Quantum computing, norms and polynomials

Escudero Gutiérrez, F.

Chapter 5

Towards Aaronson and Ambainis conjecture via Fourier completely bounded polynomials

5.1 Introduction

Understanding the quantum query complexity of Boolean functions $f : D \to \{-1, 1\}$, where D is a subset of $\{-1, 1\}^n$, has been a crucial task of quantum information science [Amb18]. Many celebrated quantum algorithms show an advantage in terms of query complexity, for example in unstructured search [Gro96], period finding [Sho97], Simon's problem [Sim97], NAND-tree evaluation [FGG07] and element distinctness [Amb07]. However, these advantages are limited to be polynomial in the case of total functions (those with $D = \{-1, 1\}^n$), while they can be exponential for highly structured problems (informally, this means that $|D| = o(2^n)$), such as for Simon's problem [Sim97], period finding [Sho97] or k-fold forrelation [AA15, Tal20, BS21, SSW21]. It is widely believed that a lot of structure is necessary for superpolynomial speedups[1]. The following folklore conjecture, which has circulated since the late 90s,

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[1] Recently, Yamakawa and Zhandry showed that superpolynomial speedups can be attained in unstructured search problems. That does not contradict that structure is needed to achieve super-polynomial speedups in decision problems, which are those modeled by Boolean functions [YZ22].

but was first formally posed by Aaronson and Ambainis [AA09], formalizes this idea.

Conjecture 5.1 (Folklore)**.** The biases of t-query quantum algorithms can be simulated with error at most ε on at least a $(1-\delta)$-fraction of the inputs using $\mathrm{poly}(t, 1/\varepsilon, 1/\delta)$ classical queries.

In other words, it is believed that quantum query algorithms can be approximated almost everywhere by classical query algorithms with only a polynomial overhead.

A route towards proving Conjecture 5.1 was designed by Aaronson and Ambainis using that the bias of quantum query algorithms are polynomials. Indeed, Beals et al. [BBC+01], proved that the bias of a t-query quantum algorithm is a bounded polynomial $p : \{-1, 1\}^n \to \mathbb{R}$ of degree at most $2t$. Based on this observation, Aaronson and Ambainis conjectured in [AA09] that every bounded polynomial of bounded degree has an influential variable.

Conjecture 5.2 (Aaronson-Ambainis (AA))**.** Let $p : \{-1, 1\}^n \to \mathbb{R}$ be a polynomial of degree at most t with $\|p\|_\infty \leq 1$. Then, p has a variable with influence at least $\mathrm{poly}(\mathrm{Var}[p], 1/t)$.

The argument of [AA09, Theorem 7] to show that Conjecture 5.2 would imply Conjecture 5.1 works as follows. Let p the bounded polynomial of degree at most $2t$ that represents the bias of t query quantum algorithm. Say that we want to approximate $p(y)$ for some $y \in \{-1, 1\}^n$. First, query an influential variable i of y. Then, the restricted polynomial $p|_{x(i)=y(i)}$ would also be a bounded polynomial of degree at most $2t$, so we can query again an influential variable. Given that the influences of these variables are big, after a *small* number of queries the remaining polynomial would have a low variance, so if we output its expectation it would be close to $p(y)$ with high probability.

A few reductions to other conjectures have been made. The first one is that is sufficient to prove the conjecture for one-block decoupled polynomials [OZ15]. Very recently, Lovett and Zhang stated two conjectures related to fractional certificate complexity that, if true, would imply the AA conjecture [LZ22]. Also recently, Austin et al. showed a connection of the AA conjecture with cryptography: they proved that if the AA conjecture is false, then there is a secure key agreement in the quantum random oracle model that cannot be broken classically [ACC+22]. The most recent work in this line of research is the one by Bhattacharya, who showed that the conjecture is true for random restrictions of the polynomial [Bha25]. Regarding particular cases, it is only known to be true in a few scenarios: Boolean functions

$f : \{-1, 1\}^n \to \{-1, 1\}$ [Mid05, OSSS05, JZ11], symmetric polynomials [Iva19], multilinear forms whose Fourier coefficients are all equal in absolute value [Mon12] and block-multilinear completely bounded polynomials [BSdW22].

The last result is relevant in this context because Arunachalam, Briët and Palazuelos showed that the biases of quantum query algorithms are polynomials that are not only bounded, but also completely bounded [ABP19] (see Theorem 3.6). This is a more restricted normalization condition, which can be informally understood as the polynomial taking bounded values when evaluated not only on bounded scalars, but also on bounded matrix inputs. This way, one could try to use this extra condition to prove results about quantum query algorithms.

This idea was first put in practice by Bansal, Sinha and de Wolf [BSdW22]. They showed that the AA conjecture holds for completely bounded block-multilinear forms, which implies an almost everywhere classical simulation result, similar to Conjecture 5.1, for the amplitudes of certain quantum query algorithms. These algorithms query different (non-controlled) bit strings on every query, while Conjecture 5.1 concerns algorithms that query the same controlled bit string on every query.

Results of this chapter

We follow that line of work and use the characterization of [ABP19] to design a route towards Conjecture 5.1. Our first result is a new presentation of that characterization that is more convenient for our purposes. To do this we introduce the Fourier completely bounded t-norms ($\|\cdot\|_{\mathrm{fcb},t}$), which are relaxations of the supremum norm. In these norms we not only take the supremum of the values that the polynomial takes over Boolean strings as in Eq. (2.12), but also on matrix inputs that behave like Boolean strings. We will not include formal definitions in the introduction, but we illustrate the concept of having *Boolean behavior of degree t* with an example. For $m \in \mathbb{N}$, we denote the $m \times m$ real matrices by M_m. Say that $t = 4$ and $n = 6$, then if a pair of vectors $u, v \in \mathbb{R}^m$ and a string of matrices $A \in (M_m)^6$ have Boolean behaviour of degree 4, they satisfy, for instance,

$$\langle u, A(1)A(1)A(2)A(3)v \rangle = \langle u, A(5)A(2)A(3)A(5)v \rangle,$$

because they should simulate the relation $x(1)x(1)x(2)x(3) = x(5)x(2)x(3)x(5)$ satisfied by any Boolean string $\{-1, 1\}^6$. As the reader might guess, (u, v, A) will have Boolean behavior of degree t if it simulates the relations of $\mathbb{F}_2^n$ that involve product of t of the canonical generators.

Using the Fourier expansion of polynomials defined on the Boolean hypercube we will introduce a natural way of evaluating polynomial in matrix inputs that have Boolean behavior, which allows us to introduce the Fourier completely bounded t-norm.

Definition 5.3. (Informal version of Definition 5.10) Let $p : \{-1, 1\}^n \to \mathbb{R}$ be a polynomial of degree at most t. Its *Fourier completely bounded t-norm* is given by

$$\|p\|_{\text{fcb},t} := \sup |p(u, v, A)| \tag{5.1}$$

where the supremum is taken over all (u, v, A) that have Boolean behavior of degree t.

After a reinterpretation of the semidefinite programs proposed in [GL19] to characterize quantum query complexity, based on [ABP19], we show that the Fourier completely bounded t-norms are those that characterize quantum query algorithms.

Theorem 5.4. *Let $p : \{-1, 1\}^n \to \mathbb{R}$. Then, p is the bias of a t-query quantum algorithm if and only if its degree is at most $2t$ and $\|p\|_{\text{fcb},2t} \leq 1$.*

This new presentation of the main result of [ABP19] is more compact than the original one. It is presented directly in terms of polynomials of the Boolean hypercube, does not involve a minimization over possible completely bounded extensions of p as in Definition 4.7, and eludes the use of tensors/multilinear forms.

