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Abstract

For countably infinite IFSs on $\mathbb{R}^2$ consisting of affine contractions with diagonal linear parts, we give conditions under which the affinity dimension is an upper bound for the Hausdorff dimension and a lower bound for the lower box-counting dimension. Moreover, we identify a family of countably infinite IFSs for which the Hausdorff and the affinity dimension are equal, and which have full dimension spectrum. The corresponding self-affine sets are related to restricted digit sets for signed Lüroth expansions.

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1. Introduction and statement of main results

The dimension theory of self-affine sets generated by finite iterated function systems (IFSs) has been developed since the 1980s, when it was investigated for which types of sets the Hausdorff and box-counting dimensions coincide, see for example [Bed84, McM84]. In 1988 Falconer [Fal88] introduced the affinity dimension, an expression which purely depends on the singular values of the linear parts of the affine maps in the IFS. It turns out that for finitely generated self-affine sets in $\mathbb{R}^D$, the affinity dimension is an upper bound for the upper box-counting dimension, which is known to be an upper bound for the Hausdorff dimension. Moreover, Falconer proved that the Hausdorff dimension is almost surely (with respect to the translation vectors of the affine maps in the IFS) equal to the minimum of *D* and the affinity dimension. Fraser [Fra12] later introduced a modified affinity dimension and showed that for a class of finitely generated box-like self-affine sets satisfying the rectangular open set condition (ROSC) the modified affinity dimension coincides with the box-counting and packing dimensions. More recently, Morris [Mor18] gave a simple description of the affinity dimension of self-affine sets in case the linear parts of the contractions consist of diagonal and anti-diagonal matrices. In the diagonal case, under the condition that each of the canonical projections of the IFS is exponentially separated, Rapaport [Rap23] showed that the Hausdorff dimension of the self-affine set coincides with the minimum of its affinity dimension and *D*. The authors of [KR14, Jur21] considered the affinity dimension for infinite affine IFSs that are *irreducible*, meaning the linear parts of the affine maps do not all preserve a common proper non-trivial linear subspace. They showed that the Hausdorff and affinity dimensions of the corresponding self-affine sets coincide. Outside of these classes of self-affine sets the Hausdorff and affinity dimensions do not necessarily coincide. For instance the Hausdorff dimension of the self-affine set constructed by taking the cross product of the middle $\frac{1}{2}$-Cantor set and the middle $\frac{7}{8}$-Cantor set equals $\frac{3}{4}$, whereas its affinity dimension is 1.[4]

Conformal infinite IFSs have been studied since the seminal work of Mauldin and Urbański [MU96]. One of the differences between finite and infinite conformal IFSs, highlighted in [MU96], is that even under the open set condition (OSC), the Hausdorff and box-counting dimensions of their limit sets need not be equal. Moreover, in contrast to the finite setting, the limit set of an infinite IFS need not be compact.

In this article we generalise some of the above dimension theory results to a class of non-irreducible non-conformal infinite IFSs, for which the projections are not necessarily exponentially separated. More precisely, for a countable collection $\{L_i : i \in I\}$ of diagonal 2×2 matrices over $\mathbb{R}$ of the form

$$L_i = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}, \quad |a_i|, |b_i| \in (0, 1), \tag{1.1}$$

and for $r > 0$, we set

$$P_I(r) = \begin{cases} \max\left\{\sum_{i \in I} |a_i|^r, \sum_{i \in I} |b_i|^r\right\} & \text{if } 0 < r \leqslant 1, \\ \max\left\{\sum_{i \in I} |a_i| \cdot |b_i|^{r-1}, \sum_{i \in I} |b_i| \cdot |a_i|^{r-1}\right\} & \text{if } 1 < r \leqslant 2, \\ \sum_{i \in I} |a_i \cdot b_i|^{r/2} & \text{if } r > 2. \end{cases} \tag{1.2}$$

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[4] We thank Ian Morris for providing this example.

The *affinity dimension* $d(L_i \mid i \in I)$ of $\{L_i : i \in I\}$ is defined by

$$d(L_i \mid i \in I) = \inf \left\{ r > 0 : \sum_{m \in \mathbb{N}} \sum_{u \in I^m} \varphi^r(L_u) < \infty \right\},$$

where $\varphi^r(L_u) = \varphi^r(L_{(u_1,\ldots,u_m)})$ is the singular value function of the matrix product $L_u = L_{u_1} \cdots L_{u_m}$, see (2.2). For countably infinite alphabets I we find the following analogue of [Mor18, corollary 2].

Theorem 1.1. *Suppose we have a countable alphabet I and a collection $\{L_i : i \in I\}$ of diagonal 2×2 matrices, as given in (1.1), with $\sup_{i \in I} \max\{|a_i|, |b_i|\} < 1$, then*

$$\inf \left\{ r > 0 : \sum_{m \in \mathbb{N}} \sum_{u \in I^m} \varphi^r(L_u) < \infty \right\} = \inf \{ r > 0 : P_I(r) \leqslant 1 \}.$$

Our next result compares different notions of dimension to the affinity dimension and identifies a large class of infinite IFSs for which the affinity dimension gives a lower bound for the lower box-counting dimension and an upper bound for the Hausdorff dimension.

Theorem 1.2. *Let I be a countably infinite alphabet and F be the self-affine set of an IFS $\{A_i : i \in I\}$ on $[0,1]^2$, where each A_i is an affine map with linear part L_i as in (1.1), and $\sup_{i \in I} \max\{|a_i|, |b_i|\} < 1$. Then the following hold.*

(a) $\dim_{\mathcal{H}}(F) \leqslant \min\{2, d(L_i \mid i \in I)\}$
(b) *If there exists a finite alphabet $I_1 \subseteq I$ such that $\dim_B(F_{I_2}) = d(L_i \mid i \in I_2)$ for all finite alphabets I_2 with $I_1 \subseteq I_2 \subseteq I$, where F_{I_2} is the limit set of $\{A_i : i \in I_2\}$, then $d(L_i \mid i \in I) \leqslant \underline{\dim}_B(F) \leqslant \overline{\dim}_B(F) = \dim_P(F)$.*

Let π_1, π_2 denote the canonical projections onto the first and second coordinate, respectively. The conditions in theorem 1.2(ii) hold, for example, when F satisfies the ROSC, see section 2.2, and there exists a finite subalphabet $I_1 \subseteq I$ such that either

(i) $\dim_B(\pi_1(F_{I_1})) = \dim_B(\pi_2(F_{I_1})) = 1$, or
(ii) $\dim_B(\pi_1(F_{I_1})) = 1$ and $|a_i| \geqslant |b_i|$ for each $i \in I$.

This is a consequence of [Fra12, corollaries 2.6 and 2.7] combined with (a) if $I_1 \subseteq I_2 \subseteq I$, then $F_{I_1} \subseteq F_{I_2} \subseteq F_I$, and (b) if $\{A_i : i \in I\}$ is an infinite IFS satisfying the ROSC, then $\{A_i : i \in I'\}$ satisfies the ROSC for $I' \subseteq I$.

Remark 1.3. The proof of theorem 1.2(ii) uses what is commonly referred to as an *exhausting principle*, meaning that when taking a sequence of nested finite subsets $I_1 \subset I_2 \subset \ldots \subset I$ satisfying $\bigcup_{n \in \mathbb{N}} I_n = I$, then $(d(L_i \mid i \in I_n))_{n \in \mathbb{N}}$ is a non-decreasing sequence satisfying $\sup_{n \in \mathbb{N}} d(L_i \mid i \in I_n) = d(L_i \mid i \in I)$. This can be used to compute a numerical approximation of the affinity dimension $d(L_i \mid i \in I)$ and we examine the rate of convergence of $d(L_i \mid i \in I_n)$ to $d(L_i \mid i \in I)$ in section 3.2. In tandem with theorem 1.2(i), this also yields an upper bound for $\dim_{\mathcal{H}}(F)$.

Next we provide a family of planar self-affine sets for which we can simplify the affinity dimension even further. Considering this family is motivated by questions on number expansions with restrictions on their digits. A famous example of a restricted digit set is the middle

third Cantor set, which is the set of numbers in $[0,1]$ that have a ternary expansion without the digit 1. For non-integer base expansions, results on restricted digit sets are considered, for example, in [KSS95, PS95, Lal97, DK09]. For continued fractions, which have infinite digit sets, restricted digit sets have been extensively studied since the work of Jarnik [Jar28] and Good [Goo41]. The infinite IFSs we will be concerned with relate to another type of number expansions with an infinite digit set, namely *Lüroth expansions* [Lür83]. For $x \in (0,1]$, these are expressions of the form

$$x = \sum_{n \in \mathbb{N}} \frac{d_n - 1}{\prod_{i=1}^n d_i (d_i - 1)}, \tag{1.3}$$

where $d_n \in \mathbb{N}_{\geqslant 2}$ for $n \in \mathbb{N}$. There are many known results on Lüroth expansions, for instance, concerning level sets defined in terms of the frequencies of digits or sets of points with growth rate restrictions on the digits, see for example [BI09, FLMW10, LWY18, AGR21, Zho22, FZ23, BK24, HSZ24].

Lüroth expansions can be obtained from the infinite IFS $\{h_d : d \in \mathbb{N}_{\geqslant 2}\}$ where $h_d : [0,1] \to [1/d, 1/(d-1)]$ is defined by $h_d(x) = (x + d - 1)/(d(d-1))$. If x has a Lüroth expansion as in (1.3) with digit sequence $(d_n)_{n \in \mathbb{N}}$, then

$$x = \lim_{n \to \infty} h_{d_1} \circ h_{d_2} \circ \cdots \circ h_{d_n} (0).$$

Over the years several generalisations of the Lüroth number system have been proposed. In particular, the authors of [KKK90, KKK91] considered alternating Lüroth expansions, which are very similar to the ones in (1.3) but the terms in the series alternate in sign, hence the name. In [BBDK96] it was shown that alternating Lüroth expansions have better approximation properties than Lüroth expansions and a family of number systems was described that interpolate between the Lüroth and alternating Lüroth systems. The corresponding expansions, which we call *signed Lüroth expansions*, are of the form

$$\sum_{n \in \mathbb{N}} (-1)^{\sum_{i=1}^{n-1} s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i (d_i - 1)}, \tag{1.4}$$

where $s_n \in \{0,1\}$ and $d_n \in \mathbb{N}_{\geqslant 2}$ for $n \in \mathbb{N}$, and where we set $\sum_{i=1}^0 s_i = 0$. In [KM22] it was shown that Lebesgue almost all numbers $x \in [0,1]$ have uncountably many different signed Lüroth expansions and a one-parameter family of number systems in $\mathbb{R}^2$ was introduced that generate, for each x, all possible signed Lüroth expansions.

The system from [KM22] is related to an infinite affine IFS as follows. For each parameter $p \in (0,1)$ consider the IFS $\{A_{s,d}^p : (s,d) \in \{0,1\} \times \mathbb{N}_{\geqslant 2}\}$ where $A_{s,d}^p : [0,1]^2 \to [0,1]^2$ is defined, for $(w,x) \in [0,1]^2$, by

$$A_{s,d}^p(w,x) = \left(L_{s,d}^p(w,x)^\top + v_{s,d}^p \right)^\top \quad \text{with} \quad L_{s,d}^p = \begin{pmatrix} p^{1-s}(1-p)^s & 0 \\ 0 & (-1)^s \frac{1}{d(d-1)} \end{pmatrix}$$

$$\text{and} \quad v_{s,d}^p = \begin{pmatrix} sp \\ \frac{1}{d-s} \end{pmatrix}. \tag{1.5}$$

For $p \in (0,1)$ and $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 2}$ we let $\mathcal{F}_J^p$ denote the self-affine set of the non-irreducible IFS $\{A_{s,d}^p : (s,d) \in J\}$. See figure 1 for illustrations of approximations of the sets $\mathcal{F}_J^p$ with $p = \frac{1}{2}$ and for J equal to $J_1 = \{(0,2),(1,2)\}$, $J_2 = \{(0,2),(1,2),(0,3)\}$ and $J_3 = \{0,1\} \times \{2,4,6\}$.
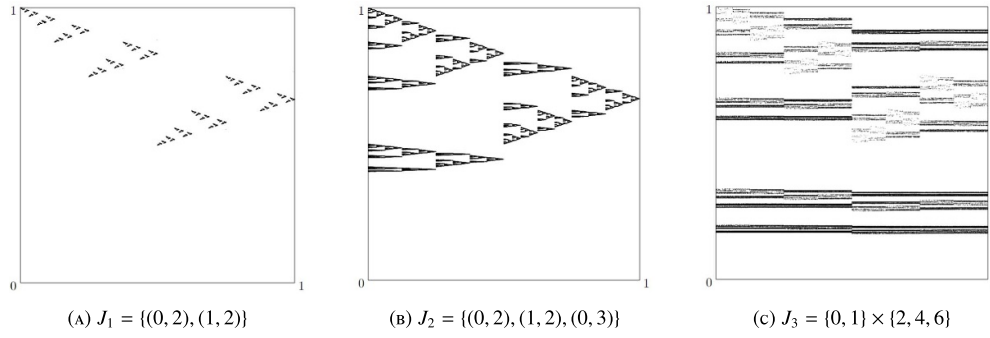
(A) $J_1 = \{(0,2),(1,2)\}$ (B) $J_2 = \{(0,2),(1,2),(0,3)\}$ (C) $J_3 = \{0,1\} \times \{2,4,6\}$

Figure 1. The self-affine sets $\mathcal{F}^p_{J_1}, \mathcal{F}^p_{J_2}$ and $\mathcal{F}^p_{J_3}$ for $p = \frac{1}{2}$.

If x has an expansion of the form (1.4) with sign sequence $(s_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ and digit sequence $(d_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}_{\geqslant 2}$, then $x = \pi_2(\lim_{n\to\infty} A^p_{s_1,d_1} \circ A^p_{s_2,d_2} \circ \cdots \circ A^p_{s_n,d_n}((0,0)))$. The maps $A^p_{0,d}$ correspond to the Lüroth system in the sense that for each digit sequence $(d_n)_{n\in\mathbb{N}}$,

$$\pi_2\left(\lim_{n\to\infty} A^p_{0,d_1} \circ A^p_{0,d_2} \circ \cdots \circ A^p_{0,d_n}((0,0))\right) = \lim_{n\to\infty} h_{d_1} \circ h_{d_2} \circ \cdots \circ h_{d_n}(0).$$

Similarly, the maps $A^p_{1,d}$ correspond to the alternating Lüroth system from [KKK90, KKK91]. The collection of all signed Lüroth expansions is then obtained from all possible compositions of the Lüroth and alternating Lüroth systems, and the parameter $p \in (0,1)$ governs the weight that is put on each of them, or the probability with which the maps $A^p_{s,d}$ are chosen in such a composition.

