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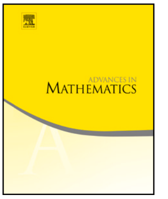

Iterated function systems of affine expanding and contracting maps on the unit interval

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We analyze the two-point motions of iterated function systems on the unit interval generated by expanding and contracting affine maps, where the expansion and contraction rates are determined by a pair (M, N) of integers.

This dynamics depends on the Lyapunov exponent. For a negative Lyapunov exponent we establish synchronization, meaning convergence of orbits with different initial points. For a vanishing Lyapunov exponent we establish intermittency, where orbits are close for a set of iterates of full density, but are intermittently apart. For a positive Lyapunov exponent we show the existence of an absolutely continuous stationary measure for the two-point dynamics.

For nonnegative Lyapunov exponent and pairs (M, N) that are multiplicatively dependent integers, we provide explicit expressions for absolutely continuous stationary measures of the two-point motions. These stationary measures are infinite σ-finite measures in the case of zero Lyapunov exponent. For varying Lyapunov exponent we find here a phase transition for the system of two-point motions, in which the support of the stationary measure explodes with intermittent dynamics and an infinite stationary measure at the transition point.

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1. Introduction

In this article we introduce a natural and simple toy model of iterated function systems on the interval with affine expanding and contracting maps and explore its dynamics. We focus in particular on the dynamics of two orbits simultaneously, the so-called two-point motions. Our set-up is as follows. Given a pair (M, N) of integers $M, N \geq 2$, let

$$f_0 : [0,1) \to [0,1);\ x \mapsto Nx \pmod 1$$

be the N-adic map and let

$$f_i : [0,1) \to [0,1);\ x \mapsto (x+i-1)/M, \qquad 1 \leq i \leq M,$$

be M contracting maps. Fig. 1 depicts the graphs for a few values of (M, N).

For a sequence $\omega = (\omega_0, \omega_1, \dots) \in \{0, 1, \dots, M\}^{\mathbb{N}}$, write

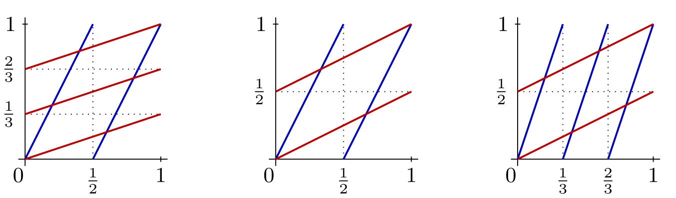
$$f_\omega^n = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} \circ f_{\omega_0}, \tag{1.1}$$

for n compositions of maps from $\{f_0, f_1, \dots, f_M\}$ with $f_\omega^0 = \mathrm{id}$ equal to the identity mapping. We consider orbits $(f_\omega^n(x))_{n \geq 1}$ for points $x \in [0,1)$, where the $\omega_i \in \{0, 1, \dots, M\}$ are picked independently and identically distributed with probabilities p_i. Throughout the article we make the following assumption on the probability vector $\mathbf{p} = (p_0, \dots, p_M)$: Choose the map f_0 with probability $0 < p_0 < 1$ and all maps f_i, $1 \leq i \leq M$, with equal probability $p_i = \frac{1-p_0}{M}$. So the randomness depends on a single parameter $p_0 \in (0,1)$ and the probability vector $\mathbf{p}$ is of the special form

$$\mathbf{p} = \left(p_0, \frac{1-p_0}{M}, \dots, \frac{1-p_0}{M} \right). \tag{1.2}$$

Let ν denote the $\mathbf{p}$-Bernoulli measure on $\{0, 1, \dots, M\}^{\mathbb{N}}$. Let λ denote the Lebesgue measure.

We are interested in results on the two-point motions $(f_\omega^n(x), f_\omega^n(y))_{n \geq 0}$ for $x, y \in [0,1)$ and $\omega \in \{0, 1, \dots, M\}^{\mathbb{N}}$. Statistical properties of such two-point motions are obtained by studying the iterated function system on $[0,1)^2$ generated by the maps



(a) $(3,2)$ (b) $(2,2)$ (c) $(2,3)$

Fig. 1. Graphs of $f_0, \dots, f_M$ for $(M, N) = (3,2), (2,2), (2,3)$.

$$f_i^{(2)}(x,y) = (f_i(x), f_i(y)), \qquad 0 \leq i \leq M. \tag{1.3}$$

We note that two-point motions in contexts of stochastic differential equations are considered in work by Baxendale, see in particular [12–14]. For compositions of independent random diffeomorphisms it is investigated in [42].

Here we consider two types of results. Firstly, we investigate the asymptotics of the distances $|f_\omega^n(x) - f_\omega^n(y)|$ when $n \to \infty$. Below we show that, with the probability vector $\mathbf{p}$ from (1.2), the Lebesgue measure is a stationary measure for the iterated function system $\{f_i\,;\, 0 \leq i \leq M\}$ on $[0,1)$. (We note that Lebesgue measure is not always the unique stationary measure, examples of non-uniqueness can be deduced from [17].) In this sense we treat conservative systems and one expects points from typical orbits to lie uniformly distributed in the unit interval. However, we will see that different values of M and N, or different values of p_0, lead to significant differences in the behavior of two orbits with different initial conditions under the same composition of maps. We distinguish three different types of dynamical behavior, the occurrence of which hinges on the sign of the Lyapunov exponent

$$L_{p_0} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln f'_{\omega_i} = p_0 \ln(N) - (1-p_0)\ln(M). \tag{1.4}$$

This limit exists almost surely and equals the given constant by the strong law of large numbers. The following theorem assembles our main results on the asymptotics of $|f_\omega^n(x) - f_\omega^n(y)|$.

Theorem 1.1. *Let $M, N \geq 2$ be integers and $0 < p_0 < 1$ be given. For the iterated function system $\{f_0, f_1, \dots, f_M\}$ and probability vector $\mathbf{p}$ as in (1.2), we have the following.*

(i) *Suppose $L_{p_0} < 0$. Then*

$$\lim_{n\to\infty} |f_\omega^n(x) - f_\omega^n(y)| = 0$$

for all $x, y \in [0,1)$ and ν-almost all ω.

(ii) *Suppose $L_{p_0} = 0$. Then for every $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \frac{1}{n} |\{0 \leq i < n\,;\, |f_\omega^i(x) - f_\omega^i(y)| < \varepsilon\}| = 1$$

for all $x, y \in [0,1)$ and ν-almost all ω, while for any small $\beta > 0$, any $x, y \in [0,1)$ and ν-almost all ω either $|f_\omega^n(x) - f_\omega^n(y)| = 0$ for some n or $|f_\omega^n(x) - f_\omega^n(y)| > \beta$ for infinitely many values of n.

(iii) *Suppose $L_{p_0} > 0$. Then*

$$P(\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \left|\{0 \leq i < n\,;\, |f_\omega^i(x) - f_\omega^i(y)| < \varepsilon\}\right|$$

exists for $\nu \times \lambda$*-almost all* (ω, x, y)*, and*

$$\lim_{\varepsilon \to 0} P(\varepsilon) = 0.$$

This theorem combines statements of Theorem 3.1, Theorem 3.4, Theorem 3.5 and Theorem 3.15 below.

To put the results in a broader context, we comment on the phenomena of synchronization, intermittency and instability, observed in the three different cases.

$L_{p_0} < 0$**:** In this case the contraction wins from the expansion and synchronization occurs, which means that orbits from different initial points in $[0, 1)$ converge to each other almost surely. This is comparable to synchronization by noise [3,48]. Related synchronization results have been obtained in diverse settings, see e.g. [12,22,30,46]. Kleptsyn and Volk [40] treat such a phenomenon in the context of smooth monotone interval maps forced by transitive subshifts of finite type. Closely related is [26] that provides cases of synchronization for iterated function systems generated by interval diffeomorphisms. Synchronization by noise in random logistic maps is considered in [1,4,54].

$L_{p_0} = 0$**:** In this neutral case a phenomenon reminiscent of intermittency arises. Intermittency, first studied in [50], refers to the phenomenon where a dynamical system shows sudden transitions from a long period of exhibiting one type of dynamical behavior to a period of another type of dynamics. Recently this was analyzed in the context of random dynamics for the random Gauss-Rényi map [6,10,35,37,59], random LSV maps [7,8], random logistic maps [1,5] and more general families in [31,38].

In the setting of Theorem 1.1, orbits of different initial points are intermittently very close together or some distance apart. The set of iterates for which orbits are close has full density, but the complement is still an infinite set. A similar mechanism arises in iterated function systems of interval diffeomorphisms [26], or more generally for skew product systems with interval diffeomorphisms as fiber maps [32]. In both papers one gets a singular distribution of orbit points instead of the uniform distribution that we find.

$L_{p_0} > 0$**:** Here the expansion wins from the contraction and orbits tend to diverge from each other. Random interval maps with a condition on average expansion have been studied extensively, see e.g. [2,16,20,36,44,47]. Following the definition from [47] the expanding on average condition would correspond to $\frac{p_0}{N} + (1 - p_0)M < 1$, which does not align with the condition that $L_{p_0} > 0$. Hence we do not rely on these expansions on average results here. The two-point maps are connected to Jablonski maps [52], and in the positive Lyapunov exponent case to research on invariant measures for random Jablonski maps [9,11,33,39].

Our second set of main results concerns invariant measures of the iterated function system from (1.3), again with the probability vector $\mathbf{p}$ from (1.2), under an additional assumption on the expansion and contraction factors M and N. Two integers $M, N \geq 1$ are called *multiplicatively dependent* if they are powers of the same natural number, i.e., if $M = \kappa^\ell$ and $N = \kappa^k$ for some integers $\kappa > 1$ and $k, \ell \geq 1$. Here we always take k, ℓ to be relatively prime. This condition is equivalent to $\ln(N)/\ln(M) = k/\ell \in \mathbb{Q}$. If M, N are not multiplicatively dependent, they are called multiplicatively independent. Some of the difficulties in the analysis for Theorem 1.1 are caused by the points of discontinuity of f_0 and can be circumvented in case M, N are multiplicatively dependent. This leads to the following theorem.

Write $\Delta = \{(x, x)\ ;\ x \in [0, 1)\}$ for the diagonal in $[0, 1)^2$ and write $\Delta_\varepsilon = \{(x, y) \in [0, 1)^2\ ;\ |y - x| < \varepsilon\}$ for the ε-neighborhood of Δ. We let $a(\varepsilon) \sim b(\varepsilon)$ stand for $a(\varepsilon)/b(\varepsilon)$ bounded and bounded away from zero as $\varepsilon \to 0$.

Theorem 1.2. *Let $M, N \geq 2$ be integers and $0 < p_0 < 1$ be given. Assume that M and N are multiplicatively dependent with $N = \kappa^k$ and $M = \kappa^\ell$. For the iterated function system $\{f_0^{(2)}, f_1^{(2)}, \ldots, f_M^{(2)}\}$ and probability vector $\mathbf{p}$ as in (1.2), we have the following.*

(i) *Suppose $L_{p_0} < 0$. Then the iterated function system of two-point maps admits Lebesgue measure on Δ as stationary measure.*
(ii) *Suppose $L_{p_0} = 0$. Then the iterated function system of two-point maps admits Lebesgue measure on Δ as stationary measure. Furthermore, it admits an infinite σ-finite absolutely continuous stationary measure of full topological support.*
(iii) *Suppose $L_{p_0} > 0$. Then the iterated function system of two-point maps admits Lebesgue measure on Δ as stationary measure. Furthermore, it admits an absolutely continuous stationary probability measure $\mu^{(2)}$ of full topological support and with*

$$\mu^{(2)}(\Delta_\varepsilon) \sim \varepsilon^{-\ln(\nu_1)/\ln(\kappa)},$$

where ν_1 is the unique real solution in $(0, 1)$ to $p_0 z^{k+\ell} - z^\ell + 1 - p_0 = 0$. The density of $\mu^{(2)}$ is bounded precisely if $\nu_1 \kappa < 1$.

The measure $\mu^{(2)}(\Delta_\varepsilon)$ from Theorem 1.2(iii) quantifies the proportion of iterates that typical orbits $f_\omega^n(x)$ and $f_\omega^n(y)$ are close. This theorem combines statements of Corollary 2.2, Theorem 3.1, Theorem 3.7, Theorem 3.10 and Remark 3.13 below. Further results of a similar flavor, in particular with explicit expressions for stationary measures, or for stationary measures in case of multiplicatively independent pairs (M, N), are found in Section 3.

We stress that the original iterated function system $\{f_i\ ;\ 0 \leq i \leq M\}$ behaves independently of p_0 in the sense that Lebesgue measure on the interval $[0, 1)$ is stationary for all values of p_0. The above theorem however makes clear that, depending on the parameters, the corresponding two-point motions show a range of different behaviors. In particular the theorem describes a bifurcation or phase transition in the iterated function system of two-point motions as the Lyapunov exponent crosses zero for varying p_0. See also [18] for the notion of stochastic n-point bifurcation. This phase transition involves a discontinuous change of the support of the stationary measure of the two-point motion (an explosion of its support) and an infinite stationary measure at the bifurcation point. Transitions that involve a Lyapunov exponent crossing zero have been considered in different contexts such as stochastic differential equations [13], random dynamical systems [27,49,62,63], in studies of noise-induced order such as [23,24,43], and in settings with skew product systems such as [60].

The article is outlined as follows. In the next section we introduce preliminaries on random dynamics, we prove that Lebesgue measure is stationary for the iterated function systems $\{f_0, f_1, \dots, f_M\}$ and we introduce several extensions of these systems that are useful in later parts of the text. In particular we explain a connection to a class of heterochaos baker maps in three dimensions, similar to the map introduced in [52] as a model for heterogeneous chaos. Since the first appearance of our paper as a preprint there have been several investigations of, in particular, ergodic properties of heterochaos baker maps [55–58]. See [51] for further information on heterochaos baker maps.

In Section 3 we study the iterated function system $\{f_0^{(2)}, f_1^{(2)}, \dots, f_M^{(2)}\}$ and derive our main results. The section is divided into three parts depending on the sign of L_{p_0}. All parts come with their own techniques. The case of a vanishing Lyapunov exponent uses theory of random walks involving stopping times with time dependent stopping criteria. This material is developed in Appendix A. We end the article with a short description of possible future extensions of this research.

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2. Skew product systems

2.1. Lebesgue measure is stationary

As usual an approach using a skew product system aids to describe the iterated function system as a single dynamical system, and to use the machinery of dynamical systems theory and ergodic theory. Write $\Sigma = \{0, \dots, M\}^{\mathbb{N}}$ for the space of one-sided infinite sequences of symbols in $\{0, \dots, M\}$, endowed with the product topology obtained from the discrete topology on $\{0, \dots, M\}$. Elements $\omega \in \Sigma$ will be written as $\omega = (\omega_i)_{i\in\mathbb{N}}$. Let $\sigma : \Sigma \to \Sigma$ be the *left shift operator* defined by

$$(\sigma\omega)_i = \omega_{i+1}, \quad i \geq 0.$$

Write $[a_0 \cdots a_k]$ for the *cylinder*

$$[a_0 \cdots a_k] = \{\omega \in \Sigma\,;\, \omega_j = a_j,\, 0 \leq j \leq k\}.$$

We equip Σ with the Borel σ-algebra. Given any $0 < p_0 < 1$ and the corresponding positive probability vector $\mathbf{p}$ as specified in (1.2), we let $\nu = \nu_{\mathbf{p}}$ denote the Bernoulli measure on Σ that is defined on the cylinder sets by

$$\nu([a_0 \cdots a_k]) = \prod_{j=0}^{k} p_{a_j}.$$

The measure ν is an ergodic invariant measure for the shift map σ.

Define the skew product system $F : \Sigma \times [0,1) \to \Sigma \times [0,1)$ by

$$F(\omega, x) = (\sigma\omega, f_{\omega_0}(x)).$$

We use the notation $F^k(\omega, x) = (\sigma^k\omega, f_\omega^k(x))$ for iterates, where f_ω^k is as defined in (1.1). We also write $f_\eta^k(x)$ for elements $\eta = \eta_0 \cdots \eta_{m-1} \in \{0, \dots, M\}^m$, called *words*, with $k \le m$. With slight abuse of notation we will use λ to denote the one-, two- and three-dimensional Lebesgue measure, since the meaning will be clear from the context.