Given that the Fourier completely bounded t-norms are at least the supremum norm[2], Theorem 5.4 suggests that Conjecture 5.2 may be more general than necessary. Hence, we propose the following weaker conjecture, that would also imply Conjecture 5.1.

Conjecture 5.5. *Let $p : \{-1, 1\}^n \to \mathbb{R}$ be a polynomial of degree at most t with $\|p\|_{\text{fcb},t} \leq 1$. Then, p has a variable with influence at least $\text{poly}(\text{Var}[p], 1/t)$.*

Using a generalization through *creation* and *annihilation* operators of the construction used by Varopoulos to rule out a von Neumann's inequality for degree 3 polynomials [Var74], we can prove a particular case of Conjecture 5.5.

Theorem 5.6. *Let $t \in \mathbb{N}$. Let $p : \{-1, 1\}^n \to \mathbb{R}$ be a homogeneous polynomial of degree t and with $\|p\|_{\text{fcb},t} \leq 1$. Then, the maximum influence of p is at least $\text{Var}[p]^2$.*

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[2]From the results of [BP19] it can be inferred that there is a sequence of polynomials p_n of degree 3 such that $\|p_n\|_{\text{fcb},3}/\|p_n\|_\infty \to_n \infty$.

The proof of the homogeneous case does not straightforwardly generalize (see Remark 5.20), but it suggests a way to solve the general case (see Remark 5.21). In particular, we propose Question 5.22 (that reminds of tensor networks and almost-quantum correlations), which if answered affirmatively would imply Conjecture 5.5.

Theorem 5.6 is the first result concerning the AA conjecture whose constant has no dependence on the degree (to prove Conjecture 5.1 we could afford a polynomial dependence on the degree). Also, it requires considerably fewer algebraic constraints than the other particular cases for which we know AA conjecture to hold. In addition, thanks to Theorem 5.4, it can be interpreted directly in terms of quantum query algorithms.

Corollary 5.7. *Let $t \in \mathbb{N}$. Let $\mathcal{A}$ be a t-query quantum algorithm whose bias is a homogeneous polynomial $p : \{-1, 1\}^n \to \mathbb{R}$ of degree $2t$. Then, the maximum influence of p is at least $\mathrm{Var}[p]^2$.*

With a similar construction as the one we used for Theorem 5.6, we can reprove the results of [BSdW22] regarding the influence of block-multilinear completely bounded polynomials. These polynomials have a particular algebraic structure and also a normalization condition when evaluated on matrix inputs (see Section 5.4.1 below).

Theorem 5.8. *Let $t \in \mathbb{N}$. Let $p : \{-1, 1\}^{n \times t} \to \mathbb{R}$ be a block-multilinear degree t polynomial with $\|p\|_{\mathrm{cb}} \leq 1$. Then, p has a variable of influence at least $(\mathrm{Var}[p]/t)^2$. What is more, if p is homogeneous of degree t, then it has a variable of influence at least $\mathrm{Var}[p]^2$.*

Theorem 5.8 corresponds to [BSdW22, Theorem 1.4], where Bansal et al. proved the same result but with influences at least $\mathrm{Var}[p]^2/[e(t+1)^4]$ in the general case and with $\mathrm{Var}[p]^2/(t+1)^2$ in the homogeneous degree t case. Their proofs involve evaluating p in *random infinite dimensional matrix inputs*, which they can control using ideas of free probability. However, our proof evaluates p in explicit finite dimensional matrix inputs, is shorter and obtains better constants. In particular, our constant for the homogeneous case is optimal.

5.2   The Fourier completely bounded t-norms

There is a vast theory concerning the properties of multilinear maps $T : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ that are completely bounded, i.e., bounded when they are extended to matrix domains [Pau03]. However, to the best of our knowledge, there is no notion of being

completely bounded for polynomials $p : \{-1,1\}^n \to \mathbb{R}$ defined on the Boolean hypercube. Here, we propose a matrix notion of behaving like a Boolean string. Then, using the Fourier expansion of these polynomials we define the evaluation of the polynomials on these matrix inputs that behave like Boolean strings. Finally, we introduce the Fourier completely bounded t-norms and prove a few of their properties.

We recall that every $p : \{-1,1\}^n \to \mathbb{R}$ can be written as

$$p(x) = \sum_{S \subseteq [n]} \hat{p}(S) \prod_{i \in S} x(i), \tag{5.2}$$

where $\hat{p}(S)$ are the Fourier coefficients of p. We say that p has degree at most t if $\hat{p}(S) = 0$ for every $|S| > t$, where $|S|$ denotes the cardinality of S.

We will be interested on simulating the behavior of bit strings $x \in \{-1,1\}^n \times \{1\}$ with one extra frozen variable[3]. Given $t \in \mathbb{N}$ and $\mathbf{i}, \mathbf{j} \in [n+1]^t$ we say that $\mathbf{i} \sim \mathbf{j}$, if

$$x(i_1) \ldots x(i_d) = x(j_1) \ldots x(j_d) \text{ for every } x \in \{-1,1\}^n \times \{1\}. \tag{5.3}$$

In other words, if we define

$$S_{\mathbf{i}} := \{k \in [n] : \ k \text{ occurs an odd number of times in } \mathbf{i}\},$$

then $\mathbf{i} \sim \mathbf{j}$ if and only if $S_{\mathbf{i}} = S_{\mathbf{j}}$. Note that $n+1$ does not belong to these sets $S_{\mathbf{i}}$. Given $S \subseteq [n]$ with $|S| \leq t$, we write $[\mathbf{i}^S]$ to denote the equivalence class of indices $\mathbf{i}$ such that $S_{\mathbf{i}} = S$.

Definition 5.9. Let $n, t, m \in \mathbb{N}$. Let $u, v \in S^{m-1}$ and let $A \in (B_m)^n$. We say that (u, v, A) has *Boolean behavior of degree t* if

$$\langle u, A(i_1) \ldots A(i_d) v \rangle = \langle u, A(j_1) \ldots A(j_d) v \rangle$$

for all $\mathbf{i}, \mathbf{j} \in [n+1]^t$ such that $\mathbf{i} \sim \mathbf{j}$. We call $\mathscr{BB}^t$ to the set of (u, v, A) with Boolean behavior of degree t.

Informally, having Boolean behavior of degree t means that the relations of Eq. (5.3) and some normalization conditions are satisfied. In particular, for any bit string $x \in \{-1,1\}^n \times \{1\}$ and any $t \in \mathbb{N}$, we have that $(1, 1, x)$ has Boolean behavior of degree t.

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[3]The extra variable set to 1 is there because quantum query algorithms query a controlled bit string. A non-controlled version, which would not require that extra variable.

Also note that given $t \in \mathbb{N}$, for every $S \subseteq [n]$ with $|S| \leq t$ there is at least one $\mathbf{i} \in [n+1]^t$ such that $S_{\mathbf{i}} = S$. Thus, given (u, v, A) with Boolean behavior of degree t, for every $|S| \leq t$ the product $\prod_{i \in S} x(i)$ can be simulated (in a unique manner) by $\langle u, A(i_1^S) \ldots A(i_d^S) v \rangle$. In particular, this means that for a polynomial p of degree at most t, we can define through Eq. (5.2) an evaluation of p on every (u, v, A) that has Boolean behavior of degree t, which leads to the definition Fourier completely bouded t-norm.

Definition 5.10. Let $p : \{-1, 1\}^n \to \mathbb{R}$ be a polynomial of degree at most t. Then, its *Fourier completely bounded t-norm* is defined by

$$\|p\|_{\mathrm{fcb},t} = \sup_{(u,v,A) \in \mathscr{B}\mathscr{B}^t} \sum_{S \subseteq [n], |S| \leq t} \hat{p}(S) \langle u, A(i_1^S) \ldots A(i_d^S) v \rangle.$$

The rest of the section is devoted to prove a few results concerning the Fourier completely bounded t-norms. First of all we show that, indeed, they are norms.