For $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 2}$, the projection onto the second coordinate of the limit set of the IFS $\{A^p_{s,d} : (s,d) \in J\}$ contains precisely those numbers $x \in (0,1]$ that have a signed Lüroth expansion in which only digits $(s,d) \in J$ occur. Thereby, selecting different sets J corresponds to placing different restrictions on the digits in the expansions. In this article we examine the geometry of the *restricted digit sets*

$$
\begin{aligned}
F_J &= \{x \in (0,1] : x \text{ has a signed Lüroth expansion with all digits in } J\} \\
&= \Big\{x \in (0,1] : \text{ there exists } ((s_n,d_n))_{n\in\mathbb{N}} \in J^{\mathbb{N}} \text{ with} \\
&\qquad x = \pi_2\left(\lim_{n\to\infty} A^p_{s_1,d_1} \circ A^p_{s_2,d_2} \circ \cdots \circ A^p_{s_n,d_n}((0,0))\right)\Big\},
\end{aligned}
\tag{1.6}
$$

as well as the geometry of the self-affine sets $\mathcal{F}^p_J$. Note, F_J does not depend on the parameter p, which is why we have omitted it from the notation. Similar to [BF23, theorem 4.3], in theorem 4.5 we obtain expressions for the Hausdorff, upper box-counting and packing dimensions of F_J. Moreover, we use results from [RGU16] to obtain an expression for the Hausdorff dimension of non-autonomous variants of F_J, where the set J describing the restriction can change at each time step. We use these results in tandem with the results of [Mar54] to show the following.

Theorem 1.4. *Let I_0 and I_1 denote two non-empty subsets of $\mathbb{N}_{\geqslant 2}$ and let $J = (\{0\} \times I_0) \cup (\{1\} \times I_1)$. For $p \in (0,1)$,*

$$\dim_{\mathcal{H}}(\mathcal{F}_J^p) \geqslant 1 + \inf\left\{ r \in (0,1] : \left(\sum_{d_0 \in I_0}\left(\frac{1}{d_0(d_0-1)}\right)^r\right)^p \left(\sum_{d_1 \in I_1}\left(\frac{1}{d_1(d_1-1)}\right)^r\right)^{1-p} \leqslant 1 \right\}.$$

(1.7)

Further, if $I_0 = I_1 = I \subseteq \mathbb{N}_{\geqslant 2}$ and $p \in (0,1)$, then

$$\dim_{\mathcal{H}}(\mathcal{F}_J^p) = d(L_{s,d}^p \mid (s,d) \in J) = 1 + \inf\left\{ r \in (0,1] : \sum_{d \in I}\left(\frac{1}{d(d-1)}\right)^r \leqslant 1 \right\}. \qquad (1.8)$$

In particular, if I is finite, then $\dim_{\mathcal{H}}(\mathcal{F}_J^p) = \dim_P(\mathcal{F}_J^p) = \dim_B(\mathcal{F}_J^p) = d(L_{s,d}^p \mid (s,d) \in J)$.

Theorem 1.4 shows that our family of examples includes finitely and infinitely generated planar self-affine sets for which the Hausdorff and affinity dimensions coincide, despite not being irreducible and despite having canonical projections to the *x*-coordinate that are not necessarily exponentially separated, complementing the work of [Fra12, KR14, Jur21, Rap23]. In the finite case this implies that the Hausdorff, packing, box-counting and affinity dimensions all coincide.

Families of finitely generated self-affine sets where the Hausdorff, packing, box-counting and affinity dimensions coincide were already shown to exist in [MS19]. For planar self-affine sets generated by irreducible infinite IFSs it was shown in [KR14, Jur21] that the Hausdorff and affinity dimensions coincide. Our class of self-affine sets provides new examples where these equalities hold. Further, in [Mor18] it was proven that the modified affinity dimension from [Fra12] can be simplified when F is the limit set of a finitely generated affine IFS for which the linear parts of the affine maps consist of diagonal and anti-diagonal matrices as long as $\dim_B \pi_1(F) = \dim_B \pi_2(F)$. On our way to proving theorem 1.4, we show that one can drop the condition $\dim_B \pi_1(F) = \dim_B \pi_2(F)$ at the cost of having only diagonal matrices and still obtain the same simplification, see proposition 3.2.

In the above, we discussed dimension results of the sets F_J and $\mathcal{F}_J^p$ for fixed sets $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 2}$. An interesting related question is, given a real number $y \in [0,2]$ can we find $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 2}$ such that the dimension of $\mathcal{F}_J^p$ equals y? This question is related to the *Texan conjecture* [Hen96, MU99], which concerns the density of the dimensions of bounded type continued fraction sets in $[0,1]$. Its resolution [KZ06] has generated a wealth of results and questions on the topological structure of the dimension spectrum of infinite IFSs. In [CLU19] it was shown that the dimension spectra of conformal graph directed Markov systems are compact and perfect and that the IFS resulting from the complex continued fraction algorithm has full dimension spectrum. These results were built on in [Jur21], where examples of non-irreducible infinite IFSs consisting of affine maps whose dimension spectrum is neither compact nor perfect were given. We show that the self-affine sets $\mathcal{F}_J^p$ have full Hausdorff dimension spectrum.

Theorem 1.5. *For $p \in (0,1)$ we have $\{\dim_{\mathcal{H}}(\mathcal{F}_J^p) : J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 2}\} = [0,2]$.*

1.1. Outline

In section 2 we introduce some notation and recall the necessary preliminaries. Section 3 concerns the results on the affinity dimension of countable collections of diagonal 2×2 matrices

and contains the proofs of theorems 1.1 and 1.2 as well as a brief analysis on numerical estimates for the affinity dimension. In section 4 we discuss the Hausdorff, upper box-counting and packing dimensions of the self-affine subsets F_J of $\mathbb{R}$ and the Hausdorff dimension of certain non-autonomous versions of F_J. These results will then be used in section 5 to prove theorems 1.4 and 1.5.

2. Preliminaries

2.1. Symbolic dynamics

An *alphabet I* is a countable set of symbols, which we call *digits*, equipped with the discrete topology. A *word u* with digits in I is a finite concatenation of digits $u = (i_1, \ldots, i_n)$ for some $n \in \mathbb{N}$ and $i_j \in I$ for all $j \in \{1, 2, \ldots, n\}$. We let I^m denote the set of all words of length m with digits in I and set $I^+ = \bigcup_{m \in \mathbb{N}} I^m$. Let $I^{\mathbb{N}} = \{(i_k)_{k \in \mathbb{N}} : i_k \in I \text{ for all } k \in \mathbb{N}\}$ denote the set of all (one-sided) infinite sequences with elements in I and endow $I^{\mathbb{N}}$ with the product topology. With this topology the space $I^{\mathbb{N}}$ is metrisable and in the case that I is finite $I^{\mathbb{N}}$ is also compact. For $i, j \in \mathbb{N}$ with $i \leqslant j$ and $\omega = (\omega_k)_{k \in \mathbb{N}} \in I^{\mathbb{N}}$ we let $\omega_{[i,j]} = (\omega_i, \omega_{i+1}, \ldots, \omega_j) \in I^{j-i+1}$. We use the same notation if $v \in I^k$ and $1 \leqslant i \leqslant j \leqslant k$ for some $k \in \mathbb{N}$.

2.2. Self-similar sets, self-affine sets and separation conditions

Fix $D \in \mathbb{N}$ and let X denote a non-empty compact subset of $\mathbb{R}^D$. If I is a countable alphabet, a family $\Phi = \{\phi_i : i \in I\}$ of (non-trivial) contractions $\phi_i : X \to X$ is called an *iterated function system* (*IFS*). We call Φ a *finite IFS* if I is a finite alphabet and an *infinite IFS* if I is a countably infinite alphabet. For any $m \in \mathbb{N}$ and any finite word $u = (u_1, u_2, \ldots, u_m) \in I^+$ we let

$$\phi_u = \phi_{u_1} \circ \phi_{u_2} \circ \cdots \circ \phi_{u_m}, \tag{2.1}$$

and for $\omega = (\omega_k)_{k \in \mathbb{N}} \in I^{\mathbb{N}}$ we observe that $(\phi_{\omega_{[1,k]}}(X))_{k \in \mathbb{N}}$ forms a nested sequence of non-empty compact sets with decreasing diameters. By the Cantor Intersection Theorem, $\bigcap_{k \in \mathbb{N}} \phi_{\omega_{[1,k]}}(X)$ is a singleton and we denote its only element by $\pi(\omega)$. We call the map $\pi : I^{\mathbb{N}} \to X$ the *projection map* of Φ and refer to $\pi(I^{\mathbb{N}})$, the image of $I^{\mathbb{N}}$ under π, as the *limit set* of Φ. When I is finite the natural action of Φ on the set of compact non-empty subsets of X, defined via $\Phi(A) = \bigcup_{i \in I} \phi_i(A)$, is a contraction with respect to the Hausdorff metric. By the Banach contraction mapping principle there exists a unique non-empty set E satisfying $\Phi(E) = E$. Moreover, in this setting, $E = \pi(I^{\mathbb{N}})$.

Independent of I being finite or countably infinite, if the contractions of Φ are all *similarities*, that is, if for all $i \in I$ there exists $c_i \in (0, 1)$ with $|\phi_i(x) - \phi_i(y)| = c_i |x - y|$ for all $x, y \in X$, then we call the limit set of Φ *self-similar*. If the contractions of Φ are *affine*, that is, if for each $i \in I$ there exists a $D \times D$ matrix L_i whose singular values lie in $(0, 1)$ and a vector $v_i \in \mathbb{R}^D$ with $\phi_i((x_1, \ldots, x_D)) = (L_i(x_1, \ldots, x_D)^\top + v_i)^\top$, for all $(x_1, \ldots, x_D) \in X$, then we call the limit set of Φ *self-affine*.

Two natural separation conditions we will use are the OSC and the ROSC. We say that Φ satisfies the *OSC* if there exists a non-empty open subset U of X such that $\phi_i(U) \subseteq U$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for all distinct $i, j \in I$. Such sets U will be called *feasible open sets for Φ*. For self-affine sets we sometimes require a slightly stronger separation condition, namely that the OSC is satisfied with $U = (a_1, b_1) \times \cdots \times (a_D, b_D)$, for some $a_1, \ldots, a_D, b_1, \ldots, b_D \in \mathbb{R}$ with $a_k < b_k$ for all $k \in \{1, \ldots, D\}$. We refer to this latter separation condition as the *ROSC*.

Another separation condition, which has been shown to have powerful consequences, is for an IFS to be exponentially separated, which is defined as follows. For two affine maps $\psi_1, \psi_2 \colon \mathbb{R} \to \mathbb{R}$ with $\psi_i(x) = r_i x + c_i$, for $i \in \{1, 2\}$ and $x \in \mathbb{R}$, we write

$$\rho(\psi_1, \psi_2) = \begin{cases} \infty & \text{if } r_1 \neq r_2, \\ |c_1 - c_2| & \text{otherwise.} \end{cases}$$

An affine IFS $\Psi = \{\psi_i : i \in I\}$ is said to be *exponentially separated* if there exist a constant $c > 0$ and an infinite set $Q \subset \mathbb{N}$ such that $\rho(\psi_\omega, \psi_\nu) \geqslant c^n$ for all $n \in Q$ and all distinct $\omega, \nu \in I^n$. A related condition is for Ψ to have *exact overlaps*, meaning there exist an $n \in \mathbb{N}$ and distinct $\omega, \nu \in I^n$ satisfying $\psi_\omega = \psi_\nu$. Indeed, it is easily verifiable that Ψ has no exact overlaps whenever it is exponentially separated. On the other hand, Ψ is exponentially separated whenever it has no exact overlaps and r_i and c_i are algebraic for all $i \in I$.

The above discussions relate to the *autonomous* setting; the contractions in (2.1) are chosen from the same IFS at each time step. A more general setting is the *non-automonous* setting, which is where the IFS is allowed to vary at each time step. Formally, a *non-automous self-similar IFS (NSIFS)* consists of a sequence $\Phi = (\Phi^{(n)})_{n \in \mathbb{N}}$ of self-similar IFSs $\Phi^{(n)} = \{\phi_i^{(n)} : i \in I^{(n)}\}$ defined on a common non-empty compact set $X \subseteq \mathbb{R}^D$.

As in the autonomous case we observe, for $(\omega_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} I^{(n)}$, that $(\phi_{\omega_1}^{(1)} \circ \cdots \circ \phi_{\omega_n}^{(n)}(X))_{n \in \mathbb{N}}$ forms a nested sequence of non-empty compact sets with decreasing diameters. Therefore $\bigcap_{n \in \mathbb{N}} \phi_{\omega_1}^{(1)} \circ \cdots \circ \phi_{\omega_n}^{(n)}(X)$ is a singleton. As above, let us denote the element of this singleton by $\pi(\omega)$. We refer to the map $\pi : \prod_{n \in \mathbb{N}} I^{(n)} \to X$ as the *projection map* of Φ, and call the image $\pi(\prod_{n \in \mathbb{N}} I^{(n)})$ of $\prod_{n \in \mathbb{N}} I^{(n)}$ under π the *limit set* of Φ. Further, we say that the NSIFS Φ satisfies the OSC if $\phi_i^{(n)}(\mathrm{int}(X)) \cap \phi_j^{(n)}(\mathrm{int}(X)) = \emptyset$ for all $n \in \mathbb{N}$ and all distinct $i, j \in I^{(n)}$.

In order to obtain dimension estimates on the limit set of an NSIFS, we will assume the OSC and additionally the *uniform contraction condition*. The latter means there exists an $\eta \in (0, 1)$ such that for each $j \in \mathbb{N}$ we have $c_{\omega_j}^{(j)} c_{\omega_{j+1}}^{(j+1)} \cdots c_{\omega_{j+m}}^{(j+m)} \leqslant \eta^m$ for all sufficiently large $m \in \mathbb{N}$, where $\omega_k \in I^{(k)}$ and where $c_{\omega_k}^{(k)}$ denotes the contraction ratio of the similarity $\phi_{\omega_k}^{(k)}$ for each $k \in \mathbb{N}$. For further details on NSIFSs we refer the reader to [RGU16].

