Logarithmic double ramification cycles

Holmes, D.S.T.; Molcho, S.; Pandharipande, R.; Pixton, A.; Schmitt, J.

Citation
Holmes, D. S. T., Molcho, S., Pandharipande, R., Pixton, A., & Schmitt, J. (2025). Logarithmic double ramification cycles. *Inventiones Mathematicae*, *240*, 35–121. doi:10.1007/s00222-025-01318-z

Version:	Publisher's Version
License:	Licensed under Article 25fa Copyright Act/Law (Amendment Taverne)
Downloaded from:	https://hdl.handle.net/1887/4285494

Note: To cite this publication please use the final published version (if applicable).

Inventiones mathematicae (2025) 240:35–121
https://doi.org/10.1007/s00222-025-01318-z

Logarithmic double ramification cycles

D. Holmes[1] · S. Molcho[2] · R. Pandharipande[3] · A. Pixton[4] · J. Schmitt[3]

Received: 5 October 2022 / Accepted: 10 January 2025 / Published online: 10 February 2025
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2025

Abstract

Let $A = (a_1, \ldots, a_n)$ be a vector of integers which sum to $k(2g-2+n)$. The double ramification cycle $\mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})$ on the moduli space of curves is the virtual class of an Abel-Jacobi locus of pointed curves $(C, x_1, \ldots, x_n)$ satisfying

$$\mathcal{O}_C\Big(\sum_{i=1}^{n} a_i x_i\Big) \simeq \big(\omega_C^{\log}\big)^k .$$

The Abel-Jacobi construction requires log blow-ups of $\overline{\mathcal{M}}_{g,n}$ to resolve the indeterminacies of the Abel-Jacobi map. Holmes (J. Inst. Math. Jussieu 2019) has shown that $\mathsf{DR}_{g,A}$ admits a canonical lift $\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$ to the logarithmic Chow ring, which is the limit of the intersection theories of all such blow-ups. The main result of the paper is an explicit formula for $\mathsf{logDR}_{g,A}$ which lifts Pixton's formula for $\mathsf{DR}_{g,A}$. The central idea is to study the universal Jacobian over the moduli space of curves (following Caporaso (Am. Math. Soc. 7(3):589–660 1994), Kass-Pagani (Trans. Am. Math. Soc. 372:4851–4887 2019), and Abreu-Pacini (Adv. Math. 378:107520 2021)) for certain stability conditions. Using the criterion of Holmes-Schwarz (Algebr. Geom. 9(5):574–605 2022), the universal double ramification theory of Bae-Holmes-Pandharipande-Schmitt-Schwarz (Acta Math. 230(2):205–319 2023) applied to the universal line bundle determines the logarithmic double ramification cycle. The resulting formula, written in the language of piecewise polynomials, depends upon the stability condition (and admits a wall-crossing study). Several examples of logarithmic and higher double ramification cycles are computed.

Contents

1 Introduction 37
1.1 Double ramification cycles 37
1.2 Logarithmic double ramification cycles 38
1.3 Log modifications and cone stacks 39
1.4 Log intersection theory 42
1.5 Resolving the Abel-Jacobi map 42

Extended author information available on the last page of the article

1.6 Moduli of line bundles on stable curves . . . 44
1.7 Formula for logDR . . . 49
1.7.1 Tautological classes in the logarithmic Chow ring . . . 49
1.7.2 The subdivision associated to a stability condition . . . 50
1.7.3 The formula for logDR . . . 53
1.8 Example in genus 1 . . . 55
1.9 Relations in $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$ . . . 55
1.10 Conventions . . . 56
2 Logarithmic geometry . . . 56
2.1 Overview . . . 56
2.2 Artin fans and cone stacks . . . 56
2.3 Tropicalization . . . 57
2.4 Subdivisions . . . 61
2.5 Piecewise linear functions . . . 63
3 Logarithmic curves . . . 65
3.1 Definitions . . . 65
3.2 Subdivisions of log curves . . . 66
3.3 Abel-Jacobi theory on a tropical curve . . . 67
3.4 Algebraizing tropical Abel-Jacobi theory . . . 69
4 Stability conditions . . . 70
4.1 Definitions . . . 70
4.2 Nondegenerate stability conditions . . . 72
4.2.1 Moduli of θ-stable bundles . . . 72
4.2.2 Proof of Theorem 35 . . . 74
4.3 A graph of genus 2 . . . 80
5 Theorem A . . . 84
5.1 Universal constructions . . . 84
5.2 Almost twistability . . . 85
5.3 Category of twists . . . 85
5.4 Proof of Theorem A . . . 87
6 Piecewise polynomials and strata classes . . . 89
6.1 Overview . . . 89
6.2 Relating piecewise polynomials and graph sums . . . 89
6.2.1 Normally decorated strata classes . . . 89
6.2.2 Piecewise polynomials . . . 90
6.2.3 Extension . . . 91
6.2.4 Commutation . . . 94
6.2.5 Explicit correspondence . . . 98
6.3 Pixton's formula in terms of piecewise polynomials . . . 100
6.3.1 Contributions . . . 100
6.3.2 The universal DR cycle . . . 102
7 Theorems B and C . . . 104
7.1 Families of curves . . . 104
7.2 Geometric construction of a logarithmic class on S . . . 105
7.3 The functions $\mathfrak{P}$ and $\mathfrak{L}$ . . . 105
7.4 The formula . . . 107

7.5 Proofs of Theorems B and C 108
8 Genus 1 calculations 109
8.1 Form of the answer 109
8.2 θ-stability 109
8.3 The formula 110
9 Computational tools and higher double ramification cycles 114
9.1 A Sage implementation of logDR 114
9.2 Double-double ramification cycles 116
References 118
Authors and Affiliations 120

1 Introduction

1.1 Double ramification cycles

Let $\mathcal{M}_{g,n}$ be the moduli space of nonsingular curves of genus g with n distinct marked points over $\mathbb{C}$. Given a vector of integers $A = (a_1, \ldots, a_n)$ satisfying

$$\sum_{i=1}^{n} a_i = 0\,,$$

we can define a substack of $\mathcal{M}_{g,n}$ by

$$\left\{ (C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n} \;\middle|\; \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i x_i\Big) \simeq \mathcal{O}_C \right\}. \tag{1}$$

From the point of view of relative Gromov-Witten theory, the most natural compactification of the substack (1) is the space $\overline{\mathcal{M}}^{\sim}_{g,A}$ of stable maps to *rubber* [40, 54, 55]: stable maps to $\mathbb{CP}^1$ relative to 0 and ∞ modulo the $\mathbb{C}^*$-action on $\mathbb{CP}^1$.

The rubber moduli space carries a natural virtual fundamental class $\left[\overline{\mathcal{M}}^{\sim}_{g,A}\right]^{\mathrm{vir}}$ of dimension $2g-3+n$. The pushforward via the canonical morphism

$$\epsilon : \overline{\mathcal{M}}^{\sim}_{g,A} \to \overline{\mathcal{M}}_{g,n}$$

is the *double ramification cycle* on the moduli space of stable curves,

$$\epsilon_* \left[\overline{\mathcal{M}}^{\sim}_{g,A}\right]^{\mathrm{vir}} = \mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})\,. \tag{2}$$

The double ramification cycle $\mathsf{DR}_{g,A}$ can also be defined via log stable maps (and was motivated in part by Symplectic Field Theory [32]).

The classical approach to the locus (1) in $\mathcal{M}_{g,n}$ is via Abel-Jacobi theory for the universal curve. However, the Abel-Jacobi map does not extend over the boundary

$$\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$$

of the moduli space of stable curves. Approaches by Marcus-Wise [56] and Holmes

[41], motivated by log geometry, provide a partial resolution of the Abel-Jacobi map which is sufficient to define a double ramification cycle.

We fix for the remainder of the paper an integer k and a vector of integers $A = (a_1, \ldots, a_n)$ satisfying

$$\sum_{i=1}^{n} a_i = k(2g - 2 + n)\,.$$

The Abel-Jacobi construction in fact yields a more general *k-twisted double ramification cycle* associated to the vector A,

$$\mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})\,, \tag{3}$$

and related to the substack

$$\left\{(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n} \;\middle|\; \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i x_i\Big) \simeq (\omega_C^{\log})^{\otimes k}\right\}\,,$$

where $\omega_C^{\log} = \omega_C\left(\sum_{i=1}^{n} x_i\right)$ is the *log canonical line bundle* of C. The cycle (3) agrees with definition (2) in the $k = 0$ case.