Proposition 2.1. *Let $0 < p_0 < 1$. Then the corresponding product measure $\mu := \nu \times \lambda$ on $\Sigma \times [0,1)$ is an invariant probability measure for F.*

Proof. For invariance it suffices to consider product sets $A = [a_0 \cdots a_j] \times J$ of cylinder sets $[a_0 \cdots a_j]$ and intervals J. Note that for each $x \in [0,1)$ there are N inverse images in $f_0^{-1}\{x\}$ and there is a unique $1 \le j \le M$ for which an inverse image $y \in [0,1)$ with $f_j(y) = x$ exists. One immediately computes that

$$\begin{aligned}\mu(F^{-1}(A)) &= \mu\left(\bigcup_{i=0}^{M} [i\, a_0 \cdots a_j] \times f_i^{-1}(J)\right)\\ &= \nu([a_0 \cdots a_j]) \left[p_0 \lambda(f_0^{-1}(J)) + \frac{1-p_0}{M} \sum_{i=1}^{M} \lambda(f_i^{-1}(J))\right]\\ &= \nu([a_0 \cdots a_j]) \left[p_0 N \frac{\lambda(J)}{N} + \frac{1-p_0}{M} M \lambda(J)\right]\\ &= \mu(A). \quad \square\end{aligned}$$

Note that the proof of Proposition 2.1 uses the specifics of the probability vector $\mathbf{p}$.

Invariance of μ for F implies that λ is a *stationary measure* for the iterated function system $\{f_i\,;\, 0 \le i \le M\}$ with probability vector $\mathbf{p}$ in the sense that

$$\lambda = \sum_{i=0}^{M} p_i (f_i)_* \lambda.$$

Here $(f_i)_*$ stands for the push forward measure $(f_i)_*\lambda(A) = \lambda(f_i^{-1}(A))$. Therefore, a direct consequence of Proposition 2.1 above is the following.

Corollary 2.2. *The diagonal* $\Delta = \{(x,x)\,;\, x \in [0,1)\}$ *is an invariant set for the iterated function system* $\{f_i^{(2)}\,;\, 0 \le i \le M\}$ *from* (1.3) *with probability vector* $\mathbf{p}$ *and Lebesgue measure restricted to* $\Delta = \{(x,x)\,;\, x \in [0,1)\}$ *is a stationary measure.*

Below we will also verify the ergodicity of the measure μ for the skew product F. Instead of writing that μ is ergodic, we also say that the corresponding stationary measure λ is ergodic to mean the same. The proofs of ergodicity provided in the next section are different for the three cases identified in Theorem 1.1. They use a map that is isomorphic to F as well as an extension of this map. Later we will also use a multivalued map. For easy reference we use the remainder of this section to introduce all these different maps.

2.2. One- two- and three-dimensional piecewise affine maps

We first conjugate the shift map to an expanding interval map. Write

$$r_i = \sum_{j=0}^{i-1} p_j, \qquad 0 \le i \le M+1.$$

This gives $0 = r_0 < r_1 < \cdots < r_M < r_{M+1} = 1$. Define the expanding interval map $L : [0,1] \to [0,1]$ by setting

$$L(w) = \begin{cases} \dfrac{w}{p_0}, & 0 \le w < p_0, \\ \\ \dfrac{M(w - r_i)}{1 - p_0}, & r_i \le w < r_{i+1},\ 1 \le i \le M. \end{cases} \tag{2.1}$$

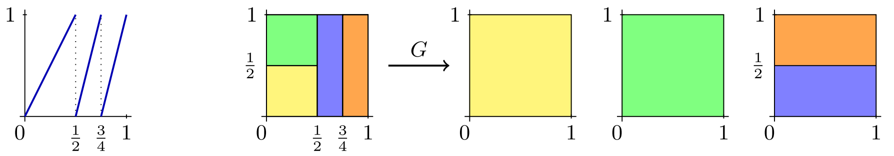
See Fig. 2(a) for an example. Then the map $h : \Sigma \to [0,1]$ given by

$$h(\omega) = \sum_{i=0}^{\infty} \prod_{j=0}^{i-1} p_{\omega_j} r_{\omega_i} \tag{2.2}$$

satisfies $h \circ \sigma = L \circ h$ and $h_* \nu = \lambda$. There is only a countable set of codes in Σ on which h is not injective. So, as h is invertible after removing sets of zero measure, it defines a measurable isomorphism. From this we see that the skew product map F is measurably isomorphic to $G : [0,1)^2 \to [0,1)^2$ given by

$$G(w,x) = \begin{cases} \left(\dfrac{w}{p_0},\ Nx \pmod 1 \right), & 0 \le w < p_0, \\ \\ \left(\dfrac{M(w - r_i)}{1 - p_0},\ \dfrac{x + i - 1}{M} \right), & r_i \le w < r_{i+1}, 1 \le i \le M. \end{cases} \tag{2.3}$$

See Fig. 2(b) for an example.



(a) L (b) G

Fig. 2. Graphs of L and G for $(M, N) = (2, 2)$ and $p_0 = \frac{1}{2}$. G maps the colored areas in the unit square on the left to the areas of the same color on the right. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Consider the invertible extension $\Gamma : [0, 1)^3 \to [0, 1)^3$ of the maps $L : [0, 1) \to [0, 1)$ from (2.1) and $G : [0, 1)^2 \to [0, 1)^2$ from (2.3) given by

$$\Gamma(w, x, y) = \begin{cases} \left(\dfrac{w}{p_0}, Nx - j, \dfrac{p_0(y + j)}{N}\right), & 0 \le w < p_0, \\ & \frac{j}{N} \le x < \frac{j+1}{N}, 0 \le j < N, \\ \left(\dfrac{M(w - r_i)}{1 - p_0}, \dfrac{x + i - 1}{M}, (1 - p_0)y + p_0\right), & r_i \le w < r_{i+1}, 1 \le i \le M. \end{cases} \tag{2.4}$$

For $M = N = 2$ and $p_0 = \frac{1}{2}$, we get

$$\Gamma(w, x, y) = \begin{cases} \left(2w, 2x - j, \dfrac{y + j}{4}\right), & 0 \le w < 1/2, \\ & \frac{j}{2} \le x < \frac{j+1}{2}, j = 0, 1, \\ \left(4w - (2 + i), \dfrac{x + i}{2}, \dfrac{y + 1}{2}\right), & \frac{1}{2} + \frac{i}{4} \le w < \frac{1}{2} + \frac{i+1}{4}, i = 0, 1, \end{cases}$$

a graphical depiction of which is shown in Fig. 3. This particular map is somewhat reminiscent of the two-dimensional baker map B on $[0, 1)^2$ given by

$$B(w, x) = \begin{cases} \left(2w, \dfrac{x}{2}\right), & 0 \le w < \frac{1}{2}, \\ \left(2w - 1, \dfrac{x + 1}{2}\right), & \frac{1}{2} \le w < 1, \end{cases}$$

which has an expanding and a contracting direction, or more specific, a positive Lyapunov exponent $\ln(2)$ and a negative Lyapunov exponent $-\ln(2)$. The iterated function systems that we analyze in this article thus inspire three-dimensional analogues of the baker map. As mentioned in the introduction, similar maps feature in [51,52] in studies of heterogeneous chaos.

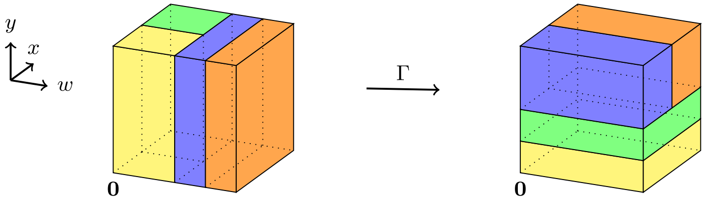


Fig. 3. The map G for $M = N = 2$ and $p_0 = \frac{1}{2}$ maps the regions on the left to the regions on the right according to the colors.

The map Γ is invertible, the inverse being given by

$$\Gamma^{-1}(w,x,y) = \begin{cases} \left(p_0 w, \dfrac{x+j}{N}, \dfrac{Ny}{p_0} - j\right), & \frac{jp_0}{N} \le y < \frac{(j+1)p_0}{N}, 0 \le j < N, \\ \left(\dfrac{(1-p_0)(w+i)}{M} + p_0, Mx - i, \dfrac{y-p_0}{1-p_0}\right), & p_0 \le y < 1, \\ & \frac{i}{M} \le x < \frac{i+1}{M}, 0 \le i < M. \end{cases}$$

Note that all maps L, G and Γ have Lebesgue measure, with appropriate dimension, as invariant measure. We have the following relation between Γ and F (for the purpose of the statement considered on compact spaces).

Lemma 2.3. *The skew product* $F : \Sigma \times [0,1] \to \Sigma \times [0,1]$ *is a factor of the three-dimensional map* $\Gamma : [0,1]^3 \to [0,1]^3$.

Proof. Recall the definition of the isomorphism $h : \Sigma \to [0,1]$ between the map $L : [0,1] \to [0,1]$ and the left shift $\sigma : \Sigma \to \Sigma$ from (2.2). Let $\pi_{w,x} : [0,1]^3 \to [0,1]^2$, $(w,x,y) \to (w,x)$ be the canonical projection onto the first two coordinates. One easily verifies that the map $(h^{-1}, \mathrm{id}) \circ \pi_{w,x} : [0,1]^3 \to \Sigma \times [0,1]$ (up to sets of measure zero) is surjective, measurable, measure preserving and satisfies $F \circ ((h^{-1}, \mathrm{id}) \circ \pi_{w,x}) = ((h^{-1}, \mathrm{id}) \circ \pi_{w,x}) \circ \Gamma$, thus constituting a factor map. □

2.3. Multivalued maps

It is sometimes helpful to consider an associate iterated function system of multivalued maps. Write $\mathbb{K}$ for the class of nonempty compact subsets of $[0,1)$. Define the multivalued map $F_1 : \mathbb{K} \to \mathbb{K}$ by

$$F_1(A) = \bigcup_{i=1}^{M} f_i(A), \tag{2.5}$$

in which the contracting maps $f_1, \ldots, f_M$ are combined. We also write

$$F_0(A) = f_0(A). \tag{2.6}$$

We can then look at the iterated function system generated by F_0 and F_1. A composition F_η^n with $\eta \in \{0,1\}^{\mathbb{N}}$ is a multivalued map. As all maps $f_1, \ldots, f_M$ that make up F_1 have the same constant derivative $1/M$, and f_0 has constant derivative N, we can speak of $(F_\eta^n)'$. The graphs in F_η^n are equally spaced line pieces with constant slope $(F_\eta^n)'$. The number of elements in the set $F_\eta^n(\{x\})$ is independent of $x \in [0,1)$, so $\#F_\eta^n(\{x\}) = \#F_\eta^n(\{0\})$, and $F_\eta^n(\{0\})$ is always of the form

$$S_j := \{i/M^j \,;\, 0 \le i < M^j\}$$

for some $j \ge 0$. Let $\Pi : \Sigma \to \Sigma_2$ be the projection that maps all symbols $1, \ldots, M$ to 1; so $\Pi(\omega) = \eta$ with

$$\eta_i = \begin{cases} 0, & \omega_i = 0, \\ 1, & \omega_i \in \{1, \ldots, M\}. \end{cases}$$

We use the next lemma in the section on intermittency.

Lemma 2.4. *Let $\eta = \eta_0 \cdots \eta_{n-1} \in \{0,1\}^n$, $n \ge 1$, and let $j \ge 0$ be such that $F_\eta^n(\{0\}) = S_j$. Then for each $i \ne k$,*

$$\nu(\{\omega \in \Pi^{-1}[\eta]\,;\, f_\omega^n(0) = i/M^j\}) = \nu(\{\omega \in \Pi^{-1}[\eta]\,;\, f_\omega^n(0) = k/M^j\}).$$

Proof. Set $\gamma = \#\{0 \le i \le n-1\,;\, \eta_i = 1\}$ for the number of occurrences of the digit 1 in η. Note that $\Pi^{-1}[\eta]$ is the disjoint union of M^γ cylinders of length n in Σ. Note also that $F_1(S_0) = S_1$ and $F_1(S_l) = S_{l+1}$ for any $l > 0$, while $F_0(S_l) \subseteq S_l$. There are $i \ne k$ such that $f_0(i/M^l) = f_0(k/M^l)$ if and only if there is an i such that $f_0(i/M^l) = 0$, so such that $Ni/M^l \in \mathbb{N}$, if and only if N and M share a common prime factor. Hence, if M and N are relatively prime, then $F_\eta^n(\{0\}) = S_\gamma$ and for each i,

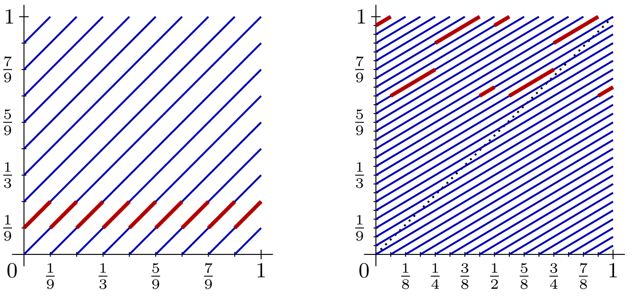
$$\nu(\{\omega \in \Pi^{-1}[\eta]\,;\, f_\omega^n(0) = i/M^\gamma\}) = p^{n-\gamma}\Big(\frac{1-p}{M}\Big)^\gamma.$$

Suppose N and M are not relatively prime. The map f_0 wraps the unit interval around itself N times with constant expansion factor. So, for any $0 \le m < l$ for which $f_0(S_l) = S_m$ it follows that for each i_1, i_2,

$$\#\{0 \le k \le M^l - 1\,;\, f_0(k/M^l) = i_1\} = \#\{0 \le k \le M^l - 1\,;\, f_0(k/M^l) = i_2\}.$$

Since all cylinders $[\omega_0 \cdots \omega_{n-1}] \subseteq \Pi^{-1}[\eta]$ have equal ν-measure, this implies the lemma. □

Properties of graphs of f_ω^n and F_η^n in relation to each other are illustrated in Figs. 4 and 10.



(a) $(3,3), \eta = (11010010)$ (b) $(3,2), \eta = (1010010)$

Fig. 4. Left picture: a plot of the graphs of F_η^8 for $(M,N) = (3,3)$ and $\eta = (11010010)$. The red graph is the graph of f_ω^8 for $\omega = (12020020)$. Right picture: a plot of F_η^7 for $(M,N) = (3,2)$ and $\eta = (1010010)$. The red graph is the graph of f_ω^7 for $\omega = (3020020)$.

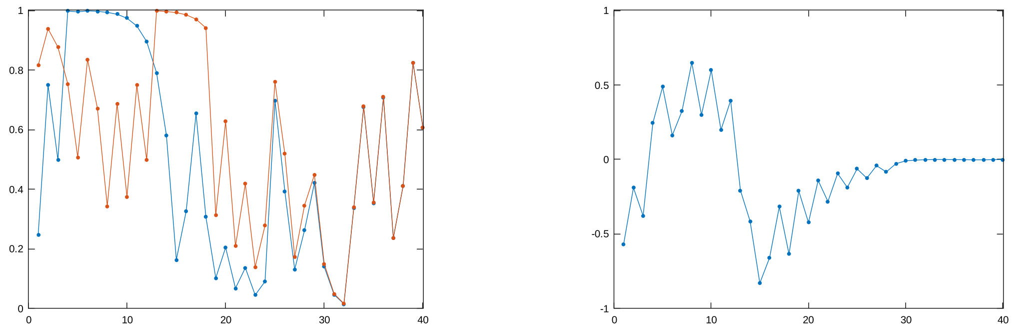


Fig. 5. Two time series of f_ω^n for two different initial points, for $(M,N) = (3,2)$ and $p_0 = \frac{1}{2}$. The signed difference between the two, depicted in the right panel, shows convergence of the orbits to each other.

3. Two-point dynamics

This central section treats the dynamics of the skew product systems for different values of (M,N) and p_0, focusing on convergence and divergence of orbits and statistical properties of orbits. We treat separately the cases with $L_{p_0} < 0$, $L_{p_0} = 0$ and $L_{p_0} > 0$. Note that for $p_0 = 1/2$, this is the same as $M > N$, $M = N$ and $M < N$, respectively.

3.1. $L_{p_0} < 0$ *(synchronization)*

If the contraction is stronger than the expansion, one may expect the orbits of nearby points to converge to each other under identical compositions. The numerical observation in Fig. 5 illustrates this.

We will establish such convergence in fact uniformly on $[0,1)$. The discontinuities of f_0 form an obstacle in the analysis, since nearby points are mapped a positive distance apart if they are on different sides of a point of discontinuity of f_0. Iterates f_ω^n may have many discontinuities on $[0,1)$, as the graph in Fig. 6 illustrates. A Borel-Cantelli

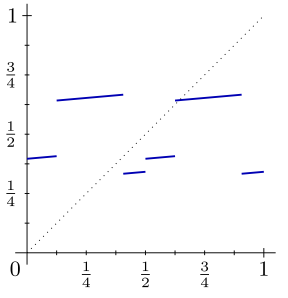


Fig. 6. The graph of an iterate f_ω^{12} for $(M, N) = (3, 2)$ and some ω. Although the slope $(f_\omega^{12})'$ is small, the map is not a contraction on $[0, 1)$ because of the discontinuities.

argument (see for instance [21] for the Borel-Cantelli lemmas) however makes clear that orbits are only infrequently very close to the points of discontinuity of f_0, which allows to prove the following result.