2.3. Box-counting, Hausdorff, and (modified) affinity dimensions

In this section we introduce the notions of dimension which we will mainly be concerned with, namely the box-counting, Hausdorff, affinity and modified affinity dimensions. We will also touch on the packing dimension, but since we do not use its definition directly we omit it. For more information on these notions of dimension we refer the reader to [Fal97, Fal14].

Fix $D \in \mathbb{N}$. The *lower* and *upper box-counting dimensions* of a bounded set $F \subseteq \mathbb{R}^D$ are defined by

$$\underline{\dim}_B(F) = \liminf_{\delta \to 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \qquad \text{and} \qquad \overline{\dim}_B(F) = \limsup_{\delta \to 0} \frac{\log(N_\delta(F))}{-\log(\delta)}$$

respectively, where $N_\delta(F)$ denotes the smallest cardinality of a δ-cover of F, or alternatively, the number of closed squares in a δ-mesh whose intersection with F is non-empty. When $\underline{\dim}_B(F)$ and $\overline{\dim}_B(F)$ are equal we refer to the common value as the *box-counting dimension* of F and denote it by $\dim_B(F)$.

Let F be as above and let s and δ denote two non-negative real numbers. We define the δ-*approximate to the s-dimensional Hausdorff outer measure of F* to be

$\mathcal{H}_\delta^s(F) = \inf\left\{\sum_{i\in\mathbb{N}}\mathrm{diam}(U_i)^s : F \subseteq \bigcup_{i\in\mathbb{N}} U_i \text{ and } 0 \leqslant \mathrm{diam}(U_i) < \delta\right\}$, where $\mathrm{diam}(U_i)$ denotes the diameter of U_i. The *s-dimensional Hausdorff outer measure of F* is given by $\mathcal{H}^s(F) = \lim_{\delta\to 0}\mathcal{H}_\delta^s(F)$ and the *Hausdorff dimension* of F is $\dim_{\mathcal{H}}(F) = \inf\{s \geqslant 0 : \mathcal{H}^s(F) = 0\}$, which coincides with the value $\sup\{s \geqslant 0 : \mathcal{H}^s(F) = \infty\}$. Note that for a bounded set $F \subseteq \mathbb{R}^D$ these dimensions satisfy the relations $\dim_{\mathcal{H}}(F) \leqslant \underline{\dim}_B(F) \leqslant \overline{\dim}_B(F)$ and $\dim_{\mathcal{H}}(F) \leqslant \dim_P(F) \leqslant \overline{\dim}_B(F)$, where $\dim_P(F)$ denotes the packing dimension of F. However, in general, there is no relationship between the lower box-counting and packing dimensions of a given set.

Let $M_D(\mathbb{R})$ denote the collection of $D \times D$ matrices over $\mathbb{R}$. Given $L \in M_D(\mathbb{R})$ and $i \in \{1,2,\ldots,D\}$, we denote the *i*-th largest singular value of L, including multiplicities, by $\alpha_i(L)$ and we define the *singular value function* φ^r by

$$\varphi^r(L) = \begin{cases} \alpha_1(L)\alpha_2(L)\cdots\alpha_{\lceil r\rceil-1}(L)(\alpha_{\lceil r\rceil}(L))^{r-\lceil r\rceil+1} & \text{if } r \in (0,D], \\ |\det(L)|^{r/D} & \text{if } r > D, \end{cases} \tag{2.2}$$

where $\lceil r\rceil = \min\{k \in \mathbb{Z} : k \geqslant r\}$. It is through this function that for finite alphabets I the *affinity dimension* $d(L_i \mid i \in I)$ of a collection of matrices $\{L_i\}_{i\in I}$ was defined by Falconer in [Fal88] by setting

$$d(L_i \mid i \in I) = \inf\left\{r \in (0,D] : \sum_{m\in\mathbb{N}}\sum_{u\in I^m}\varphi^r(L_u) < \infty\right\},$$

where for $u = (u_1,\ldots,u_m) \in I^m$ with $m \in \mathbb{N}$ we set $L_u = L_{u_1}\cdots L_{u_{m-1}}L_{u_m}$.

The modified affinity dimension, introduced in [Fra12] by Fraser, for box-like self-affine sets is a variant of Falconer's affinity dimension that relies on knowledge of the dimensions of the projection of the given self-affine set F onto the coordinate axes. Moreover, it is defined only when the ambient space is $\mathbb{R}^2$. Before defining the modified affinity dimension, we introduce some further notation.

Let $\Phi = \{\phi_i : i \in I\}$ be a finite IFS containing affine maps $\phi_i : [0,1]^2 \to [0,1]^2$ defined, for $i \in I$ and $(w,x) \in \mathbb{R}^2$, by $\phi_i(w,x) = (L_i(w,x)^\top + v_i)^\top$ where $L_i \in M_2(\mathbb{R})$ is a diagonal matrix as in (1.1) and $v_i \in \mathbb{R}^2$ is a translation vector. Assume that Φ satisfies the ROSC and let $F \subseteq \mathbb{R}^2$ denote the limit set of Φ. Observe that, under our assumptions, the projections $\pi_1(F)$ and $\pi_2(F)$ are self-similar subsets of $\mathbb{R}$ generated by finite IFSs. Assuming that these systems satisfy the OSC, both $\dim_B(\pi_1(F))$ and $\dim_B(\pi_2(F))$ exist, see for instance [Hut81]. For $u \in I^+$ we define $\pi_u : \mathbb{R}^2 \to \mathbb{R}$ by

$$\pi_u = \begin{cases} \pi_1 & \text{if } \mathrm{diam}\left(\pi_1\left(\phi_u\left([0,1]^2\right)\right)\right) \geqslant \mathrm{diam}\left(\pi_2\left(\phi_u\left([0,1]^2\right)\right)\right), \\ \pi_2 & \text{if } \mathrm{diam}\left(\pi_1\left(\phi_u\left([0,1]^2\right)\right)\right) < \mathrm{diam}\left(\pi_2\left(\phi_u\left([0,1]^2\right)\right)\right), \end{cases} \tag{2.3}$$

and set $r(u) = \dim_B(\pi_u(F))$. For $r > 0$ and $u \in I^+$ the *modified singular value function* φ^r_{mod} of L_u is defined by

$$\varphi^r_{\mathrm{mod}}(L_u) = \alpha_1(L_u)^{r(u)}\alpha_2(L_u)^{r-r(u)}. \tag{2.4}$$

Note that these definitions are simplified slightly compared to the original definitions in [Fra12], as we will only consider affine contractions with diagonal linear parts. As a consequence each box-like set in the present article will be of *separated type*, meaning each contraction maps horizontal lines to horizontal lines.

In [Fra12] it was shown that for a finite IFS the *modified pressure function* $P_{\mathrm{mod}} \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ given by

$$P_{\mathrm{mod}}(r) = \lim_{n \to \infty} \left(\sum_{u \in I^n} \varphi_{\mathrm{mod}}^r(L_u) \right)^{1/n} \tag{2.5}$$

is well defined and strictly decreasing in r. Furthermore, it was shown that there exists a unique $t \in \mathbb{R}_{>0}$, which we will refer to as the *modified affinity dimension* of F, satisfying $P_{\mathrm{mod}}(t) = 1$, and that under the given assumptions we have $\dim_B(F) = \dim_P(F) = t$. More recently, Rapaport [Rap23] has shown that if one replaces the ROSC with the assumptions that (i) there exists $i \in I$ with $|a_i| \neq |b_i|$, and (ii) $\Psi_1 = \{\pi_1 \circ \phi_i : i \in I\}$ and $\Psi_2 = \{\pi_2 \circ \phi_i : i \in I\}$ are exponentially separated, then $\dim_{\mathcal{H}}(F) = \min\{d(L_i | i \in I), 2\}$. This generalises [Hoc14, theorem 1.5], in which it was shown that if $\Psi = \{\psi_i : i \in I\}$ is an IFS on $\mathbb{R}$ where for each $i \in I$, $\psi_i(x) = r_i x + c_i$ with r_i and c_i algebraic, then either Ψ has exact overlaps, or the limit set F of Ψ satisfies

$$\dim_{\mathcal{H}}(F) = \min\{t, 1\} \tag{2.6}$$

where t is the unique positive real solution satisfying $\sum_{i \in I} r_i^t = 1$.

3. Affinity dimensions for infinite affine IFSs with diagonal linear parts

3.1. Proof of theorems 1.1 and 1.2

In this section we prove theorems 1.1 and 1.2. Following this we show, in proposition 3.2, that in our setting the modified affinity dimension from [Fra12] can be simplified. We begin with the proof of theorem 1.1 where we utilise ideas from [Mor18].

Proof of theorem 1.1. Let I be a countable alphabet and $\{L_i : i \in I\}$ a collection of diagonal 2×2 matrices, as given in (1.1), with $\sup_{i \in I} \max\{|a_i|, |b_i|\} < 1$. Note that the singular value function $\varphi^r(L)$ of a matrix L, as defined in (2.2), is non-negative, strictly decreasing and continuous in r, so by the root test,

$$d(L_i \mid i \in I) = \inf \left\{ r > 0 : \limsup_{m \to \infty} \left(\sum_{u \in I^m} \varphi^r(L_u) \right)^{1/m} \leqslant 1 \right\}.$$

For a diagonal matrix $L = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and any $r > 0$ define the matrix

$$L^{(r)} = \begin{pmatrix} a^{(r)} & 0 \\ 0 & b^{(r)} \end{pmatrix} = \begin{cases} \begin{pmatrix} |a|^r & 0 \\ 0 & |b|^r \end{pmatrix} & \text{if } 0 < r \leqslant 1, \\[12pt] \begin{pmatrix} |a| \cdot |b|^{r-1} & 0 \\ 0 & |b| \cdot |a|^{r-1} \end{pmatrix} & \text{if } 1 < r \leqslant 2, \\[12pt] \begin{pmatrix} |ab|^{r/2} & 0 \\ 0 & |ab|^{r/2} \end{pmatrix} & \text{if } 2 < r. \end{cases}$$

Next, we show $\varphi^r(L) = \|L^{(r)}\|$ for each $r > 0$, where $\|\cdot\|$ denotes the operator norm on $M_2(\mathbb{R})$:

- For $0 < r \leqslant 1$ we have $\varphi^r(L) = \alpha_1(L)^r = \max\{|a|^r, |b|^r\} = \|L^{(r)}\|$.
- For $1 < r \leqslant 2$, since $\alpha_1(L) \geqslant \alpha_2(L)$ by definition, we have $\frac{\alpha_1(L)}{\alpha_2(L)} \geqslant 1$ and hence $(\frac{\alpha_1(L)}{\alpha_2(L)})^{r-1} \leqslant \frac{\alpha_1(L)}{\alpha_2(L)}$, or equivalently $\alpha_1(L) \cdot \alpha_2(L)^{r-1} \geqslant \alpha_2(L) \cdot \alpha_1(L)^{r-1}$. This implies that $\varphi^r(L) = \max\{|a| \cdot |b|^{r-1}, |b| \cdot |a|^{r-1}\} = \|L^{(r)}\|$.
- For $r > 2$ we have $\varphi^r(L) = |\det(L)|^{r/2} = |ab|^{r/2} = \|L^{(r)}\|$.

Observe that $(LK)^{(r)} = L^{(r)} K^{(r)}$ for diagonal matrices L and K. Therefore, if we return to our collection of matrices $\{L_i : i \in I\}$, we can unambiguously set $L_u^{(r)} = L_{i_1}^{(r)} \cdots L_{i_m}^{(r)} = (L_{i_1} \cdots L_{i_m})^{(r)}$ for each $u = (i_1, \ldots, i_m) \in I^m$. In particular, it follows that

$$\varphi^r(L_u) = \|L_u^{(r)}\| \quad \text{for any } u \in I^m \text{ and } r > 0. \tag{3.1}$$

Suppose that for some $r > 0$ the series $\sum_{i \in I} L_i^{(r)}$ does not converge in $(M_2(\mathbb{R}), \|\cdot\|)$. In other words, suppose that $P_I(r) = \max\{\sum_{i \in I} a_i^{(r)}, \sum_{i \in I} b_i^{(r)}\} = \infty$. In this case, (3.1) implies for $m \in \mathbb{N}$ that

$$\sum_{u \in I^m} \varphi^r(L_u) = \sum_{u \in I^m} \|L_u^{(r)}\| = \sum_{i_1, \ldots, i_m \in I} \max\left\{ a_{i_1}^{(r)} \cdots a_{i_m}^{(r)}, b_{i_1}^{(r)} \cdots b_{i_m}^{(r)} \right\}$$

$$\geqslant \max\left\{ \sum_{i_1, \ldots, i_m \in I} a_{i_1}^{(r)} \cdots a_{i_m}^{(r)}, \sum_{i_1, \ldots, i_m \in I} b_{i_1}^{(r)} \cdots b_{i_m}^{(r)} \right\}$$

$$= \max\left\{ \sum_{i \in I} a_i^{(r)}, \sum_{i \in I} b_i^{(r)} \right\}^m = \infty.$$

Hence, $\limsup_{m \to \infty} \left(\sum_{u \in I^m} \varphi^r(L_u) \right)^{1/m} = P_I(r)$. If on the other hand the series $\sum_{i \in I} L_i^{(r)}$ converges, then

$$\left\| \sum_{i \in I} L_i^{(r)} \right\| = \max\left\{ \sum_{i \in I} a_i^{(r)}, \sum_{i \in I} b_i^{(r)} \right\} = P_I(r) < \infty.$$

For $L \in M_2(\mathbb{R})$ let $|L|$ denote the sum of the absolute values of the components of L, and observe that $|\cdot|$ is a norm on $M_2(\mathbb{R})$. If L and K are non-negative diagonal matrices we have $|L + K| = |L| + |K|$ and so by the continuity of norms we have $|\sum_{i=1}^{\infty} K_i| = \sum_{i=1}^{\infty} |K_i|$ for any sequence $(K_i)_i$ of non-negative diagonal matrices. Note, for a diagonal matrix

$$L = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and $m \in \mathbb{N}$ we have $\|L\| = \max\{|a|, |b|\} = \max\{|a|^m, |b|^m\}^{1/m} = \|L^m\|^{1/m}$. Since $M_2(\mathbb{R})$ is a finite-dimensional vector space, the norms $\|\cdot\|$ and $|\cdot|$ are equivalent. Therefore, there exists $c > 0$ such that $c^{-1}|L| \leqslant \|L\| \leqslant c|L|$ for any $L \in M_2(\mathbb{R})$. Combining the above yields the following for $m \in \mathbb{N}$,

$$\sum_{u \in I^m} \varphi^r(L_u) = \sum_{u \in I^m} \left\| L_u^{(r)} \right\| \leqslant \sum_{u \in I^m} c \left| L_u^{(r)} \right| = c \left| \sum_{u \in I^m} L_u^{(r)} \right| \leqslant c^2 \left\| \sum_{u \in I^m} L_u^{(r)} \right\|$$

$$= c^2 \left\| \left(\sum_{i \in I} L_i^{(r)} \right)^m \right\| = c^2 \left\| \sum_{i \in I} L_i^{(r)} \right\|^m. \tag{3.2}$$

Likewise

$$\sum_{u \in I^m} \varphi^r(L_u) \geqslant c^{-2} \left\| \sum_{i \in I} L_i^{(r)} \right\|^m. \tag{3.3}$$

Equations (3.2) and (3.3) together imply

$$\limsup_{m \to \infty} \left(\sum_{u \in I^m} \varphi^r(L_u) \right)^{1/m} = \left\| \sum_{i \in I} L_i^{(r)} \right\| = P_I(r).$$

$\square$

We now prove theorem 1.2. For (i), that is, to show that the affinity dimension is always an upper bound for the Hausdorff dimension, we follow steps similar to those in the proof for [Fal88, proposition 5.1].