Eliashberg posed the question of computing $\mathsf{DR}_{g,A}$ in 2001. A complete formula for $\mathsf{DR}_{g,A}$ in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ was conjectured by Pixton in 2014 and proven in [47] for $k = 0$ and in [10] for general k. Pixton's formula expresses $\mathsf{DR}_{g,A}$ directly as a sum over stable graphs Γ indexing the boundary strata of $\overline{\mathcal{M}}_{g,n}$. The contribution of each stable graph Γ is the constant term of a polynomial naturally associated to the combinatorics of Γ and A.

We refer the reader to [10, Sect. 0], [47, Sect. 0], and [70, Sect. 5] for more leisurely introductions to the subject of double ramification cycles. For a sampling of the development and application of the theory in a variety of directions, see [8, 10, 15, 18–20, 24, 28–30, 35, 42, 44, 45, 47, 48, 62, 64, 66, 72, 75, 78].

1.2 Logarithmic double ramification cycles

The definition of the double ramification cycle by Holmes [41] yields cycle classes on iterated blow-ups of boundary strata of the moduli space $\overline{\mathcal{M}}_{g,n}$ which push forward to $\mathsf{DR}_{g,A}$ on $\overline{\mathcal{M}}_{g,n}$. The *logarithmic Chow ring* $\mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$ describes the intersection theory on all suitable blow-ups of $\overline{\mathcal{M}}_{g,n}$, and Holmes' construction naturally yields a *logarithmic double ramification cycle*

$$\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})\,.$$

The definition of $\mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$ is reviewed in Sects. 1.3–1.4 below. The class $\mathsf{logDR}_{g,A}$, a refinement of $\mathsf{DR}_{g,A}$, plays a fundamental role in logarithmic Gromov-Witten theory (and has recently been applied in [26] to the study of double Hurwitz numbers and in [74] to the log Gromov-Witten theory of toric varieties).

The starting point of our paper is the following question: *can Pixton's formula for* $\mathsf{DR}_{g,A}$ *be lifted to a formula for* $\mathsf{logDR}_{g,A}$ *in the logarithmic cycle theory of the*

moduli space of curves? Our main result is a formula for such a lift obtained by applying the universal theory of [10] to the universal Jacobian over the moduli space of curves with respect to certain stability conditions (using the criterion of [43]).

The difficulties which arise in computing $\mathsf{logDR}_{g,A}$ via the original definition of Holmes are explained in Sect. 1.5. Our new approach using stability conditions for line bundles on nodal curves is presented in Sect. 1.6. The main results about the logarithmic double ramification cycle are given in an abstract form (Theorem A) in Sect. 1.6 and in an explicit form (Theorem B) in Sect. 1.7. Pixton's double ramification cycle relations are lifted to $\mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$ by Theorem C presented in Sect. 1.9.

We introduce the main constructions, ideas, and results of the paper in Sects. 1.3-1.7. The reader should be able to obtain a full overview of our argument by reading these sections. The body of the paper contains a detailed presentation with complete proofs.

1.3 Log modifications and cone stacks

We begin by discussing *log modifications* of $\overline{\mathcal{M}}_{g,n}$,

$$\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}\,,$$

which are birational morphisms used in the definition of the logarithmic Chow ring. Log modifications generalize the notion of an iterated blow-up of boundary strata. Some background in log geometry is presented in Sects. 2 and 3.

The most convenient way to describe a log modification of $\overline{\mathcal{M}}_{g,n}$ is via the *cone stack* $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ associated to the pair $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$. As in the case of toric geometry, where every normal toric variety X has an associated fan Σ_X, the cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ is essentially a cone complex describing the combinatorics of the boundary stratification of $\overline{\mathcal{M}}_{g,n}$. However, as the name suggests, in contrast to the toric case, the presence of automorphisms on the stack $\overline{\mathcal{M}}_{g,n}$ forces us to work with a cone stack (in the sense of [25]) instead of a usual cone complex.[1] Just as toric modifications $\widehat{X} \to X$ of a toric variety X are in bijective correspondence with subdivisions $\widetilde{\Sigma} \to \Sigma_X$ of the associated fan, log modifications of $\overline{\mathcal{M}}_{g,n}$ are in bijective correspondence with subdivisions $\widetilde{\Sigma} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ of the cone stack.

The boundary strata of $\overline{\mathcal{M}}_{g,n}$ are indexed by *stable graphs* Γ: decorated graphs (with possible loops and multi-edges) describing the topological type of the generic stable curve $(C, x_1, \ldots, x_n)$ parameterized by the strata. For the precise definition of a stable graph, we refer the reader to [39, Appendix A]. See Fig. 1 for an illustration.

The cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ is constructed from cones[2] associated to stable graphs. For a stable graph Γ with edge set $E(\Gamma)$, the associated cone σ_Γ is defined by

$$\sigma_\Gamma = (\mathbb{R}_{\geq 0})^{E(\Gamma)} = \{\ell : E(\Gamma) \to \mathbb{R}_{\geq 0}\}\,.$$

[1] Some authors use the term *stacky fan* for the cone stack to emphasize the analogy to toric geometry, see [16, 17, 27].

[2] We follow the definition of [2, §2.1]: cones are always strongly convex, rational polyhedral, and generate the vector space spanned by their ambient lattice. See Sect. 2.2 for a careful discussion.

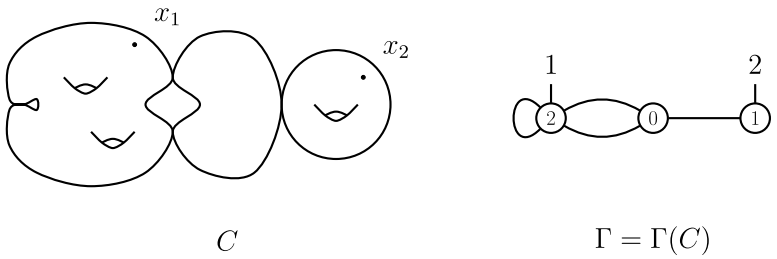


Fig. 1 A curve $(C, x_1, x_2) \in \overline{\mathcal{M}}_{5,2}$ and the associated stable graph Γ

An element of σ_Γ is an assignment of nonnegative lengths $\ell(e)$ to the edges e of the graph Γ.

If a stable graph Γ' is obtained from Γ by *contracting* a subset $E_0 \subset E(\Gamma)$ of the edges of Γ, we view the corresponding cone $\sigma_{\Gamma'}$ as the face of σ_Γ defined by the conditions that $\ell(e_0) = 0$ for $e_0 \in E_0$. To make the face construction more flexible, recall the notion of a *morphism of stable graphs*

$$\varphi : \Gamma \to \Gamma' ,$$

encoding a particular way that Γ' is obtained from Γ by contracting a subset of edges. See [39, Appendix A] for the definition of these morphisms (which are called Γ'-structures on Γ in [39]).[3] Part of the data of φ is an injective map $\varphi_E : E(\Gamma') \to E(\Gamma)$ identifying the edges of Γ' as edges of Γ not contracted by φ. We then obtain a natural map of cones

$$\iota_\varphi : \sigma_{\Gamma'} \to \sigma_\Gamma , \quad \ell' \mapsto \left(\ell : e \mapsto \begin{cases} \ell'(\varphi_E^{-1}(e)) & \text{for } e \in \varphi_E(E(\Gamma')). \\ 0 & \text{otherwise.} \end{cases} \right)$$

representing $\sigma_{\Gamma'}$ as a face of σ_Γ.

The cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ is defined as the direct limit (in the 2-categorical sense)

$$\Sigma_{\overline{\mathcal{M}}_{g,n}} = \varinjlim_{\Gamma \in \mathcal{G}_{g,n}} \sigma_\Gamma \tag{4}$$

over the category $\mathcal{G}_{g,n}$ of stable graphs (with morphisms $\varphi : \Gamma \to \Gamma'$ as above and associated morphisms $\iota_\varphi : \sigma_{\Gamma'} \to \sigma_\Gamma$ of the corresponding cones).

The limit (4) is formally defined as a cone stack in the sense of [25] where $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ is denoted $\mathcal{M}_{g,n}^{\mathsf{trop}}$, see in particular [25, Sects. 3.3–3.4].[4] However, all the additional data that we will require (subdivisions and piecewise polynomial functions on fans) will be defined on the individual cones σ_Γ. It will *not* be necessary for the reader to recall the machinery of cone stacks to follow the remainder of Sect. 1.