Proposition 5.11. *Let $t \in \mathbb{N}$. Then, $\|\cdot\|_{\mathrm{fcb},t}$ is a norm in the space of polynomials $p : \{-1, 1\}^n \to \mathbb{R}$ of degree at most t.*

Proof. It clearly satisfies the triangle inequality and is homogeneous. Also, if $p = 0$ then $\|p\|_{\mathrm{fcb},t} = 0$, and vice versa, because $\|p\|_\infty \leq \|p\|_{\mathrm{fcb},t}$. $\qquad\square$

One nice property of these norms is that they can be computed as semidefinite programs.

Proposition 5.12. *Let $p : \{-1, 1\}^n \to \mathbb{R}$ be a polynomial of degree at most t. Then, its Fourier completely bounded t-norm can be written as the following SDP*

$$\|p\|_{\mathrm{fcb},t} = \sup \sum_{S \in [n], |S| \leq t} \hat{p}(S) \langle u, v_{\mathbf{i}^S} \rangle, \tag{5.4}$$

$$u, v, v_{\mathbf{i}} \in \mathbb{R}^m, \ m \in \mathbb{N}, \ \mathbf{i} \in [n+1]^s, \ s \in [t],$$

$$\langle u, v_{\mathbf{i}} \rangle = \langle u, v_{\mathbf{j}} \rangle, \text{ if } \mathbf{i} \sim \mathbf{j}, \ \mathbf{i}, \mathbf{j} \in [n+1]^t, \tag{5.5}$$

$$\langle u, u \rangle = \langle v, v \rangle = 1, \tag{5.6}$$

$$\mathrm{Gram}_{\substack{\mathbf{j} \in [n+1]^s \\ s \in [t-1]_0}} \{v_{i\mathbf{j}}\} \preccurlyeq \mathrm{Gram}_{\substack{\mathbf{j} \in [n+1]^s \\ s \in [t-1]_0}} \{v_{\mathbf{j}}\}, \text{ for } i \in [n+1], \tag{5.7}$$

where we by $v_{\mathbf{j}}$ with $\mathbf{j} \in [n+1]^0$ we mean v, Gram denotes the gram matrix and the symbol '$\preccurlyeq$' the usual matrix inequality.

Proof. Let $\|\|p\|\|$ be the expression on the right-hand side of Eq. (5.4). Note that Eq. (5.5) represents the relations of bit strings of Eq. (5.3), while Eqs. (5.6) and (5.7) encode normalization conditions.

On the one hand, every $(u, v, A) \in \mathscr{B}\mathscr{B}^t$ defines a feasible instance for $\|\|p\|\|$ through

$$v_{\mathbf{i}} := A(i_1) \dots A(i_s)v$$

for every $\mathbf{i} \in [n+1]^s$ and every $s \in [t]$. Given that the value of this instance is

$$\sum_{S \subseteq [n], |S| \leq t} \widehat{p}(S) \langle u, A(i_1^S) \dots A(i_d^S)v \rangle$$

we have that $\|\|p\|\| \geq \|p\|_{\text{fcb},t}$.

On the other hand, let $u, v, v_{\mathbf{i}} \in \mathbb{R}^m$ be a feasible instance of $\|\|p\|\|$. For $i \in [n+1]$ define $A(i) \in M_m$ as the linear map from $\mathbb{R}^m$ to $\mathbb{R}^m$ that takes $v_{\mathbf{j}}$ to $v_{i\mathbf{j}}$ for every $\mathbf{j} \in [n+1]^s$ and every $s \in [t-1]_0$, and it is extended to the orthogonal complement as 0. First of all, we should check that this is a correct definition, meaning that for every $\lambda \in \mathbb{R}^m$, with $m = (n+1)^{t-1} + \dots + (n+1)^0$, we have that

$$\sum_{\mathbf{j}} \lambda_{\mathbf{j}} v_{\mathbf{j}} = 0 \implies \sum_{\mathbf{j}} \lambda_{\mathbf{j}} v_{i\mathbf{j}} = 0.$$

Indeed, we can prove something stronger:

$$\left(\sum_{\mathbf{j}} \lambda_{\mathbf{j}} v_{i\mathbf{j}}\right)^{\mathsf{T}} \sum_{\mathbf{j}'} \lambda_{\mathbf{j}'} v_{i\mathbf{j}'} = \lambda^{\mathsf{T}} \text{Gram}_{\substack{\mathbf{j} \in [n+1]^s \\ s \in [t-1]_0}, \{v_{i\mathbf{j}}\}} \lambda \leq \lambda^{\mathsf{T}} \text{Gram}_{\substack{\mathbf{j} \in [n+1]^s \\ s \in [t-1]_0}, \{v_{\mathbf{j}}\}} \lambda$$

$$= \left(\sum_{\mathbf{j}} \lambda_{\mathbf{j}} v_{\mathbf{j}}\right)^{\mathsf{T}} \sum_{\mathbf{j}'} \lambda_{\mathbf{j}'} v_{\mathbf{j}'}.$$

The above calculation also proves that the $A(i)$'s are contractions, and thanks to Eq. (5.5) it follows that (u, v, A) has Boolean behavior of degree t. Finally, note that the value of this (u, v, A) for $\|p\|_{\text{fcb},t}$ is the same as the value of $(u, v, v_{\mathbf{j}})$ for $\|\|p\|\|$, so $\|p\|_{\text{fcb},t} \geq \|\|p\|\|$. $\qquad \square$

Given $t, t' \in \mathbb{N}$ with $t' > t$ and a polynomial $p : \{-1, 1\}^n \to \mathbb{R}$ of degree at most t, $\|p\|_{\text{fcb},t}$ and $\|p\|_{\text{fcb},t'}$ have different definitions, but they are comparable. In particular, we prove that the Fourier completely bounded t-norms are not increasing.

Proposition 5.13. *Let $p : \{-1,1\}^n \to \mathbb{R}$ be a polynomial of degree at most t. Then,*

$$\|p\|_{\mathrm{fcb},t+1} \leq \|p\|_{\mathrm{fcb},t}.$$

Remark 5.14. Proposition 5.13 is coherent with Theorem 5.4 (proved below), because allowing more queries to quantum algorithms only increases their power. Theorem 5.4 also suggests that $\|p\|_{\mathrm{fcb},n} = \|p\|_\infty$ should hold, because n quantum queries should be enough to output any bounded polynomial. If true, alongside Propositions 5.12 and 5.13, it would mean that $(\|p\|_{\mathrm{fcb},t})_{t \in [n]}$ is a decreasing hierarchy of SDPs that tend to $\|p\|_\infty$.