Proof of theorem 1.2. For Part (i), since $F \subseteq \mathbb{R}^2$, we naturally have $\dim_{\mathcal{H}}(F) \leqslant 2$. Therefore, it is sufficient to show that $\dim_{\mathcal{H}}(F) \leqslant d(L_i \mid i \in I)$. To this end, let $\delta > 0$ be given. By the assumption that $\sup_{i \in I} \max\{|a_i|, |b_i|\} < 1$ there exists some integer k_δ such that for all sequences $u \in I^{k_\delta}$ we have $\alpha_1(L_u), \alpha_2(L_u) < \delta$. Now take any $m \geqslant k_\delta$ and recall that $F \subseteq \bigcup_{u \in I^m} A_u([0,1]^2)$. For each $u \in I^m$, $A_u([0,1]^2)$ is a rectangle with side lengths $\alpha_1(L_u)$ and $\alpha_2(L_u)$. This rectangle can be covered by $\lceil \alpha_1(L_u)/\alpha_2(L_u) \rceil$ squares of side length $\alpha_2(L_u)$, and hence also by this many circles of diameter $\sqrt{2}\alpha_2(L_u)$. Note that, since $\alpha_1(L_u)/\alpha_2(L_u) \geqslant 1$, we have $\lceil \alpha_1(L_u)/\alpha_2(L_u) \rceil \leqslant 2\alpha_1(L_u)/\alpha_2(L_u)$. For every $0 < r \leqslant 2$ we have

$$\mathcal{H}^r_{\sqrt{2}\delta}(F) \leqslant \sum_{u \in I^m} 2 \frac{\alpha_1(L_u)}{\alpha_2(L_u)} \left(\sqrt{2}\alpha_2(L_u) \right)^r = 2 \left(\sqrt{2} \right)^r \sum_{u \in I^m} \alpha_1(L_u) \alpha_2(L_u)^{r-1} \leqslant 4 \sum_{u \in I^m} \varphi^r(L_u).$$

Since this holds for all $m \geqslant k_\delta$, and since k_δ diverges to infinity as δ tends to zero,

$$0 \leqslant \mathcal{H}^r(F) \leqslant 4 \limsup_{m \to \infty} \sum_{u \in I^m} \varphi^r(L_u),$$

for each $0 < r \leqslant 2$. Now for any r satisfying $\sum_{m=1}^{\infty} \sum_{u \in I^m} \varphi^r(L_u) < \infty$ we have $\limsup_{m \to \infty} \sum_{u \in I^m} \varphi^r(L_u) < \infty$ and so $\mathcal{H}^r(F) < \infty$. Thus, $\dim_{\mathcal{H}}(F) = \inf\{r \geqslant 0 : \mathcal{H}^r(F) < \infty\} \leqslant \inf\{r > 0 : \sum_{m=1}^{\infty} \sum_{u \in I^m} \varphi^r(L_u) < \infty\} = d(L_i \mid i \in I)$. This concludes the proof of theorem 1.2 (i).

For Part (ii) observe that $P_I(r) = \sup_{I' \subseteq I \text{ finite}} P_{I'}(r)$ for all $r > 0$ with $P_I(r)$ as defined in (1.2). By assumption, for a finite subset I_2 with $I_1 \subseteq I_2 \subseteq I$, we have $d(L_i \mid i \in I_2) = \dim_B(F_{I_2}) \leqslant 2$. By theorem 1.1, $P_{I_2}(r + \varepsilon) \leqslant 1$ for all $\varepsilon > 0$ and all $r \geqslant 2$, which implies $d(L_i \mid i \in I) = \inf\{r > 0 : P_I(r) \leqslant 1\} \leqslant 2$.

Next, we show that

$$d(L_i \mid i \in I) = \sup_{I' \subseteq I \text{ finite}} d(L_i \mid i \in I'), \tag{3.4}$$

from which we will conclude the required result. To this end, observe that each of the series in the definition of P_I has positive terms, and thus, $P_{I_1}(r) \leqslant P_{I_2}(r)$ for $r > 0$ and $I_1 \subseteq I_2 \subseteq I$. Therefore,

$$d(L_i \mid i \in I) = \inf\left\{r > 0 : \sup_{I' \subseteq I \text{ finite}} P_{I'}(r) \leqslant 1\right\} = \inf\{r > 0 : P_{I'}(r) \leqslant 1 \text{ for all finite } I' \subseteq I\}.$$

Write $Z = \sup_{I' \subseteq I \text{ finite}} d(L_i \mid i \in I') = \sup_{I' \subseteq I \text{ finite}} \inf\{r > 0 : P_{I'}(r) \leqslant 1\}$. For each finite sub-set $I' \subseteq I$ we have $Z \geqslant \inf\{r > 0 : P_{I'}(r) \leqslant 1\}$, so since $P_{I'}(r)$ is strictly decreasing in r we also have $P_{I'}(Z) \leqslant 1$. As this holds for all finite $I' \subseteq I$, it follows that $d(L_i \mid i \in I) \leqslant Z$. Further, for each $r > d(L_i \mid i \in I)$ we have $P_{I'}(r) \leqslant 1$ for each finite $I' \subseteq I$ and hence $r \geqslant Z$, from which we conclude that $d(L_i \mid i \in I) = Z$.

For each $I_1 \subseteq I$ we have $F_{I_1} \subseteq F$, where F_{I_1} is as in our hypotheses of theorem 1.2(ii). By the monotonicity of both the lower box-counting and affinity dimensions, and by (3.4), we have that

$$\underline{\dim}_B(F) \geqslant \sup_{I_2 \subseteq I \text{ finite}} \underline{\dim}_B(F_{I_2}) = \sup_{I_1 \subseteq I_2 \subseteq I \text{ finite}} \dim_B(F_{I_2}) = \sup_{I_1 \subseteq I_2 \subseteq I \text{ finite}} d(L_i \mid i \in I_2)$$

$$= \sup_{I_2 \subseteq I \text{ finite}} d(L_i \mid i \in I_2) = d(L_i \mid i \in I).$$

Since the affine maps we consider are bi-Lipschitz, it follows from [MU96, theorem 3.1] that

$$\dim_P(F) = \overline{\dim}_B(F) = \dim_P(\overline{F}) = \overline{\dim}_B(\overline{F}).$$

Thus, under our assumptions, $d(L_i \mid i \in I) \leqslant \underline{\dim}_B(F) \leqslant \overline{\dim}_B(F) = \dim_P(F)$. $\square$

Remark 3.1. In [Fal88, theorem 5.4] and [Fra12, theorem 2.4] it is shown that the affinity and modified affinity dimensions of a finite affine IFS is an upper bound for the upper box-counting dimension of the associated self-affine set. This result relies on the fact that the singular values of the affine maps in a finite IFS are uniformly bounded from below by a positive constant. Such a lower bound on the singular values does not exist in general for infinite IFSs. Thus, the proofs of the aforementioned theorems do not naturally generalise to the case of infinite affine IFSs.

In [Fra12] it was shown that for a class of finitely generated planar box-like self-affine sets the box-counting and packing dimensions are bounded above by (and when the ROSC is satisfied equal to) the modified affinity dimension, which is the unique $t \in \mathbb{R}_{>0}$ solving $P_{\text{mod}}(t) = 1$, see (2.5). In [Mor18, proposition 5], for a similar class of finitely generated planar box-like self-affine sets F, and under the assumption that $\dim_B \pi_1(F) = \dim_B \pi_2(F)$, a simple expression for P_{mod} was obtained. Next, we show that in the case where we have only diagonal matrices the same simplification of P_{mod} can be obtained no matter the values of $\dim_B \pi_1(F)$ and $\dim_B \pi_2(F)$.

Proposition 3.2. *Let I be a finite alphabet and let F be the limit set of an IFS $\{A_i : i \in I\}$ where each A_i is an affine contraction on $[0,1]^2$ with linear part $L_i \in M_2(\mathbb{R})$ as given in (1.1). Set $r_1 = \dim_B(\pi_1(F))$ and $r_2 = \dim_B(\pi_2(F))$ and assume $\{A_i : i \in I\}$ satisfies the ROSC. Under these assumptions, $\dim_B(F) = \dim_P(F) = t$, where $t \in \mathbb{R}_{>0}$ is the unique solution to*

$$\max\left\{\sum_{i \in I} |a_i|^{r_1} |b_i|^{t-r_1}, \sum_{i \in I} |b_i|^{r_2} |a_i|^{t-r_2}\right\} = 1.$$

Proof. Let $u = (i_1, \ldots, i_m) \in I^m$ for some $m \in \mathbb{N}$. The singular values of L_u are $a(u) = \prod_{k=1}^{m} |a_{i_k}|$ and $b(u) = \prod_{k=1}^{m} |b_{i_k}|$. Recall that $r(u) = \dim_B(\pi_u(F))$, where π_u is as in (2.3). Since for $j \in \{1, 2\}$, $r(u)$ equals r_j when $\alpha_1(L_u)$ corresponds to the contraction in the jth coordinate, we obtain for each $r > 0$ that, with φ^r_{mod} as in (2.4),

$$\varphi^r_{\mathrm{mod}}(L_u) = \begin{cases} a(u)^{r_1} b(u)^{r-r_1} & \text{if } a(u) \geqslant b(u), \\ b(u)^{r_2} a(u)^{r-r_2} & \text{if } a(u) < b(u), \end{cases} \quad \text{and set}$$

$$\mathfrak{L}_u^{(r)} = \begin{pmatrix} |a(u)|^{r_1} |b(u)|^{r-r_1} & 0 \\ 0 & |b(u)|^{r_2} |a(u)|^{r-r_2} \end{pmatrix}.$$

Since $P_{\mathrm{mod}}(\dim_B F) = 1$ by [Fra12] and $\dim_B F \leqslant r_1 + r_2$, we need only consider the case $r \in (0, r_1 + r_2]$. For $r > 0$,

$$\frac{a(u)^{r_1} b(u)^{r-r_1}}{b(u)^{r_2} a(u)^{r-r_2}} = \left(\frac{a(u)}{b(u)} \right)^{r_1+r_2-r},$$

and so for $r \in (0, r_1 + r_2]$ we have $\frac{a(u)^{r_1} b(u)^{r-r_1}}{b(u)^{r_2} a(u)^{r-r_2}} \geqslant 1$ when $a(u) \geqslant b(u)$, and $\frac{a(u)^{r_1} b(u)^{r-r_1}}{b(u)^{r_2} a(u)^{r-r_2}} < 1$ otherwise. Thus,

$$\varphi^r_{\mathrm{mod}}(L_u) = \max \left\{ a(u)^{r_1} b(u)^{r-r_1}, \; b(u)^{r_2} a(u)^{r-r_2} \right\} = \|\mathfrak{L}_u^{(r)}\|,$$

for $r \in (0, r_1 + r_2]$. Following the same steps as in the proof of theorem 1.1,

$$\max \left\{ \sum_{i \in I} |a_i|^{r_1} |b_i|^{r-r_1}, \; \sum_{i \in I} |b_i|^{r_2} |a_i|^{r-r_2} \right\} = \left| \sum_{i \in I} \mathfrak{L}_i^{(r)} \right| = \limsup_{m \to \infty} \left(\sum_{u \in I^m} \varphi^r_{\mathrm{mod}}(L_u) \right)^{1/m}$$

$$= P_{\mathrm{mod}}(r),$$

for each $r \in (0, r_1 + r_2]$. The required result now follows from an application of [Fra12, theorem 2.4]. $\qquad\square$

3.2. Numerical estimates for the affinity dimension

As noted in remark 1.3 and shown in (3.4), when taking a sequence of nested finite subsets $I_1 \subset I_2 \subset \ldots \subset I$ which satisfy $\bigcup_{n \in \mathbb{N}} I_n = I$, then $(d(L_i \mid i \in I_n))_{n \in \mathbb{N}}$ is a non-decreasing sequence satisfying $\sup_{n \in \mathbb{N}} d(L_i \mid i \in I_n) = d(L_i \mid i \in I)$. Recall that theorem 1.1 gives the equality $d(L_i \mid i \in I) = \inf \{ r > 0 : P_I(r) \leqslant 1 \}$ under the mild condition $\sup_{i \in I} \max \{|a_i|, |b_i|\} < 1$. Thus, to obtain the speed of convergence of $d(L_i \mid i \in I_n)$ to $d(L_i \mid i \in I)$ one needs to compute $\inf \{ r > 0 : P_I(r) \leqslant 1 \} - \inf \{ r > 0 : P_{I_n}(r) \leqslant 1 \}$. Depending on the data, effective approximations of this difference are possible.

Example 3.3. Let $\gamma \in (0, 1)$ be fixed and suppose we have a collection $\{L_i : i \in I\}$ of diagonal 2×2 matrices, as given in (1.1), with $I = \mathbb{N}$ and $a_i = \gamma^i$ and $b_i < a_i$, for all $i \in \mathbb{N}$, so that $\sup_{i \in I} \max \{|a_i|, |b_i|\} < 1$. In this case, for $r \in (0, 1)$, the pressure $P_I(r)$ is the geometric series with ratio γ^r. Moreover, if $I_n = \{1, 2, \ldots, n\}$, then $P_{I_n}(r)$ is the n-th partial sum of the geometric series with ratio γ^r, for $r \in (0, 1)$ and $n \in \mathbb{N}$. With this observation at hand, an elementary calculation reveals

$$0 \leqslant \inf \{ r > 0 : P_I(r) \leqslant 1 \} - \inf \{ r > 0 : P_{I_n}(r) \leqslant 1 \} \leqslant -\log(2/\beta_n)/\log(\gamma), \tag{3.5}$$

where β_n denotes the *n*-th multinacci number, namely the unique real solution in the interval $(1,2)$ of $x^{-n} + x^{-n+1} + \cdots + x^{-1} = 1$. Indeed, if $\gamma = 1/\beta_2 = (-1 + \sqrt{5})/2$ and $n = 20$, then the error given in (3.5) is less than 10^{-6}.