[3] See also [76, Definition 2.5] for another exposition of stable graphs and morphisms.

[4] $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ is not the same as the generalised cone complex of [2], which can be seen as a kind of coarse moduli space for $\Sigma_{\overline{\mathcal{M}}_{g,n}}$.

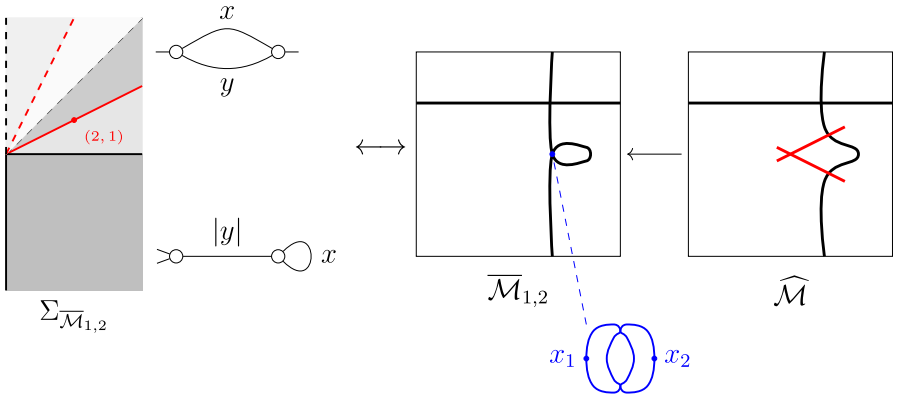


Fig. 2 The cone stack $\Sigma_{\overline{\mathcal{M}}_{1,2}}$ with a subdivision (in red) and the corresponding log modification $\widehat{\mathcal{M}}$ of $\overline{\mathcal{M}}_{1,2}$, which replaces the self-intersection of the boundary divisor of irreducible curves with a chain of rational curves of length 2. For the cone stack (on the left), we draw the double cover of the upper (stacky) cone, and correspondingly, the local pictures around the self-intersection of δ_{irr} on the right represent an étale double cover of the neighbourhood of this point. If X and Y are the étale local coordinates around the node corresponding to x and y, then the three affine patches of the subdivision are given by $\mathbb{C}[X, Y, t]/(X - tY^2)$, $\mathbb{C}[X, Y, u, v]/(Y^2 - uX, X^2 - vY, XY - uv)$, and $\mathbb{C}[X, Y, s]/(Y - sX^2)$ respectively (Color figure online)

A *subdivision* $\widetilde{\Sigma} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ is a cone stack specified by a collection $(\widetilde{\Sigma}_\Gamma)_{\Gamma \in \mathcal{G}_{g,n}}$ of fans satisfying:

(i) each $\widetilde{\Sigma}_\Gamma$ is a collection of finitely many rational polyhedral cones in $(\mathbb{R}_{\geq 0})^{E(\Gamma)}$, and has total support equal to $\sigma_\Gamma = (\mathbb{R}_{\geq 0})^{E(\Gamma)}$,
(ii) the fans $\widetilde{\Sigma}_\Gamma$ are compatible with morphisms $\varphi : \Gamma \to \Gamma'$ in the sense that

$$\iota_\varphi^{-1}(\widetilde{\Sigma}_\Gamma) = \widetilde{\Sigma}_{\Gamma'}\,.$$

The subdivision $\widetilde{\Sigma} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ determines a proper birational morphism $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$, which we call a *log modification*.

The essential idea is that the fan $\widetilde{\Sigma}_\Gamma$ determines, in étale local coordinates, the toric modifications near the boundary stratum associated to Γ. The compatibility of the subdivisions $\widetilde{\Sigma}_\Gamma$ with face maps ensures that these modifications glue to a global birational morphism. A detailed definition is given in Sect. 2.4. We illustrate a subdivision of the cone stack $\Sigma_{\overline{\mathcal{M}}_{1,2}}$ and the associated log modification, in Fig. 2.

There is an equivalence of categories between log modifications and the subdivisions of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$. Under the equivalence, the blow-up of $\overline{\mathcal{M}}_{g,n}$ along a normal closed stratum corresponds to the star subdivision along the barycenter of the corresponding cone. The full iterated boundary blow-up of $\overline{\mathcal{M}}_{g,n}$, called the explosion in [64, Sect. 5], precisely corresponds to the full iterated star subdivision of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$. It is however not straightforward in general to describe the log modification corresponding to a given subdivision in terms of more familiar algebro-geometric operations.

1.4 Log intersection theory

We can now define the logarithmic Chow ring of $\overline{\mathcal{M}}_{g,n}$:

$$\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n}) = \varinjlim_{\widetilde{\mathcal{M}}} \mathsf{CH}^*(\widetilde{\mathcal{M}})\,. \tag{5}$$

The direct limit is taken over those log modifications $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ where the domain $\widetilde{\mathcal{M}}$ is a nonsingular Deligne-Mumford stack. There exists a morphism

$$\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$$

precisely if the associated subdivision $\widetilde{\Sigma}$ refines the subdivision $\widetilde{\Sigma}'$. Given such a morphism, there is a pullback map $\mathsf{CH}^*(\widetilde{\mathcal{M}}') \to \mathsf{CH}^*(\widetilde{\mathcal{M}})$ which is used to define the above direct limit.

A singular Deligne-Mumford stack $\mathcal{M}$ has an operational Chow ring $\mathsf{CH}^*_{\mathsf{op}}(\mathcal{M})$, as defined in [79, Sect. 5]. Since operational classes admit pullbacks under arbitrary maps, the limit (5) can be taken over *all* log modifications $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ using operational Chow theory:

$$\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n}) = \varinjlim_{\text{all } \widetilde{\mathcal{M}}} \mathsf{CH}^*_{\mathsf{op}}(\widetilde{\mathcal{M}})\,. \tag{6}$$

Definitions (5) and (6) agree since every log modification can be further modified to desingularize the domain and since operational and classical Chow groups agree for nonsingular Deligne-Mumford stacks [79, Proposition 5.6].

Viewing $\overline{\mathcal{M}}_{g,n}$ as the trivial log modification of itself, there exists a canonical algebra morphism,

$$\mathsf{CH}^*(\overline{\mathcal{M}}_{g,n}) \to \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})\,,$$

which is injective, since an inverse map of $\mathbb{Q}$-vector spaces

$$\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n}) \to \mathsf{CH}^*(\overline{\mathcal{M}}_{g,n})$$

is given by proper pushforward under the maps $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$. The second map fails to be a ring morphism in general. For more details on logarithmic Chow rings and intersection theory see [13, 43, 62, 64].

1.5 Resolving the Abel-Jacobi map

We describe now Holmes' construction [41] of the log double ramification cycle in $\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})$. Let

$$\mu : \mathcal{J}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

be the (noncompact) universal Jacobian of line bundles of multidegree 0 on stable curves of genus g with n marked points. Recalling that $A = (a_1, \ldots, a_n)$ is a vector

of integers with sum $k(2g-2+n)$, the Abel-Jacobi map aj_A is the rational map defined by

$$\mathsf{aj}_A : \overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{J}_{g,n}\,, \quad \mathsf{aj}_A([C,x_1,\ldots,x_n]) = (\omega_C^{\log})^{\otimes k}\Big(-\sum_{i=1}^{n} a_i x_i\Big)\,. \tag{7}$$

In [41], Holmes constructs a universal birational map $\mathcal{U}^{\diamond}_{g,A} \to \overline{\mathcal{M}}_{g,n}$ on which the Abel-Jacobi map (7) can be extended. The space $\mathcal{U}^{\diamond}_{g,A}$ sits as an open substack in a *noncanonical* log modification

$$\rho : \overline{\mathcal{M}}^{\diamond}_{g,A} \to \overline{\mathcal{M}}_{g,n}\,, \tag{8}$$

so we have a diagram

$$\begin{array}{ccccc} & & \rho^*\mathcal{J}_{g,n} & \longrightarrow & \mathcal{J}_{g,n} \\ & \overset{\mathsf{aj}_A}{\nearrow} & \downarrow & & \downarrow \\ \mathcal{U}^{\diamond}_{g,A} & \subseteq & \overline{\mathcal{M}}^{\diamond}_{g,A} & \xrightarrow{\rho} & \overline{\mathcal{M}}_{g,n} \end{array}$$

Denoting by $e \subseteq \rho^*\mathcal{J}_{g,n}$ the preimage of the zero section of the universal Jacobian, the inverse image $\mathsf{aj}_A^{-1}(e) \subset \mathcal{U}^{\diamond}_{g,A}$ is proper (compact). The refined intersection product $\mathsf{aj}_A^*([e])$ then defines a cycle class in $\mathsf{CH}^g(\overline{\mathcal{M}}^{\diamond}_{g,n})$ which represents $\mathsf{logDR}_{g,A}$.