Proof of Proposition 5.13. Let (u, v, A) have Boolean behavior of degree $t + 1$. Then,

$$(\widetilde{u}, \widetilde{v}, \widetilde{A}) = (u, \frac{A(n+1)v}{\|A(n+1)v\|}, A) \tag{5.8}$$

has Boolean behavior of degree t. Also, given that $t + 1 > t$, we have that for every $S \subseteq [n]$ with $|S| \leq t$, there exists $\mathbf{i} \in [n+1]^{t+1}$ such that $S_\mathbf{i} = S$, $i_{t+1} = n + 1$, and

$$\langle u, A(i_1) \dots A(i_{t+1})v \rangle = \|A(n+1)v\| \langle \widetilde{u}, \widetilde{A}(i_1) \dots \widetilde{A}(i_d)\widetilde{v} \rangle. \tag{5.9}$$

This way,

$$
\begin{aligned}
\|p\|_{\mathrm{fcb},t+1} &= \sup_{(u,v,A) \in \mathscr{BB}^{t+1}} \sum_{S \subseteq [n], |S| \leq t} \widehat{p}(S) \langle u, A(i_1^S) \dots A(i_{t+1}^S)v \rangle \\
&= \sup_{(u,v,A) \in \mathscr{BB}^{t+1}} \|A(n+1)v\| \sum_{S \subseteq [n], |S| \leq t} \widehat{p}(S) \langle \widetilde{u}, \widetilde{A}(i_1^S) \dots \widetilde{A}(i_d^S)\widetilde{v} \rangle \\
&\leq \sup_{(u,v,A) \in \mathscr{BB}^{t+1}} \sum_{S \subseteq [n], |S| \leq t} \widehat{p}(S) \langle \widetilde{u}, \widetilde{A}(i_1^S) \dots \widetilde{A}(i_d^S)\widetilde{v} \rangle \\
&\leq \sup_{(u',v',A') \in \mathscr{BB}^{t}} \sum_{S \subseteq [n], |S| \leq t} \widehat{p}(S) \langle u', A'(i_1^S) \dots A'(i_d^S)v' \rangle \\
&= \|p\|_{\mathrm{fcb},t},
\end{aligned}
$$

where in the second line we have used Eq. (5.9), and in the third line that $\|A(n+1)v\| \leq 1$, and in the fourth that $(\widetilde{u}, \widetilde{v}, \widetilde{A})$ has Boolean behavior of degree t. $\square$

The next proposition states that $\|\cdot\|_{\mathrm{fcb},t}$ does not increase after restrictions, which is a relevant feature to ensure that Conjecture 5.5 implies Conjecture 5.1. Given a polynomial $p : \{-1,1\}^n \to \mathbb{R}$ and $i \in [n]$, the *restriction of p to the i-th variable*

being set to $y \in \{-1, 1\}$ is the polynomial $q : \{-1, 1\}^{n-1} \to \mathbb{R}$ (whose variables we index with $x(1), \ldots, x(i-1), x(i+1), \ldots, x(n)$ for convenience) defined by $q(x) :=$ $p(x(1), \ldots, x(i-1), y, x(i+1), \ldots, x(n)).$

Proposition 5.15. *Let $p : \{-1, 1\}^n \to \mathbb{R}$ be a polynomial of degree at most t and let $i \in [n]$. Let $q : \{-1, 1\}^{n-1} \to \mathbb{R}$ be the restriction of p to the i-th variable being set to $y \in \{-1, 1\}$. Then,*

$$\|q\|_{\mathrm{fcb},t} \leq \|p\|_{\mathrm{fcb},t}.$$

Proof. Consider a pair of vectors and a string of matrices $(u, v, A(1), \ldots, A(i-1), A(i+1), \ldots, A(n+1))$ with Boolean behavior of degree t. Define $\widetilde{u} := u$, $\widetilde{v} := v$ and $\widetilde{A}(j)$ for $j \in [n+1]$ as

$$\widetilde{A}(j) = \begin{cases} A(j) & \text{if } j \neq i, \\ yA(n+1) & \text{if } j = i. \end{cases}$$

It can be verified that $(\widetilde{u}, \widetilde{v}, \widetilde{A}(1), \ldots, \widetilde{A}(n+1))$ has Boolean behavior of degree t. Now note that for every $S \subseteq [n] - \{i\}$, it is satisfied that

$$\widehat{q}(S) = \widehat{p}(S) + y\widehat{p}(S \cup \{i\}). \tag{5.10}$$

Also, for every $S \subseteq [n] - \{i\}$ with $|S| \leq t - 1$, it is satisfied that

$$\langle \widetilde{u}, \widetilde{A}(j_1^S) \ldots \widetilde{A}(j_d^S) \widetilde{v} \rangle = y \langle \widetilde{u}, \widetilde{A}(j_1^{S \cup \{i\}}) \ldots \widetilde{A}(j_d^{S \cup \{i\}}) \widetilde{v} \rangle. \tag{5.11}$$

Thus,

$$
\begin{aligned}
\|q\|_{\mathrm{fcb},t} &= \sup_{\substack{(u,v,A(j))\in\mathscr{BB}^t \\ j\in[n+1]-\{i\}}} \sum_{S\subseteq[n]-\{i\},|S|\le t} \widehat{q}(S)\langle u, A(j_1^S)\ldots A(j_t^S)v\rangle \\
&= \sup_{\substack{(u,v,A(j))\in\mathscr{BB}^t \\ j\in[n+1]-\{i\}}} \sum_{S\subseteq[n]-\{i\},|S|\le t} \widehat{p}(S)\langle u, A(j_1^S)\ldots A(j_d^S)v\rangle \\
&\quad + \sum_{S\subseteq[n]-\{i\},|S|\le t-1} y\widehat{p}(S\cup\{i\})\langle u, A(j_1^S)\ldots A(j_d^S)v\rangle \\
&= \sup_{\substack{(u,v,A(j))\in\mathscr{BB}^t \\ j\in[n+1]-\{i\}}} \sum_{S\subseteq[n]-\{i\},|S|\le t} \widehat{p}(S)\langle \widetilde{u}, \widetilde{A}(j_1^S)\ldots \widetilde{A}(j_d^S)\widetilde{v}\rangle \\
&\quad + \sum_{S\subseteq[n]-\{i\},|S|\le t-1} \widehat{p}(S\cup\{i\})\langle \widetilde{u}, \widetilde{A}(j_1^{S\cup\{i\}})\ldots \widetilde{A}(j_d^{S\cup\{i\}})\widetilde{v}\rangle \\
&\le \sup_{\substack{(u',v',A'(j))\in\mathscr{BB}^t \\ j\in[n+1]}} \sum_{S\subseteq[n],|S|\le t} \widehat{p}(S)\langle u', A'(j_1^S)\ldots A'(j_d^S)v'\rangle \\
&= \|p\|_{\mathrm{fcb},t},
\end{aligned}
$$

where in the second line we have used Eq. (5.10), in the fourth line Eq. (5.11), and in
the sixth line that $(\tilde{u}, \tilde{v}, \tilde{A})$ has Boolean behavior.   □

5.3   Quantum query algorithms are Fourier completely bounded polynomials

Now we are ready to prove Theorem 5.4, that fully characterizes quantum query
algorithms in terms of the Fourier completely bounded t-norms.

Theorem 5.4. *Let $p : \{-1,1\}^n \to \mathbb{R}$. Then, p is the bias of a t-query quantum
algorithm if and only if its degree is at most $2t$ and $\|p\|_{\mathrm{fcb},2t} \le 1$.*

To prove Theorem 5.4 we reinterpret the semidefinite programs of [GL19], based
on [ABP19].

Theorem 5.16 (Gribling-Laurent). *Let $p : \{-1,1\}^n \to \mathbb{R}$. Then, p is the bias of
t-query quantum algorithm if and only if its degree is at most $2t$ and the value of the*

following semidefinite program is at most 0,

$$\max \ -w + \sum_{x \in \{-1,1\}^n} \frac{p(x)\phi(x)}{2^n} \tag{5.12}$$

$$\text{s.t. } w \geq 0, \ m \in \mathbb{N}, \ A_s \in (B_m)^{n+1}, \ u, v \in \mathbb{R}^m, \ s \in [2t],$$

$$\|\phi\|_1 = 1, \ \|u\|^2 = \|v\|^2 = w,$$

$$\hat{\phi}(S_{\mathbf{i}}) = \langle u, A_1(i_1) \ldots A_{2t}(i_{2t})v \rangle, \ \mathbf{i} \in [n+1]^{2t},$$

where $\|\phi\|_1 = \sum_{x \in \{-1,1\}^n} \frac{|\phi(x)|}{2^n}$.