One can also obtain bounds in the following more general settings.

Proposition 3.4. *Suppose we have a collection $\{L_i : i \in I\}$ of diagonal 2×2 matrices, as given in (1.1), with $\sup_{i \in I} \max\{|a_i|, |b_i|\} < 1$ and with I countably infinite. Further, suppose that the associated pressure function P_I is continuous at $r = \inf\{r > 0 : P_I(r) \leqslant 1\}$. For a sequence of nested finite subsets $I_1 \subset I_2 \subset \ldots \subset I$ satisfying $\bigcup_{n \in \mathbb{N}} I_n = I$, let $r_n = \inf\{r > 0 : P_{I_n}(r) \leqslant 1\}$.*

(i) *If $r \leqslant 1$, then for all $n \in \mathbb{N}$*

$$0 \leqslant r - r_n \leqslant \max\left\{\frac{\log\left(1 - \sum_{i \in I \setminus I_n} |a_i|^r\right)}{\log\left(\max_{i \in I_n} |a_i|\right)}, \frac{\log\left(1 - \sum_{i \in I \setminus I_n} |b_i|^r\right)}{\log\left(\max_{i \in I_n} |b_i|\right)}\right\}. \tag{3.6}$$

If one can determine $N \in \mathbb{N}$ such that $\max\{\sum_{i \in I \setminus I_N} |a_i|^{r_N}, \sum_{i \in I \setminus I_N} |b_i|^{r_N}\} < 1$, then r can be replaced by r_n in the right hand side of (3.6) leading to an error bound which holds for all $n \geqslant N$.

(ii) *If $r \in (1,2]$, then for all $n \in \mathbb{N}$ with $r_n \geqslant 1$*

$$0 \leqslant r - r_n \leqslant \max\left\{\frac{\log\left(1 - \sum_{i \in I \setminus I_n} |a_i|^{r-1}|b_i|\right)}{\log\left(\max_{i \in I_n} |a_i|\right)}, \frac{\log\left(1 - \sum_{i \in I \setminus I_n} |a_i||b_i|^{r-1}\right)}{\log\left(\max_{i \in I_n} |b_i|\right)}\right\}. \tag{3.7}$$

If one can determine $N \in \mathbb{N}$ such that $\max\{\sum_{i \in I \setminus I_N} |a_i|^{r_N-1}|b_i|, \sum_{i \in I \setminus I_N} |b_i|^{r_N-1}|a_i|\} < 1$, then r can be replaced by r_n in the right hand side of (3.7) leading to an error bound which holds for all $n \geqslant N$ with $r_n \geqslant 1$.

Proof. Suppose we are in the setting of (i). For all $n \in \mathbb{N}$,

$$\sum_{i \in I} |a_i|^r = \sum_{i \in I_n} |a_i|^{r_n} |a_i|^{r-r_n} + \sum_{i \in I \setminus I_n} |a_i|^r \leqslant A_n^{r-r_n} \cdot P_{I_n}(r_n) + \sum_{i \in I \setminus I_n} |a_i|^r = A_n^{r-r_n} + \sum_{i \in I \setminus I_n} |a_i|^r,$$

where $A_n = \max_{i \in I_n} |a_i|$. Similarly, if $B_n = \max_{i \in I_n} |b_i|$, then $\sum_{i \in I} |b_i|^r \leqslant B_n^{r-r_n} + \sum_{i \in I \setminus I_n} |b_i|^r$. Thus, using the assumption that P_I is continuous at r, it follows that

$$1 = P_I(r) = \max\left\{\sum_{i \in I} |a_i|^r, \sum_{i \in I} |b_i|^r\right\} \leqslant \max\left\{A_n^{r-r_n} + \sum_{i \in I \setminus I_n} |a_i|^r, B_n^{r-r_n} + \sum_{i \in I \setminus I_n} |b_i|^r\right\}, \tag{3.8}$$

from which we deduce (3.6). That r can be replaced by r_n in (3.6) if $\max\{\sum_{i \in I \setminus I_N} |a_i|^{r_N}, \sum_{i \in I \setminus I_N} |b_i|^{r_N}\} < 1$ follows from combining $\sum_{i \in I \setminus I_n} |a_i|^r \leqslant \sum_{i \in I \setminus I_n} |a_i|^{r_n}$ and $\sum_{i \in I \setminus I_n} |b_i|^r \leqslant \sum_{i \in I \setminus I_n} |b_i|^{r_n}$ with (3.8). Analogously, one can obtain the desired bounds in setting (ii). $\square$

Remark 3.5. One can obtain a bound for $r - r_n$ in a similar manner to proposition 3.4 when $r_n \leqslant 1 < r \leqslant 2$. However, this bound will include an extra term related to the behaviour of $|a_i/b_i|$ and $|b_i/a_i|$, which can become unruly depending on the values of a_i and b_i. Specifically, the bound will take the form

$$\max \left\{ \frac{\log\left(\left(1 - \sum_{i \in I \setminus I_n} |a_i|^{r-1} |b_i|\right) \min_{i \in I_n} \{|a_i/b_i|\} \right)}{\log\left(\max_{i \in I_n} |a_i|\right)}, \right.$$
$$\left. \frac{\log\left(\left(1 - \sum_{i \in I \setminus I_n} |a_i| |b_i|^{r-1}\right) \min_{i \in I_n} \{|b_i/a_i|\} \right)}{\log\left(\max_{i \in I_n} |b_i|\right)} \right\}.$$

Example 3.6. Let us consider the IFS $\{A_{s,d}^p : (s,d) \in I\}$, where $A_{s,d}^p$ is as defined in (1.5) and $I = \{0,1\} \times \{t^2 : t \in \mathbb{N}_{\geqslant 2}\}$. For $n \in \mathbb{N}_{\geqslant 2}$, let $I_n = \{0,1\} \times \{2^2, 3^2, \ldots, n^2\}$. In this case, using the bounds obtained in proposition 3.4(ii), and setting $p = 1/2$, we obtain:

$$r_n = 1.439\,08, \quad 0 \leqslant r - r_n \leqslant 0.001\,699\,17 \quad \text{and thus} \quad r \in [1.439\,08, 1.440\,78] \quad \text{for } n = 2\,000,$$
$$r_n = 1.439\,59, \quad 0 \leqslant r - r_n \leqslant 0.000\,290\,52 \quad \text{and thus} \quad r \in [1.439\,59, 1.439\,88] \quad \text{for } n = 20\,000.$$

Example 3.7. Let $k \in \mathbb{N}$ and let us consider the IFS $\{A_{s,d}^p : (s,d) \in I\}$ where $A_{s,d}^p$ is as defined in (1.5) and $I = \{0,1\} \times \{kt : t \in \mathbb{N}_{\geqslant 2}\}$. For $n \in \mathbb{N}_{\geqslant 2}$, let $I_n = \{0,1\} \times \{2k, 3k, \ldots, nk\}$. In this case, using proposition 3.4(ii), and setting $p = 1/2$, we obtain the following numerical values for r_n and upper bounds for $r - r_n$.

	$k = 2$		$k = 3$		$k = 4$		$k = 5$	
	r_n	$r - r_n$	r_n	$r - r_n$	r_n	$r - r_n$	r_n	$r - r_n$
$n = 2000$	1.66708	0.039 5558	1.595 78	0.117 1740	1.558 45	0.342 2140	1.534 09	—
$n = 4000$	1.669 95	0.028 9186	1.600 33	0.085 4776	1.564 03	0.206 8390	1.540 38	—
$n = 6000$	1.671 26	0.024 2520	1.602 51	0.072 3308	1.566 76	0.165 8090	1.543 49	0.618 041
$n = 8000$	1.672 07	0.021 4616	1.603 89	0.064 6306	1.568 50	0.144 2860	1.545 49	0.405 103
$n = 10000$	1.672 63	0.019 5459	1.604 86	0.059 3979	1.569 75	0.130 5270	1.546 93	0.328 481
$n = 12000$	1.673 05	0.018 1224	1.605 61	0.055 5312	1.570 71	0.120 7590	1.548 04	0.285 562
$n = 14000$	1.673 38	0.017 0087	1.606 20	0.052 5152	1.571 48	0.113 3560	1.548 94	0.257 252
$n = 16000$	1.673 65	0.016 1050	1.606 69	0.050 0723	1.572 12	0.107 4900	1.549 68	0.236 825
$n = 18000$	1.673 88	0.015 3518	1.607 11	0.048 0378	1.572 67	0.102 6890	1.550 32	0.221 211
$n = 20000$	1.674 07	0.014 7108	1.607 47	0.046 3068	1.573 14	0.098 6614	1.550 88	0.208 785

As examples 3.3, 3.6 and 3.7 demonstrate the accuracy to which we can estimate the difference $r - r_n$ depends on the given IFS, and more so on the size and the decay rate of the singular values of the linear parts of the affine transformations of the given IFS.

4. Box-counting and Hausdorff dimensions of F_J

Here we collect and develop results which allow us to compute the box-counting and Hausdorff dimensions of the sets F_J from (1.6). These results are utilised in our proofs of theorems 1.4 and 1.5. Recall for $(s,d) \in \{0,1\} \times \mathbb{N}_{\geqslant 2}$ the definition of the maps $A_{s,d}^p : [0,1]^2 \to [0,1]^2$ from (1.5). We will be interested in their second coordinates, which are the maps

$\phi_{s,d} : [0,1] \to [1/d, 1/(d-1)]$ given by

$$\phi_{s,d}(x) = \frac{(-1)^s x}{d(d-1)} + \frac{1}{d-s}.$$

(Note that the maps $\phi_{0,d}$ correspond to the maps h_d from the introduction.) For each $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 2}$, the IFS $\Phi = \{\phi_{s,d} : (s,d) \in J\}$ on $[0,1]$ consists of similarities and its limit set is precisely the set F_J. It is a consequence of [KM22] that each number in $(\frac{1}{2}, 1]$ has at least one signed Lüroth expansion for which the corresponding digit pairs $(s_n, d_n) \in \{0,1\} \times \mathbb{N}_{\geqslant 2}$ satisfy $d_n = 2$ for all $n \in \mathbb{N}$. Consequently, if $(0,2), (1,2) \in J$, then the restricted digit set corresponding to J contains $(\frac{1}{2}, 1]$, yielding

$$\dim_{\mathcal{H}}(F_J) = \dim_B(F_J) = 1. \tag{4.1}$$

In all other cases we split our analysis of F_J into the cases where J is finite and where J is countably infinite.

4.1. Restricted digit sets with finite alphabets for signed Lüroth expansions

Throughout this section we assume that J is a finite subset of $\{0,1\} \times \mathbb{N}_{\geqslant 2}$ and that $\{(0,2), (1,2)\} \not\subseteq J$. Since the set F_J is self-similar it follows from [Hut81, theorem 5.3(1)] that, in case the IFS $\Phi = \{\phi_{s,d} : (s,d) \in J\}$ satisfies the OSC, the Hausdorff and box-counting dimensions of F_J equal the unique $r \in \mathbb{R}_{>0}$ satisfying

$$\sum_{(s,d) \in J} \left(\frac{1}{d(d-1)} \right)^r = 1. \tag{4.2}$$

More generally, if one replaces the OSC by Φ having no exact overlaps, then by [Hoc14, theorem 1.5] the Hausdorff and box-counting dimensions of F_J equal $\min\{r, 1\}$, see also (2.6).

   When $(0,2), (1,2) \notin J$ one can show that the OSC is satisfied with feasible open set $(0, \frac{1}{2})$ and, moreover, that we do not have exact overlaps. Below we discuss examples of sets J containing just one of $(0,2)$ or $(1,2)$ where the OSC is satisfied, example 4.1, and where the OSC is not satisfied, example 4.2.

Example 4.1. Let $d \in \mathbb{N}_{\geqslant 3}$ and consider the set $J = \{(0,2), (0,d), (1,d)\}$. The IFS $\{\phi_{s,d} : (s,d) \in J\}$ satisfies the OSC with feasible open set $U = \bigcup_{k=0}^{\infty} \phi_{0,2}^k ((\frac{1}{d}, \frac{1}{d-1}))$, but neither $(0,1)$ nor $(0, 1/2)$ are feasible open sets. Here $\phi_{0,2}^0$ is defined to be the identity. To see that U is a feasible open set for the OSC, we observe, by construction, that $\phi_{0,2}(U) \subseteq U$, and that $\phi_{s,d}(U) \subseteq \phi_{s,d}((0,1)) = (\frac{1}{d}, \frac{1}{d-1}) \subseteq U$ for $s \in \{0,1\}$. It remains to show that $\phi_{0,2}(U)$, $\phi_{0,d}(U)$ and $\phi_{1,d}(U)$ are pairwise disjoint. Since $\phi_{0,2}(U) \subseteq (\frac{1}{2}, 1)$, and since $\phi_{0,d}(U)$ and $\phi_{1,d}(U)$ are subsets of $(\frac{1}{d}, \frac{1}{d-1}) \subseteq (0, \frac{1}{2})$, it suffices to verify that $\phi_{0,d}(U) \cap \phi_{1,d}(U) = \emptyset$. To this end, we define $U_1 = U \cap [0, \frac{1}{2}] = (\frac{1}{d}, \frac{1}{d-1})$ and $U_2 = U \cap [\frac{1}{2}, 1] = \bigcup_{k=1}^{\infty} \phi_{0,2}^k ((\frac{1}{d}, \frac{1}{d-1}))$. Observe that the injective maps $\phi_{0,d}$ and $\phi_{1,d}$ satisfy $\phi_{0,d}(x) = \phi_{1,d}(y)$ if and only if $y = 1 - x$, in which case exactly one of x and y is an element of U_1 and exactly one of x and y is an element of U_2. As such, $\phi_{0,d}(U) \cap \phi_{1,d}(U) = \emptyset$ if and only if $U_2 \cap (1 - U_1) = \emptyset$, where by $1 - U_1$ we mean the open interval $(1 - \frac{1}{d-1}, 1 - \frac{1}{d})$. Recalling that $\phi_{0,2}(x) = \frac{1}{2}x + \frac{1}{2}$ for $x \in [0,1]$, we have $\phi_{0,2}^k(x) = \frac{1}{2^k}x + \sum_{j=1}^k \frac{1}{2^j}$ for $k \in \mathbb{N}$. Thus, for $k \in \mathbb{N}$, we have that $\phi_{0,2}^k(\frac{1}{d-1}) < \phi_{0,2}^{k+1}(\frac{1}{d})$ if and only if $d^2 - 2d - 1 > 0$, but this latter inequality holds since $d \geqslant 3$. This in tandem with the fact that $\phi_{0,2}$ is strictly increasing implies that, for $k, l \in \mathbb{N}$ with $k > l$, the open intervals

$\phi_{0,2}^k((\frac{1}{d}, \frac{1}{d-1}))$ and $\phi_{0,2}^l((\frac{1}{d}, \frac{1}{d-1}))$ are disjoint, and if $x \in \phi_{0,2}^k((\frac{1}{d}, \frac{1}{d-1}))$ and $y \in \phi_{0,2}^l((\frac{1}{d}, \frac{1}{d-1}))$, then $x > y$. It remains to show that $1 - U_1$ lies strictly in between two such consecutive intervals.