Theorem 3.1. *Consider $L_{p_0} < 0$. For all $x, y \in [0, 1)$,*

$$\lim_{n\to\infty} |f_\omega^n(y) - f_\omega^n(x)| = 0$$

for ν-almost all $\omega \in \Sigma$.

Proof. Let ζ be a number with $e^{L_{p_0}} < \zeta < 1$. With $a_i = \ln(f_i')$, we can write

$$(f_\omega^n)' = e^{\sum_{i=0}^{n-1} a_{\omega_i}}.$$

Recall from (1.4) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} a_{\omega_i} = L_{p_0} < \ln(\zeta) < 0,$$

for ν-almost all ω. So, for ν-almost all ω, $\sum_{i=0}^{n-1} a_{\omega_i} - n\ln(\zeta)$ converges to $-\infty$ and therefore $e^{\sum_{i=0}^{n-1} a_{\omega_i}}/\zeta^n$ goes to zero as $n \to \infty$. We find that for ν-almost all ω,

$$\max\{(f_\omega^n)'/\zeta^n, n \geq 0\}$$

exists. Hence, for ν-almost all $\omega \in \Sigma$, there exists a $C_\omega > 0$ so that

$$(f_\omega^n)' \leq C_\omega \zeta^n.$$

For any $\tilde{\varepsilon} > 0$ one can choose $C > 1$ and a set $\Omega_C \subset \Sigma$ of measure $\nu(\Omega_C) > 1 - \tilde{\varepsilon}$, so that $(f_\omega^n)' \leq C\zeta^n$ for all $\omega \in \Omega_C$. Let $n_1 = n_1(\tilde{\varepsilon})$ be such that $C\zeta^{n_1} < 1/n_1^2$.

Write

$$B(r)=\bigcup_{i=1}^{N-1}\left[\frac{i}{N}-r,\frac{i}{N}+r\right] \tag{3.1}$$

for the r-neighborhood of the set $\mathcal{C}=\{1/N,\dots,(N-1)/N\}$ of discontinuity points of f_0. Let $E_n=\Sigma\times B(1/n^2)$. By the F-invariance of μ it holds that $\mu(F^{-n}(E_n))=\mu(E_n)=1/n^2$, so by the Borel-Cantelli lemma, we get for μ-almost all $(\omega,x)\in\Sigma\times[0,1)$ that $F^n(\omega,x)\in E_n$ for at most finitely many n. Hence, the set of points

$$B=\{(\omega,x)\in\Sigma\times[0,1)\,;\,\exists\, n_0=n_0(\omega,x)\ \text{ s.t. } f_\omega^n(x)\notin B\left(1/n^2\right)\ \text{ for all } n\geq n_0\}$$

satisfies $\mu(B)=1$.

Write

$$\mathcal{C}^*=\bigcup_{n\geq 0}\ \bigcup_{\substack{i_1\cdots i_n\in\\ \{0,1,\dots,M\}^n}} f^{-1}_{i_1\dots i_n}(\mathcal{C})$$

for the set of points in $[0,1)$ that are eventually mapped to $\mathcal{C}$ by some $\omega\in\Sigma$. As $\mathcal{C}^*$ is a countable set,

$$\mu\big((\Omega_C\times([0,1)\setminus\mathcal{C}^*))\cap B\big)>1-\tilde{\varepsilon},$$

which means that we can find an $x\in[0,1)\setminus\mathcal{C}^*$, such that

$$\nu\big(\{\omega\in\Sigma\,;\,(\omega,x)\in(\Omega_C\times([0,1)\setminus\mathcal{C}^*))\cap B\}\big)>0. \tag{3.2}$$

Fix such a point x. Then for any ω in the set from (3.2) there is, by continuity, an open interval J_ω with $x\in J_\omega$, such that

$$f_\omega^n(J_\omega)\cap\mathcal{C}=\emptyset,\qquad \text{for all } n<\max\{n_0,n_1\}.$$

By the choice of (ω,x) we get for all $n\geq\max\{n_0,n_1\}$ that $f_\omega^n(x)\notin B(1/n^2)$ and $(f_\omega^n)'\leq C\zeta^n<1/n^2$. Hence, we recursively obtain that for all $n\geq\max\{n_0,n_1\}$ the set $f_\omega^n(J_\omega)$ is an interval and

$$\lambda(f_\omega^n(J_\omega))\leq\frac{\lambda(f_\omega^{n-1}(J_\omega))}{n^2}\leq\frac{1}{n^2},$$

so that $f_\omega^n(J_\omega)\cap\mathcal{C}=\emptyset$. Moreover, for every ω in the set from (3.2) there is an $n\geq 1$, such that

$$\Big(x-\frac{1}{n},x+\frac{1}{n}\Big)\subseteq J_\omega.$$

So we can find an $n_2 \geq 1$ such that the set

$$\hat{\Omega} := \left\{ \omega \in \Sigma\,;\, (\omega, x) \in (\Omega_C \times ([0,1) \setminus \mathcal{C}^*)) \cap B \text{ and } \left(x - \frac{1}{n_2}, x + \frac{1}{n_2} \right) \subseteq J_\omega \right\}$$

satisfies $\nu(\hat{\Omega}) > 0$.

For each $t \geq 1$ and $\eta \in \{1, \ldots, M\}^t$, the set $f_\eta^t([0,1))$ is an interval of length $1/M^t$. The union of these intervals, varying over all $\eta \in \{1, \ldots, M\}^t$ for fixed t, covers $[0,1)$. Hence, there exist $t \in \mathbb{N}$ and $\eta \in \{1, \ldots, M\}^t$ with $f_\eta^t([0,1)) \subset \left(x - \frac{1}{n_2}, x + \frac{1}{n_2}\right)$. Then for each concatenated sequence $\tilde{\omega} = \eta\omega$, with $\omega \in \hat{\Omega}$, and each $n \geq 1$ the image $f_{\tilde{\omega}}^n([0,1))$ is an interval with $\lim_{n\to\infty} \lambda(f_{\tilde{\omega}}^n([0,1))) = 0$. Hence, we have found a set $\Psi = \eta\hat{\Omega} \subset \Sigma$ with $\nu(\Psi) > 0$, so that for any $y, z \in [0,1)$,

$$\lim_{n\to\infty} |f_{\tilde{\omega}}^n(y) - f_{\tilde{\omega}}^n(z)| = 0. \tag{3.3}$$

The following argument concludes the proof of Theorem 3.1 from (3.3). Assume that there is a set $\Xi \subseteq \Sigma$ with $\nu(\Xi) > 0$ of ω for which $f_\omega^n([0,1))$ is not contained in an interval of length shrinking to 0. We will derive a contradiction from this. By the Lebesgue density theorem we can take a density point ξ of Ξ, meaning

$$\lim_{j\to\infty} \frac{\nu([\xi_1 \cdots \xi_j] \cap \Xi)}{\nu([\xi_1 \cdots \xi_j])} = 1.$$

(The Lebesgue density theorem is formulated for Lebesgue measure on the interval, but transfers to Bernoulli measure on Σ, compare (2.1).) Then

$$\lim_{j\to\infty} \nu(\sigma^j([\xi_1 \cdots \xi_j] \cap \Xi)) = 1$$

and moreover,

$$\sigma^j([\xi_1 \cdots \xi_j] \cap \Xi) \subset \Xi.$$

This contradicts the construction of the set Ψ with $\nu(\Psi) > 0$, since $\Psi \cap \Xi = \emptyset$.

Alternatively, and perhaps more elegantly, one could proceed from (3.3) as suggested by the referee as follows. Let $\Omega' \subset \Sigma$ be the set of $\omega \in \Sigma$ for which $\lim_{n\to\infty} |f_\omega^n(y) - f_\omega^n(x)| = 0$ holds for all $x, y \in [0,1)$. Note that $\sigma^{-1}(\Omega') \subset \Omega'$. Since σ preserves the measure ν, we obtain $\sigma^{-1}(\Omega') = \Omega'$ up to a set of measure zero. By ergodicity of σ we find $\nu(\Omega') \in \{0,1\}$. Now (3.3) shows that we cannot have $\nu(\Omega') = 0$. □

Next we set out to prove that the product measure $\mu = \nu \times \lambda$ is ergodic for F. To do so we use the system Γ from Section 2.2. The proof relies on the statements on the dynamics in Theorem 3.1.

Theorem 3.2. *Consider $L_{p_0} < 0$. The measure μ is an ergodic invariant measure for F.*

Proof. Ergodicity of Lebesgue measure for Γ will be established by exploiting invertibility of Γ and using a Hopf argument as in [61, Section 4.2.6]. From Proposition 2.3 it then follows that μ is ergodic for F.

Let φ be a continuous function on $[0,1)^3$ and consider the time averages

$$\varphi^+(u) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\Gamma^i(u)),$$

$$\varphi^-(u) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(\Gamma^{-i}(u)).$$

As Lebesgue measure is invariant, we find that there is a set $V \subset [0,1)^3$ of full Lebesgue measure, so that for $u \in V$, the two limits exist and are equal (see [61, Section 3.2.3]):

$$\varphi^+(u) = \varphi^-(u), \quad \text{for Lebesgue almost all } u.$$

For $u \in [0,1)^3$, write

$$W^s(u) = \{v \in [0,1)^3\,;\, v = u + (0,x,y) \text{ for some } x, y\}$$

and

$$W^u(u) = \{v \in [0,1)^3\,;\, v = u + (w,0,0) \text{ for some } w\}.$$

Using Theorem 3.1 we get that for u in a set of full Lebesgue measure, if $v \in W^s(u)$ then $|\Gamma^n(u) - \Gamma^n(v)| \to 0$ as $n \to \infty$ and therefore $\varphi^+(u) = \varphi^+(v)$. Likewise for u in a set of full Lebesgue measure, if $v \in W^u(u)$ then $|\Gamma^{-n}(u) - \Gamma^{-n}(v)| \to 0$ as $n \to \infty$ and therefore $\varphi^-(u) = \varphi^-(v)$. We conclude that for u in a set $U \subset V$ of full Lebesgue measure, φ^+ is constant along $W^s(u)$ and φ^- is constant along $W^u(u)$.

As in [61, Lemma 4.2.17], using Fubini's theorem one sees that there is a set Y of full Lebesgue measure in $[0,1)^3$ so that for given $u, v \in Y$ there are $u', v' \in Y \cap U$ with $u' \in W^s(u)$, $v' \in W^s(v)$ and moreover $v' \in W^u(u')$. It follows that for such points $u, v \in Y \cap U$,

$$\varphi^-(u) = \varphi^+(u) = \varphi^+(u') = \varphi^-(u') = \varphi^-(v') = \varphi^+(v') = \varphi^+(v) = \varphi^-(v).$$

Hence, φ^+ and φ^- exist and are constant on a set of full Lebesgue measure. □

As a corollary, typical orbits of the iterated function system $\{f_i\,;\, 0 \le i \le M\}$ are uniformly distributed on $[0,1)$. This is made explicit in the following result. Let $A \subset [0,1)$ and write χ_A for its characteristic function:

$$\chi_A(x) = \begin{cases} 0, & x \notin A, \\ 1, & x \in A. \end{cases}$$

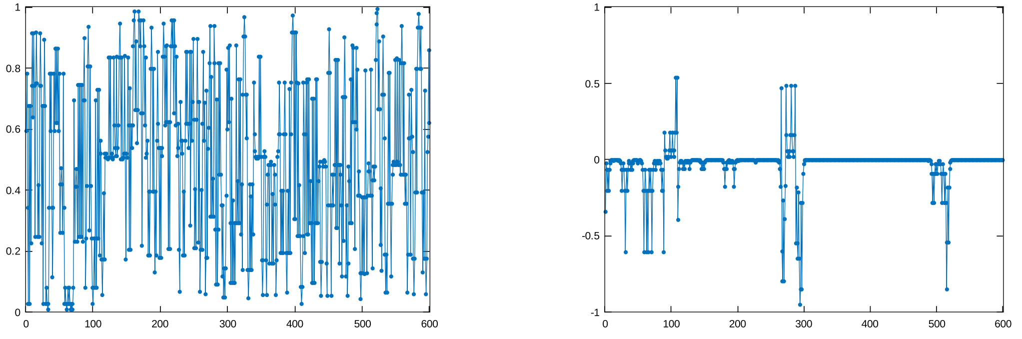


Fig. 7. A time series of $f_\omega^n(x)$ for $(M,N) = (3,3)$. The right panel shows the signed difference with another time series with the same ω.

Proposition 3.3. *Consider $L_{p_0} < 0$. For a Borel set A with $\lambda(A) > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(f_\omega^i(x)) = \lambda(A)$$

for μ-almost all $(\omega, x) \in \Sigma \times [0,1)$.

Proof. The measure μ is ergodic for F by Theorem 3.2. The proposition therefore follows from an application of Birkhoff's ergodic theorem to the integrable function $\chi_A \circ \pi_2$, where $\pi_2 : \Sigma \times [0,1) \to [0,1)$ is the canonical projection on the second coordinate of F. $\square$

3.2. $L_{p_0} = 0$ (intermittency)

In the case where expansion and contraction balance each other, a phenomenon reminiscent of intermittency arises. In Fig. 7 the panel on the right shows signed distances between two orbits with different starting points but identical ω. One sees that the orbits are mostly close together with occasional bursts where the orbits diverge. We make this statement quantitative and provide proofs below. The reader is invited to compare the results with [26] on iterated functions systems of interval diffeomorphisms. The novelty in the setting here is the use of an expanding map, which enables having Lebesgue measure as stationary measure and a uniform distribution of orbit points.

Theorem 3.4. *Consider $L_{p_0} = 0$. For every $\varepsilon > 0$, for all $x, y \in [0,1)$,*

$$\lim_{n\to\infty} \frac{1}{n} \left| \{0 \le i < n\, ;\, |f_\omega^i(x) - f_\omega^i(y)| < \varepsilon\} \right| = 1$$

for ν-almost all $\omega \in \Sigma$.

We describe the strategy in words before giving the formal proof. As in the proof of Theorem 3.1, we will rely on a Borel-Cantelli type argument. The proof also uses statements on random walks on the line that are developed in Appendix A.

Let $\varepsilon > 0$ be small. The main strategy is to subdivide time into periods where we are sure that the distances $|f^i_\omega(x) - f^i_\omega(y)|$ are small enough (smaller than ε) followed by periods where the distances might be too big. Then we proceed by showing that the latter periods take up only a negligible part of time. To do this we fix a word $\zeta_1 \cdots \zeta_D \in \{1, \dots, M\}^D$ with D large enough so that $1/M^D < \varepsilon$. Now start with the entire interval $[0, 1)$ and iterate under f^n_ω until an iterate $n_1 = n_1(\omega)$ with $n_1 > D$ and $\sigma^{n_1 - D+1}\omega \in [\zeta_1 \cdots \zeta_D]$. So the final D symbols $\omega_{n_1-D+1} \cdots \omega_{n_1}$ equal $\zeta_1 \cdots \zeta_D$. Then $f^{n_1+1}_\omega([0, 1))$ is contained in the interval

$$J = f^D_\zeta([0, 1)), \tag{3.4}$$

which has length smaller than ε. Hence, we arrived at a time n_1+1 for which $|f^{n_1+1}_\omega(x) - f^{n_1+1}_\omega(y)| < \varepsilon$. The expected stopping time, that is, the average value of $\omega \mapsto n_1(\omega)$, is finite.

Now we are interested in the number of iterates it takes until the image $f^n_\omega([0, 1))$, $n > n_1 + 1$, is no longer contained in an interval of size ε. As in the proof of Theorem 3.1 the discontinuities of f_0 pose a difficulty here. To control possible intersections of $f^n_\omega([0, 1)) \subset f^{n-(n_1+1)}_{\sigma^{n_1+1}\omega}(J)$ with the set $\mathcal{C}$ of critical points, we take a slightly different approach. Suppose p is such that $1/(p + 1)^2 < \varepsilon < 1/p^2$. We iterate instead until the image $f^{m_1+1}_\omega([0, 1))$, $m_1 > n_1$, is no longer contained in an interval of size $1/(p + m_1 - n_1)^2$. Note that this criterion depends on the number of iterates m_1. This defines an iterate $m_1 = m_1(\sigma^{n_1}\omega)$ that marks the end of a period of time where we are sure that the images $f^n_\omega([0, 1))$ are small enough. The expected stopping time, the average value of $m_1 - n_1$, will be shown to be infinite.

Finally, we continue the procedure and obtain a sequence $0 = m_0 < n_1 < m_1 < n_2 < m_2 < \cdots$ of stopping times. Here

(1) $f^{n_i+1}_\omega([0, 1))$ is contained in an interval of length ε. This is guaranteed by stopping after the appearance of a specific word $\omega_{n_i-D+1} \cdots \omega_{n_i} = \zeta_1 \cdots \zeta_D$, so that we find $f^{n_i+1}_\omega([0, 1)) \subset J$;

(2) $f^{m_i+1-n_i}_{\sigma^{n_i}\omega}(J)$ is not (more accurately, with the conditions we use it can no longer be guaranteed to be) contained in an interval of length $1/(p + m_i - n_i)^2$ (which is smaller than ε). The use of interval lengths that are decreasing in $m_i - n_i$, instead of working with a fixed interval length ε, is done to control and be able to avoid intersections with critical points.