While the construction of the blow-up (8) is not canonical nor even explicit,[5] the resulting logarithmic cycle class

$$\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})\,,$$

is well-defined by [41, Theorem 1.2]. The most basic properties are:

- If $n=0$ and $A=\emptyset$, then $\mathsf{logDR}_{g,\emptyset} = (-1)^g\lambda_g$, where the top Chern class of the Hodge bundle λ_g is pulled back from $\mathsf{CH}^g(\overline{\mathcal{M}}_g)$, see [62].
- If $k=0$, the class $\mathsf{logDR}_{g,A}$ pushes forward to the standard double ramification cycle $\mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n})$ defined via the moduli space of rubber maps by [41, Theorem 1.3].

Since the $n=0$ case is solved, we will always assume $n \geq 1$.

In order to calculate $\mathsf{logDR}_{g,A}$ using the above construction via Abel-Jacobi theory, several difficulties must be overcome:

(i) Since the construction of the blow-up $\rho : \overline{\mathcal{M}}^{\diamond}_{g,A} \to \overline{\mathcal{M}}_{g,n}$ in [41] is only implicit (depending upon noncanonical choices), a direct study of $\overline{\mathcal{M}}^{\diamond}_{g,A}$ is difficult.

(ii) The class $\mathsf{aj}_A^*([e])$ arises from the geometry of the Abel-Jacobi map on the open set $\mathcal{U}^{\diamond}_{g,A}$, not the global geometry of $\overline{\mathcal{M}}^{\diamond}_{g,A}$, and so is not directly accessible via intersection theory on $\overline{\mathcal{M}}^{\diamond}_{g,A}$.

[5] An abstract resolution of singularities is required.

(iii) Even if the above issues could be overcome, in what language would the answer be expressed?

Our solution to both (i) and (ii) is to find geometrically meaningful models for $\overline{\mathcal{M}}^{\diamond}_{g,A}$ via the moduli spaces of line bundles on quasi-stable curves. The noncanonical aspect of $\overline{\mathcal{M}}^{\diamond}_{g,A}$ is not completely lost. There is still a choice of stability condition needed to define the moduli space of line bundles, but the stability condition is the only choice. By applying the main result of [43] together with the formula of [10] for the universal double ramification cycle, we can calculate $\mathsf{logDR}_{g,A}$. For (iii), the answer is expressed in the subring of tautological classes in $\mathsf{logCH}^{\star}(\overline{\mathcal{M}}_{g,n})$, suggested by D. Ranganathan[6] and developed in [43, 62, 64]. Tautological classes include κ_1, the cotangent line classes ψ_i, and classes coming from the algebra of piecewise polynomials on the cone stack associated to $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$.

As discussed above, the formula for $\mathsf{logDR}_{g,A}$ depends upon the choice of an appropriate[7] stability condition. The dependence can be useful: special stability conditions can be selected to simplify the formula depending upon the properties of the vector A. The wall-crossing study of the formula leads to relations in $\mathsf{logCH}^{g}(\overline{\mathcal{M}}_{g,n})$ when different stability conditions calculate the same class.

1.6 Moduli of line bundles on stable curves

A basic difficulty in writing a formula for the logarithmic double ramification cycle is the noncompactness of the universal Jacobian $\mathcal{J}_{g,n}$ of multidegree 0 line bundles on stable curves. As a consequence, the universal space $\mathcal{U}^{\diamond}_{g,A}$ is also not compact. Our approach here is to view the rational map

$$\mathsf{aj}_A : \overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{J}_{g,n}$$

as taking values in a *compactified* Jacobian and to resolve the indeterminacies of aj_A. Such resolutions of indeterminancies turn out to be proper and provide modular compactifications of $\mathcal{U}^{\diamond}_{g,A}$ of precisely the type needed to compute $\mathsf{logDR}_{g,A}$.

The construction of compactifications of the moduli spaces of line bundles on stable curves goes back at least to [21, 69]. In the past decades, there has been a continuous study of these spaces, see [5, 14, 22, 23, 33, 34, 49, 57, 58]. A short summary of what we need for our approach to the logarithmic double ramification cycle is presented here.

Since $\mathcal{J}_{g,n}$ is not compact, we can find 1-parameter families of nonsingular curves carrying line bundles of degree 0 degenerating to a stable nodal curve where the line bundle fails to degenerate to a line bundle of degree 0 on every irreducible component of the nodal curve. The issue is not simply about multidegrees. Searching for limits among all line bundles of total degree 0 does not suffice and, in fact, further

[6] In the lecture by D. Ranganathan in the *Algebraic Geometry and Moduli Zoominar* at ETH Zürich in April 2020.

[7] The formula is well-defined for all *nondegenerate* stability conditions, but calculates $\mathsf{logDR}_{g,A}$ only for *small* stability conditions. The precise definitions are given in Sect. 4.

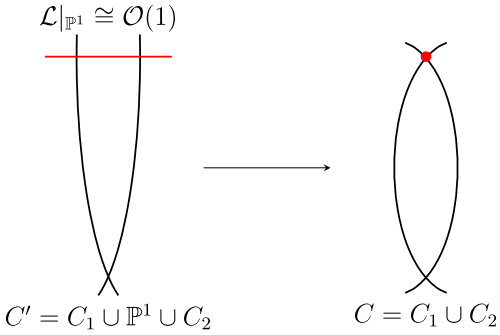


Fig. 3 A quasi-stable curve C' with stabilization C, and an admissible line bundle $\mathcal{L}$ on it

complicates the geometry: the Picard scheme $\mathcal{P}ic\,(\mathcal{C}_{g,n}/\overline{\mathcal{M}}_{g,n})$ of line bundles of all multidegrees on stable curves

$$\mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

is neither separated nor universally closed.

In order to obtain a compactification of $\mathcal{J}_{g,n}$, we must include limits that are not line bundles on stable curves. There are two equivalent approaches. The first is to consider the moduli space of all rank 1 torsion free sheaves on stable curves. The second, which we will follow, is to consider the moduli space of *admissible* line bundles on *quasi-stable* curves.

A flat family of nodal curves $\mathcal{C} \to \mathcal{S}$ is *quasi-stable* if the relative dualizing sheaf $\omega_{\mathcal{C}/\mathcal{S}}$ is nef and chains of unstable components have length at most 1. A line bundle $\mathcal{L}$ on $\mathcal{C}$ is *admissible* if $\mathcal{L}$ has degree 1 on each unstable component of $\mathcal{C}$. The data of an admissible line bundle on a quasi-stable curve is equivalent to the data of a rank 1 torsion free sheaf on the associated stabilization,

$$\mathsf{st} : \mathcal{C} \to \mathcal{C}^{\mathrm{st}},$$

by assigning to an admissible line bundle $\mathcal{L}$ the rank 1 torsion free sheaf $\mathsf{st}_*\mathcal{L}$, see [34] and Fig. 3 for an illustration.

The resulting moduli space parametrizing either admissible line bundles on quasi-stable curves or rank 1 torsion free sheaves on stable curves is the *universal Jacobian* which is universally closed over the moduli space of stable curves, but not separated.

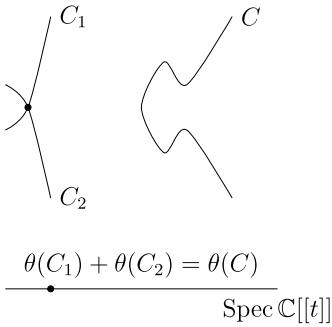
The failure of separation of the universal Jacobian occurs because of the possibility of twisting $\mathcal{L}$ by divisors $\mathcal{T} \subset \mathcal{C}$ supported over nodal curves. Twisting defines an equivalence relation on the *universal* Jacobian by declaring

$$(C, \mathcal{L}) \sim (C, \mathcal{L} \otimes \mathcal{O}(\mathcal{T}))\,.$$

To find compactifications of the Jacobian $\mathcal{J}_{g,n}$ which are proper and separated, we must choose a unique representative in each equivalence class of twistings. Stability conditions exactly make such choices.