For this, set $k = \lceil \log_2(d-1) \rceil - 1 \in \mathbb{N}$, and note that for this k we have $\frac{1}{2^{k+1}} \leqslant \frac{1}{2^{k+1}} \frac{2^{k+1}+1}{d} = (1 + \frac{1}{2^{k+1}})\frac{1}{d}$, and hence $1 - \frac{1}{d} \leqslant \frac{1}{2^{k+1}d} + 1 - \frac{1}{2^{k+1}}$. Since $d-1$ is an integer, we have $k = \lceil \log_2(d - 2 + 1) \rceil - 1 = \lfloor \log_2(d-2) \rfloor$ and so we also have $2^k \leqslant d - 2$. Equivalently, we have $\frac{1}{d-1} \leqslant \frac{1}{2^k+1} = \frac{2^k}{2^k(2^k+1)} = \frac{1}{2^k}(\frac{1}{2^k} + 1)^{-1}$, or $(\frac{1}{2^k} + 1)\frac{1}{d-1} \leqslant \frac{1}{2^k}$. This yields

$$\phi_{0,2}^k\left(\frac{1}{d-1}\right) = \frac{1}{2^k(d-1)} + 1 - \frac{1}{2^k} \leqslant 1 - \frac{1}{d-1} < 1 - \frac{1}{d} \leqslant \frac{1}{2^{k+1}d} + 1 - \frac{1}{2^{k+1}} = \phi_{0,2}^{k+1}\left(\frac{1}{d}\right),$$

from which we conclude that $1 - U_1$ lies strictly between $\phi_{0,2}^k((\frac{1}{d}, \frac{1}{d-1}))$ and $\phi_{0,2}^{k+1}((\frac{1}{d}, \frac{1}{d-1}))$ and hence $(1 - U_1) \cap U_2 = \emptyset$.

Example 4.2. By [Hut81, theorem 5.3(1)], if J is such that $\sum_{(s,d) \in J} \frac{1}{d(d-1)} > 1$, then since $F_J \subseteq [0,1]$, the OSC is not satisfied. This is the case, for instance, when J contains as a strict subset either $\{(0,2), (0,3), (1,3), (0,4), (1,4)\}$ or $\{(1,2), (0,3), (1,3), (0,4), (1,4)\}$.

We conclude this section by considering the non-autonomous setting in the case both digits $(0,2)$ and $(1,2)$ are omitted entirely. For $\mathbb{J} = (J_k)_{k \in \mathbb{N}}$ a sequence of finite subsets $J_k \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 3}$, let $F_{\mathbb{J}}$ denote the limit set of the NSIFS $(\{\phi_{s,d} : (s,d) \in J_k\})_{k \in \mathbb{N}}$ acting on $[0, \frac{1}{2}]$. The set $F_{\mathbb{J}}$ coincides with a generalised type of restricted digit set

$$F_{\mathbb{J}} = \{x \in [0,1] : x \text{ has a signed Lüroth expansion with digits } (s_k, d_k) \text{ in } J_k \text{ for each } k \in \mathbb{N}\}.$$

Such sets are of particular interest in relation to various questions on the growth rate of the digits d_k, as studied for Lüroth expansions in for instance [JR12, CWW13, AGR21]. We obtain the following result.

Proposition 4.3. *If the sequence $\mathbb{J} = (J_k)_{k \in \mathbb{N}}$ with $J_k \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 3}$ is of sub-exponential growth, that is, each set J_k is finite and $\lim_{k \to \infty} \frac{1}{k} \log \#J_k = 0$, then*

$$\dim_{\mathcal{H}}(F_{\mathbb{J}}) = \inf \left\{ r \in (0,1] : \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \log \left(\sum_{(s,d) \in J_k} \left(\frac{1}{d(d-1)} \right)^r \right) < 0 \right\}.$$

Proof. Since each $\phi_{s,d}$ is a similarity, the IFS $\{\phi_{s,d} : (s,d) \in J_k\}$ is conformal. Moreover, $|\phi_{s,d}'(x)| = \frac{1}{d(d-1)} \leqslant \frac{1}{6} < 1$ for each $x \in [0, \frac{1}{2}]$ and hence $(\{\phi_{s,d} : (s,d) \in J_k\})_{k \in \mathbb{N}}$ is uniformly contracting. Therefore, by [RGU16, theorem 1.1], the Hausdorff dimension of $F_{\mathbb{J}}$ equals $\inf\{r > 0 : \underline{P}(r) < 0\}$. Here $\underline{P}$ is the *lower pressure function* defined by

$$\underline{P}(r) = \liminf_{m \to \infty} \frac{1}{m} \log \sum_{(s_1,d_1) \in J_1, \ldots, (s_m,d_m) \in J_m} \|(\phi_{s_1,d_1} \circ \cdots \circ \phi_{s_m,d_m})'\|_\infty^r,$$

and $\|\cdot\|_\infty$ denotes the supremum norm. For $(s_1,d_1) \in J_1, \ldots, (s_m,d_m) \in J_m$ it holds that

$$\|(\phi_{s_1,d_1} \circ \cdots \circ \phi_{s_m,d_m})'\|_\infty = \prod_{k=1}^m \frac{1}{d_k(d_k - 1)}$$

and so the lower pressure function becomes

$$
\begin{aligned}
\underline{P}(r) &= \liminf_{m\to\infty}\frac{1}{m}\log\sum_{(s_1,d_1)\in J_1,\dots,(s_m,d_m)\in J_m}\prod_{k=1}^{m}\left(\frac{1}{d_k\,(d_k-1)}\right)^r\\
&= \liminf_{m\to\infty}\frac{1}{m}\log\left(\prod_{k=1}^{m}\sum_{(s,d)\in J_k}\left(\frac{1}{d\,(d-1)}\right)^r\right)\\
&= \liminf_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m}\log\left(\sum_{(s,d)\in J_k}\left(\frac{1}{d\,(d-1)}\right)^r\right).
\end{aligned}
$$

$\square$

For each subset $J\subseteq\{0,1\}\times\mathbb{N}_{\geqslant 2}$ and each sequence $\mathbf{s}=(s_k)_{k\in\mathbb{N}}\in\{0,1\}^{\mathbb{N}}$ we define the subset $F_{J,\mathbf{s}}$ of F_J containing the numbers $x\in[0,1]$ for which there exists a sequence $(d_k)_{k\in\mathbb{N}}\in\mathbb{N}_{\geqslant 2}^{\mathbb{N}}$ such that $((s_k,d_k))_{k\in\mathbb{N}}$ lies in $J^{\mathbb{N}}$ and gives a signed Lüroth expansion for x.

Proposition 4.4. *Let I_0 and I_1 be finite subsets of $\mathbb{N}_{\geqslant 2}$, let $p\in(0,1)$ and let μ_p denote the p-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ with $\mu_p(\{\mathbf{s}=(s_1,s_2,\dots)\in\{0,1\}^{\mathbb{N}}:s_1=0\})=p$. If $J=(\{0\}\times I_0)\cup(\{1\}\times I_1)$, then for μ_p-almost every sequence $\mathbf{s}\in\{0,1\}^{\mathbb{N}}$ it holds that $\dim_{\mathcal{H}}(F_{J,\mathbf{s}})=t$, where $t\in\mathbb{R}_{>0}$ is the unique solution to*

$$
\left(\sum_{d_0\in I_0}\left(\frac{1}{d_0\,(d_0-1)}\right)^t\right)^p\left(\sum_{d_1\in I_1}\left(\frac{1}{d_1\,(d_1-1)}\right)^t\right)^{1-p}=1. \tag{4.3}
$$

Proof. Let $\mathbf{s}=(s_k)_{k\in\mathbb{N}}\in\{0,1\}^{\mathbb{N}}$ and set $\mathbb{J}=(J_k)_{k\in\mathbb{N}}$ where $J_k=\{s_k\}\times I_{s_k}$ for $k\in\mathbb{N}$. By construction the set $F_{J,\mathbf{s}}$ coincides with the set $F_{\mathbb{J}}$. Following similar arguments to proposition 4.3, we have

$$
\dim_{\mathcal{H}}(F_{J,\mathbf{s}})=\inf\left\{r\in(0,1]:\liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\left(\sum_{d\in I_{s_k}}\left(\frac{1}{d(d-1)}\right)^r\right)<0\right\}.
$$

For $n\in\mathbb{N}$ define $\tau_0(\mathbf{s},n)=\#\{1\leqslant k\leqslant n:s_k=0\}$ and observe

$$
\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\log\left(\sum_{d\in I_{s_k}}\left(\frac{1}{d(d-1)}\right)^r\right)&=\frac{\tau_0\,(\mathbf{s},n)}{n}\log\left(\sum_{d_0\in I_0}\left(\frac{1}{d_0\,(d_0-1)}\right)^r\right)\\
&\quad+\frac{n-\tau_0\,(\mathbf{s},n)}{n}\log\left(\sum_{d_1\in I_1}\left(\frac{1}{d_1\,(d_1-1)}\right)^r\right).
\end{aligned}
$$

Applying the Birkhoff Ergodic theorem, where the dynamics is driven by the left-shift map on $\{0,1\}^{\mathbb{N}}$ and where we take the indicator function on the set $\{\mathbf{s}=(s_1,s_2,\dots)\in\{0,1\}^{\mathbb{N}}:s_1=0\}$ for the observable, we obtain $\lim_{n\to\infty}\frac{\tau_0(\mathbf{s},n)}{n}=p$ for μ_p-almost every $\mathbf{s}\in\{0,1\}^{\mathbb{N}}$. Hence, for such $\mathbf{s}$,

$$\dim_{\mathcal{H}}(F_{J,\mathbf{s}})$$

$$= \inf \left\{ r \in (0,1] : p \log \left(\sum_{d_0 \in I_0} \left(\frac{1}{d_0(d_0-1)} \right)^r \right) + (1-p) \log \left(\sum_{d_1 \in I_1} \left(\frac{1}{d_1(d_1-1)} \right)^r \right) < 0 \right\}$$

$$= \inf \left\{ r \in (0,1] : \left(\sum_{d_0 \in I_0} \left(\frac{1}{d_0(d_0-1)} \right)^t \right)^p \left(\sum_{d_1 \in I_1} \left(\frac{1}{d_1(d_1-1)} \right)^t \right)^{1-p} < 1 \right\}.$$

$\square$

4.2. Restricted digit sets with infinite alphabets for signed Lüroth expansions

We first consider the case when $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 3}$ and then turn to the case when exactly one of $(0,2)$ or $(1,2)$ lies in J.

Theorem 4.5. *If J is a countably infinite subset of $\{0,1\} \times \mathbb{N}_{\geqslant 3}$, then*

$$\dim_{\mathcal{H}}(F_J) = \inf \left\{ r > 0 : \sum_{(s,d) \in J} \left(\frac{1}{d(d-1)} \right)^r \leqslant 1 \right\} \ \ and$$

$$\dim_P(F_J) = \overline{\dim}_B(F_J) = \max \left\{ \dim_{\mathcal{H}}(F_J), \overline{\dim}_B \left(\left\{ \frac{1}{d-s} : (s,d) \in J \right\} \right) \right\}.$$

Proof. For any $(s,d) \in J$ we have $\phi_{s,d}([0,1]) = [\frac{1}{d}, \frac{1}{d-1}] \subseteq [0,\frac{1}{2}]$ and so the limit set of the IFS $\{\phi_{s,d} : (s,d) \in J\}$ on $[0,1]$ coincides with that of the restricted IFS $\{\phi_{s,d}|_{[0,1/2]} : (s,d) \in J\}$ on $[0,\frac{1}{2}]$ and satisfies the OSC with feasible open set $(0,\frac{1}{2})$. One readily checks that the restricted IFS satisfies the conditions of [MU96, corollary 3.17] and thus that $\dim_{\mathcal{H}}(F_J) = \inf\{r > 0 : P(r) \leqslant 0\}$. Here $P : \mathbb{R}_{>0} \to \mathbb{R} \cup \{\infty\}$ is the *pressure function* defined, for $r > 0$, by

$$P(r) = \lim_{m \to \infty} \frac{1}{m} \log \left(\sum_{(s_1,d_1),\ldots,(s_m,d_m) \in J} \|(\phi_{s_1,d_1} \circ \cdots \circ \phi_{s_m,d_m})'\|_\infty^r \right).$$

Since $\|(\phi_{s_1,d_1} \circ \cdots \circ \phi_{s_m,d_m})'\|_\infty = \prod_{k=1}^m \frac{1}{d_k(d_k-1)}$ for any $(s_1,d_1),\ldots,(s_m,d_m) \in J$, this becomes

$$P(r) = \lim_{m \to \infty} \frac{1}{m} \log \sum_{(s_1,d_1),\ldots,(s_m,d_m) \in J} \prod_{k=1}^m \left(\frac{1}{d_k(d_k-1)} \right)^r$$

$$= \lim_{m \to \infty} \frac{1}{m} \log \left(\sum_{(s,d) \in J} \left(\frac{1}{d(d-1)} \right)^r \right)^m = \log \sum_{(s,d) \in J} \left(\frac{1}{d(d-1)} \right)^r. \tag{4.4}$$

Therefore, $P(r) \leqslant 0$ if and only if $\sum_{(s,d) \in J} (\frac{1}{d(d-1)})^r \leqslant 1$, yielding the result for the Hausdorff dimension.