The natural number m_i is the first integer beyond n_i for which this holds, and n_i is the first integer beyond $m_{i-1} + D$ (we let $m_0 = 0$) with $\sigma^{n_i-D+1}\omega \in [\zeta_1 \cdots \zeta_D]$. As $f^{j-n_i}_{\sigma^{n_i}\omega}(J)$ contains $f^j_\omega([0, 1))$, we have that during iterates $n_i + 1 \le j \le m_i$, $f^j_\omega([0, 1))$ is contained in an interval of length ε. Note that this does not mean that $f^j_\omega([0,1))$ is itself an interval. We find that the $n_i - m_{i-1}$ have finite expectation and the $m_i - n_i$ have infinite expectation. This is combined to prove the occurrence of intermittency.

Proof of Theorem 3.4. Fix an $\varepsilon > 0$ and a word $\zeta_1 \cdots \zeta_D \in \{1,\dots,M\}^D$ with D large enough so that $1/M^D < \varepsilon$. Let $J = f^D_\zeta([0,1))$, then $\lambda(J) < \varepsilon$. Let $p \in \mathbb{N}$ satisfy $1/(p+1)^2 < \varepsilon < 1/p^2$ and let x_J denote the midpoint of J. Recall the definition of r-neighborhoods $B(r)$ of the points of discontinuity $\mathcal{C}$ from (3.1). We seek estimates for the stopping time

$$W(\omega) = \min\left\{ n > 0\,;\ (f^n_\omega)' > \frac{1}{(p+n)^2\varepsilon} \quad \text{or} \quad f^n_\omega(x_J) \in B\Big(\frac{1}{(p+n)^2}\Big) \quad \text{with } \omega_n = 0 \right\}. \tag{3.5}$$

To understand the conditions, note that if $f^i_\omega(J) \cap \mathcal{C} = \emptyset$ for $0 \le i < n$ and $(f^n_\omega)' \le \frac{1}{(p+n)^2\varepsilon}$, then $f^n_\omega(J)$ is an interval with $\lambda(f^n_\omega(J)) < \frac{1}{(p+n)^2}$. Further, if for $0 \le i < n$ we have $f^i_\omega(x_J) \notin B\big(\frac{1}{(p+i)^2}\big)$ when $\omega_i = 0$, and $(f^i_\omega)' \le \frac{1}{(p+i)^2\varepsilon}$, then $f^i_\omega(J) \cap \mathcal{C} = \emptyset$ for $0 \le i < n$. Consequently, these conditions plus $(f^n_\omega)' \le \frac{1}{(p+n)^2\varepsilon}$ imply that $f^n_\omega(J)$ is an interval with $\lambda(f^n_\omega(J)) < \frac{1}{(p+n)^2}$.

Hence, $W(\omega)$ is such that for all for $n < W(\omega)$ the set $f^n_\omega(J)$ is an interval with $\lambda(f^n_\omega(J)) \le \frac{1}{(p+n)^2} < \varepsilon$.

In the following analysis we first look at the derivatives of compositions, so at the first condition in (3.5). For this, consider the process for $d_n = (f^n_\omega)'$, given by $d_0 = 1$ and

$$d_{n+1} = \begin{cases} N d_n, & \omega_{n+1} = 0, \\ d_n/M, & \omega_{n+1} \in \{1,\dots,M\}. \end{cases}$$

For each $n \ge 0$, let $z_n = -\ln(d_n)$. For z_n we obtain the random walk given by $z_0 = 0$ and

$$z_{n+1} = \begin{cases} z_n - \ln(N), & \omega_{n+1} = 0, \\ z_n + \ln(M), & \omega_{n+1} \in \{1,\dots,M\}, \end{cases}$$

for $n \ge 0$. Recall that $L_{p_0} = 0$ means $p_0 \ln(N) - (1-p_0)\ln(M) = 0$. The average step size for this random walk, equal to $-L_{p_0}$, is zero. The criterion $d_n > \frac{1}{(p+n)^2\varepsilon}$, which is the first condition appearing in the definition (3.5) of $W(\omega)$, is equivalent to $z_n < -\ln\big(\frac{1}{(p+n)^2}\big) + \ln(\varepsilon)$. Therefore, we are interested in the stopping time

$$\begin{aligned} W_1(\omega) &= \min\left\{ n > 0\,;\, z_n < -\ln\Big(\frac{1}{(p+n)^2}\Big) + \ln(\varepsilon) \right\} \\ &= \min\left\{ n > 0\,;\, (f^n_\omega)' > \frac{1}{(p+n)^2\varepsilon} \right\}, \end{aligned}$$

which satisfies $W_1(\omega) \geq W(\omega)$. By Lemma A.3 in the appendix, the average of the stopping time $W_1(\omega)$ is infinite:

$$\int_{\Sigma} W_1(\omega)\, d\nu(\omega) = \infty.$$

If for each $u > 0$ we set

$$C_u = \{\omega \in \Sigma \,;\, \omega_i \in \{1, \dots, M\} \text{ for } 0 \leq i < u\}, \tag{3.6}$$

then we also have $\int_{C_u} W_1(\omega)\, d\nu(\omega) = \infty$ for any $u > 0$. This holds since the first u iterates give contractions and the ω_i's are independent.

To study the second condition in the definition of $W(\omega)$, write $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ and let $\Pi : \Sigma \to \Sigma_2$ be as in Section 2.3, i.e., projecting all symbols $1, 2, \dots, M$ to 1. Consider iterates F_η^n with F_0, F_1 as defined in (2.5), (2.6). With $\eta = \Pi(\omega)$ we have $(F_\eta^n)' = (f_\omega^n)'$. The calculated stopping time $W_1(\omega)$ is thus identical for any symbol sequence in $\Pi^{-1}\{\eta\}$ and we may write $W_1(\eta)$.

Fix $\eta \in \Sigma_2$ and let $n = W_1(\eta)$. The multivalued map F_η^i is built from affine graphs with slope $(F_\eta^i)'$, stacked in an equidistant fashion (see for example Fig. 4). Let

$$\Xi_i = \left\{\omega \in \Pi^{-1}([\eta_0 \cdots \eta_{n-1}])\,;\, f_\omega^i(x_J) \notin B\Big(\frac{1}{(p+i)^2}\Big) \text{ whenever } \omega_i = 0\right\}$$

and set

$$\Xi = \cap_{i=0}^{n-1} \Xi_i.$$

Since $n = W_1(\eta)$, the set Ξ contains all sequences $\omega \in \Pi^{-1}([\eta_0 \cdots \eta_{n-1}])$ that satisfy both conditions from the definition of $W(\omega)$ in (3.5) up to the stopping time $W_1(\omega)$, so for which $W(\omega) = W_1(\omega)$. We next show that for certain η this collection is large enough to conclude that $\int_\Sigma W(\omega) d\nu(\omega) = \infty$.

From $(F_\eta^i)' = (f_\omega^i)' \leq \frac{1}{(p+i)^2\varepsilon}$ for $0 \leq i < n$, we get that there are at least $\lceil \varepsilon(p+i)^2 \rceil$ different graphs in F_η^i. Let ξ_i be the number of points in $F_\eta^i(\{x_J\})$ lying in $B(1/(p+i)^2)$ and write $\psi_i = \xi_i / \# F_\eta^i(\{x_J\})$ for the proportion of points from $F_\eta^i(\{x_J\})$ contained in $B(1/(p+i)^2)$. A closed interval of length b contains at most $\lceil b\,(\# F_\eta^i(\{x_J\}) + 1) \rceil$ points of $F_\eta^i(\{x_J\})$. It follows that there is a constant $K > 0$, independent of i and η, with

$$\psi_i \leq K/(p+i)^2.$$

We conclude that ψ_i is summable. So there exists a $u > 0$ with $\sum_{i=u}^{\infty} \psi_i < 1/2$. Here u can be taken uniformly in η, as the above estimates are uniform in η.

Now let $\eta \in [1^u]$ with $W_1(\eta) = n$. This implies that $W(\omega) > u$ for each ω from $\Pi^{-1}([\eta_0, \dots, \eta_{n-1}])$. From Lemma 2.4 it follows that

$$\psi_i = \frac{\nu(\Pi^{-1}([\eta_0 \cdots \eta_{n-1}]) \setminus \Xi_i)}{\nu(\Pi^{-1}([\eta_0 \cdots \eta_{n-1}]))},$$

assuming $\omega_i = 0$. Since $\eta_i = 1$ for all $0 \leq i < u$ and $\sum_{i=u}^{\infty} \psi_i < 1/2$ we then have

$$\frac{\nu(\Xi)}{\nu(\Pi^{-1}([\eta_0 \cdots \eta_{n-1}]))} \geq 1/2. \tag{3.7}$$

Hence

$$\begin{aligned}\int_{C_u} W(\omega)\, d\nu(\omega) &\geq \int_{\{\omega \in C_u\, ;\, W(\omega) = W_1(\omega)\}} W(\omega)\, d\nu(\omega) \\ &\geq \sum_{\eta \in [1^u], W_1(\eta) = n, n \geq u} n\, \nu(\Pi^{-1}([\eta_0 \cdots \eta_{n-1}]))/2 \\ &= \infty.\end{aligned}$$

In the second estimate we used (3.7) which says that for each $\eta \in [1^u]$ with $S(\eta) = n$, at least half of $\Pi^{-1}([\eta_0 \cdots \eta_{n-1}])$ counts in the integral. We conclude

$$\int_{\Sigma} W(\omega)\, d\nu(\omega) = \infty.$$

Define the stopping time

$$V(\omega) = \min\{n \geq D - 1\, ;\ \sigma^{n-D+1}\omega \in [\zeta_1 \cdots \zeta_D]\},$$

where $\zeta_1 \cdots \zeta_D$ is the word fixed at the beginning of the proof. For ν-almost every $\omega \in \Sigma$ the stopping time $V(\omega)$ is finite. So for ν-almost every $\omega \in \Sigma$ it takes a finite number of iterates k before $f_\omega^k([0,1)) \subseteq f_\zeta^D([0,1)) = J$ and thus $\lambda(f_\omega^k([0,1))) < \varepsilon$.

It is well known that the average of the stopping time $V(\omega)$ is bounded:

$$\int_{\Sigma} V(\omega)\, d\nu(\omega) < \infty.$$

Combining the knowledge on the stopping times $V(\omega)$ and $W(\omega)$, we get for ν-almost all $\omega \in \Sigma$ an infinite sequence of stopping times $0 < n_1 < m_1 < n_2 < m_2 < \cdots$ with

$$\begin{aligned} n_i &= V(\sigma^{m_{i-1}}\omega), \\ m_i &= W(\sigma^{n_i}\omega) \end{aligned}$$

(where we set $m_0 = 0$). By the strong law of large numbers, see [21, Theorems 2.4.1 and 2.4.5] (for finite and infinite expectations respectively) we have that for ν-almost all $\omega \in \Sigma$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(n_i - m_{i-1}) < \infty,$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(m_i - n_i) = \infty.$$

This implies the theorem (see the calculation in the proof of [5, Theorem 4]). □

Theorem 3.5. *Consider $L_{p_0} = 0$. Let β be a small positive number. Let $x, y \in [0, 1)$. Then for ν-almost all $\omega \in \Sigma$, either $|f^n_\omega(x) - f^n_\omega(y)| = 0$ for some n or $|f^n_\omega(x) - f^n_\omega(y)| > \beta$ for infinitely many values of n.*

Proof. The proof follows from the following observation. If $x < y$ are close, then as long as f^i_ω, $1 \le i < n$, is continuous on the interval $[x, y]$, we have

$$|f^n_\omega(x) - f^n_\omega(y)| = (f^n_\omega)'|x - y|.$$

The values $z_n = -\ln((f^n_\omega)')$ are given by a random walk $z_0 = 0$ and

$$z_{n+1} = \begin{cases} z_n - \ln(N), & \omega_{n+1} = 0, \\ z_n + \ln(M), & \omega_{n+1} \in \{1, \dots, M\}, \end{cases}$$

for $n \ge 0$. Because $p_0 \ln(N) - (1 - p_0)\ln(M) = 0$, this random walk is recurrent. So $(f^n_\omega)' = e^{-z_n}$ takes on arbitrarily large values. □

Modifying the proof of Theorem 3.2 allows to prove ergodicity of μ from Proposition 2.1 also in case $L_{p_0} = 0$.

Theorem 3.6. *Consider $L_{p_0} = 0$. The measure $\mu = \nu \times \lambda$ is an ergodic invariant measure for F.*

Proof. The proof follows that of Theorem 3.2, replacing the statement of Theorem 3.1 by the statement and arguments of Theorem 3.4. The proof of Theorem 3.4 calculates stopping times to get the statement on the dynamics of the x-coordinate. We must incorporate the y-coordinate. The following observations show how this works.

Write $\pi_w, \pi_x, \pi_y : [0, 1)^3 \to [0, 1)$ for the coordinate projection to the w-coordinate, x-coordinate, and y-coordinate, respectively. First consider a $\zeta \in \Sigma$ with $\zeta_i \in \{1, \dots, M\}$ for $0 \le i < D$ for some large D. Then $J = f^D_\zeta([0, 1))$ is a small interval, see (3.4). Recall the definition of the isomorphism h between the left shift σ and the expanding interval map L from (2.2). Note that $\pi_y\Gamma^D(h(\zeta), x, y)$ is independent of x and $H = \pi_y\Gamma^D(h(\zeta), x, [0, 1))$ is an interval of length $\lambda(H) = (1 - p_0)^D$. This is small for D large.

Next, whenever $u, v \in [0, 1)^3$ with $\pi_w u = \pi_w v$ and $\pi_x\Gamma(u), \pi_x\Gamma(v)$ are close to each other, then $\pi_y\Gamma$ contracts the distance between the points with a uniform contraction factor. So if $|\pi_x\Gamma^i(u) - \pi_x\Gamma^i(v)|$ stays small, then also $|\pi_y\Gamma^i(u) - \pi_y\Gamma^i(v)|$ stays small. □

As in Proposition 3.3 we conclude that typical orbits are uniformly distributed. Recall from (1.3) the definition of the two-point maps $f_i^{(2)} : [0,1)^2 \to [0,1)^2$ given by

$$f_i^{(2)}(x,y) = (f_i(x), f_i(y)),\ 0 \le i \le M.$$

In Corollary 2.2 we established that Lebesgue measure on the diagonal $\Delta = \{(x,x)\ ;\ x \in [0,1)\} \subset [0,1)^2$ is stationary for the iterated function system on $[0,1)^2$ generated by $f_i^{(2)}$ with probabilities p_i, $0 \le i \le M$. In the theorem below we write $\lambda|_A$ for two-dimensional Lebesgue measure restricted to A. The theorem gives, for multiplicatively dependent M, N, an explicit expression for an infinite stationary measure of full topological support, with a density that diverges along the diagonal Δ.

