}_\Lambda$. In either case, one can demonstrate that the singly generated dynamical system is indeed *global*, henceforth $\mathcal{G}_\Lambda$ is isomorphic to the action groupoid $\Omega_0 \rtimes \mathbb{Z}$ given by such dynamics.

The coarse-geometric model and robustness

One prominent feature of topological phases and their invariants is their *robustness* under disorder. One way to illustrate this is by showing that such topological phases live inside the K-theory of a sufficiently robust observable C*-algebra, such that a physically reasonable perturbation will not change the K-theory class of a topological phase. The insight of Kubota [Kub17] is to consider the universal C*-algebra generated by all short-range, locally–finite-rank Hamiltonians, which are Roe C*-algebras.

The Roe C*-algebras are central objects in coarse geometry, a subject that studies large-scale properties of spaces, in the sense that two spaces yield isomorphic Roe C*-algebras if they look the same from a large distance. This happens, in particular, to the Euclidean space $\mathbb{R}^d$ and a Delone set $\Lambda \subseteq \mathbb{R}^d$ inside it. The coarse-geometric model of an aperiodic material is thus given by the Roe C*-algebra $C^*_{\mathrm{Roe}}(\Lambda)$ of a Delone set Λ, where Λ is considered as a discrete metric space.

The robustness of Roe C*-algebras on a periodic lattice has been demonstrated in detail by Ewert and Meyer [EM19]. More precisely, Ewert and Meyer considered the canonical inclusion map $\iota\colon C^*(\mathbb{Z}^d) \hookrightarrow C^*_{\mathrm{Roe}}(\mathbb{Z}^d)$ and its induced maps $K_i(\iota)$ in K-theory. The kernel of $K_i(\iota)$ are interpreted as *weak* topological phases, which all come from "stacking" topological phases along a coordinate direction. The *strong* topological phases are, in contrast, those K-theory classes in $K_i(C^*(\mathbb{Z}^d))$ that survive under $K_i(\iota)$.

The results in Chapter 6 provide the aperiodic counterpart of the above story. That is, we describe the robustness of topological phases supported on a Delone set Λ, and compare the groupoid model $C^*(\mathcal{G}_\Lambda)$ with the Roe C*-algebra model $C^*_{\mathrm{Roe}}(\Lambda)$. By a characterisation of the Roe C*-algebra of closed subsets in $\mathbb{R}^d$ in [EM19, Theorem 4], the regular representations π_ω of the groupoid C*-algebra $C^*(\mathcal{G}_\Lambda)$ maps into a Roe C*-algebra $C^*_{\mathrm{Roe}}(\omega)$, hence serves as a candidate comparison map.

We are also able to characterise which topological phases belong to the image of this comparison map, in other words, being "strong". Bourne and Mesland has constructed an unbounded Kasparov $C^*(\mathcal{G}_\Lambda) \otimes C\ell_{0,d}$-$C(\Omega_0)$–module ${}_d\lambda_{\Omega_0}$ called the *bulk cycle*. This can be viewed as a family of "position" operators parameterised by the hull Ω_0. Localisation of the bulk cycle at every point $\omega \in \Omega_0$ gives a position spectral triple, which maps into the Roe C*-algebra $C^*_{\mathrm{Roe}}(\omega)$ of the Delone set corresponding to ω . We show that the strong topological phases are precisely those that are not killed by position spectral triples.