The equality of the packing and upper box-counting dimensions follows from [MU96, theorem 3.1] and the formula for the upper box-counting dimension is a consequence of [MU99, theorem 2.11]. $\square$

Remark 4.6. Theorem 4.5 in tandem with [BF23, theorem 3.5 and corollary 3.6] yields that if J is a countably infinite subset of $\{0,1\} \times \mathbb{N}_{\geqslant 3}$, then for $\theta \in [0,1]$,

$$\max \left\{ \dim_{\mathcal{H}}(F_J), \, \underline{\dim}_\theta \left(\left\{ \frac{1}{d-s} : (s,d) \in J \right\} \right) \right\}$$
$$\leqslant \underline{\dim}_\theta (F_J)$$
$$\leqslant \overline{\dim}_\theta (F_J) = \max \left\{ \dim_{\mathcal{H}}(F_J), \, \overline{\dim}_\theta \left(\left\{ \frac{1}{d-s} : (s,d) \in J \right\} \right) \right\},$$

where $\underline{\dim}_\theta$ and $\overline{\dim}_\theta$ respectively denote the lower and upper intermediate dimensions. Moreover, applying the same analysis as in [BF23, theorem 4.3(2)] to the sets F_J yields that the maps $\theta \mapsto \underline{\dim}_\theta(F_J)$ and $\theta \mapsto \overline{\dim}_\theta(F_J)$ are continuous at $\theta = 0$. For a formal definition of the lower and upper intermediate dimensions we refer the reader to [FFK20], where they were first introduced, and where it was noted that $\underline{\dim}_\theta(F) = \overline{\dim}_\theta(F) = \dim_{\mathcal{H}}(F)$ when $\theta = 0$, and $\underline{\dim}_\theta(F) = \underline{\dim}_B(F)$ and $\overline{\dim}_\theta(F) = \overline{\dim}_B(F)$ when $\theta = 1$, for any bounded set $F \subseteq \mathbb{R}^D$.

We treat some examples where the limit set F_J has equal Hausdorff and box-counting dimensions as well as examples where the Hausdorff dimension is strictly smaller than the box-counting dimension.

Example 4.7. Suppose $J = (\{0,1\} \times \mathbb{N}_{\geqslant 3}) \setminus S$ for some finite set S. For any $r \leqslant \frac{1}{2}$ we have

$$\sum_{(s,d)\in J} \left(\frac{1}{d(d-1)} \right)^r \geqslant \sum_{(s,d)\in J} \left(\frac{1}{d^2} \right)^r \geqslant \sum_{(s,d)\in J} \frac{1}{d} = 2\sum_{d=3}^{\infty} \frac{1}{d} - \sum_{(s,d)\in S} \frac{1}{d} = \infty.$$

It therefore follows from theorem 4.5 that $\dim_{\mathcal{H}}(F_J) = \inf\{r : \sum_{(s,d)\in J} (\frac{1}{d(d-1)})^r \leqslant 1\} \geqslant \frac{1}{2}$. Further, we have that $\frac{1}{2} = \overline{\dim}_B(\{\frac{1}{n} : n \in \mathbb{N}\}) \geqslant \overline{\dim}_B(\{\frac{1}{d-s} : (s,d) \in J\})$, so by theorem 4.5 we have $\dim_{\mathcal{H}}(F_J) = \dim_B(F_J)$.

Example 4.8. If $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 3}$ is such that $\{\frac{1}{d-s} : (s,d) \in J\} = \{\frac{1}{n^k} : n \in \mathbb{N}_{\geqslant 2}\}$ for some $k \in \mathbb{N}$, then $\dim_B(\{\frac{1}{d-s} : (s,d) \in J\}) = \frac{1}{k+1}$, see for instance [Fal14, example 3.1]. Theorem 4.5 in tandem with remark 4.6 then implies $\dim_B(F_J)$ exists and equals $\max\{\dim_{\mathcal{H}}(F_J), \frac{1}{1+k}\}$.

For instance, if $J = \{(0, n^k) : n \in \mathbb{N}_{\geqslant 2}\}$ for some $k \in \{2,3,4,5,6\}$, then $\frac{2k}{k+1} \leqslant \frac{2 \cdot 6}{6+1} = \frac{12}{7}$, and so for each $r \leqslant \frac{1}{k+1}$,

$$\sum_{(s,d)\in J} \left(\frac{1}{d(d-1)} \right)^r \geqslant \sum_{n=2}^{\infty} \left(\frac{1}{n^k(n^k-1)} \right)^{\frac{1}{k+1}} \geqslant \sum_{n=2}^{\infty} \frac{1}{n^{\frac{2k}{k+1}}} \geqslant \sum_{n=2}^{\infty} \frac{1}{n^{\frac{12}{7}}} = \zeta\left(\frac{12}{7}\right) - 1 > 1,$$

with ζ denoting the Riemann ζ-function. It follows that $\dim_{\mathcal{H}}(F_J) \geqslant \frac{1}{k+1}$ and hence $\dim_{\mathcal{H}}(F_J) = \dim_B(F_J)$.

If instead we take $J = \{(1, n^k + 1) : n \in \mathbb{N}_{\geqslant 2}\}$ for some integer $k \geqslant 7$, then for any $r > \frac{1}{2k}$ we have

$$\sum_{(s,d)\in J} \left(\frac{1}{d(d-1)} \right)^r = \sum_{n=2}^{\infty} \left(\frac{1}{(n^k+1)n^k} \right)^r \leqslant \sum_{n=2}^{\infty} \frac{1}{n^{2kr}} = \zeta(2kr) - 1.$$

In particular, for any $r \geqslant \frac{1}{k+1} - \frac{1}{100k}$ we have that $\zeta(2kr) - 1 \leqslant \zeta\left(\frac{2k}{k+1} - \frac{1}{50}\right) - 1 \leqslant \zeta\left(\frac{7}{4} - \frac{1}{50}\right) - 1 < 1$ and hence $\dim_{\mathcal{H}}(F_J) \leqslant \frac{1}{k+1} - \frac{1}{100k} < \frac{1}{k+1} = \dim_B(F_J)$.

Remark 4.9. Whenever $\{\phi_{s,d} : (s,d) \in J\}$, with J countably infinite, satisfies the OSC with a feasible open set consisting of finitely many open intervals, one could attempt to show that the result of theorem 4.5 holds by representing the system as an infinitely generated conformal graph-directed system in the sense of [MU03] and by applying the results therein. However, if J is infinite and contains either $(0,2)$ or $(1,2)$, but not both, the OSC can only be satisfied with feasible open set $(0,1)$ or with a feasible open set consisting of an infinite union of disjoint open intervals. The former is the case when for each $d \in \mathbb{N}_{\geqslant 2}$ the alphabet J contains at most one of the digits $(0,d)$ and $(1,d)$, while the latter is the case whenever there is at least one digit $d \in \mathbb{N}_{\geqslant 2}$ for which $(0,d),(1,d) \in J$.

To see this, note that if $\{\phi_{s,d} : (s,d) \in J\}$ satisfies the OSC with feasible open set U, then for each $(s,d) \in J$, the open set U must have a non-empty intersection with $\phi_{s,d}([0,1]) = [\frac{1}{d}, \frac{1}{d-1}]$. Hence, assuming J is infinite and U is a finite union of open intervals, then one of these intervals must be of the form $(0,\varepsilon)$ for some $\varepsilon \in (0,1]$. If $(1,2) \in J$ this means U must also contain the interval $\phi_{1,2}((0,\varepsilon)) = (1 - \frac{1}{2}\varepsilon, 1)$. If instead $(0,2) \in J$ then U must contain $\bigcup_{k \in \mathbb{N}} \phi_{0,2}^k((0,\varepsilon)) = \bigcup_{k \in \mathbb{N}} (1 - \frac{1}{2^k}, \frac{1}{2^k}\varepsilon + 1 - \frac{1}{2^k})$. In both cases the assumption that U is a finite union of open intervals yields that U must contain intervals $(0,\varepsilon)$ and $(\delta,1)$ for some $\varepsilon \in (0,1]$ and $\delta \in [0,1)$. However, if $d \in \mathbb{N}_{\geqslant 2}$ is such that $(0,d),(1,d) \in J$ then $\phi_{0,d}((0,\varepsilon)) \cap \phi_{1,d}((\delta,1)) = (\frac{1}{d}, \frac{1}{d(d-1)}\varepsilon + \frac{1}{d}) \cap (\frac{1}{d}, \frac{1}{d-1} - \frac{1}{d(d-1)}\delta) \neq \emptyset$, meaning U is not a feasible open set for the IFS $\{\phi_{s,d} : (s,d) \in J\}$ to satisfy the OSC.

As a corollary to theorem 4.5 and [CLU19, corollary 6.8] we obtain in the following result that the dimension spectra of the IFSs $\{\phi_{0,d} : d \in \mathbb{N}_{\geqslant 2}\}$ and $\{\phi_{1,d} : d \in \mathbb{N}_{\geqslant 2}\}$ are full, which we utilise in the proof of theorem 1.5.

Corollary 4.10. *For $s \in \{0,1\}$ we have $\{\dim_{\mathcal{H}}(F_J) : J \subseteq \{s\} \times \mathbb{N}_{\geqslant 2}\} = [0,1]$.*

Proof. By [CLU19, corollary 6.8], it is sufficient to verify that (i) $\sum_{d \in \mathbb{N}_{\geqslant k+1}} \left(\frac{1}{d(d-1)}\right)^t \geqslant \left(\frac{1}{k(k-1)}\right)^t$, for all $k \in \mathbb{N}_{\geqslant 3}$ and $t \in (0,1)$ and (ii) $\inf\{t > 0 : P(t) \leqslant 0\} = 1$ with P as in (4.4) and $J = \{s\} \times \mathbb{N}_{\geqslant 2}$. Part (i) follows from the fact that, for $k \in \mathbb{N}_{\geqslant 3}$, the map $t \mapsto \sum_{d \in \mathbb{N}_{\geqslant k+1}} \left(\frac{k(k-1)}{d(d-1)}\right)^t$, where it is well defined on $(0,1]$, is monotonically decreasing, and that $\sum_{d \in \mathbb{N}_{\geqslant k+1}} \frac{k(k-1)}{d(d-1)} = k - 1 \geqslant 1$. Part (ii) follows from $t \mapsto \sum_{d \in \mathbb{N}_{\geqslant 2}} \left(\frac{1}{d(d-1)}\right)^t$ being monotonically decreasing and $\sum_{d \in \mathbb{N}_{\geqslant 2}} \frac{1}{d(d-1)} = 1$. $\qquad \square$

5. Hausdorff and affinity dimensions of $\mathcal{F}_J^p$

We now consider the self-affine sets $\mathcal{F}_J^p$ generated by the IFSs $\{A_{s,d}^p : (s,d) \in J\}$ on $[0,1]^2$ for countable alphabets $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 2}$. Notably, each IFS $\{A_{s,d}^p : (s,d) \in J\}$ satisfies the ROSC and the linear part of each affine map $A_{s,d}^p$ is the diagonal matrix

$$L_{s,d}^p = \begin{pmatrix} p^{1-s}(1-p)^s & 0 \\ 0 & (-1)^s \frac{1}{d(d-1)} \end{pmatrix}.$$

Applying theorem 1.1 to this setting yields the following expression for the affinity dimension of $\{L_{s,d}^p : (s,d) \in J\}$.

Proposition 5.1. *Let $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 2}$ be a finite or countably infinite alphabet satisfying $\pi_1(J) = \{0,1\}$ and let $p \in (0,1)$ be arbitrary. The affinity dimension $d(L_{s,d}^p \mid (s,d) \in J)$ of*

$\{L_{s,d}^p : (s,d) \in J\}$ *lies in* $[1,2]$ *and equals*

$$\inf\left\{r \in (1,2] : \max\left\{\sum_{(s,d)\in J} p^{1-s}(1-p)^s \left(\frac{1}{d(d-1)}\right)^{r-1}, \sum_{(s,d)\in J}(p^{1-s}(1-p)^s)^{r-1}\left(\frac{1}{d(d-1)}\right)\right\} \leqslant 1\right\}. \tag{5.1}$$

If (a) $\sum_{(s,d)\in J}\frac{1}{d(d-1)} \leqslant 1$ *or (b)* $J = \{0,1\} \times I$ *for some* $I \subseteq \mathbb{N}_{\geqslant 2}$, *or (c)* $p = \frac{1}{2}$, *then this formula simplifies to*

$$d(L_{s,d}^p \mid (s,d) \in J) = \inf\left\{r \in (1,2] : \sum_{(s,d)\in J} p^{1-s}(1-p)^s \left(\frac{1}{d(d-1)}\right)^{r-1} \leqslant 1\right\}.$$

Proof. Since by assumption $\pi_1(J) = \{0,1\}$, we have for each $0 < r < 1$ that

$$\max\left\{\sum_{(s,d)\in J}(p^{1-s}(1-p)^s)^r, \sum_{(s,d)\in J}\left(\frac{1}{d(d-1)}\right)^r\right\}$$
$$\geqslant \sum_{(s,d)\in J}(p^{1-s}(1-p)^s)^r > \sum_{(s,d)\in J} p^{1-s}(1-p)^s \geqslant 1.$$

This implies that $\inf\{r > 0 : \max\{\sum_{(s,d)\in J}(p^{1-s}(1-p)^s)^r, \sum_{(s,d)\in J}(\frac{1}{d(d-1)})^r\} \leqslant 1\} \geqslant 1$, and so by theorem 1.1 the affinity dimension of $\{L_{s,d}^p : (s,d) \in J\}$ is at least 1. Further, for $r \geqslant 2$ we have

$$\sum_{(s,d)\in J}\left(\frac{p^{1-s}(1-p)^s}{d(d-1)}\right)^{r/2} \leqslant \sum_{(s,d)\in J}\frac{p^{1-s}(1-p)^s}{d(d-1)} \leqslant \sum_{d\in\pi_2(J)}\frac{p+1-p}{d(d-1)} \leqslant \sum_{d\in\mathbb{N}_{\geqslant 2}}\frac{1}{d(d-1)} = 1.$$

Thus, $\inf\{r > 2 : \sum_{(s,d)\in J}(\frac{p^{1-s}(1-p)^s}{d(d-1)})^{r/2} \leqslant 1\} = 2$, and so theorem 1.1 implies the affinity dimension is at most 2 and equals the quantity given in (5.1).