Theorem 3.7. *Consider $L_{p_0} = 0$ and (M,N) with N, M multiplicatively dependent: $N = \kappa^k$ and $M = \kappa^\ell$. Then the iterated function system generated by the two-point maps $f_i^{(2)}$, $0 \le i \le M$, admits a σ-finite infinite absolutely continuous stationary measure.*

Proof. We look for an invariant measure $m^{(2)}$ of the form $m^{(2)} = \sum_{h=0}^{\infty} m_h$ with

$$m_h = b_h \sum_{j=0}^{\kappa^h - 1} \kappa^h\ \lambda|_{[j/\kappa^h,(j+1)/\kappa^h)^2}\ , \tag{3.8}$$

where b_h can be read as the mass assigned to $\bigcup_{j=0}^{\kappa^h-1} \left[j/\kappa^h, (j+1)/\kappa^h\right)^2$, which is the union of the squares on the diagonal of size $\frac{1}{\kappa^{2h}}$ determined by κ-adic neighbors i/κ^{2h}, $(i+1)/\kappa^{2h}$. So $m_h([0,1)^2) = b_h$ and for the total measure we have

$$m^{(2)}([0,1)^2) = \sum_{h=1}^{\infty} m_h([0,1)^2) = \sum_{h=1}^{\infty} b_h.$$

Consider the push-forward map of measures given by

$$\mathcal{P}m = p_0 \left(f_0^{(2)}\right)_* m + \sum_{j=1}^{M} \frac{1-p_0}{M} \left(f_j^{(2)}\right)_* m. \tag{3.9}$$

Then $\mathcal{P}$ maps $\sum_{i=0}^{\infty} m_i$ to $\sum_{i=0}^{\infty} \hat{m}_i$, with

$$\begin{aligned}
\hat{m}_0 &= p_0(m_0 + \cdots + m_k),\\
\hat{m}_1 &= p_0 m_{k+1},\\
\vdots &= \ \vdots\\
\hat{m}_{\ell-1} &= p_0 m_{k+\ell-1},\\
\hat{m}_\ell &= (1-p_0)m_0 + p_0 m_{k+\ell},
\end{aligned}$$

$$\vdots = \vdots$$

$$\hat{m}_{\ell+j} = (1-p_0)m_j + p_0 m_{j+k+\ell},$$

$$\vdots = \vdots$$

A measure of the sought form $\sum_{h=0}^{\infty} m_h$ with m_h as in (3.8) is determined by the sequence of numbers $(b_i)_{i\in\mathbb{N}}$. Write

$$\mathcal{M}(\mathbb{N}) = \{(b_i)_{i\in\mathbb{N}} \ ; \ b_i \geq 0\},$$

which can be identified with the set of σ-finite measures on $\mathbb{N}$. The push-forward map $\mathcal{P}$ from (3.9) induces a map $\mathcal{Q} : \mathcal{M}(\mathbb{N}) \to \mathcal{M}(\mathbb{N})$. To make this explicit, suppose $\mathcal{P}$ maps $\sum_{i=0}^{\infty} m_i$ to $\sum_{i=0}^{\infty} \hat{m}_i$, and $(b_i)_{i\in\mathbb{N}}$ is given by (3.8) and likewise $(\hat{b}_i)_{i\in\mathbb{N}}$ corresponds to $\sum_{i=0}^{\infty} \hat{m}_i$. Denoting $\mathbf{b} = (b_i)_{i\in\mathbb{N}}$, $\hat{\mathbf{b}} = (\hat{b}_i)_{i\in\mathbb{N}}$, then

$$\mathcal{Q}(\mathbf{b}) = \hat{\mathbf{b}}.$$

Identifying the union of squares $\cup_{j=0}^{\kappa^h-1}[j/\kappa^h, (j+1)/\kappa^h)^2$ with the integer h, $\mathcal{Q}$ becomes the push-forward operator associated to the random walk on $\mathbb{N}$ given by

$$x_{n+1} = \begin{cases} \max\{0, x_n - \ell\}, & \omega_{n+1} = 0, \\ x_n + k, & \omega_{n+1} \in \{1, \dots, M\}. \end{cases} \tag{3.10}$$

As k and ℓ are relatively prime, by Bézout's identity there are integers α, β with $\alpha k + \beta \ell = 1$. Noting this, it follows from $L_{p_0} = 0$ that (3.10) is recurrent. It is in fact null-recurrent, and not positively recurrent, since $L_{p_0} = 0$ implies that the expected return time to a site is infinite (see [41, Section 2.3]). By [19] there is a unique infinite stationary measure for (3.10). This gives the fixed point $\mathcal{Q}(\mathbf{b}) = \mathbf{b}$ with the required property that $\sum_{h=0}^{\infty} b_h = \infty$. □

Stationary measures for the two-point maps may not be unique, as Example 3.8 shows.

Example 3.8. Take $M = N = 3$. Let Λ be the middle third Cantor set on $[0,1]$. Take the product set $\Upsilon_1 = \Lambda^2$ in $[0,1]^2$ and for $i \geq 1$ define recursively

$$\Upsilon_{i+1} = \overline{\bigcup_{j=1}^{3} f_j^{(2)} \left(\Upsilon_i \cap [0,1)^2\right)}$$

(these are closed sets; the definition involves taking closures as the maps f_i are defined on the left closed, right open interval $[0,1)$). One can characterize Υ_i as the set of points (x,y) that in a ternary representation admit an expansion

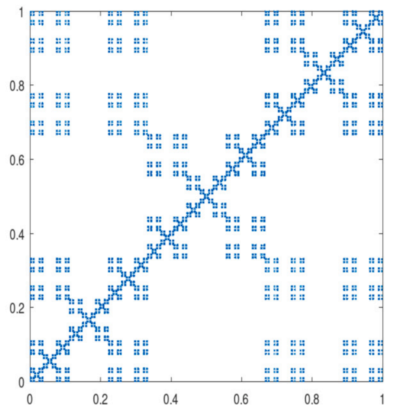


Fig. 8. Depicted is the invariant set $\cup_{i=1}^{\infty} \Upsilon_i$ for the two-point maps with $M = N = 3$. In the case $p_0 = 1/2$, so that $L_{p_0} = 0$, the two-point maps admit an infinite stationary measure supported on $\cup_{i=1}^{\infty} \Upsilon_i$.

$$(0.a_1 a_2 \dots a_{i-1} b_1 b_2 \dots, 0.a_1 a_2 \dots a_{i-1} c_1 c_2 \dots)$$

with $a_j \in \{0, 1, 2\}$, $1 \le j < i$, and $b_j, c_j \in \{0, 2\}$, $1 \le j$. The first i digits in the expansions for x and y are identical. The sets Υ_i accumulate onto the diagonal as $i \to \infty$.

Now Υ_1 is essentially invariant under $f_0^{(2)}$, or more precise,

$$\overline{f_0^{(2)}(\Upsilon_1 \cap [0,1)^2)} = \Upsilon_1.$$

Further,

$$\overline{f_0^{(2)}(\Upsilon_{i+1} \cap [0,1)^2)} = \Upsilon_i,$$
$$\overline{f_j^{(2)}(\Upsilon_i \cap [0,1)^2)} \subset \Upsilon_{i+1}, \qquad j = 1, 2, 3.$$

If $p_0 = 1/2$, so that $L_{p_0} = 0$, then the reasoning of Theorem 3.7 provides an infinite stationary measure $m^{(2)}$ on $[0,1)^2$ of the form

$$m^{(2)} = \sum_{h=0}^{\infty} m_h,$$

where m_h is supported on Υ_h and satisfies $m_h(\Upsilon_h) = 1$ (see Fig. 8). Details are left to the reader.

In the same setting, still with $M = N = 3$, the map f_0 has a periodic orbit of period 2; $x_2 = f_0(x_1)$, $x_1 = f_0(x_2)$. The set $\Upsilon_1 = (x_1, x_2) \cup (x_2, x_1)$ is invariant under $f_0^{(2)}$. Define recursively $\Upsilon_{i+1} = f_1^{(2)}(\Upsilon_i) \cup f_2^{(2)}(\Upsilon_i) \cup f_3^{(2)}(\Upsilon_i)$. The union $\cup_{i=1}^{\infty} \Upsilon_i$ consists of isolated points that accumulate onto the diagonal, and is invariant for the iterated function system of two-point maps. One can find a stationary measure that assigns positive measure to its points.

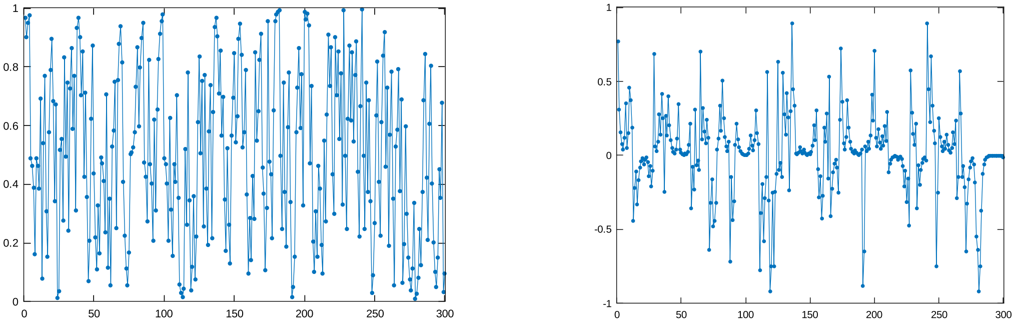


Fig. 9. A time series of $f^n_\omega(x)$ for $p_0 = 1/2$ and $(M, N) = (2, 3)$. The right panel shows a signed difference with another time series with the same ω.

3.3. $L_{p_0} > 0$ *(divergence)*

A goal of this section is to explain the outcome of numerical experiments such as depicted in Fig. 9. From a dynamics point of view this is done in Theorem 3.15 below.

First we establish the ergodicity of the product measure $\mu = \nu \times \lambda$, which proceeds by connecting to Theorem 3.2.

Theorem 3.9. *Consider* $L_{p_0} > 0$. *The measure* μ *is an ergodic invariant measure for* F.

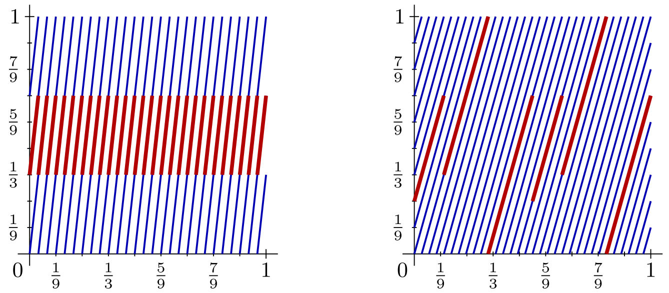
Proof. Recall the definition of the invertible map $\Gamma : [0, 1)^3 \to [0, 1)^3$ from (2.4) with inverse

$$\Gamma^{-1}(w, x, y) = \begin{cases} \left(p_0 w, \frac{x+j}{N}, \frac{Ny}{p_0} - j\right), & \frac{jp_0}{N} \le y < \frac{(j+1)p_0}{N},\ 0 \le j < N, \\ \left(\frac{(1-p_0)(w+i)}{M}, Mx - i, \frac{y-p_0}{1-p_0}\right), & p_0 \le y < 1, \\ & \frac{i}{M} \le x < \frac{i+1}{M},\ 0 \le i < M. \end{cases}$$

As in the proof of Theorem 3.2 one proves that three-dimensional Lebesgue measure is ergodic for Γ^{-1}, with the difference that now the map is expanding in the direction of y and contracting in the direction of w. Lebesgue measure is therefore also ergodic for Γ. Reasoning as for Theorem 3.2, it follows that μ is ergodic for F. $\square$

As a corollary we have that typical orbits are uniformly distributed, see Proposition 3.3.

Before we formulate and prove the result that provides the last part of Theorem 1.1, we first focus on stationary measures for the iterated function system generated by the two-point maps. We will give two results. Firstly, for multiplicatively dependent M, N we provide an explicit expression for a stationary measure $m^{(2)}$ that is absolutely continuous with respect to Lebesgue and has full topological support and that, contrary to the case $L_{p_0} = 0$ (see Theorem 3.7), is a finite measure. Fig. 11 shows a numerical approximation of the density function of this stationary measure for $p_0 = 1/2$ and



(a) $(3,3), \eta = (10010010)$ (b) $(3,2), \eta = (0010010)$

Fig. 10. Left picture: a plot of the graphs of F^8_η for $(M,N) = (3,3)$ and $\eta = (10010010)$. The red graph is the graph of f^8_ω for $\omega = (20020020)$. Right picture: a plot of the graphs of F^7_η for $(M,N) = (3,2)$ and $\eta = (0010010)$. The red graph is the graph of f^7_ω for $\omega = (0020030)$.

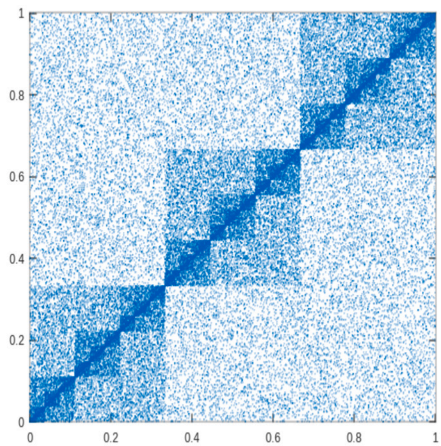


Fig. 11. A plot of the stationary distribution for the two-point motion by a numerically computed histogram of an orbit, for $p_0 = 1/2$ and $(M,N) = (3,9)$.

$(M,N) = (3,9)$. Secondly, we prove the existence of such a measure for all pairs (M,N) without identifying an explicit expression. The proof of the second result will again run into the difficulties caused by the discontinuities of f_0, see Fig. 10 that includes a plot of the graphs of F^7_η with F_0, F_1 introduced in (2.5), (2.6).

If $N = \kappa^k$ and $M = \kappa^\ell$, then $L_{p_0} > 0$ reads $p_0 k \ln(\kappa) - (1-p_0)\ell \ln(\kappa) > 0$. We will use that this implies

$$\frac{\ell}{k+\ell} < p_0 < 1. \tag{3.11}$$

Theorem 3.10. *Consider $L_{p_0} > 0$ and (M,N) with N, M multiplicatively dependent: $N = \kappa^k$ and $M = \kappa^\ell$. Then the iterated function system generated by the two-point maps $f^{(2)}_i$, $0 \le i \le M$, admits an absolutely continuous stationary measure $m^{(2)}$ of the form*

$$m^{(2)} = \sum_{h=0}^{\infty} b_h \sum_{j=0}^{\kappa^h - 1} \kappa^h \ \lambda|_{[j/\kappa^h,(j+1)/\kappa^h)^2} \,,$$

with b_h satisfying the recurrence equation

$$b_{j+k+\ell} = \frac{1}{p_0} b_{j+\ell} - \frac{1-p_0}{p_0} b_j, \quad j \geq 0,$$

and a suitable initial condition on $b_0, \dots, b_{k+\ell-1}$.

Moreover, with ν_1 being the unique real solution in $(0,1)$ to $p_0 \zeta^{k+\ell} - \zeta + 1 - p_0 = 0$,

$$\lim_{h \to \infty} b_h / \nu_1^h$$

exists and is a positive number.

Proof. We take the setup of the proof of Theorem 3.7, which we briefly repeat. We look for an invariant measure $m^{(2)}$ of the form $m^{(2)} = \sum_{h=0}^{\infty} m_h$ with

$$m_h = b_h \sum_{j=0}^{\kappa^h - 1} \kappa^h \ \lambda|_{[j/\kappa^h,(j+1)/\kappa^h)^2} \,. \tag{3.12}$$

Consider the push-forward map $\mathcal{P}$ of measures from (3.9). A measure of the sought for form $\sum_{h=0}^{\infty} m_h$ with m_h as in (3.12) is determined by the sequence of numbers $(b_i)_{i\in\mathbb{N}}$. The push-forward map $\mathcal{P}$ from (3.9) induces a map $\mathcal{Q} : \mathcal{M}_1(\mathbb{N}) \to \mathcal{M}_1(\mathbb{N})$, where

$$\mathcal{M}_1(\mathbb{N}) = \left\{ (b_i)_{i\in\mathbb{N}} \ ; \ b_i \geq 0, \sum_{i=0}^{\infty} b_i = 1 \right\}.$$

As noted in the proof of Theorem 3.7, $\mathcal{Q}$ is the push-forward operator associated to the random walk on $\mathbb{N}$ given by

$$x_{n+1} = \begin{cases} \max\{0, x_n - k\}, & \omega_{n+1} = 0, \\ x_n + \ell, & \omega_{n+1} \in \{1, \dots, M\}. \end{cases} \tag{3.13}$$

As $L_{p_0} > 0$, this is a positive recurrent random walk. Hence $\mathcal{Q}$ admits a fixed point in $\mathcal{M}_1(\mathbb{N})$.

Having established the existence of a fixed point $\mathcal{Q}(\mathbf{b}) = \mathbf{b}$, we continue with calculations that will result in expressions for b_h. The stationary measure $m^{(2)}$ satisfies $\mathcal{P}m^{(2)} = m^{(2)}$. For the coefficients b_h, $h \geq 0$, this gives equations

$$\begin{aligned} b_k &= \frac{1-p_0}{p_0} b_0 - b_1 - \cdots - b_{k-1}, \\ b_{k+1} &= \frac{1}{p_0} b_1, \end{aligned}$$

$$
\begin{aligned}
\vdots &= \vdots \\
b_{k+\ell-1} &= \frac{1}{p_0} b_{\ell-1}, \\
b_{k+\ell} &= \frac{1}{p_0} b_\ell - \frac{1-p_0}{p_0} b_0, \\
\vdots &= \vdots \\
b_{j+k+\ell} &= \frac{1}{p_0} b_{j+\ell} - \frac{1-p_0}{p_0} b_j, \\
\vdots &= \vdots
\end{aligned}
$$

The recurrence equation $b_{j+k+\ell} = \frac{1}{p_0} b_{j+\ell} - \frac{1-p_0}{p_0} b_j$ that appears here, is equivalent to the linear system

$$
\begin{pmatrix} b_{h+1} \\ b_{h+2} \\ \vdots \\ b_{h+\ell+1} \\ \vdots \\ b_{h+k+\ell-1} \\ b_{h+k+\ell} \end{pmatrix}
=
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\
-\frac{1-p_0}{p_0} & 0 & 0 & \cdots & \frac{1}{p_0} & \cdots & 0
\end{pmatrix}
\begin{pmatrix} b_h \\ b_{h+1} \\ \vdots \\ b_{h+\ell} \\ \vdots \\ b_{h+k+\ell-2} \\ b_{h+k+\ell-1} \end{pmatrix},
$$

$h \geq 0$. Denote the above matrix by A. Its characteristic equation is $p_0 \zeta^{\ell+k} - \zeta^\ell + 1 - p_0 = 0$, so that

$$
p_0(\zeta^{k+\ell} - 1) = \zeta^\ell - 1. \tag{3.14}
$$

We claim that for any $p_0 < 1$ the zeros of the characteristic equation, thus the eigenvalues of A, are as follows.