Remark 5.17. Theorem 5.16 corresponds to [GL19, Equation (24)]. There, the authors not only ask for the $A_s(i)$ to be contractions, but also unitaries. However, that extra restriction does not change the value of the semidefinite program because we can always block-encode a contraction in the top left corner of an unitary (see for instance [AAI+16, Lemma 7]). We also want to remark that $A_s(i)$ can be taken to be equal to $A_{s'}(i)$ for every $s, s' \in [2t]$ and every $i \in [n+1]$, as this extra restriction does not change value of the semidefinite program. Indeed, let (u, v, A_s, w, ϕ) be part of feasible instance of Eq. (5.12). Define now

$$\tilde{u} := u \otimes e_1,$$

$$\tilde{v} := v \otimes e_{2t+1},$$

$$A(i) := \sum_{s \in [2t]} A_s(i) \otimes e_s e_s^{\mathsf{T}},$$

where $\{e_s\}_{s \in [2t+1]}$ is an orthonormal basis of $\mathbb{R}^{2t+1}$. Then,

$$\langle u, A_1(i_1) \ldots A_d(i_{2t})v \rangle = \langle \tilde{u}, \tilde{A}(i_1) \ldots \tilde{A}(i_{2t})\tilde{v} \rangle,$$

for every $\mathbf{i} \in [n+1]^{2t}$. Hence, $(\tilde{u}, \tilde{v}, \tilde{A}, w, \phi)$ is a feasible instance for Eq. (5.12) that attains the same value as (u, v, A_s, w, ϕ).

Proof of 5.4. Thanks to Theorem 5.16 and Remark 5.17, we know that p is the output of t-query quantum algorithm if and only if its degree is at most $2t$ and the following

constraint is satisfied

$$\sum_{x \in \{-1,1\}^n} \frac{p(x)\phi(x)}{2^n} \leq w \tag{5.13}$$

$$\text{s.t. } w \geq 0, \ m \in \mathbb{N}, \ A \in (B_m)^{n+1}, \ u, v \in \mathbb{R}^m,$$

$$\|\phi\|_1 = 1, \tag{5.14}$$

$$\|u\|^2 = \|v\|^2 = w,$$

$$\widehat{\phi}(S_{\mathbf{i}}) = \langle u, A(i_1) \dots A(i_{2t})v \rangle, \ \mathbf{i} \in [n+1]^{2t}.$$

Now, note that if (u, v, A, ϕ, w) satisfies all conditions of Eq. (5.13) except for Eq. (5.14), then $(u/\sqrt{\|\phi\|_1}, v/\sqrt{\|\phi\|_1}, A, \phi/\|\phi\|_1, w/\|\phi\|_1)$ would be a feasible instance. Furthermore, given that

$$\sum_{x \in \{-1,1\}^n} \frac{p(x)\phi(x)}{2^n} \leq w \iff \frac{1}{\|\phi\|_1} \sum_{x \in \{-1,1\}^n} \frac{p(x)\phi(x)}{2^n} \leq \frac{w}{\|\phi\|_1},$$

we can write Eq. (5.13) forgetting about the normalization condition of Eq. (5.14). In other words, Eq. (5.13) is equivalent to

$$\sum_{x \in \{-1,1\}^n} \frac{p(x)\phi(x)}{2^n} \leq w \tag{5.15}$$

$$\text{s.t. } w \geq 0, \ m \in \mathbb{N}, \ A \in (B_m)^{n+1}, \ u, v \in \mathbb{R}^m,$$

$$\|u\|^2 = \|v\|^2 = w, \tag{5.16}$$

$$\widehat{\phi}(S_{\mathbf{i}}) = \langle u, A(i_1) \dots A(i_{2t})v \rangle, \ \mathbf{i} \in [n+1]^{2t}.$$

In addition, by homogeneity we can assume $w = 1$, as if (u, v, A, ϕ, w) is a feasible instance, then $(u/\sqrt{w}, v/\sqrt{w}, A, \phi/w, 1)$ also is, and Eq. (5.15) is satisfied for the first instance if and only if is satisfied for the second instance. Also note, that if (u, v, A) are part of a feasible instance of Eq. (5.15), then it automatically has Boolean behavior of degree $2t$, and any (u, v, A) defines a feasible instance for Eq. (5.15). Finally, by Parseval's identity we can rewrite $\sum_{x \in \{-1,1\}^n} \frac{p(x)\phi(x)}{2^n}$ as $\sum_{S \subseteq [n]} \widehat{p}(S)\widehat{\phi}(S)$. Putting altogether we get that p is the output of t-query quantum algorithm if and only if its

degree is at most $2t$ and

$$\sum_{S \subseteq [n], |S| \leq 2t} \widehat{p}(S) \langle u, A(i_1^S) \dots A(i_{2t}^S) v \rangle \leq 1$$

s.t. (u, v, A) has Boolean behavior of degree $2t$,

which is the same as saying that $\|p\|_{\mathrm{fcb}, 2t} \leq 1$. $\qquad\qquad\square$

5.4 Aaronson and Ambainis conjecture for (Fourier) completely bounded polynomials

In this section we prove Theorem 5.6 and Theorem 5.8. Both are based on the construction used by Varopoulos to disprove a degree 3 von Neumann's inequality [Var74].

5.4.1 AA conjecture for block-multilinear completely bounded polynomials

Before proving Theorem 5.8, we shall specify what is a block-multilinear completely bounded polynomial. A *block-multilinear polynomial* of degree t is a polynomial $p : \{-1, 1\}^{n \times t} \to \mathbb{R}$ such that if we divide the variables $x \in \{-1, 1\}^{n \times t}$ in t blocks of n coordinates each, then the every of the monomials of p has at most one coordinate of each of the blocks. In other words, the block-multilinear polynomials of degree t are those that can be written as

$$p(x_1, \dots, x_d) = \hat{p}(\emptyset) + \sum_{s \in [t]} \sum_{\substack{\mathbf{b} \in [t]^s \\ b_1 < \dots < b_s}} \sum_{\mathbf{i} \in [n]^s} \hat{p}(\{(b_1, i_1), \dots, (b_s, i_s)\}) x_{b_1}(i_1) \dots x_{b_s}(i_s),$$

$$(5.17)$$

for every $(x_1, \dots, x_d) \in (\{-1, 1\}^n)^t$. For this kind of polynomials, there is a very natural way of evaluating them in matrix inputs,

$$p(A_1, \dots, A_d) = \hat{p}(\emptyset)\mathrm{Id}_m + \sum_{s \in [t]} \sum_{\substack{\mathbf{b} \in [t]^s \\ b_1 < \dots < b_s}} \sum_{\mathbf{i} \in [n]^s} \hat{p}(\{(b_1, i_1), \dots, (b_s, i_s)\}) A_{b_1}(i_1) \dots A_{b_s}(i_s),$$

$$(5.18)$$

for every $A_s \in (M_m)^n$, $s \in [t]$ and $m \in \mathbb{N}$. The *completely bounded norm* of a block-multilinear polynomial is defined as[4]

$$\|p\|_{\mathrm{cb}} := \sup\{\|p(A_1, \ldots, A_d)\| : m \in \mathbb{N}, \ A_s \in (B_m)^n, \ s \in [t]\}. \tag{5.19}$$

Concerning these polynomials, we can show the following.