Theorem E (Position spectral triples detect strong topological phases, Theorem 6.32)**.** *For every $\omega \in \Omega_0$, The following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{KK}(C\ell_{n,0}, C^*(\mathcal{G}_\Lambda)) & \xrightarrow{{}_d\lambda_{\Omega_0}} & \mathrm{KK}(C\ell_{n,d}, C(\Omega_0)) \\
\downarrow \pi^N_\omega & \searrow \xi^{\mathrm{Gpd}}_{\omega,N} & \downarrow \mathrm{ev}_\omega \\
\mathrm{KK}(C\ell_{n,0}, C^*_{\mathrm{Roe}}(\omega)) & \xrightarrow[\sim]{\xi^{\mathrm{Roe}}_\omega} & \mathrm{KK}(C\ell_{n,d}, \mathbb{R}).
\end{array}$$

where $\xi^{\mathrm{Roe}}_\omega$ is the spectral triple over $C^*_{\mathrm{Roe}}(\omega)$ constructed from the position operator on $\ell^2(\omega)$, ${}_d\lambda_{\Omega_0}$ is the bulk cycle, $\mathrm{ev}_\omega \colon C(\Omega_0) \to \mathbb{C}$ is the evaluation map at $\omega \in \Omega_0$, and $\xi^{\mathrm{Gpd}}_\omega$ is the composition $\mathrm{ev}_\omega \circ {}_d\lambda_{\Omega_0}$.

We also explain that "stacking" aperiodic topological phases along another Delone set yields trivial topological invariants. This extends the observation of Ewert and Meyer on weak topological phases:

Theorem F (Stacked topological phases are weak, Theorem 6.41)**.** *Let $\xi^{\mathrm{Gpd}}_{\omega\times\ell,N}$ be the spectral triple over $C^*(\mathcal{G}_{\Lambda\times L}) \otimes \mathbb{M}_N(\mathbb{R})$ defined in (6.20). Then for each n, its induced map*

$$\mathrm{KK}(C\ell_{n,0}, C^*(\mathcal{G}_{\Lambda\times L})) \to \mathrm{KK}(C\ell_{n,p+q}, \mathbb{R})$$

vanishes on the image of

$$\varphi^{\mathrm{Gpd}}_* \colon \mathrm{KK}(C\ell_{n,0}, C^*(\mathcal{G}_\Lambda)) \to \mathrm{KK}(C\ell_{n,0}, C^*(\mathcal{G}_{\Lambda\times L})).$$

Symmetry-breaking topological phases

The last chapter of the dissertation comes from an idea of Guo Chuan Thiang, and is an ongoing joint project [LT25a]. We look back at *periodic* models of materials, i.e. the underlying Delone set is a lattice group $\Lambda = \mathbb{Z}^d$. The lattice Λ is a microscopically distinguished object, corresponding to a *choice* of the perfect crystalline structure of the material. However, *macroscopic* physical observables are usually independent of the choice of this lattice. This is the core philosophy of coarse geometry, and we shall offer a *symmetry-breaking* viewpoint towards it.

The symmetry-breaking approach describes the physical system by an inductive limit C*-algebra $C^*_{\mathrm{Roe}}(X)^{\mathfrak{S}}$, called the *symmetry-breaking Roe C*-algebra*, coming from *coarsening* the lattice Λ, i.e. replacing Λ by its finite-index sublattices. All such sublattices are isomorphic to $\mathbb{Z}^d$, whereas the connecting maps between their equivariant Roe C*-algebras may have a non-trivial effect. This corresponds to several $\mathbb{Q}$-components in the K-theory of the symmetry-breaking Roe C*-algebra on $\mathbb{R}^d$, which we interpret as *weak* topological phases:

Theorem G (Theorem 8.51)**.** *The* K*-theory of* $C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}}$ *is given by*

$$\mathrm{K}_i(C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}}) \simeq \begin{cases} \mathbb{Q}^{2^{d-1}-1} \oplus \mathbb{Z} & \text{if } i-d \text{ is even;} \\ \mathbb{Q}^{2^{d-1}} & \text{if } i-d \text{ is odd.} \end{cases}$$

We may use the symmetry-breaking Roe C*-algebra $C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}}$ as the observable C*-algebra on a periodic lattice. A class in $\mathrm{K}_*(C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}})$ represents a *weak* topological phase if it consists of only $\mathbb{Q}$-components, and a *strong* topological phase if it consists of only a $\mathbb{Z}$-component. Our definition of weak or strong topological phases coincide with those in the coarse-geometric sense:

Theorem H (Theorem 8.54)**.** *Let* $\psi\colon C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}} \hookrightarrow C^*_{\mathrm{Roe}}(\mathbb{R}^d)$ *be the natural inclusion. Its induced map*

$$\mathrm{K}_*(\psi)\colon\ \mathrm{K}_*(C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}}) \to \mathrm{K}_*(C^*_{\mathrm{Roe}}(\mathbb{R}^d))$$

is a surjective group homomorphism, which is zero on all $\mathbb{Q}$*-components of* $\mathrm{K}_*(C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}})$*, and restricts to an isomorphism on the unique* $\mathbb{Z}$*-component. As a consequence, every weak topological phase vanishes in the image of* $\mathrm{K}_*(\psi)$*.*

1.3. Structure of the thesis

The present dissertation comprises of two parts. In the first part, Chapters 2 to 4, we study the spectral truncation method of index pairings via E-theory. In the second part, Chapters 5 to 8, we elaborate on several aspects of topological phases of matter.

Chapter 2 provides an introduction to bivariant K-theories. We characterise KK-theory and E-theory by their universal properties. Then we describe the Fredholm picture of KK-theory introduced by Kasparov, and the asymptotic morphism picture of E-theory introduced by Connes and Higson. The index pairing is a special case of the Kasparov or E-theory product, which is discussed briefly in the last section.

Chapter 3 provides a concrete realisation of the universal comparison functor $\natural\colon \mathfrak{KK} \to \mathfrak{E}$ from KK-theory to E-theory, which exists by their universal properties. We recall the unbounded picture of KK-theory, and use it to explicitly construct the functor $\natural$ in Section 3.4. Special cases of this construction seems to be known to experts, whereas an explicit general statement is missing in the literature.

Chapter 4 contains the main result of the first part, which is contained in joint work [LM25]. We describe the index pairing between an odd K-theory class and an odd unbounded Kasparov module by a pair of quasi-projections, supported on a submodule obtained from a finite spectral truncation. We achieve this by pairing the K-theory class with an asymptotic morphism determined by the unbounded Kasparov module. We interpret the spectral localiser of Loring and Schulz-Baldes as an instance of such an index pairing.

Chapter 5 gives an overview of the mathematical objects of the second part: the observable C*-algebras of aperiodic topological materials. We describe the groupoid (dynamical) model and the Roe C*-algebra (coarse-geometric) model of aperiodic materials.

Chapter 6 studies the robustness of topological phases on aperiodic lattices. We show that the (localised) regular representations of the groupoid C*-algebra model maps into the Roe C*-algebra model. Furthermore, such representations induce maps in K-theory. We show that the strong topological phases in the groupoid model are detected by position spectral triples, and that topological phases coming from stacking along another Delone set are always weak in the coarse-geometric sense. The result is contained in my recent preprint [Li25].

Chapter 7 gives several alternative dynamical descriptions of the groupoid model in the one-dimensional case. We describe the groupoid model in dimension one as the Renault–Deaconu groupoid of a topological graph, and as a semigroup crossed product by a wide subsemigroup of its inverse semigroup of local bisections. We interpret the Cuntz–Pimsner model of Bourne and Mesland as a topological graph C*-algebra in the sense of Katsura, and strengthen their result using the two viewpoints described above.

Chapter 8 constructs an inductive limit C*-algebra, called the symmetry-breaking Roe C*-algebra, as an observable C*-algebra that prescribes the symmetry-breaking process of lattices. The K-theory of this C*-algebra accommodates both strong and weak topological phases, and distinguishes them by their divisibility. This allows us to interpret strong topological phases as those that are invariant under a symmetry-breaking process, corresponding physically to the independence of the underlying sample of the material. This chapter initiates from an idea of Guo Chuan Thiang, and is an joint ongoing project [LT25a] with Thiang.

A list of notation and symbols and a list of alphabetical indices are provided at the end of this dissertation.