(1) A has a simple eigenvalue at 1, which is the only eigenvalue on the unit circle. There are $k-1$ eigenvalues outside the unit circle and there are ℓ eigenvalues inside the unit circle. Write them as

$$
\eta_0 = 1, \qquad \eta_1, \ldots, \eta_{k-1} \in \{z \in \mathbb{C} \; ; \; |z| > 1\}, \qquad \nu_1, \ldots, \nu_\ell \in \{z \in \mathbb{C} \; ; \; |z| < 1\};
$$

(2) The eigenvalue with largest modulus among $\{\nu_1, \ldots, \nu_\ell\}$ is single, real and positive. Let $\nu_1 \in (0,1)$ be this eigenvalue.

To prove the statements on the eigenvalues, write $S_r(a) = \{z \in \mathbb{C} \; ; \; |z-a| = r\}$ for the circle in the complex plane of radius r and center a. Consider $\zeta \in \mathbb{C}$ with $|\zeta| = r$. Then

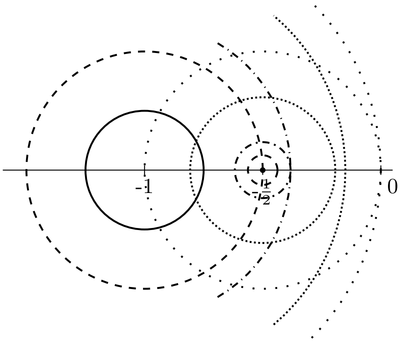


Fig. 12. The circles $S_{p_0 r^{k+l}}(-p_0)$ and $S_{r^l}(-1)$ for the values $p_0 = -\frac{1}{2}$, $N = 4$, $M = 2$ and $r = \frac{1}{4}$ (solid), $r = \frac{1}{2}$ (dashed), $r = \hat{r} = \frac{\sqrt{5}-1}{2}$ (dash dotted), $r = \frac{85}{100}$ (densely dotted) and $r = 1$ (loosely dotted).

$$p_0(\zeta^{k+\ell} - 1) \in S_{p_0 r^{k+\ell}}(-p_0), \qquad \zeta^\ell - 1 \in S_{r^\ell}(-1).$$

Solutions to (3.14) with $|\zeta| = r$ can occur only if these two circles $S_{p_0 r^{k+\ell}}(-p_0)$ and $S_{r^\ell}(-1)$ intersect, so we consider their mutual position.

First consider the situation that $p_0 = 1$. For $r = 1$ the resulting circles $S_{p_0}(-p_0)$ and $S_1(-1)$ are identical and for any other value of r the circles do not intersect. The solutions to (3.14) are therefore given by $\{e^{2\pi(j/k)i}, 0 \leq j < k\}$. Together with the solution at 0, with multiplicity ℓ, these are the solutions to (3.14) with $p_0 = 1$.

Now assume $\frac{\ell}{k+\ell} < p_0 < 1$, see (3.11). For $r = 1$, $S_{p_0}(-p_0)$ and $S_1(-1)$ intersect only at the origin. As k and ℓ are relatively prime, and therefore also $k+\ell$ and ℓ are relatively prime, $\zeta = 1$ is the unique solution with $|\zeta| = 1$ to (3.14).

Now consider $\frac{\ell}{k+\ell} < p_0 < 1$ with $r < 1$. Write $h_0(r) = p_0(r^{k+\ell} - 1)$ and $h_1(r) = r^\ell - 1$. Then $-1 = h_1(0) < h_0(0) = -p_0$, while $0 = h_1(1) = h_0(1)$. The function $h_0'(r)/h_1'(r) = p_0 \frac{k+\ell}{\ell} r^k$ is monotone increasing on $[0, 1]$, from 0 at $r = 0$ to $p_0 \frac{k+\ell}{\ell} > 1$ at $r = 1$. There is therefore a unique solution $\hat{r} = \hat{r}(p_0)$ in $(0, 1)$ to $h_0(r) = h_1(r)$, see Fig. 12 for an illustration. Moreover, $\hat{r} \to 0$ as $p_0 \to 1$ and $\hat{r} \to 1$ as $p_0 \to \frac{\ell}{k+\ell}$.

This means that $S_{p_0 \hat{r}^{k+\ell}}(-p_0)$ is tangent to $S_{\hat{r}^\ell}(-1)$ (at $\hat{r}^\ell - 1$). For $\hat{r} < r < 1$, $S_{p_0 r^{k+\ell}}(-p_0) \cap S_{r^\ell}(-1) = \emptyset$. We conclude that the real solution $\nu_1 = \hat{r}$ is the solution to (3.14) of largest modulus of solutions inside the unit disc. Again as k and ℓ are relatively prime, it is an isolated solution, and other solutions inside the unit disc have smaller modulus.

Since eigenvalues depend continuously on p_0, and 1 is an isolated eigenvalue and the only eigenvalue on the unit circle for $\frac{\ell}{k+\ell} < p_0 < 1$, eigenvalues can not cross the unit circle when varying p_0. So for any $\frac{\ell}{k+\ell} < p_0 < 1$, there are ℓ eigenvalues inside the circle of radius $\hat{r}(p_0)$, there is an isolated eigenvalue at 1, and the remaining $k-1$ eigenvalues lie outside the unit circle. This concludes the proof of the statements on the solutions to (3.14).

A solution to the equations for b_h is determined by an initial vector $b_0, \ldots, b_{k+\ell-1}$. For higher indices h, b_h is given by the recurrence equation. For a solution with b_h

converging to 0 as $h \to \infty$, we need the initial vector to be contained in the sum of the (generalized) eigenspaces corresponding to the contracting eigenvalues $\nu_1, \dots, \nu_\ell$. Recall that we already know there is such a solution to the equations.

The initial condition can not be contained in the range of $(A - \nu_1)$ (thus must have a component in the direction of the eigenvector corresponding to the unique real positive eigenvalue ν_1, when decomposing in a basis of generalized eigenvectors), since otherwise b_h can not be positive for all h. As ν_1 is the eigenvalue of largest modulus of all eigenvalues inside the unit circle, this implies b_h/ν_1^h converges to a positive value as $h \to \infty$. □

Example 3.8 is relevant to the context of Theorem 3.10: there may be various stationary measures that are not absolutely continuous.

Remark 3.11. In the above proof we concluded the existence of a fixed point of $\mathcal{Q}$ in $\mathcal{M}_1(\mathbb{N})$ from positive recurrence of (3.13). Here we connect to a different approach. Consider the diffeomorphism $h : \mathbb{R} \to (0, 1)$ given by $h(x) = \frac{e^x}{1+e^x}$. The random walk (3.13) considered on $\mathbb{R}$ is topologically conjugated through h with the iterated function system

$$y_{n+1} = \begin{cases} \max\left\{0, \frac{e^{-\ell} y_n}{1+(e^{-\ell}-1)y_n}\right\}, & \omega_{n+1} = 0, \\ \frac{e^k y_n}{1+(e^k-1)y_n}, & \omega_{n+1} = 1 \end{cases} \tag{3.15}$$

on $(0, 1)$. By continuous extension we have 1 as a common fixed point for the two maps generating (3.15). Moreover, the iterated function systems have a positive Lyapunov exponent L_{p_0} at 1. Following the reasoning of [26, Lemma 3.2] (see also [25, Proposition 4.1]; it amounts to following a Krylov-Bogolyubov procedure on a suitable closed class of measures), the iterated function system (3.15) admits a stationary measure supported on $h(\mathbb{N})$. Hence $\mathcal{Q}$ admits a fixed point in $\mathcal{M}_1(\mathbb{N})$.

Example 3.12. We work out the general result of Theorem 3.10 in two special cases.

(1) Pairs (M, N) with $N = M^k$ correspond to $\kappa = M$ and $\ell = 1$. By Theorem 3.10, the iterated function system generated by the two-point maps $f_i^{(2)}$, $0 \le i \le M$, admits an absolutely continuous stationary measure $m^{(2)}$ of the form

$$m^{(2)} = \sum_{h=0}^{\infty} b_h \sum_{j=0}^{M^h-1} M^h \; \lambda|_{[j/M^h,(j+1)/M^h)^2},$$

with $b_h = \nu_1^h$ for ν_1 the unique real solution in $(0, 1)$ to

$$p_0 \zeta^{k+1} - \zeta + 1 - p_0 = 0.$$

For $k = 1$ the solution ν_1 is given by $\nu_1 = \frac{1-p_0}{p_0}$. For $k = 2$ it is given by $\nu_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\frac{1-p_0}{p_0}}$.

(2) The second special case we consider is of pairs (M, N) with $M = N^\ell$. This corresponds to $\kappa = N$ and $k = 1$. Then the iterated function system generated by the two-point maps $f_i^{(2)}$, $0 \le i \le M$, admits an absolutely continuous stationary measure $m^{(2)}$ of the form

$$m^{(2)} = \sum_{h=0}^{\infty} b_h \sum_{j=0}^{N^h-1} N^h \, \lambda|_{[j/N^h,(j+1)/N^h)^2}$$

with

$$\begin{aligned} b_0 &= 1, \\ b_j &= (1-p_0)/p_0^j, && 1 \le j \le \ell, \\ b_{j+1} &= \frac{1}{p_0} b_j - \frac{1-p_0}{p_0} b_{j-\ell}, && j \ge \ell. \end{aligned}$$

We have $b_h \sim \nu_1^h$, where ν_1 is the unique solution in $(0, 1)$ to $p_0\zeta^{\ell+1} - \zeta^\ell + 1 - p_0 = 0$. For $\ell = 1$, this gives $\nu_1 = \frac{1-p_0}{p_0}$. For $\ell = 2$, $\nu_1 = \frac{1-p_0+\sqrt{(1-p_0)(1+3p_0)}}{2p_0}$.

Remark 3.13. Writing $m^{(2)}$ obtained in Theorem 3.10 as

$$m^{(2)} = \sum_{h=0}^{\infty} b_h \kappa^h \sum_{j=0}^{\kappa^h-1} \lambda|_{[j/\kappa^h,(j+1)/\kappa^h)^2} \, ,$$

and noting $b_h \sim \nu_1^h$, it is clear that its density is bounded if $\nu_1 \kappa < 1$.

The two-point maps $f_i^{(2)}$ are examples of Jablonski maps [34]. In the literature, see [9,15,33], it is proved that random Jablonski maps admit an absolutely continuous stationary measure under an expansion on average condition. In our setting this gives that the iterated function system generated by $\{f_i^{(2)}\}$, $0 \le i \le M$, admits an absolutely continuous stationary measure if $\frac{p_0}{N} + (1 - p_0)M < 1$. Under this condition the stationary measure has bounded Tonelli variation. We apply [11] to get an absolutely continuous stationary measure under the condition $L_{p_0} > 0$. This may not have bounded Tonelli variation, compare also Remark 3.13 and [47, Section 4]. In contrast to Theorem 3.10, here we do not have an explicit expression for the density function.

Theorem 3.14. *Consider $L_{p_0} > 0$. The iterated function system on $[0, 1)^2$ generated by $f_i^{(2)}$, $0 \le i \le M$, admits an absolutely continuous stationary probability measure $m^{(2)}$. Furthermore, $m^{(2)}$ has full topological support.*

Proof. We will apply [11] that considers skew product maps with an invertible base map. For this reason we take the map $\hat{F}^{(2)} : \{0, \ldots, M\}^{\mathbb{Z}} \times [0,1)^2 \to \{0, \ldots, M\}^{\mathbb{Z}} \times [0,1)^2$ given by

$$\hat{F}^{(2)}(\omega, x, y) = (\sigma\omega, f^{(2)}_{\omega_0}(x, y)).$$

The base map of the skew product system, that is, σ acting on $\{0, \ldots, M\}^{\mathbb{Z}}$, is invertible. The $\mathbf{p}$-Bernoulli measure on $\{0, \ldots, M\}^{\mathbb{Z}}$ with $\mathbf{p}$ as in (1.2), which we also denote by ν as in the one-sided case, is an ergodic invariant probability measure, so we fit the setting considered in [11]. Under the condition $L_{p_0} > 0$, [11, Theorem 4.2 and Remark 5] provides a family $\mu^{(2)}_\omega$ of random absolutely continuous invariant measures on $[0,1)^2$. Invariant here means

$$\left(f^{(2)}_{\omega_0}\right)_* \mu^{(2)}_\omega = \mu^{(2)}_{\sigma\omega}, \qquad \text{for } \nu\text{-almost all } \omega.$$

Furthermore, the measure $\hat{\mu}^{(2)}$ with marginal ν on $\{0, \ldots, M\}^{\mathbb{Z}}$ and fiber measures $\mu^{(2)}_\omega$ on $\{\omega\} \times [0,1)^2$, is invariant under $\hat{F}^{(2)}$.

Let $\hat{\Pi} : \{0, \ldots, M\}^{\mathbb{Z}} \times [0,1)^2 \to \Sigma \times [0,1)^2$ be the natural coordinate projection

$$\hat{\Pi}((\omega_i)_{i\in\mathbb{Z}}, x, y) = ((\omega_i)_{i\in\mathbb{N}}, x, y).$$

Then $\mu^{(2)} = \hat{\Pi}_* \left(\hat{\mu}^{(2)}\right)$ is an invariant measure for $F^{(2)}$. To show that $\mu^{(2)}$ is an absolutely continuous measure, take a set $A \subset \Sigma \times [0,1)^2$ of zero measure for $\nu \times \lambda$. We wish to show that $\mu^{(2)}(A) = 0$. Now $\hat{\Pi}^{-1}(A)$ has zero measure for $\nu \times \lambda$ on $\{0, \ldots, M\}^{\mathbb{Z}} \times [0,1)^2$. So $\hat{\Pi}^{-1}(A) \cap \left(\{\omega\} \times [0,1)^2\right)$ has zero Lebesgue measure for almost all $\omega \in \{0, \ldots, M\}^{\mathbb{Z}}$. It thus has zero measure for $\mu^{(2)}_\omega$, for almost all $\omega \in \{0, \ldots, M\}^{\mathbb{Z}}$, by absolute continuity of $\mu^{(2)}_\omega$. Hence A has zero measure for $\mu^{(2)}$.

By [45, Theorem 3.1 and Corollary 3.1], $\mu^{(2)}$ is an invariant product measure, so of the form $\mu^{(2)} = \nu \times m^{(2)}$.

By iterating under the expanding map $f^{(2)}_0$ we recognize that $m^{(2)}$ has full topological support. Namely, take any open set $O \subset [0,1)^2$. Now $\left(f^{(2)}_0\right)^n$ maps rectangles

$$R^n_{ij} = [i/N^n, (i+1)/N^n) \times [j/N^n, (j+1)/N^n)$$

onto $[0,1)^2$. Take a set of positive $m^{(2)}$ measure. As $m^{(2)}$ is absolutely continuous, we can take a Lebesgue density point of this set. For n large and the rectangle R^n_{ij} containing this Lebesgue density point, $\left(f^{(2)}_0\right)^{-n}(O) \cap R^n_{ij}$ has positive $m^{(2)}$ measure. The topological support of $m^{(2)}$ therefore intersects O. □

We also have the following related dynamical statement, showing that orbits may stick close together for some iterates, but then diverge again. Recall that Δ_ε denotes the ε-neighborhood $\{(x,y) \in [0,1)^2 \ ; \ |x - y| < \varepsilon\}$ of the diagonal Δ in $[0,1)^2$.

Theorem 3.15. *Consider $L_{p_0} > 0$. Let $0 < t < 1$. There is a set of points (ω, x, y) in $\Sigma \times [0,1)^2$ of full $\nu \times \lambda$-measure, for which*

$$P(\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \left|\{0 \le i < n\,;\, \left|f^i_\omega(x) - f^i_\omega(y)\right| < \varepsilon\}\right|$$

exists. Moreover,

$$\lim_{\varepsilon\to 0} P(\varepsilon) = 0.$$

If M and N are multiplicatively dependent with $N = \kappa^k$, $M = \kappa^\ell$, then

$$P(\varepsilon) \sim \varepsilon^{-\ln(\nu_1)/\ln(\kappa)},$$

where ν_1 is the unique solution in $(0,1)$ to $p_0\zeta^{k+\ell} - \zeta + 1 - p_0 = 0$.

Proof. The statement for general pairs (M, N) follows from [16, Proposition 4.1], see also [11, Theorem 4.5], combined with Theorem 3.14.