Theorem 5.8. *Let $t \in \mathbb{N}$. Let $p : \{-1, 1\}^{n \times t} \to \mathbb{R}$ be a block-multilinear degree t polynomial with $\|p\|_{\mathrm{cb}} \leq 1$. Then, p has a variable of influence at least $(\mathrm{Var}[p]/t)^2$. What is more, if p is homogeneous of degree t, then it has a variable of influence at least $\mathrm{Var}[p]^2$.*

Remark 5.18. With our proof of the homogeneous case of Theorem 5.8 we can show that for the case of $p : \{-1, 1\}^{n \times t} \to \mathbb{R}$ being a homogeneous degree t block-multinear polynomial we have the following non-commutative root influence inequality

$$\|p\|_{\mathrm{cb}} \geq \sum_{i \in [n]} \sqrt{\mathrm{Inf}_{s,i}[p]}, \tag{5.20}$$

for any $s \in [t]$. This improves [BSdW22, Theorem 1.4] in two ways. First, we can allow s to be any number in $[t]$, while they only prove the result of $s \in \{1, t\}$. Second, they prove a weaker statement that depends on t, namely,

$$\|p\|_{\mathrm{cb}} \geq \sum_{i \in [n]} \frac{\sqrt{\mathrm{Inf}_{s,i}[p]}}{\sqrt{e(t+1)}},$$

for $s \in \{1, t\}$.

Remark 5.19. Given that $p(x_1, \ldots, x_d) = x_1(1) \ldots x_d(1)$ is a homogeneous degree t block-multilinear completely bounded polynomial with $\mathrm{Var}[p]^2 = \mathrm{MaxInf}[p] = 1$, we have that the homogeneous case of Theorem 5.8 is optimal.

Proof of the homogeneous degree t case of Theorem 5.8. Let p be a homogeneous degree t block-multilinear polynomial. Let $s \in [t]$. We label the coordinates by (r, i), where $r \in [t]$ indicates the block, and $i \in [n]$. Our goal is defining $A \in (B_m)^n$ and $f_\emptyset, e_\emptyset \in S^{m-1}$ such that

$$\langle f_\emptyset, A(i_1) \ldots A(i_d) e_\emptyset \rangle = \frac{\widehat{p}(\{(1, i_1), \ldots, (t, i_d)\})}{\sqrt{\mathrm{Inf}_{s, i_s}[p]}}. \tag{5.21}$$

---

[4]We abuse notation here, as, for the case of homogeneous block-multilinear polynomials, this definition conflicts with the one given in Definition 4.7. For the rest of the chapter, we will use the one in Eq. (5.18).

5.4. Aaronson and Ambainis conjecture for (Fourier) completely bounded polynomials

Once we are there, we can prove the announced root-influence inequality Eq. (5.20). Indeed,

$$
\begin{aligned}
\|p\|_{\mathrm{cb}} &\geq \sum_{i_1,\ldots,i_d \in [n]} \widehat{p}(\{(1,i_1),\ldots,(t,i_d)\}) \langle f_\emptyset, A(i_1)\ldots A(i_d) e_\emptyset \rangle \\
&= \sum_{i_1,\ldots,i_d \in [n]} \widehat{p}(\{(1,i_1),\ldots,(t,i_d)\}) \frac{\widehat{p}(\{(1,i_1),\ldots,(t,i_d)\})}{\sqrt{\mathrm{Inf}_{s,i_s}[p]}} \\
&= \sum_{i_s \in [n]} \frac{1}{\sqrt{\mathrm{Inf}_{s,i_s}[p]}} \underbrace{\sum_{i_1,\ldots,i_{s-1},i_{s+1},i_d \in [n]} \widehat{p}(\{(1,i_1),\ldots,(t,i_d)\})^2}_{\mathrm{Inf}_{s,i}[p]} \\
&= \sum_{i \in [n]} \sqrt{\mathrm{Inf}_{s,i}[p]}.
\end{aligned}
$$

Finally, the statement about the maximal influence quickly follows from the root-influence inequality

$$
\|p\|_{\mathrm{cb}} \geq \sum_{i \in [n]} \sqrt{\mathrm{Inf}_{s,i}[p]} \geq \sum_{i \in [n]} \frac{\mathrm{Inf}_{s,i}[p]}{\sqrt{\mathrm{MaxInf}[p]}} = \frac{\mathrm{Var}[p]}{\sqrt{\mathrm{MaxInf}[p]}},
$$

which after rearranging yields

$$
\mathrm{MaxInf}[p] \geq \left(\frac{\mathrm{Var}[p]}{\|p\|_{\mathrm{cb}}} \right)^2.
$$

Hence, it suffices to design $(f_\emptyset, e_\emptyset, A) \in S^{m-1} \times S^{m-1} \times (B_m)^n$ satisfying Eq. (5.21). Let $\mathcal{S} := \{\{(r,i_r),\ldots,(t,i_t)\} : i_r,\ldots,i_t \in [n], \ s+1 \leq r \leq t\}$ and $\mathcal{S}' := \{\{(1,i_1),\ldots,(r,i_r)\} : i_1,\ldots,i_r \in [n], \ r \leq s-1\}$. Let $m := 2+|\mathcal{S}|+|\mathcal{S}'|$. Let $\{e_\emptyset, e_S, f_\emptyset, f_{S'} : S \in \mathcal{S}, \ S' \in \mathcal{S}'\}$ be an orthonormal basis of $\mathbb{R}^m$, and define $A(i) \in M_m$ by

$$
\begin{aligned}
A(i)e_S &:= e_{S \cup \{(t-|S|,i)\}}, \ \text{for} \ 0 \leq |S| \leq t-s-1, \ S \in \mathcal{S}, \\
A(i)e_S &:= \sum_{\substack{S' \in \mathcal{S}' \\ |S'|=s-1}} \frac{\widehat{p}(S' \cup S \cup \{(s,i)\})}{\sqrt{\mathrm{Inf}_{s,i}(p)}} f_{S'}, \ \text{for} \ |S| = t-s, \ S \in \mathcal{S}, \\
A(i)f_{S'} &:= \delta_{(|S'|,i) \in S'} f_{S'-\{(|S'|,i)\}}, \ S' \in \mathcal{S}'.
\end{aligned}
$$

We claim that $(f_\emptyset, e_\emptyset, A(i))$ satisfies Eq. (5.21). This is because the first applications of the $A(i)$'s act like a *creation* operator and the last as *annihilation* operators. The first $t-s-1$ of the matrices on $e_\emptyset$ create a vector that stores the indices of these

first $t - s - 1$ applications, namely

$$A(s+1) \ldots A(t)e_\emptyset = e_{\{((s+1,i_{s+1}),\ldots,(t,i_d))\}}.$$

The $t - s$ application has a unique behavior, as it maps the previous vector to a superposition of f. vectors, namely

$$A(i_s)e_{\{((s+1,i_{s+1}),\ldots,(t,i_d))\}} = \sum_{\substack{S' \in \mathcal{S}' \\ |S'|=s-1}} \frac{\widehat{p}(S' \cup ((s,i_s),\ldots,(t,i_d)))}{\sqrt{\mathrm{Inf}_{s,i}(p)}} f_{S'}.$$

Finally, the last $s - 1$ applications of the matrices act like annihilation operators, meaning that

$$A(i_1) \ldots A(i_{s-1})f_{S'} = \delta_{S',((1,i_1),\ldots,(s-1,i_{s-1}))} f_\emptyset.$$

Putting everything together we conclude that indeed Eq. (5.21) is satisfied.