Following [49], a *stability condition* θ of type (g, n) and degree d is a rule which assigns a rational number to every irreducible component of every stable curve $(C, x_1, \ldots, x_n)$ of genus g with n marked points and satisfies two properties:

(i) the sum of the values of θ over the irreducible components of C equals d,
(ii) θ is additive under all partial smoothings:



We may also view a stability condition θ in combinatorial terms: θ is an assignment of a rational value to every vertex of every stable graph G of type (g, n) which sums to d and which is additive with respect to contractions of edges.

There is a *trivial stability condition* θ^{tr} of degree 0 given by

$$\theta^{\mathsf{tr}}(D) = 0$$

for every irreducible component $D \subset C$. A more interesting example (introduced in [21, 57]) is the *canonical stability condition* θ^K of degree $2g - 2 + n$ defined by

$$\theta^K(D) = \mathsf{deg}\left[\omega_C\Big(\sum_{i=1}^{n} x_i\Big)\Big|_D\right]$$

for an irreducible component $D \subset C$. The value of θ^K depends on the genus of D and the number nodes and markings of C which lie on D.

Fix a stability condition θ of degree d. An admissible line bundle $\mathcal{L}$ of degree d on a quasi-stable curve C is *θ-stable* (respectively, *θ-semistable*) if and only if, for every proper subcurve[8] $\emptyset \subsetneq \underline{C} \subsetneq C$ with neither $\underline{C}$ nor its complement consisting entirely of unstable components, we have

$$\theta(\underline{C}) - \frac{E(\underline{C}, \underline{C}^c)}{2} \;<\; (\leq)\; \mathsf{deg}(\mathcal{L}|_{\underline{C}}) \;<\; (\leq)\; \theta(\underline{C}) + \frac{E(\underline{C}, \underline{C}^c)}{2},$$

where $E(\underline{C}, \underline{C}^c)$ is the number of intersection points of $\underline{C}$ with the complementary subcurve $\underline{C}^c \subset C$ and

$$\theta(\underline{C}) = \sum_{D \subset \underline{C}} \theta(D)$$

is the sum of the values of θ over all irreducible components $D \subset \underline{C}$.[9] In other words,

[8] We will later phrase these inequalities in combinatorial terms using the dual graph of C.

[9] The stability condition θ extends to quasi-stable curves by assigning weight 0 to all unstable components.

an admissible line bundle $\mathcal{L}$ is θ-semistable if the degree of $\mathcal{L}$ restricted to each subcurve $\underline{C} \subset C$ is close enough to the assigned value $\theta(\underline{C})$.

The connection with the above discussion about compactifications of the Jacobian can be seen as follows. A nontrivial twist by $\mathcal{O}(\mathcal{T})$ in families changes the degree on $\underline{C}$ by at least $E(\underline{C}, \underline{C}^c)$ for some subcurve $\underline{C} \subset C$. Thus, there can be at most one representative of the equivalence class of a line bundle which is θ-stable. A stability condition θ is *nondegenerate* if every θ-semistable line bundle on every quasi-stable curve C is stable.

Let θ be a nondegenerate stability condition of type (g, n). By a result of [49] (see also [58] for an earlier, alternative perspective), there exists a moduli stack $\mathcal{P}^{\theta}_{g,n}$ of θ-stable admissible line bundles on quasi-stable curves, satisfying the following properties:

(i) The stack $\mathcal{P}^{\theta}_{g,n}$ is proper, nonsingular, and of dimension $4g - 3 + n$.
(ii) There is natural morphism $\mu : \mathcal{P}^{\theta}_{g,n} \to \overline{\mathcal{M}}_{g,n}$.
(iii) The stack $\mathcal{P}^{\theta}_{g,n}$ carries a universal quasi-stable curve with a universal admissible line bundle.[10]

In general, neither the trivial stability condition θ^{tr} nor the canonical stability condition θ^K is nondegenerate. However, in case $n \geq 1$, explicit perturbations can be easily constructed in both cases and seen to be nondegenerate (as explained in [49] and reviewed in Sect. 4).

A stability condition θ of degree 0 is *small* if line bundles of multidegree 0 on the universal curve $\mathcal{C}_{g,n}$ are θ-stable. Nondegenerate perturbations of θ^{tr} sufficiently close to θ^{tr} are small. For our formula for the logarithmic double ramification cycle, we will require θ to be a small nondegenerate stability condition of degree 0.

Recall that $A = (a_1, \ldots, a_n)$ is a vector of integers summing to $k(2g - 2 + n)$. Given a small nondegenerate stability condition θ of degree 0 and type (g, n), there is a rational Abel-Jacobi section of the morphism μ,

$$\mathsf{aj}_A : \overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{P}^{\theta}_{g,n} ,$$

defined (as before) by

$$\mathsf{aj}_A([C, x_1, \ldots, x_n]) = (\omega_C^{\log})^{\otimes k}\Big(- \sum_{i=1}^{n} a_i x_i \Big) .$$

As the stability condition θ is nondegenerate, the space $\mathcal{P}^{\theta}_{g,n}$ is proper. The stability condition θ then canonically[11] determines a log modification

$$\rho : \overline{\mathcal{M}}^{\theta}_{g,A} \to \overline{\mathcal{M}}_{g,n} \tag{9}$$

which resolves the Abel-Jacobi map,

$$\mathsf{aj} : \overline{\mathcal{M}}^{\theta}_{g,A} \to \mathcal{P}^{\theta}_{g,n} , \qquad \mathsf{aj} = \mathsf{aj}_A \circ \rho .$$

[10] The condition $n \geq 1$ is used for the existence of the universal line bundle.

[11] The subdivision $\widetilde{\Sigma}^{\theta} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ which determines the log modification ρ is defined in Sect. 1.7.2. The resolution of the Abel-Jacobi map $\mathsf{aj} : \overline{\mathcal{M}}^{\theta}_{g,A} \to \mathcal{P}^{\theta}_{g,n}$ is constructed in Sect. 4, see Definition 37.

Over $\overline{\mathcal{M}}^{\theta}_{g,A}$, we have a universal family

$$\pi : \mathcal{C}^{\theta} \to \overline{\mathcal{M}}^{\theta}_{g,A}$$

which is a quasi-stable model of the pullback $\overline{\mathcal{M}}^{\theta}_{g,A} \times_{\overline{\mathcal{M}}_{g,n}} \mathcal{C}$ of the universal stable curve, and an admissible line bundle

$$\mathcal{L}^{\theta} \to \mathcal{C}^{\theta}$$

obtained from the universal admissible line bundle on the moduli stack of θ-stable sheaves. Since θ is small, $\mathcal{J}_{g,n} \subset \mathcal{P}^{\theta}_{g,n}$ is an open substack. In fact, $\mathcal{U}^{\diamond}_{g,A}$ is an open substack of $\overline{\mathcal{M}}^{\theta}_{g,A}$, and $\overline{\mathcal{M}}^{\theta}_{g,A}$ can play the role of $\overline{\mathcal{M}}^{\diamond}_{g,A}$ in the definition of [41]. The class

$$\mathsf{aj}^{*}_{A}[e] \in \mathsf{CH}^{g}_{\mathsf{op}}(\overline{\mathcal{M}}^{\theta}_{g,A})$$

therefore represents $\mathsf{logDR}_{g,A}$.