Extra reasoning is needed to prove the statement for multiplicatively dependent M and N. Write

$$S_h = \sum_{j=0}^{\kappa^h - 1} [j/\kappa^h, (j+1)/\kappa^h)^2.$$

Take a point $(x_0, y_0) \in [0,1)^2$, which we consider to lie in $S_0 = [0,1)^2$. Iterate $(x_n, y_n) = \left(f^{(2)}_\omega\right)^n (x_0, y_0)$. If we let h_n with $h_0 = 0$ follow the random walk (3.13), so

$$h_{n+1} = \begin{cases} \max\{0, h_n - k\}, & \omega_{n+1} = 0, \\ h_n + \ell, & \omega_{n+1} \in \{1, \dots, M\}, \end{cases}$$

then we find $(x_n, y_n) \in S_{h_n}$. For the distance of (x_n, y_n) to the diagonal Δ it is irrelevant in which rectangle $[j/\kappa^{h_n}, (j+1)/\kappa^{h_n})^2)$ the point (x_n, y_n) lies, but the position inside the rectangle is. If we rescale all rectangles to $[0,1)^2$, we find a sequence of points $(\tilde{x}_n, \tilde{y}_n) \in [0,1)^2$. The point $(\tilde{x}_{n+1}, \tilde{y}_{n+1})$ can only differ from $(\tilde{x}_n, \tilde{y}_n)$ if $(\tilde{x}_n, \tilde{y}_n)$ and $(\tilde{x}_{n+1}, \tilde{y}_{n+1})$ both lie in $S_{h_n} = S_{h_{n+1}} = S_0$; in this case $(\tilde{x}_{n+1}, \tilde{y}_{n+1}) = N(\tilde{x}_n, \tilde{y}_n) \pmod 1$. Summarizing,

$$(\tilde{x}_{n+1}, \tilde{y}_{n+1}) = \begin{cases} N(\tilde{x}_n, \tilde{y}_n) \pmod 1, & S_{h_n} = S_{h_{n+1}} = S_0, \\ (\tilde{x}_n, \tilde{y}_n), & S_{h_n} \ne S_{h_{n+1}}. \end{cases}$$

As $h_n = h_{n+1} = 0$ occurs for a positive proportion of iterates, for almost all ω, and $(x,y) \mapsto N(x,y) \pmod 1$ is ergodic with respect to Lebesgue measure, for typical initial points (x_0, y_0) and almost all ω, $(\tilde{x}_n, \tilde{y}_n)$ is uniformly distributed. This implies $P(\varepsilon) = m^{(2)}(\Delta_\varepsilon)$ and the estimate $m^{(2)}(\Delta_\varepsilon) \sim \varepsilon^{-\ln(\nu_1)/\ln(\kappa)}$. □

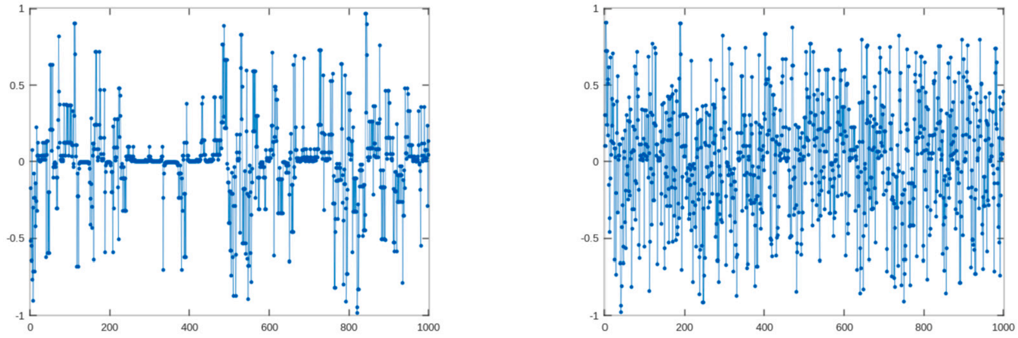


Fig. 13. Shown are time series of signed differences of two orbits, for $(M, N) = (3, 3)$. The left panel is for $p_0 = 0.6$, the right panel is for $p_0 = 0.9$. In both figures, $L_{p_0} > 0$. The stationary measure $m^{(2)}$ of the two-point maps has bounded density function for $p_0 > 3/4$: the density of $m^{(2)}$ is unbounded for the left panel and bounded for the right panel.

Fig. 13 shows time series of signed differences of two orbits, for $(M, N) = (3, 3)$ and two different values of p_0. The stationary measure has unbounded density for the left panel and bounded density for the right panel. Theorem 3.15 explains and quantifies the relative stickiness of orbits visible in the left panel. Where we associate the dynamics for zero Lyapunov exponent to intermittency, one may argue that also the occurrence of a stationary density for the two-point iterated function system that blows up at the diagonal relates to intermittency.

In addition to the above result, we have for $\nu \times \lambda$-almost all (ω, x, y),

$$\liminf_{n\to\infty} |f^n_\omega(x) - f^n_\omega(y)| = 0,$$
$$\limsup_{n\to\infty} |f^n_\omega(x) - f^n_\omega(y)| = 1,$$

for multiplicatively dependent M, N.

4. Final remarks

We conclude with brief remarks on left out topics and on generalizations that would extend the core findings of this paper to a broader framework.

4.1. Multiplicatively dependent pairs (M, N)

We note that some of our results in Section 3 admit simpler proofs in the specific and restricted case of multiplicatively dependent pairs (M, N). Especially the proofs of Theorem 3.1 and Theorem 3.4 can be simplified by applying the following Lemma 4.1, which stresses properties of graphs of f^n_ω in this case. We call intervals of the form $[i/\kappa^j, (i + 1)/\kappa^j)$, for some integers $\kappa > 1$, $j \geq 0$ and $0 \leq i < \kappa^j$, κ-adic intervals.

Lemma 4.1. *Consider (M, N) with M and N multiplicatively dependent: $N = \kappa^k$ and $M = \kappa^\ell$. Then for each $n \geq 1$ and each $\omega \in \Sigma$ the graph of f^n_ω is contained in a strip of the form $[i/\kappa^j, (i+1)/\kappa^j)$, for some integers $j \geq 0$ and $0 \leq i < \kappa^j$. Moreover, f^n_ω is a piecewise linear map with constant slope of which all branches have the interval $[i/\kappa^j, (i+1)/\kappa^j)$ as their image.*

Proof. A direct computation shows that f_i, for any $0 \leq i \leq M$, maps any κ-adic interval to a κ-adic interval. Note that, if the interval contains a discontinuity point of f_0 in its interior, then f_0 maps this interval to $[0, 1)$. □

The pictures in Figs. 4 and 10 illustrate the differences between multiplicatively dependent and multiplicatively independent pairs (M, N).

4.2. Phase transitions for general random systems

The toy model studied here shows a phase transition for the two-point motion occurring when the Lyapunov exponent crosses zero. In Theorem 1.2 we have seen that this involves an explosion of the support of the stationary measure and an infinite stationary measure at the transition point. A central question is whether this is a typical scenario for more general classes of systems.

In this article we looked at iterated function systems generated by expanding and contracting affine maps on the unit interval, determined by a pair of integers (M, N), and a probability vector $\mathbf{p}$ determined by a parameter p_0. For fixed M and N, a parameterized family of systems arises by varying the probability vector $\mathbf{p}$ more generally. It would be interesting to investigate how the dynamical properties of this family depend on $\mathbf{p}$. Where our setting has Lebesgue measure as stationary measure, a first question would be to determine a stationary measure for the iterated function system. For probability vectors other than $\mathbf{p}$ as in (1.2) one can start with constructing a stationary measure as in [16,36,44,47] and continue from there.

There are various other generalizations thinkable. One may consider different classes of affine maps. One can generalize to nonlinear maps. In [27] a somewhat similar setup is discussed for circle maps, showing that the maps need not be uniformly expanding and contracting. One can also consider higher dimensional analogs.

Finally, there is no need to restrict to iterated function systems, and one can study the same type of questions for skew product systems driven by more general noise.

4.3. d-point motions

Where we focused on 2-point motions, one may iterate more than two points by considering d-point maps $f^{(d)}_i : [0, 1)^d \to [0, 1)^d$ given by

$$f^{(d)}_i(x_1, \dots, x_d) = (f_i(x_1), \dots, f_i(x_d)).$$

This makes little difference in the cases of nonpositive Lyapunov exponent L_{p_0}, as we already analyzed the fate of the entire interval $[0,1)$ under iterations. But take for instance the setting of Theorem 3.10 with $L_{p_0} > 0$ and multiplicatively dependent (M,N). The reasoning to prove Theorem 3.10 provides, under the given assumptions, an absolutely continuous stationary measure $m^{(d)}$ for the iterated function system generated by the d-point maps $f_i^{(d)}$, $0 \le i \le M$, of the form

$$m^{(d)} = \sum_{h=0}^{\infty} b_h \sum_{j=0}^{\kappa^h - 1} \kappa^{h(d-1)} \; \lambda|_{[j/\kappa^h,(j+1)/\kappa^h)^d} \, .$$

Its density is bounded if $\nu_1 \kappa^{d-1} < 1$ (compare Remark 3.13). The transition values of p_0 where the density of $m^{(d)}$ changes from bounded to unbounded thus depend on d. This agrees with the observation that it becomes increasingly less likely to find iterates of higher numbers of points close to each other.

Appendix A. Random walks with small drift

Reductions to random walks on the half line or the line are a recurring tool for the study of the iterated function systems in this paper. This appendix develops results that are used in the main text in the study of intermittency.

A.1. Stopping times for random walks with a small negative drift

Write $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ endowed with the product topology and the Borel σ-algebra. Fix a $0 < p_0 < 1$ and write ν_2 for the $(p_0, 1-p_0)$-Bernoulli measure on Σ_2 determined on any cylinder set

$$[a_0 \cdots a_k] = \{\omega \in \Sigma_2 \, ; \, \omega_i = a_i, \, 0 \le i \le k\}, \, a_i \in \{0,1\},$$

by

$$\nu_2([a_0 \cdots a_k]) = p_0^{\#\{0 \le i \le k \, ; \, a_i = 0\}} (1-p_0)^{\#\{0 \le i \le k \, ; \, a_i = 1\}}.$$

For real numbers $L < 0$ and $R > 0$ consider the random walk given by

$$z_{n+1} = \begin{cases} z_n + L, & \omega_n = 0, \\ z_n + R, & \omega_n = 1, \end{cases} \tag{A.1}$$

so the step $L < 0$ is taken with probability p_0, the step $R > 0$ with probability $1 - p_0$. We assume that the average drift α given by

$$\alpha = p_0 L + (1-p_0) R$$

is small and negative, so $\alpha < 0$.

Consider escape from $[0, \infty)$ for the random walk (A.1). For $z_0 \in [0, \infty)$, define the stopping time

$$T = \min\{n \in \mathbb{N} \ ; \ z_n < 0\}.$$

Lemma A.1. *Assume $z_0 \in [0, R)$. Then $\int_{\Sigma_2} T(\omega)\, d\nu_2 \geq -p_0 L/|\alpha|$.*

Proof. By Wald's identity, see for instance [53, Section VII.2],

$$\int_{\Sigma_2} T(\omega)\, d\nu_2 = \frac{1}{\alpha} \int_{\Sigma_2} z_{T(\omega)} - z_0 \, d\nu_2.$$

Note that $T(\omega)$ as a function of z_0 is minimal for $z_0 = 0$. Then

$$\begin{aligned} \frac{1}{\alpha} \int_{\Sigma_2} z_{T(\omega)} - z_0 \, d\nu_2 &= \frac{1}{|\alpha|} \int_{\Sigma_2} z_0 - z_{T(\omega)} \, d\nu_2 \\ &\geq \frac{1}{|\alpha|} \int_{[0]} -z_{T(\omega)} \, d\nu_2 = -\frac{p_0 L}{|\alpha|}. \quad \square \end{aligned}$$

Next consider escape from an interval $[0, K]$ with K a large positive number. So let $z_0 \in [0, K]$ and define

$$T_K = \min\{n > 0 \ ; \ z_n < 0 \text{ or } z_n > K\}.$$

It is well known that the average stopping time to reach $(-\infty, 0) \cup (K, \infty)$ is finite:

$$\int_{\Sigma_2} T_K(\omega)\, d\nu_2(\omega) < \infty.$$

To see this it suffices to realize that escape from $[0, K]$ is guaranteed after a sufficient number of identical symbols, either 0 or 1, in ω. We will discuss the probability of escape through 0, when starting close to K, and establish that the probability of escaping through 0 can be made arbitrarily small by taking the drift α close enough to 0 and K large enough. For the next lemma we take a context of parameterized families of random walks: consider (A.1) with

$$L = L_0 + \alpha, \qquad R = R_0 + \alpha, \tag{A.2}$$

where $p_0 L_0 + (1 - p_0) R_0 = 0$. The average drift $\alpha < 0$ is taken as the parameter.

Lemma A.2. *Consider* (A.1) *with L, R given by* (A.2). *Given $\rho > 0$, there is a small $\alpha_0 < 0$ and a large $K > 0$, so that for $\alpha_0 < \alpha < 0$ and $z_0 \in (K+L, K] \subset [0, K]$, we have*

$$\nu_2(\{\omega \in \Sigma_2 \; ; \; z_{T_K} < 0\}) \leq \rho.$$

Proof. Write $\zeta_0 = 0$ and $\zeta_n = z_n - z_{n-1}$ for $n \geq 1$. For $n \geq 0$ write $S_n = \zeta_0 + \cdots + \zeta_n = z_n - z_0$. Note that for $n \geq 1$, S_n depends on $\omega_0, \ldots, \omega_{n-1}$. Consider the function

$$G_n = e^{r^* S_n},$$

where $r^* > 0$ is the solution of

$$p_0 e^{Lr^*} + (1 - p_0) e^{Rr^*} = 1.$$

By developing the exponential functions in a Taylor series, one can check that this equation has a unique solution $r^* > 0$ with $r^* \to 0$ as $\alpha \to 0$:

$$p_0 \left(1 + Lr^* + \frac{1}{2} L^2 (r^*)^2\right) + (1 - p_0) \left(1 + Rr^* + \frac{1}{2} R^2 (r^*)^2\right) = 1 + \mathcal{O}\left((r^*)^3\right)$$

yields

$$\alpha r^* + \frac{1}{2} \left(p_0 L^2 + (1 - p_0) R^2\right) (r^*)^2 = \mathcal{O}\left((r^*)^3\right)$$

and thus

$$\alpha + \frac{1}{2} \left(p_0 L^2 + (1 - p_0) R^2\right) r^* = \mathcal{O}\left((r^*)^2\right)$$

for $r^* \neq 0$, from which r^* can be solved by the implicit function theorem. Now G_n is a martingale as for any cylinder $[a_0 \ldots a_{n-1}] \subseteq \Sigma_2$,

$$\begin{aligned}
\int_{[a_0 \ldots a_{n-1}]} e^{r^* S_{n+1}} \, d\nu_2 &= \int_{[a_0 \ldots a_{n-1}]} e^{r^* S_n} e^{r^* \zeta_{n+1}} \, d\nu_2 \\
&= e^{r^* S_n} \int_{[a_0 \ldots a_{n-1}]} e^{r^* \zeta_{n+1}} \, d\nu_2 \\
&= e^{r^* S_n} \left(\int_{[a_0 \ldots a_{n-1} 0]} e^{Lr^*} \, d\nu_2 + \int_{[a_0 \ldots a_{n-1} 1]} e^{Rr^*} \, d\nu_2 \right) \\
&= e^{r^* S_n} \left(\int_{[a_0 \ldots a_{n-1}]} p_0 e^{Lr^*} + (1 - p_0) e^{Rr^*} \, d\nu_2 \right)
\end{aligned}$$

$$= \int_{[a_0 \dots a_{n-1}]} e^{r^* S_n} \, d\nu_2.$$

By Doob's optional stopping theorem, see for instance [53, Theorem VII.2.2], we see that for any large K,

$$\int_{\Sigma_2} e^{r^* S_{T_K}} \, d\nu_2 = e^{r^* S_0} = 1.$$

This gives

$$\int_{\Sigma_2} e^{r^* z_{T_K}} \, d\nu_2 = e^{z_0 r^*}.$$

Observe that $z_{T_K} \in [L, 0)$ or $z_{T_K} \in (K, K+R]$. For any large K let

$$A_K = \nu_2(\{\omega \in \Sigma_2 \; ; \; z_{T_K}(\omega) < 0\}) > 0$$

be the probability that $z_{T_K} < 0$. Let $0 < \rho < 1$. Our goal is to show that we can find α_0 and K such that $A_K \leq \rho$ for all $\alpha \in (\alpha_0, 0)$. For any $\alpha < 0$ write

$$\int_{\Sigma_2} e^{r^* z_{T_K}} \, d\nu_2 = A_K e^{c_1 r^*} + (1 - A_K) e^{K r^*} e^{c_2 r^*},$$

with

$$e^{c_1 r^*} = \frac{1}{A_K} \int_{\{\omega \in \Sigma_2 \; ; \; z_{T_K} < 0\}} e^{r^* z_{T_K}} \, d\nu_2,$$

$$e^{c_2 r^*} = \frac{1}{1 - A_K} \int_{\{\omega \in \Sigma_2 \; ; \; z_{T_K} > K\}} e^{r^* (z_{T_K} - K)} \, d\nu_2.$$

Note that $c_1 \in [L, 0]$ and $c_2 \in [0, R]$; c_1 represents the average value that z_{T_K} takes if the random walk escapes through 0 and $c_2 + K$ is the average value that z_{T_K} takes if the random walk escapes through K. We obtain

$$A_K = \frac{e^{c_2 r^*} - e^{z_0 r^*} e^{-K r^*}}{e^{c_2 r^*} - e^{c_1 r^*} e^{-K r^*}}.$$

Since $r^* \to 0$ as $\alpha \to 0$, the terms $e^{c_1 r^*}$, $e^{c_2 r^*}$ and $e^{z_0 r^*} e^{-K r^*}$ converge to 1 as $\alpha \to 0$. So we can take α_0 small so that for the corresponding r_0^*,

$$e^{c_2 r_0^*} - e^{z_0 r_0^*} e^{-K r_0^*} < \rho/2.$$

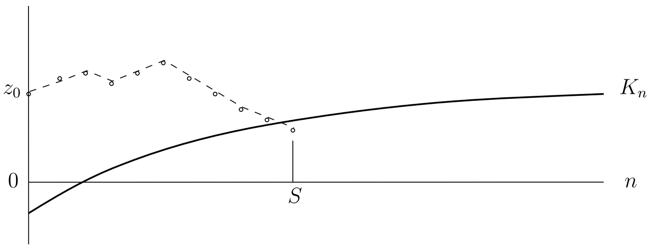


Fig. 14. Time dependent stopping levels where stopping levels K_n lie on a non constant curve.