Finally, we claim that $A(i)$ are contractions. Given that $\{e_S : 0 \le |S| \le t - s - 1,\ S \in \mathcal{S}\}$, $\{e_S : |S| = t - s,\ S \in \mathcal{S}\}$ and $\{f_{S'} : S' \in \mathcal{S}'\}$ are mapped to orthogonal spaces, we just have to check than when $A(i)$ is a contraction when it is restricted to the span of each of these 3 sets. For the first and third sets of vectors that is clear. For the second is true because for any $\lambda \in [n]^{t-s}$

$$\|A(i) \sum_{\substack{S \in \mathcal{S} \\ |S|=t-s}} \lambda_S e_S\| = \| \sum_{\substack{S \in \mathcal{S} \\ |S|=t-s}} \sum_{\substack{S' \in \mathcal{S}' \\ |S'|=s-1}} \frac{\widehat{p}(S' \cup S \cup \{(s,i)\})}{\sqrt{\mathrm{Inf}_{s,i}[p]}} \lambda_S f_{S'}\|$$

$$= \sqrt{\frac{\sum_{\substack{S' \in \mathcal{S}' \\ |S'|=s-1}} \left(\sum_{\substack{S \in \mathcal{S} \\ |S|=t-s}} \widehat{p}(S' \cup S \cup \{(s,i)\})\lambda_S \right)^2}{\mathrm{Inf}_{s,i}[p]}}$$

$$\le \sqrt{\frac{\sum_{\substack{S' \in \mathcal{S}' \\ |S'|=s-1}} \left(\sum_{\substack{S \in \mathcal{S} \\ |S|=t-s}} \widehat{p}(S' \cup S \cup \{(s,i)\})^2 \right) \left(\sum_{\substack{S \in \mathcal{S} \\ |S|=t-s}} \lambda_S^2 \right)}{\mathrm{Inf}_{s,i}[p]}}$$

$$= \sqrt{\frac{\mathrm{Inf}_{s,i}[p]}{\mathrm{Inf}_{s,i}[p]}} \sqrt{\sum_{\substack{S \in \mathcal{S} \\ |S|=t-s}} \lambda_S^2}$$

$$= \| \sum_{\substack{S \in \mathcal{S} \\ |S|=t-s}} \lambda_S e_S\|,$$

where in the inequality we have used Cauchy-Schwarz. $\qquad\square$

5.4. Aaronson and Ambainis conjecture for (Fourier) completely bounded polynomials

Proof of the general case of Theorem 5.8. Let $p : \{-1, 1\}^{n \times t} \to \mathbb{R}$ be a block-multilinear degree t polynomial. For every $s \in [t]$, let $p_{=s}$ be its degree s part. Let $D \in [t]$ be such that $\mathrm{Var}[p_{=D}] \geq \mathrm{Var}[p]/t$, which exists because $\mathrm{Var}[p] = \sum_{s \in [t]} \mathrm{Var}[p_{=s}]$. We will now divide the proof in two parts. One is showing that

$$\|p_{=D}\|_{\mathrm{cb}} \leq \|p\|_{\mathrm{cb}}, \tag{5.22}$$

and the other is proving that

$$\mathrm{MaxInf}(p_{=D}) \geq \left(\frac{\mathrm{Var}[p_{=D}]}{\|p_{=D}\|_{\mathrm{cb}}} \right)^2. \tag{5.23}$$

Once we had done that, the result will easily follow:

$$\mathrm{MaxInf}(p) \geq \mathrm{MaxInf}(p_{=D}) \geq \left(\frac{\mathrm{Var}[p_{=D}]}{\|p_{=D}\|_{\mathrm{cb}}} \right)^2 \geq \left(\frac{\mathrm{Var}[p]}{t\|p\|_{\mathrm{cb}}} \right)^2,$$

where in the second inequality we have used Eq. (5.23), and in the third we have used Eq. (5.22) and that $\mathrm{Var}[p_{=D}] \geq \mathrm{Var}[p]/t$.

First, we prove Eq. (5.22). Let $B \in B_{t+1}$ be defined by $B := \sum_{s \in [D]} e_s e_{s+1}^{\mathsf{T}}$, where $\{e_s\}_{s \in [D+1]}$ is an orthonormal basis of $\mathbb{R}^{D+1}$. Note that $\langle e_1, B^s e_{D+1} \rangle = \delta_{s,D}$ for all $s \in [t]_0$. Hence,

$$
\begin{aligned}
\|p_{=D}\|_{\mathrm{cb}} &= \sup_{\substack{u,v \in S^{m-1}, \, A \in (B_m)^n \\ m \in \mathbb{N}}} \sum_{\substack{\mathbf{b} \in [t]^D \\ b_1 < \cdots < b_D}} \sum_{\mathbf{i} \in [n]^D} \widehat{p}_{=D}(\{(b_1, i_1), \ldots, (b_D, i_D)\}) \\
&\qquad \cdot \langle u, A_{b_1}(i_1) \ldots A_{b_D}(i_D) v \rangle \\
&= \sup_{\substack{u,v \in S^{m-1}, \, A \in (B_m)^n \\ m \in \mathbb{N}}} \sum_{s \in [t]} \sum_{\substack{\mathbf{b} \in [t]^s \\ b_1 < \cdots < b_s}} \sum_{\mathbf{i} \in [n]^s} \widehat{p}(\{(b_1, i_1), \ldots, (b_s, i_s)\}) \\
&\qquad \cdot \langle u \otimes e_1, (A_{b_1}(i_1) \otimes B) \ldots (A_{b_s}(i_s) \otimes B) v \otimes e_{D+1} \rangle \\
&\leq \|p\|_{\mathrm{cb}}.
\end{aligned}
$$

Second, we prove Eq. (5.23). Let $\mathcal{S} := \{\{(b_1, i_1), \ldots, (b_{D-1}, i_{D-1})\} : \ b_s \in [t], \ b_1 < \cdots < b_{D-1}, \ i_s \in [n], \ s \in [D-1]\}$. Let $m := 2 + |\mathcal{S}|$. Let $\{v, f_\emptyset, f_S : S \in \mathcal{S}\}$ be an

orthonormal basis of $\mathbb{R}^m$. For $b \in [t]$, $i \in [n]$, define $A_b(i) \in M_m$ by

$$A_b(i)v := \sum_{\substack{S \in \mathcal{S} \\ |S| = D-1}} \frac{\widehat{p}_{=D}(S \cup \{(b,i)\})}{\sqrt{\text{MaxInf}[p_{=D}]}} f_S,$$

$$A_b(i)f_S := \delta_{(b,i) \in S} f_{S-\{(b,i)\}}, \text{ for } S \in \mathcal{S} \cup \emptyset.$$

$A_b(i)$ are contractions because they map the vectors of an orthonormal basis to orthogonal vectors without increasing their norms. Note that for $b_1 < \cdots < b_D$ and $\mathbf{i} \in [n]^D$ we have that

$$\langle f_\emptyset, A_{b_1}(i_1) \dots A_{b_D}(i_D)v \rangle = \frac{\widehat{p}_{=D}(\{(b_1, i_1), \dots, (b_D, i_D)\})}{\sqrt{\text{MaxInf}[p_{=D}]}}.$$

Thus,

$$\|p_{=D}\|_{\text{cb}} \geq \sum_{\substack{\mathbf{b} \in [t]^D \\ b_1 < \cdots < b_D}} \sum_{\mathbf{i} \in [n]^D} \widehat{p}_{=D}(\{(b_1, i_1), \dots, (b_D, i_D)\}) \langle f_\emptyset, p(A_1, \dots, A_d)v \rangle$$

$$= \frac{\text{Var}[p_{=D}]}{\sqrt{\text{MaxInf}[p_{=D}]}},$$

which after rearranging yields Eq. (5.23). $\qquad\square$

5.4.2   AA conjecture for homogeneous Fourier completely bounded polynomials

Finally, we prove a new case of the AA conjecture.