The geometry here is much more favorable than for the abstractly defined spaces $\overline{\mathcal{M}}^{\diamond}_{g,A}$. After the small nondegenerate stability condition θ has been chosen, there are *no* further choices, and we have complete understanding of the points of $\overline{\mathcal{M}}^{\theta}_{g,A}$. A modular interpretation of $\overline{\mathcal{M}}^{\theta}_{g,A}$ is described in Definition 33 of Sect. 4.2. Furthermore, $\overline{\mathcal{M}}^{\theta}_{g,A}$ naturally carries a universal double ramification cycle class

$$\mathsf{DR}^{\mathsf{op}}_{g,\mathcal{L}^{\theta}} = \varphi^{*}_{\mathcal{L}^{\theta}} \mathsf{DR}^{\mathsf{op}}_{g,\emptyset} \in \mathsf{CH}^{g}_{\mathsf{op}}(\overline{\mathcal{M}}^{\theta}_{g,A}),$$

where $\mathfrak{Pic}_g$ is the universal Picard stack (parameterizing nodal genus g curves together with a line bundle), the map

$$\varphi_{\mathcal{L}^{\theta}} : \overline{\mathcal{M}}^{\theta}_{g,A} \to \mathfrak{Pic}_g$$

is defined by the family $(\mathcal{C}^{\theta}, \mathcal{L}^{\theta})$ over $\overline{\mathcal{M}}^{\theta}_{g,A}$, and $\mathsf{DR}^{\mathsf{op}}_{g,\emptyset} \in \mathsf{CH}^{g}_{\mathsf{op}}(\mathfrak{Pic}_g)$ is the universal double ramification cycle of [10].[12] By applying the theory of almost-twistable families developed in [43], we obtain the first form of our main result (which simply states that the two natural classes on $\overline{\mathcal{M}}^{\theta}_{g,A}$ are equal).

Theorem A *Let θ be a small nondegenerate stability condition. The universal double ramification cycle associated to the line bundle $\mathcal{L}^{\theta}$ on $\mathcal{C}^{\theta} \to \overline{\mathcal{M}}^{\theta}_{g,A}$,*

$$\mathsf{DR}^{\mathsf{op}}_{g,\emptyset,\mathcal{L}^{\theta}} \in \mathsf{CH}^{g}_{\mathsf{op}}(\overline{\mathcal{M}}^{\theta}_{g,A}),$$

provides a representative for $\mathsf{logDR}_{g,A}$.

[12] The notation $\emptyset$ in $\mathsf{DR}^{\mathsf{op}}_{g,\emptyset}$ indicates the absence of marked points.

Though the claim of Theorem A is conceptual, the statement can be transformed into an explicit formula for $\mathsf{logDR}_{g,A}$ since the universal double ramification cycle has been explicitly computed in [10]. Theorem A is proven in Sect. 5.

1.7 Formula for logDR

The final step is to translate Theorem A into an explicit class in the logarithmic Chow ring using the formula for the universal double ramification cycle in [10].

1.7.1 Tautological classes in the logarithmic Chow ring

As we have seen, the logarithmic modifications $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ appearing in the definition of the logarithmic Chow ring are described by subdivisions $\widetilde{\Sigma} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$ of the cone stack. Following [43, 62, 64], we can use the same convex-geometric language to describe natural cycle classes on $\widetilde{\mathcal{M}}$ which define elements of $\mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})$.[13]

Let Σ be a fan in $\mathbb{R}^m$ with maximal cones all of dimension m (the dimension of the ambient vector space). A *strict piecewise polynomial* $f \in \mathsf{sPP}(\Sigma)$ is a continuous function $f : |\Sigma| \to \mathbb{R}$ on the support $|\Sigma| \subseteq \mathbb{R}^m$ of Σ which is given by a polynomial with rational coefficients on each maximal cone of Σ. A *piecewise polynomial function* $f \in \mathsf{PP}(\Sigma)$ is a continuous function $f : |\Sigma| \to \mathbb{R}$ which is a strict piecewise polynomial on *some* subdivision Σ' of Σ.

Given a subdivision $\widetilde{\Sigma} = (\widetilde{\Sigma}_\Gamma)_{\Gamma \in \mathcal{G}_{g,n}}$ of the cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$, a strict piecewise polynomial function

$$f = (f_\Gamma)_{\Gamma \in \mathcal{G}_{g,n}} \in \mathsf{sPP}(\widetilde{\Sigma})$$

is a collection of strict piecewise polynomials f_Γ on the fans $\widetilde{\Sigma}_\Gamma$, which are compatible with the face maps ι_φ. More precisely, for $\varphi : \Gamma \to \Gamma'$, we have $f_\Gamma \circ \iota_\varphi = f_{\Gamma'}$. A piecewise polynomial on $\widetilde{\Sigma}$ is then similarly a collection $f = (f_\Gamma)_\Gamma$ of piecewise polynomials f_Γ satisfying the same compatibility.

The (strict) piecewise polynomials on $\widetilde{\Sigma}$ form a $\mathbb{Q}$-algebra. For the log modification $\widetilde{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$ associated to $\widetilde{\Sigma}$, there are natural ring morphisms

$$\Phi : \mathsf{sPP}(\widetilde{\Sigma}) \to \mathsf{CH}^*_{\mathrm{op}}(\widetilde{\mathcal{M}}) \tag{10}$$

constructed in [43, Sect. 3.3].[14] When $\widetilde{\Sigma}$ is the cone over a simplicial complex the construction proceeds in two steps:

(i) There exists a natural map from strict piecewise *linear* functions on $\widetilde{\Sigma}$ to divisor classes on $\widetilde{\mathcal{M}}$ using the fact that rays of $\widetilde{\Sigma}$ correspond to boundary divisors of $\widetilde{\mathcal{M}}$.

[13] The language of log structures and ghost sheaves is used in [43]. Our presentation avoids ghosts and follows instead the language of fans and cones from toric geometry. The comparison of terminology is straightforward and discussed in more detail in Remark 16.

[14] Following our analogy with toric varieties, the equivariant Chow ring of a toric variety is isomorphic to the ring of strict piecewise polynomials on the associated fan, see [71].

(ii) The strict piecewise linear functions generate $\mathsf{sPP}(\widetilde{\Sigma})$ as a $\mathbb{Q}$-algebra. The unique extension of (i) defines Φ.

In general, we can subdivide $\widetilde{\Sigma}$ until it is the cone over a simplicial complex and then push forward the resulting class.

The maps Φ are compatible under subdivision of $\widetilde{\Sigma}$ and the associated birational modification of $\widetilde{\mathcal{M}}$, and thus give rise to an algebra morphism

$$\Phi : \mathsf{PP}(\Sigma_{\overline{\mathcal{M}}_{g,n}}) \to \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n}) . \tag{11}$$

An abstract construction of Φ can be found in [62, 64]. A more explicit description of Φ is presented in Sect. 6.2.5.

Definition 1 [43, Definition 3.18] The *logarithmic tautological ring*

$$\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})$$

is the subring generated by the image of Φ and the usual tautological ring[15]

$$\mathsf{R}^*(\overline{\mathcal{M}}_{g,n}) \subseteq \mathsf{CH}^*(\overline{\mathcal{M}}_{g,n}) \subseteq \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n}) .$$

In particular, the above definition gives a natural surjection

$$\mathsf{R}^*(\overline{\mathcal{M}}_{g,n}) \otimes_{\mathbb{Q}} \mathsf{PP}(\Sigma_{\overline{\mathcal{M}}_{g,n}}) \to \mathsf{logR}^*(\overline{\mathcal{M}}_{g,n}) . \tag{12}$$

By the results of [43, 62], we know $\mathsf{logDR}_{g,A} \in \mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$.

Our goal is to describe an explicit element of the left side of (12) which maps to $\mathsf{logDR}_{g,A}$. As we will see in Sect. 1.7.3, the formulas describing the element naturally involve taking exponentials of certain piecewise linear functions. Exponentiation leaves the realm of piecewise polynomials and yields *piecewise power series*. To make sense of the formulas below, we simply observe that the map Φ above naturally extends to such piecewise power series by first truncating the power series to degree (at most) $3g - 3 + n$ to obtain a piecewise polynomial.

1.7.2 The subdivision associated to a stability condition

Let θ be a nondegenerate stability condition of type (g, n) and degree 0 (as defined in Sect. 1.6). The data of the vector $A = (a_1, \ldots, a_n)$ and θ together determine a canonical subdivision

$$\widetilde{\Sigma}^{\theta} \to \Sigma_{\overline{\mathcal{M}}_{g,n}}$$

which we construct here. The subdivision $\widetilde{\Sigma}^{\theta}$ defines the log modification

$$\rho : \overline{\mathcal{M}}^{\theta}_{g,A} \to \overline{\mathcal{M}}_{g,n}$$

from (9). We will require the following notation for the construction:

[15] We refer the reader to [70] for an introduction to the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

• For a quasi-stable graph Γ, let $\vec{E}(\Gamma)$ be the set of *oriented edges* of Γ. The natural map

$$\vec{E}(\Gamma) \to E(\Gamma)$$

has fibers $\{\vec{e}, \overleftarrow{e}\}$ of size two, the two orientations on a given edge e. For $\vec{e} \in \vec{E}(\Gamma)$, we write

$$\vec{e} \to v \in V(\Gamma)$$

if $\vec{e}$ is incident to and points *towards* v. A *cycle* γ on Γ is a collection of oriented edges forming a cycle without self-intersection.