Now taking K large enough ensures that $A_K < \rho$ for these values of α_0 and K.

To show that $A_K < \rho$ for any α with $\alpha_0 < \alpha < 0$, fix such an α. We must now consider that z_n depends on α and write $z_{\alpha,n}$. Clearly, for a fixed value of z_0,

$$z_{\alpha_0,n} < z_{\alpha,n}.$$

This implies that $\nu_2(\{\omega \in \Sigma_2 \ ; \ z_{T_K} < 0\})$ of escape through 0 decreases with α. The lemma follows. □

A.2. Stopping times for random walks with time dependent levels

We stay with the random walk z_n from (A.1), but now with the no drift condition

$$p_0 L + (1 - p_0) R = 0.$$

Let $\varepsilon_n = 1/n^2$ and note that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Let also $\varepsilon > 0$ and $p \geq 1$ be such that $\varepsilon_{p+1} < \varepsilon < \varepsilon_p$. We assume that ε is small or equivalently that p is large. Define $K_n = -\ln(\varepsilon_{n+p}) + \ln(\varepsilon)$, so

$$K_n = 2\ln(n + p) + \ln(\varepsilon). \tag{A.3}$$

Note that $K_0 = 2\ln(p) + \ln(\varepsilon) < 0$ and that $\lim_{n\to\infty} K_n = \infty$. Suppose $z_0 > K_0$. We want to know the average stopping time to reach the n-dependent level K_n. Time dependent stopping levels like these have been considered in [28] and [29, Section 4.5]. We use the statements that are derived here in the study of intermittency in Section 3.2. Write

$$S = \min\{n > 0 \ ; \ z_n < K_n\}.$$

It is standard that in the case of no drift the expected stopping time to reach a point smaller than the fixed level K_0 is infinite. We will show that the expected value of S is still infinite, using the slow growth of K_n. Fig. 14 illustrates the setting.

Lemma A.3.

$$\int_{\Sigma_2} S(\omega)\, d\nu_2(\omega) = \infty.$$

Proof. The strategy of the proof is to give a lower bound for the integral from the lemma by considering, instead of the stopping level K_n, a sequence of affine stopping levels $(M_n^{(m)})_{m\geq 0}$ that have slope decreasing to 0 as $m \to \infty$. We then translate the situation of having a random walk with no drift and affine stopping levels to a random walk with small negative drift and a constant stopping level K, so that we can apply the results from the previous section. The stopping levels $M_n^{(m)}$ are obtained by taking tangent lines to K_n at suitable moments m. We will first explain this last part for an arbitrary suitable time m.

We will work with a large positive number K; a condition for K will be given in the course of the proof. As we have $z_m > K + K_m$ with positive probability, for a suitable positive integer m, to prove the lemma we may assume

$$z_0 > K + K_0.$$

Counting iterates from m on by writing $n = m + i$, $i \geq 0$, and translating the values K_n by $K_0 - K_m$ replaces K_n by $K_{i+m} + K_0 - K_m = 2\ln(i+m+p) + 2\ln(p) - 2\ln(p+m) + \ln(\varepsilon)$, $i \geq 0$. This is of the form (A.3) and shows that we may also assume that p is large in (A.3); a condition for p will also be given in the course of the proof.

We construct a stochastic process u_n built from random walks $v_n^{(m_i)}$, $m_i \leq n \leq m_{i+1}$ for certain stopping times m_i. We start with two ingredients, the introduction of stopping times S_m and $U_{m,l}$.

Definition of a stopping time S_m. Start with an integer $m \geq 0$ such that $z_m > K + K_m$. For all $n > m$ we replace the level K_n by a level $M_n = M_n^{(m)}$ depending affinely on n:

$$M_n = \alpha_m + \beta_m n$$

with α_m, β_m so that the line $x \mapsto \alpha_m + \beta_m x$ is tangent to $x \mapsto 2\ln(x + p) + \ln(\varepsilon)$ at $x = m$. This gives

$$\alpha_m = K_m - 2m/(m + p) \quad \text{and} \quad \beta_m = 2/(m + p).$$

As the graph of $x \mapsto 2\ln(x + p)$ is concave, we have $K_n \leq M_n$. Consider the stopping time

$$U := \min\{n > m \; ; \; z_n < M_n + K\} \leq \min\{n > m \; ; \; z_n < K_n + K\}.$$

Define

$$v_n^{(m)} = z_n - M_n = z_n - \beta_m(n-m) - K_m.$$

So $v_m^{(m)} = z_m - K_m$ and thus $v_m^{(m)} > K$. The sequence $v_n^{(m)}$ defines a random walk given by

$$v_{n+1}^{(m)} = z_{n+1} - \beta_m(n+1-m) - K_m = \begin{cases} v_n^{(m)} + L - \beta_m, & \omega_n = 0, \\ v_n^{(m)} + R - \beta_m, & \omega_n = 1. \end{cases}$$

Hence, the random walk $v_n^{(m)}$ has a negative drift $-\beta_m = -\frac{2}{m+p}$, which is small if p is large and depends on and is decreasing in m. The demand $z_n < M_n + K$ is equivalent to $v_n^{(m)} < K$. Write

$$S_m(\omega) = \min\{n > m \; ; \; v_n^{(m)} < K\}$$

and denote $\mathbb{E}_m(S_m) = \int_{\Sigma_2} S_m \, d\nu_2$. Note that the smallest value of $\mathbb{E}_m(S_m)$, for varying $v_m^{(m)} \geq K$, is obtained for $v_m^{(m)} = K$. As the expected stopping time is similar to the reciprocal of the average drift, see Lemma A.1,

$$\mathbb{E}_m(S_m) \geq \frac{-p_0(L-\beta_m)}{\beta_m} \geq \frac{-p_0 L}{2}(m+p). \tag{A.4}$$

Definition of a stopping time $U_{m,l}$. The second ingredient is the random walk $v_n^{(m)}$ with values inside the interval $[0, K]$. Assume $v_l^{(m)} \in (K-L, K]$ for a positive integer $l > m$ and consider a second stopping time

$$U_{m,l}(\omega) = \min\left\{n > l \; ; \; v_n^{(m)} < 0 \quad \text{or} \quad v_n^{(m)} > K\right\}.$$

Note that $v_n^{(m)} < 0$ is equivalent to $z_n < M_n$ and $v_n^{(m)} > K$ is equivalent to $z_n > M_n + K$. Write

$$\rho_m = \nu_2\left(\left\{\omega \in \Sigma_2 \; ; \; v_{U_{m,l}}^{(m)} < 0\right\}\right)$$

for the probability that $v_n^{(m)}$ crosses 0. As $v_l^{(m)} \in (K+L, K]$, by Lemma A.2 we find that ρ_m will be small for all sufficiently large m if K is large. More precisely, given any $\rho > 0$ we can choose p sufficiently small (so that the drift $-\beta_m$ is close enough to zero for all m) and K sufficiently large such that $\rho_m < \rho$ for all m or equivalently

$$\nu_2\left(\left\{\omega \in \Sigma_2 \; ; \; v_{U_{m,l}}^{(m)} > K\right\}\right) \geq 1 - \rho \tag{A.5}$$

for all m.

Construction of the process u_n. Let $m_0 = 0$, assume $u_0 > K$, and define the process u_n, $n \geq 0$, as follows. Write $l_0 = S_{m_0}$. If $l_0 < \infty$, let $m_1 = U_{m_0,l_0}$. For $m_0 \leq n \leq$

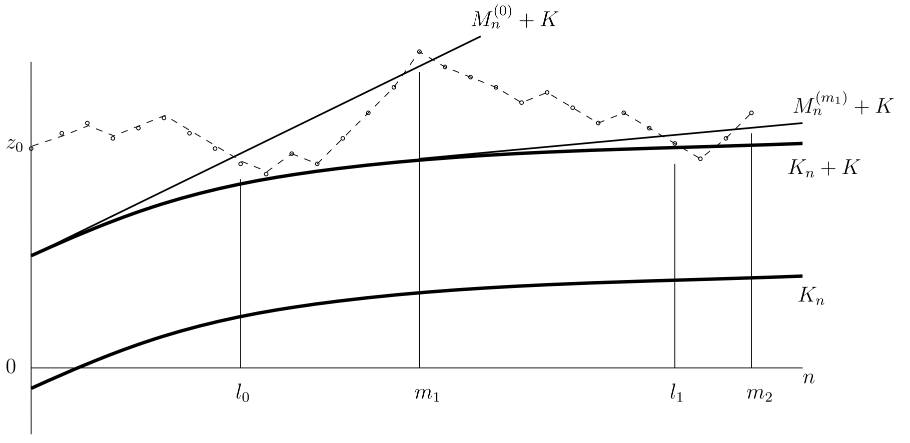


Fig. 15. A possible path z_n with indicated stopping times.

m_1 we let $u_n = v_n^{(m_0)}$. Inductively, suppose l_i is defined for $0 \le i < k$ and m_i is defined for $0 \le i \le k$. We have u_n defined for $0 \le i \le m_k$. Then set

$$l_k = S_{m_k}.$$

If $l_k < \infty$, we let $u_n = v_n^{(m_k)}$ for $m_k \le n \le l_k$ and let

$$m_{k+1} = U_{m_k,l_k}.$$

For $m_{k+1} < \infty$, we have either $v_{m_{k+1}}^{(m_k)} < 0$ or $v_{m_{k+1}}^{(m_k)} > K$. If $v_{m_{k+1}}^{(m_k)} > K$, then we let

$$u_n = v_n^{(m_k)}, \qquad l_k \le n \le m_{k+1}.$$

Note that $u_{m_{k+1}} - K_{m_{k+1}} > K$.

If some $v_{m_{i+1}}^{(m_i)} < 0$, we let $m_j = m_{i+1}$ for $j > i$. A path of the corresponding walk z_n with indicated stopping times is depicted in Fig. 15. A similar visualization of paths u_n is presented in Fig. 16.

Estimating the average of the stopping time S. Having constructed the path u_n with the sequence of stopping times m_k, we can estimate the expected value of the stopping time S. We first set the parameters. Set $c = -\frac{p_0 L}{2}$ and let $\rho > 0$ be small enough such that $(1-\rho)(1+c) > 1$. Let $\alpha_0 < 0$ and $K > 0$ be as given by Lemma A.2. Choose p so that $-\frac{2}{p} \in (\alpha_0, 0)$. Let $\frac{1}{(p+1)^2} < \varepsilon < \frac{1}{p^2}$ and let K_n, $M_n^{(m)}$ be as defined before. Note that the choice of p implies that $-\beta_m \in (\alpha_0, 0)$ for all $m \ge 0$. Let

$$E_0 = \{\omega \in \Sigma_2\,;\, l_0(\omega) < \infty \text{ and } v_{m_1}^{(m_0)}(\omega) > K\}$$

and for $n \ge 1$ let

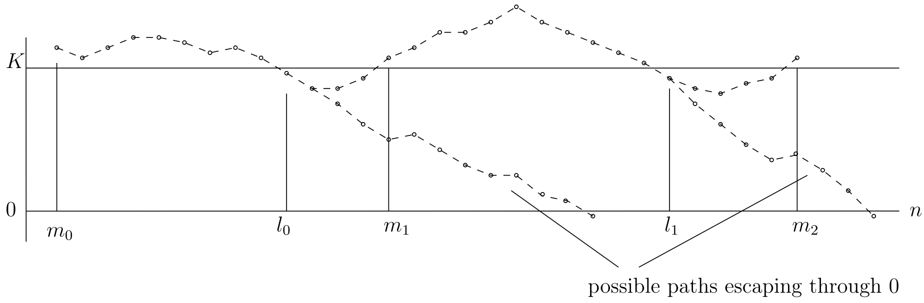


Fig. 16. Visualization of breaking up random walks u_n in parts where $u_n > K$ or $u_n \in [0, K]$.

$$E_n = \{\omega \in E_{n-1}\,;\, l_n(\omega) < \infty \text{ and } v^{(m_n)}_{m_{n+1}}(\omega) > K\}.$$

Note that $S(\omega) \geq l_0(\omega)$ for all $\omega \in \Sigma_2$ and that for $\omega \in E_n$ we have

$$S(\omega) \geq l_0(\omega) + l_1(\omega) - m_1(\omega) + \cdots + l_{n+1}(\omega) - m_{n+1}(\omega),\ n \geq 0.$$

Hence,

$$\int_{\Sigma_2} S\, d\nu_2 \geq \int_{\Sigma_2} l_0\, d\nu_2 + \sum_{n\geq 0} \int_{E_n} l_{n+1} - m_{n+1}\, d\nu_2. \tag{A.6}$$

As established in (A.4),

$$\mathbb{E}(l_0) \geq -\frac{p_0 L}{2} p = cp.$$

The set E_0 is a union of cylinders on which the time m_1 is constant. Let $\eta = \eta_0 \cdots \eta_k \in \{0,1\}^k$ be such that the cylinder $C = [\eta_0 \cdots \eta_k]$ is in E_0 with $k = m_1(\omega) =: m_1(\eta)$ for each $\omega \in C$ and write $l_0(\eta)$ for the value $l_0(\omega)$, $\omega \in C$. Then by Lemma A.1,

$$\int_C l_1 - m_1\, d\nu_2 \geq -\frac{p_0 L}{2}(m_1(\eta) + p)\nu_2(C) \geq c(l_0(\eta) + p)\nu_2(C).$$

From (A.5) we see that

$$\int_{E_0} l_1 - m_1\, d\nu_2 \geq \nu_2(E_0) c(cp + p) \geq (1-\rho)cp(1+c).$$

Similarly, let $\eta = \eta_0 \cdots \eta_k \in \{0,1\}^k$ be such that the cylinder $C = [\eta_0 \cdots \eta_k] \subseteq E_1$ with $k = m_2(\omega) =: m_2(\eta)$ for each $\omega \in C$ and write $l_0(\eta), m_1(\eta), l_1(\eta)$ for the values $l_0(\omega), m_1(\omega), l_1(\omega)$, $\omega \in C$, respectively. From Lemma A.1 we get

$$\int_C l_2 - m_2\, d\nu_2 \geq -\frac{p_0 L}{2}(m_2(\eta)+p)\nu_2(C)$$
$$\geq c(l_1(\eta)+p)\nu_2(C) \geq c(l_0(\eta)+l_1(\eta)-m_1(\eta)+p)\nu_2(C).$$

Then (A.5) gives

$$\int_{E_1} l_2 - m_2\, d\nu_2 \geq \nu_2(E_1)c(cp+c(cp+p)+p) \geq (1-\rho)^2 cp(1+c)^2.$$

Continuing, we find for each $n \geq 1$ and $\eta = \eta_0 \cdots \eta_k \in \{0,1\}^k$ for which the cylinder $C = [\eta_0 \cdots \eta_k] \subseteq E_{n-1}$ satisfies $k = m_n(\omega) =: m_n(\eta)$ for each $\omega \in C$ that

$$\int_{E_{n-1}} l_n - m_n\, d\nu_2 \geq (1-\rho)^n cp(1+c)^n.$$

Together with (A.6) and the assumption that $(1-\rho)(1+c) > 1$ this yields

$$\int_{\Sigma_2} S\, d\nu_2 \geq \sum_{i=0}^{\infty} cp(1-\rho)^i(1+c)^i = \infty. \quad \square$$

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