Theorem 5.6. *Let $t \in \mathbb{N}$. Let $p : \{-1, 1\}^n \to \mathbb{R}$ be a homogeneous polynomial of degree t and with $\|p\|_{\text{fcb},t} \leq 1$. Then, the maximum influence of p is at least $\text{Var}[p]^2$.*

Proof. Let $m := 1 + \binom{n}{0} + \cdots + \binom{n}{t-1}$. Let $\{v, f_\emptyset, f_S : S \subseteq [n], \ 1 \leq |S| \leq t-1\}$ be an orthonormal basis of $\mathbb{R}^m$. Define the matrices $A(i) \in M_m$ as

$$A(i)v := \sum_{\substack{S \ni i \\ |S| = t}} \frac{\widehat{p}(S)}{\sqrt{\text{MaxInf}[p]}} f_{S-\{i\}},$$

$$A(i)f_S := \delta_{S \ni i} f_{S-\{i\}}, \text{ for } S \subseteq [n], \ 0 \leq |S| \leq t-1,$$

for $i \in [n]$ and $A(n+1) := 0$. We claim that $(f_\emptyset, v, A(i))$ has Boolean behavior of degree t. $A(n+1)$ is clearly a contraction. For $i \in [n]$, $A(i)$ is a contraction, as it maps vectors of the orthonormal basis to orthogonal vectors without increasing the norm, because

$$\|A(i)v\|^2 = \sum_{S \ni i} \frac{\widehat{p}(S)^2}{\mathrm{MaxInf}[p]} = \frac{\mathrm{Inf}_i[p]}{\mathrm{MaxInf}[p]} \leq 1.$$

On the other hand, if $S \subseteq [n]$ satisfies $|S| \leq t-1$, then any $\mathbf{i} \in [n+1]^t$ with $S_{\mathbf{i}} = S$ either has a repeated element of $[n]$ or has an appearance of the index $n+1$, which implies that $\langle f_\emptyset, A(i_1) \ldots A(i_d) \rangle = 0 = \widehat{p}(S)$. If $|S| = t$, then any $\mathbf{i} \in [n+1]^t$ with $S_{\mathbf{i}} = S$ has t different indices in $[n]$ (corresponding to the elements of S), so in that case

$$\langle f_\emptyset, A(i_1) \ldots A(i_d)v \rangle = \frac{\widehat{p}(S)}{\sqrt{\mathrm{MaxInf}[p]}}. \tag{5.24}$$

Putting everything together we conclude that $(f_\emptyset, v, A(i))$ has Boolean behavior of degree t, so

$$\|p\|_{\mathrm{fcb},t} \geq \sum_{S \subseteq [n]} \widehat{p}(S) \langle f_\emptyset, A(i_1) \ldots A(i_d)v \rangle = \sum_{S \subseteq [n]} \frac{\widehat{p}(S)^2}{\sqrt{\mathrm{MaxInf}[p]}}$$

$$= \frac{\mathrm{Var}[p]}{\sqrt{\mathrm{MaxInf}[p]}},$$

where in the first equality we have used Eq. (5.24). After rearranging, the above expression yields

$$\mathrm{MaxInf}[p] \geq \left(\frac{\mathrm{Var}[p]}{\|p\|_{\mathrm{fcb},t}} \right)^2.$$

$\square$

Remark 5.20. Sadly, we could not extend the proof of Theorem 5.6 to the general case. Now, we aim to illustrate what would go wrong with our technique.

For example, consider a polynomial $p : \{-1, 1\}^3 \to \mathbb{R}$ with $\deg(p) = 1$ and $\|p\|_{\mathrm{fcb},3} \leq 1$. Ideally, we would want to define unit vectors u and v and contractions $A(i)$ such that for every $S \subseteq [3]$ and every $\mathbf{i} \in [\mathbf{i}^S]$ they satisfied

$$\langle u, A(i_1) A(i_2) A(i_3)v \rangle = \frac{\widehat{p}(S)}{\sqrt{\mathrm{MaxInf}[p]}}. \tag{5.25}$$

If we emulated the strategy of the proof of Theorem 5.6, then $A(1)v$ should be a *normalized* superposition of orthogonal vectors whose amplitudes are all possible $\widehat{p}(S_{\mathbf{i}})$

that have $i_3 = 1$. In particular, all $\widehat{p}(S)$ with $|S| = 1$ must be included among these amplitudes, because if $S = \{i\}$, then $S = S_{(i,1,1)}$. Hence, the *normalizing* factor of $A(1)v$ should be $\sqrt{\operatorname{Var} p}$, instead of $\sqrt{\operatorname{MaxInf}(p)}$. Note that this extra normalization comes from the fact that given that given (i_1, i_2, i_3), it may happen that $i_3 \notin S_{(i_1, i_2, i_3)}$ and $\widehat{p}(S_{(i_1, i_2, i_3)}) \neq 0$, because p is not homogeneous of degree-3. If we mimic the rest of the proof after this first step that we were forced to modify, we would reach

$$\langle u, A(i_1) \dots A(i_3)v \rangle = \frac{\widehat{p}(S)}{\sqrt{\operatorname{Var}[p]}}$$

instead of Eq. (5.25), which would lead to $\|p\|_{\text{fcb},3} \geq \sqrt{\operatorname{Var} p}$, that is trivially true, because $\|p\|_{\text{fcb},3} \geq \|p\|_\infty$ and $\|p\|_\infty \geq \sqrt{\operatorname{Var} p}$.

Remark 5.21. However, there might be a different way of, given a polynomial p of degree at most t, choosing (u, v, A) with Boolean behavior of degree t such that

$$\langle u, A(i_1) \dots A(i_d)v \rangle = \frac{\widehat{p}(S_{\mathbf{i}})}{\operatorname{poly}(t, \operatorname{MaxInf}[p])},$$

for any $\mathbf{i} \in [n+1]^t$. If that was true, one could copy and paste the proof of Theorem 5.6 and conclude Conjecture 5.5.

This reduces Conjecture 5.5 to a question with flavor of tensor networks (see [CPGSV21] for an introduction to the topic). In particular, the central questions in matrix product states theory is, given a t-tensor $T \in \mathbb{C}^{n \times \dots \times n}$, to find matrices $A_1, \dots, A_t$ of low dimension such that $T_{\mathbf{i}} = \operatorname{Tr}[A(i_1) \dots A(i_t)]$ for every $\mathbf{i} \in [n]^t$. Thus, we are asking the same question, but with a different goal: to minimize the operator norm of the matrices, instead of their dimensions.

It also has the flavor of almost-quantum correlations [NGHA15]. Almost-quantum correlations are a model for multiparty quantum mechanics that eludes tensor products and commutativity of the observables: it only imposes the commutativity on the correlations. For example, in a bipartite scenario, valid correlations would be those determined by observables $\{A_x\}_{x \in \mathcal{X}}$ and $\{B_y\}_{y \in \mathcal{Y}}$ and a state $|\psi\rangle$ such that

$$\langle \psi | A_x B_y | \psi \rangle = \langle \psi | B_y A_x | \psi \rangle, \text{ for all } x \in \mathcal{X}, y \in \mathcal{Y}.$$

In other words, almost-quatum correlations impose the commutativity conditions with respect to the *sandwiches* with $|\psi\rangle$, instead of directly imposing them to the observables, Similarly, we would like to find matrices that satisfy certain Boolean relations with respect to the product with two vectors u and v.

5.4. Aaronson and Ambainis conjecture for (Fourier) completely bounded polynomials

Question 5.22. *Given a polynomial p of degree at most t, is there $(u, v, A) \in \mathscr{B}\mathscr{B}^t$ such that*

$$\langle u, A(i_1) \dots A(i_d)v \rangle = \frac{\widehat{p}(S_{\mathbf{i}})}{\text{poly}(t, \text{MaxInf}[p])},$$

for any $\mathbf{i} \in [n+1]^t$?