Given a family $(C, \mathcal{L})$ of curves and line bundles, the change of the multidegree of $\mathcal{L}$ on a special fiber C_0 by twisting with a vertical divisor $\mathcal{T}$ (as in Sect. 1.6) is described by an acyclic flow on Γ_{C_0}. Acyclic flows appear under the name of *twists* in [36].

• A *flow* on Γ is a function $f : \vec{E}(\Gamma) \to \mathbb{Z}$ satisfying $f(\overleftarrow{e}) = -f(\vec{e})$ for all edges $e \in E(\Gamma)$. The group of flows on Γ is denoted $\mathrm{Flow}(\Gamma)$.[16] By choosing some fixed orientation on Γ, we can noncanonically identify the group $\mathrm{Flow}(\Gamma)$ with $\mathbb{Z}^{E(\Gamma)}$. A flow f is called *acyclic* if there exists no cycle γ of Γ satisfying the following conditions:

$$\forall \vec{e} \in \gamma\,,\ f(\vec{e}) \geq 0 \quad \text{and} \quad \exists \vec{e} \in \gamma\,,\ f(\vec{e}) > 0\,.$$

• A *divisor* on Γ is an element of $\mathbb{Z}^{V(\Gamma)}$. For $f \in \mathrm{Flow}(\Gamma)$, we write

$$\mathrm{div}(f) = \left(\sum_{\vec{e} \to v} f(\vec{e}) \right)_{v \in V(\Gamma)} \in \mathbb{Z}^{V(\Gamma)} \tag{13}$$

for the divisor of f. The first homology group $H_1(\Gamma)$ is precisely the kernel of the group homomorphism $\mathrm{div} \colon \mathrm{Flow}(\Gamma) \to \mathbb{Z}^{V(\Gamma)}$. There is a straightforward generalization of flows and divisors allowing values in an arbitrary abelian group instead of $\mathbb{Z}$.

• Given a length assignment $\ell : E(\Gamma) \to \mathbb{R}_{\geq 0}$, we have a *pairing*[17]

$$\langle f, g \rangle_\ell = \frac{1}{2} \sum_{\vec{e} \in \vec{E}(\Gamma)} \ell(e) f(\vec{e}) g(\vec{e}) \tag{14}$$

for $f, g \in \mathrm{Flow}(\Gamma)$.

[16] In the literature, flows are also known as *1-cocycles*, and the group is denoted $C^1(\Gamma, \mathbb{Z})$.

[17] The importance of the pairing will be seen in Lemma 26: a flow arises as the slopes of a *piecewise linear function* (Definition 24) on the metric graph (Γ, ℓ) if and only if the pairing with every element of $H_1(\Gamma)$ vanishes.

• The multidegree of the line bundle $(\omega_C^{\log})^{\otimes k}(-\sum_i a_i x_i)$ on a quasi-stable curve C with graph Γ is

$$\underline{\deg}_{k,A} = \left(k(2g(v) - 2 + n(v)) - \sum_{i \text{ at } v} a_i \right)_{v \in V(\Gamma)} \in \mathbb{Z}^{V(\Gamma)},$$

where $n(v)$ is the number of half-edges attached to v (the number of markings at v plus the number of ends of edges at v).

We can now describe the subdivision $\widetilde{\Sigma}^\theta = \{\widetilde{\Sigma}^\theta_\Gamma\}_{\Gamma \in \mathcal{G}_{g,n}}$ associated to the nondegenerate stability condition θ. Fix a stable graph Γ with associated cone

$$\sigma_\Gamma = (\mathbb{R}_{\geq 0})^{E(\Gamma)} \text{ in } \Sigma_{\overline{\mathcal{M}}_{g,n}}.$$

The fan $\widetilde{\Sigma}^\theta_\Gamma$ of the subdivision has interior cones[18] in bijective correspondence with tuples $(\widehat{\Gamma}, D, I)$ where

(i) $\widehat{\Gamma}$ is a quasi-stable graph with stabilization $\widehat{\Gamma}^s = \Gamma$,
(ii) $D \in \mathbb{Z}^{V(\widehat{\Gamma})}$ is a θ-stable multidegree,
(iii) I is an acyclic flow on $\widehat{\Gamma}$ satisfying

$$\operatorname{div}(I) = \underline{\deg}_{k,A} - D. \tag{15}$$

For each tuple $(\widehat{\Gamma}, D, I)$ as above, we will define a cone $\sigma_{\widehat{\Gamma},I} \subseteq \sigma_\Gamma$. The collection of cones $\{\sigma_{\widehat{\Gamma},I}\}$ together with their faces defines the subdivision $\widetilde{\Sigma}^\theta_\Gamma$.

Let $\sigma_{\widehat{\Gamma}} = (\mathbb{R}_{\geq 0})^{E(\widehat{\Gamma})}$ be the cone associated to the quasi-stable graph $\widehat{\Gamma}$. There is a natural map

$$\mathsf{pr} : \sigma_{\widehat{\Gamma}} \to \sigma_\Gamma, \quad \widehat{\ell} \mapsto \ell : e \mapsto \begin{cases} \widehat{\ell}(e_1) + \widehat{\ell}(e_2) & \text{if } \widehat{\Gamma} \text{ subdivides edge } e \text{ into } e_1, e_2, \\ \widehat{\ell}(e) & \text{otherwise.} \end{cases}$$

We define a subcone $\tau_{\widehat{\Gamma},I} \subseteq \sigma_{\widehat{\Gamma}}$ by the condition

$$\tau_{\widehat{\Gamma},I} = \left\{ \widehat{\ell} \in \sigma_{\widehat{\Gamma}} \;\middle|\; \langle \gamma, I \rangle_{\widehat{\ell}} = 0 \text{ for all } \gamma \in H_1(\widehat{\Gamma}) \right\},$$

where the pairing is (14) and $H_1(\widehat{\Gamma})$ is identified with a subgroup of $\mathrm{Flow}(\widehat{\Gamma})$ as explained above. The following crucial claim is proven in Lemma 38 of Sect. 4.

Claim 1 The map $\mathsf{pr}\colon \sigma_{\widehat{\Gamma}} \to \sigma_\Gamma$ induces an isomorphism from the cone $\tau_{\widehat{\Gamma},I}$ to the image

$$\sigma_{\widehat{\Gamma},I} = \mathsf{pr}(\tau_{\widehat{\Gamma},I}) \subseteq \sigma_\Gamma.$$

Moreover, the collection of cones $\{\sigma_{\widehat{\Gamma},I}\}$ together with their faces forms a fan $\widetilde{\Sigma}^\theta_\Gamma$ with support σ_Γ.

[18] An *interior cone* of $\widetilde{\Sigma}^\theta_\Gamma$ is a cone of the fan intersecting the interior $(\mathbb{R}_{>0})^{E(\Gamma)}$ of σ_Γ. The interior cones uniquely determine the fan $\widetilde{\Sigma}^\theta_\Gamma$: the fan is the collection of these interior cones together with all of their faces.

We record an important consequence of Claim 1. Let ℓ_e, $\widehat{\ell_{\widehat{e}}}$ be the coordinate functions on $\sigma_\Gamma = (\mathbb{R}_{\geq 0})^{E(\Gamma)}$, $\sigma_{\widehat{\Gamma}} = (\mathbb{R}_{\geq 0})^{E(\widehat{\Gamma})}$ giving the lengths of edges $e \in E(\Gamma)$, $\widehat{e} \in E(\widehat{\Gamma})$, viewed now as polynomials on these cones.

Claim 2 On the cone $\sigma_{\widehat{\Gamma},I}$, there exist unique linear functions $\widehat{\ell_{\widehat{e}}} = \widehat{\ell_{\widehat{e}}}(\ell)$ in the variables ℓ_e which define a section of the map $\mathsf{pr} : \tau_{\widehat{\Gamma},I} \to \sigma_{\widehat{\Gamma},I}$.

The above definition of the cones $\sigma_{\widehat{\Gamma},I}$ is rather formal. Geometric intuition for the construction can be found in the theory developed in Sects. 3 and 4, see Remark 39. A detailed example is worked out in Sect. 4.3.