For the simplification, note that whenever $p = 1/2$, we have that $p^{1-s}(1-p)^s = \frac{1}{2} \geqslant \frac{1}{d(d-1)}$ for all $(s,d) \in J$. Hence

$$\sum_{(s,d)\in J} p^{1-s}(1-p)^s \left(\frac{1}{d(d-1)}\right)^{r-1} \geqslant \sum_{(s,d)\in J}(p^{1-s}(1-p)^s)^{r-1}\frac{1}{d(d-1)},$$

for all $r \in [1,2]$, yielding (c). When $p \in (0,1)$ and $\sum_{(s,d)\in J}\frac{1}{d(d-1)} \leqslant 1$, observe that for $r \in [1,2]$,

$$\sum_{(s,d)\in J}(p^{1-s}(1-p)^s)^{r-1}\frac{1}{d(d-1)} \leqslant \sum_{(s,d)\in J}\frac{1}{d(d-1)} \leqslant 1,$$

yielding (a). For (b), by symmetry, we may assume that $p \in (0,1/2)$. For $n \in \mathbb{N}_{\geqslant 2}$ and $r \in [1,2]$, we show that

$$\frac{1}{n^{r-1}} \geqslant \left(p^{r-1} + (1-p)^{r-1}\right)\frac{1}{n}. \tag{5.2}$$

This inequality holds if and only if $n \geqslant q^{r-1} + (n-q)^{r-1}$, where $q = np$. Let $g_r \colon [0, \frac{n}{2}] \to \mathbb{R}$ be defined by $g_r(q) = q^{r-1} + (n-q)^{r-1}$, and note, by the first derivative test, that g_r is maximised at $q = \frac{n}{2}$. This implies for all $q \in (0, \frac{n}{2}]$, and hence $p \in (0, \frac{1}{2}]$, that $q^{r-1} + (n-q)^{r-1} = g_r(q) \leqslant g_r(\frac{n}{2}) = 2(\frac{n}{2})^{r-1} \leqslant 2\frac{n}{2} = n$. By the assumption that $(0,d) \in J$ implies $(1,d) \in J$ and vice versa, and using (5.2) with $n = d(d-1)$, we conclude

$$\sum_{(s,d) \in J} p^{1-s}(1-p)^s \left(\frac{1}{d(d-1)} \right)^{r-1} = \sum_{d \in I} \left(\frac{1}{d(d-1)} \right)^{r-1} \geqslant \sum_{d \in I} \frac{p^{r-1} + (1-p)^{r-1}}{d(d-1)}$$

$$= \sum_{(s,d) \in J} \left(p^{1-s}(1-p)^s \right)^{r-1} \frac{1}{d(d-1)}.$$

$\square$

Defining the maps $f_0^p(w) = pw$ and $f_1^p(w) = (1-p)w + p$ for $w \in [0,1]$ we note that $A_{s,d}^p(w,x) = (f_s^p(w), \phi_{s,d}(x))$ for each $(w,x) \in [0,1]^2$. As such, the horizontal projection $\pi_1(\mathcal{F}_J^p)$ is exactly the self-similar set of the IFS $\{f_s^p : s \in \pi_1(J)\}$. In particular, whenever $\pi_1(J) = \{0,1\}$, we have $\pi_1(\mathcal{F}_J^p) = [0,1]$ and hence $\dim_B(\pi_1(\mathcal{F}_J^p)) = 1$. In the same way, the vertical projection $\pi_2(\mathcal{F}_J^p)$ equals F_J, the self-similar set of $\{\phi_{s,d} : (s,d) \in J\}$ discussed in section 4. Under suitable conditions the dimension $\dim_B(\pi_2(\mathcal{F}_J^p))$ is given by (4.2) when J is finite. With this in mind we obtain the following result whenever $(0,2),(1,2) \notin J$.

Lemma 5.2. *If $J \subseteq \{0,1\} \times \mathbb{N}_{\geqslant 2}$ is a finite alphabet such that either $\pi_2(J) \subseteq \mathbb{N}_{\geqslant 3}$ and $\pi_1(J) = \{0,1\}$, or $\{(0,2),(1,2)\} \subseteq J$, then $\dim_B(\mathcal{F}_J^p) = \dim_P(\mathcal{F}_J^p) = d(L_{s,d}^p \mid (s,d) \in J)$ holds for all $p \in (0,1)$.*

Proof. If $\{(0,2),(1,2)\} \subseteq J$, then by construction and by (4.1), $\dim_B(\pi_1(\mathcal{F}_J^p)) = \dim_B(\pi_2(\mathcal{F}_J^p)) = 1$, and thus the result is an application of proposition 3.2 in combination with the second part of proposition 5.1. Therefore, let us consider the case when $\pi_2(J) \subseteq \mathbb{N}_{\geqslant 3}$ and $\pi_1(J) = \{0,1\}$. Set $r_1 = \dim_B(\pi_1(\mathcal{F}_J^p))$ and $r_2 = \dim_B(\pi_2(\mathcal{F}_J^p))$. By the assumption $\pi_1(J) = \{0,1\}$, we have $r_1 = 1$, and by (4.2), with the fact that $\pi_2(J) \subseteq \mathbb{N}_{\geqslant 3}$, which implies the OSC, r_2 uniquely solves $\sum_{(s,d) \in J} (\frac{1}{d(d-1)})^{r_2} = 1$. By proposition 3.2 we have $\dim_B \mathcal{F}_J^p = \dim_P \mathcal{F}_J^p = r'$, where r' solves

$$\max \left\{ \underbrace{\sum_{(s,d) \in J} p^{1-s}(1-p)^s \left(\frac{1}{d(d-1)} \right)^{r'-1}}_{=v_1(r')}, \; \underbrace{\sum_{(s,d) \in J} \left(p^{1-s}(1-p)^s \right)^{r'-r_2} \left(\frac{1}{d(d-1)} \right)^{r_2}}_{=v_2(r')} \right\} = 1.$$

Next, we show that $r' = d(L_{s,d}^p \mid (s,d) \in J)$. For this, note that $v_1(r), v_2(r)$ are decreasing in r and that

$$v_2(r_2) = \sum_{(s,d) \in J} \left(\frac{1}{d(d-1)} \right)^{r_2} = 1.$$

In particular, since $r_2 \leqslant 1$, we have $v_2(r) \leqslant 1$ for all $r \geqslant 1$. Observe that the assumption $\pi_2(J) \subseteq \mathbb{N}_{\geqslant 3}$ implies $\sum_{(s,d) \in J} \frac{1}{d(d-1)} \leqslant \sum_{(s,d) \in \{0,1\} \times \mathbb{N}_{\geqslant 3}} \frac{1}{d(d-1)} = 1$. Thus, by the second part of proposition 5.1, the number r solving $v_1(r) = 1$ satisfies $r \geqslant 1$. Since both v_1 and v_2 are decreasing in r, we deduce that $v_1(r') \geqslant v_2(r')$ giving $v_1(r') = 1$. Therefore, it follows from the second part of proposition 5.1 that $r' = d(L_{s,d}^p \mid (s,d) \in J)$. $\square$

Proof of theorem 1.4. We begin by showing (1.7) and divide the argument into two cases, when J is finite, and when J is countably infinite. To this end, let us assume that J is finite and let $E = \left\{ \mathbf{s} \in \{0,1\}^{\mathbb{N}} : \lim_{n \to \infty} \frac{\tau_0(\mathbf{s},n)}{n} = p \right\}$, where τ_0 is as in the proof of proposition 4.4. We have seen in the proof of proposition 4.4 that for all $\mathbf{s} \in E$ the Hausdorff dimension of $F_{J,\mathbf{s}}$ is independent of $\mathbf{s} \in E$ and is given by the unique t solving (4.3). As in section 2.2, let $\pi : \{0,1\}^{\mathbb{N}} \to [0,1]$ denote the projection map given by $\pi(\mathbf{s}) = \lim_{n \to \infty} f_{s_1}^p \circ \cdots \circ f_{s_n}^p(0)$. Observe that $\pi|_E : E \to \pi(E)$ is a bijection, and that $\mu_p(E) = 1$ by the Birkhoff Ergodic Theorem. Let $\xi_p : [0,1] \to [0,1]$ be defined by

$$\xi_p(w) = \begin{cases} \frac{w}{p} & \text{if } w \in [0,p], \\ \frac{w}{1-p} - \frac{p}{1-p} & \text{if } w \in (p,1], \end{cases}$$

(so f_0^p and f_1^p are the local inverses of ξ_p). Let λ denote the Lebesgue measure on the Borel σ-algebra of $[0,1]$ and denote the left-shift by $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$. The dynamical systems $([0,1], \lambda, \xi_p)$ and $(\{0,1\}^{\mathbb{N}}, \mu_p, \sigma)$ are measure theoretically isomorphic through the map π, see for instance [Fal97], and thus $\lambda(\pi(E)) = 1$.

For $w \in \pi(E)$, let $(\mathcal{F}_J^p)_w = \{x \in [0,1] : (w,x) \in \mathcal{F}_J^p\}$ be the vertical fibre of $\mathcal{F}_J^p$ based at w. Since there is a unique $\mathbf{s} = (s_k)_{k \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ with $\pi(\mathbf{s}) = w$ and each $(w,x) \in \mathcal{F}_J^p$ gives a signed Lüroth expansion of x with digit sequence $((s_k,d_k))_{k \in \mathbb{N}} \in J^{\mathbb{N}}$ via $(w,x) = \lim_{k \to \infty} (A_{s_1,d_1}^p \circ \cdots \circ A_{s_k,d_k}^p)((0,0))$, we have $(\mathcal{F}_J^p)_w = F_{J,\mathbf{s}}$. Hence, proposition 4.4 implies $\dim_{\mathcal{H}}((\mathcal{F}_J^p)_w) = \dim_{\mathcal{H}}(F_{J,\pi^{-1}(w)}) = t$, for λ-almost all $w \in [0,1]$, where $t \in \mathbb{R}_{>0}$ uniquely solves

$$\mathfrak{p}(I_0, I_1, t) = \left(\sum_{d_0 \in I_0} \left(\frac{1}{d_0(d_0-1)} \right)^t \right)^p \left(\sum_{d_1 \in I_1} \left(\frac{1}{d_1(d_1-1)} \right)^t \right)^{1-p} = 1.$$

Since t does not depend on w, and since this holds for all w in a set of positive Lebesgue measure (and hence of Hausdorff dimension 1), it is a direct consequence of [Mar54] that $\dim_{\mathcal{H}}(\mathcal{F}_J^p) \geqslant 1 + t$, yielding (1.7).

Suppose that J is countably infinite. For $n \in \mathbb{N}_{\geqslant 2}$ and $s \in \{0,1\}$ let $I_{n,s} = I_s \cap \{2,\ldots,n\}$, and let J_n denote the set $(\{0\} \times I_{n,0}) \cup (\{1\} \times I_{n,1})$. Set $t_n \in \mathbb{R}_{>0}$ to be the unique solution to $\mathfrak{p}(I_{0,n}, I_{1,n}, t_n) = 1$ and observe that $(t_n)_{n \in \mathbb{N}_{\geqslant 2}}$ is a non-decreasing sequence in $[0,1]$. This, in tandem with the fact that $\mathcal{F}_J^p \supseteq \mathcal{F}_{J_{n+1}}^p \supseteq \mathcal{F}_{J_n}^p$ for all $n \in \mathbb{N}_{\geqslant 2}$, yields $\dim_{\mathcal{H}}(\mathcal{F}_J^p) \geqslant \sup_{n \geqslant 2} \dim_{\mathcal{H}}(\mathcal{F}_{J_n}^p) \geqslant 1 + \sup_{n \geqslant 2} t_n$.

Letting $t = \sup_{n \geqslant 2} t_n$, we observe for $n \in \mathbb{N}_{\geqslant 2}$ that $\mathfrak{p}(I_{0,n}, I_{1,n}, t) \leqslant 1$. Taking the limit as n tends to infinity yields (1.7). To conclude the proof, we show (1.8). Since $I_0 = I_1 = I$, it holds by the second part of proposition 5.1 that

$$d(L_{s,d}^p \mid (s,d) \in J) = \inf \left\{ r \in (1,2) : \sum_{(s,d) \in J} p^{1-s}(1-p)^s \left(\frac{1}{d(d-1)} \right)^{r-1} \leqslant 1 \right\}$$

$$= \inf \left\{ r \in (1,2) : \sum_{d \in I} \left(\frac{1}{d(d-1)} \right)^{r-1} \leqslant 1 \right\}$$

$$= 1 + \inf \left\{ r \in (0,1] : \sum_{d \in I} \left(\frac{1}{d(d-1)} \right)^r \leqslant 1 \right\} \leqslant 2.$$

By (1.7), the assumption $I_0 = I_1 = I$ also yields

$$\dim_{\mathcal{H}}(\mathcal{F}_J^p) \geqslant 1 + \inf\left\{ r \in (0,1] : \left(\sum_{d_0 \in I}\left(\frac{1}{d_0(d_0-1)}\right)^r\right)^p \left(\sum_{d_1 \in I}\left(\frac{1}{d_1(d_1-1)}\right)^r\right)^{1-p} \leqslant 1\right\}$$

$$= 1 + \inf\left\{ r \in (0,1] : \sum_{d \in I}\left(\frac{1}{d(d-1)}\right)^r \leqslant 1\right\} = d(L_{s,d}^p \mid (s,d) \in J).$$

This in tandem with theorem 1.2(i) yields (1.8). Moreover, if I is finite, then lemma 5.2 in combination with theorem 1.2(i) gives that $\dim_{\mathcal{H}}(\mathcal{F}_J^p) = \dim_P(\mathcal{F}_J^p) = \dim_B(\mathcal{F}_J^p) = d(L_{s,d}^p \mid (s,d) \in J)$. □

Proof of theorem 1.5. Let $t \in [0,1]$ be chosen arbitrarily. For fixed $I \subseteq \mathbb{N}_{\geqslant 2}$ and $s \in \{0,1\}$ the limit set of $\{\phi_{s,d} : d \in I\}$ equals $F_{\{s\} \times I}$ whereas that of $\{A_{s,d}^p : d \in I\}$ equals $\{s\} \times F_{\{s\} \times I}$, meaning the two have equal Hausdorff dimensions. By corollary 4.10 we can find a set $I \subseteq \mathbb{N}_{\geqslant 2}$ such that $\dim_{\mathcal{H}}(\mathcal{F}_{\{0\} \times I}^p) = \dim_{\mathcal{H}}(\mathcal{F}_{\{1\} \times I}^p) = t$. With this at hand, theorems 1.4 and 4.5 together with corollary 4.10 imply $\dim_{\mathcal{H}}(\mathcal{F}_{\{0,1\} \times I}^p) = 1 + t$. □

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Data access statement

This study did not generate any new data.

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