1.7.3 The formula for logDR

Let θ be a small nondegenerate stability condition. We present here an explicit formula for the cycle

$$\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n}) .$$

To write the formula, we must first define two special strict piecewise power series $\mathfrak{P}$, $\mathfrak{L}$ on $\widetilde{\Sigma}^\theta$. The functions $\mathfrak{P}$, $\mathfrak{L}$ are uniquely determined by their restriction to the interior cones of $\widetilde{\Sigma}^\theta$. As discussed in Sect. 1.7.2, the interior cones correspond to a stable graph $\Gamma \in \mathcal{G}_{g,n}$ together with a tuple $(\widehat{\Gamma}, D, I)$. We will define the functions $\mathfrak{P}$, $\mathfrak{L}$ on the associated cone $\sigma_{\widehat{\Gamma},I} \in \widetilde{\Sigma}^\theta_\Gamma$.

• The definition of $\mathfrak{P}$ requires a sum over weightings: for a positive integer r, an *admissible weighting mod r* on $\widehat{\Gamma}$ is a flow w with values in $\mathbb{Z}/r\mathbb{Z}$ such that $\mathrm{div}(w)$ is equal modulo r to D:

$$\mathrm{div}(w) = (D \text{ reduced mod } r) \in (\mathbb{Z}/r\mathbb{Z})^{V(\widehat{\Gamma})} .$$

We define

$$\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)} = \sum_w r^{-h^1(\widehat{\Gamma})} \prod_{e \in E(\widehat{\Gamma})} \exp\left(\frac{\overline{w}(\vec{e}) \cdot \overline{w}(\overleftarrow{e})}{2} \widehat{\ell}_e \right) \in \mathbb{Q}[[\widehat{\ell}_e : e \in E(\widehat{\Gamma})]] ,$$

where the sum runs over admissible weightings w mod r. Inside the exponential, $\overline{w}(\vec{e})$ and $\overline{w}(\overleftarrow{e})$ denote the unique representative of $w(\vec{e}) \in \mathbb{Z}/r\mathbb{Z}$ and $w(\overleftarrow{e}) \in \mathbb{Z}/r\mathbb{Z}$ in $\{0, \ldots, r-1\}$.

As in [47, Appendix], for each fixed degree in the variables $\widehat{\ell}_e$, the element $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$ is polynomial in r for sufficiently large r. We denote by $\mathrm{Cont}_{(\widehat{\Gamma},D,I)}$ the polynomial in the variables $\widehat{\ell}_e$ obtained by substituting $r = 0$ into the polynomial expression for $\mathrm{Cont}^r_{(\widehat{\Gamma},D,I)}$. We define

$$\mathfrak{P}\big|_{\sigma_{\widehat{\Gamma},I}} = \mathrm{Cont}_{(\widehat{\Gamma},D,I)}\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \in \mathbb{Q}[[\ell_e : e \in E(\Gamma)]] ,$$

where we use the variable substitution $\widehat{\ell} = \widehat{\ell}(\ell)$ associated to $\sigma_{\widehat{\Gamma},I}$ from Claim 2. We claim that these functions fit together to give a well-defined strict piecewise power series $\mathfrak{P}$ on $\widetilde{\Sigma}^\theta$.

• To define $\mathfrak{L}$ on $\widetilde{\Sigma}^{\theta}$, we fix a vertex $v_0 \in V(\widehat{\Gamma})$. For every length assignment $\widehat{\ell}$ in the cone $\tau_{\widehat{\Gamma},I}$ and any vertex $v \in V(\widehat{\Gamma})$, let $\gamma_{v_0 \to v}$ be a path from v_0 to v in $\widehat{\Gamma}$. We define

$$\alpha(v) = \sum_{\vec{e} \in \gamma_{v_0 \to v}} I(\vec{e}) \cdot \widehat{\ell}_e \,, \tag{16}$$

where the sum is over the oriented edges $\vec{e}$ constituting the path $\gamma_{v_0 \to v}$. The defining equations of $\tau_{\widehat{\Gamma},I}$ imply that for $\widehat{\ell} \in \tau_{\widehat{\Gamma},I}$ the expression (16) is independent of the chosen path $\gamma_{v_0 \to v}$. We define

$$\mathfrak{L} = \sum_{v \in V(\widehat{\Gamma})} (D + \underline{\deg}_{k,A})(v) \cdot \alpha(v)\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \in \mathbb{Q}[\ell_e : e \in E(\Gamma)] \,. \tag{17}$$

The substitution of variables $\widehat{\ell} = \widehat{\ell}(\ell)$, which give the inverse of the isomorphism $\tau_{\widehat{\Gamma},I} \to \sigma_{\widehat{\Gamma},I}$ and thus have image in $\tau_{\widehat{\Gamma},I}$, ensure that the expression is independent of the choice of the paths $\gamma_{v_0 \to v}$. The expression is also independent of the base vertex v_0, which follows from the fact that the divisor $D + \underline{\deg}_{k,A}$ has total degree 0 on $\widehat{\Gamma}$.

For the $\mathsf{logDR}_{g,A}$ formula, in addition to $\mathfrak{P}$ and $\mathfrak{L}$, we will also require the tautological class

$$\eta = k^2 \kappa_1 - \sum_{i=1}^{n} a_i^2 \psi_i \in \mathsf{R}^*(\overline{\mathcal{M}}_{g,n}) \,. \tag{18}$$

Define the mixed degree logarithmic class

$$\mathbf{P}^{\theta}_{g,A} = \exp\left(-\frac{1}{2}(\eta + \Phi(\mathfrak{L}))\right) \cdot \Phi(\mathfrak{P}) \in \mathsf{logR}^*(\overline{\mathcal{M}}_{g,n}) \,, \tag{19}$$

where Φ is the extension of the map (11) to piecewise power series as described at the end of Sect. 1.7.1.

Theorem B *Let θ be a small nondegenerate stability condition. The log double ramification cycle is the degree g part of $\mathbf{P}^{\theta}_{g,A}$,*

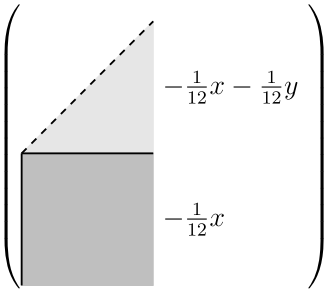
$$\mathsf{logDR}_{g,A} = \mathbf{P}^{g,\theta}_{g,A} \in \mathsf{logR}^g(\overline{\mathcal{M}}_{g,n}) \,.$$

As a consequence of Theorem B, the class

$$\mathbf{P}^{g,\theta}_{g,A} \in \mathsf{logR}^g(\overline{\mathcal{M}}_{g,n})$$

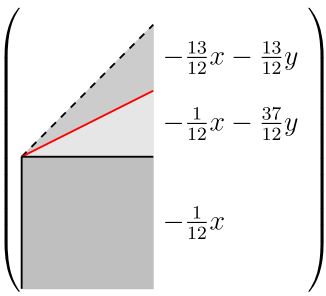
is *independent* of θ (for all small nondegenerate stability conditions θ). The definition of $\mathbf{P}^{g,\theta}_{g,A}$, however, only requires nondegeneracy of θ. The interpretation and wall-crossing analysis of $\mathbf{P}^{g,\theta}_{g,A}$ for large stability conditions θ is an interesting direction of study. A separate question is whether a formula for $\mathsf{logDR}_{g,A}$ can be written using degenerate stability conditions (such as the trivial one). Our approach requires

$$\mathsf{DR}_{1,(3,-3)} = \frac{9}{2}(\psi_1+\psi_2) + \Phi$$



$$\in \mathsf{R}^1(\overline{\mathcal{M}}_{1,2})\,,$$

$$\mathsf{logDR}_{1,(3,-3)} = \frac{9}{2}(\psi_1+\psi_2) + \Phi$$



$$\in \mathsf{logR}^1(\overline{\mathcal{M}}_{1,2})\,.$$

Fig. 4 Formulas for the cycles $\mathsf{DR}_{1,(3,-3)} = \frac{9}{2}(\psi_1+\psi_2) - \frac{1}{12}\delta_0$ and $\mathsf{logDR}_{1,(3,-3)}$, each consisting of a linear combination of classes ψ_1, ψ_2 from $\overline{\mathcal{M}}_{1,2}$ and a contribution from a piecewise linear function on a subdivision $\widetilde{\Sigma}$ of $\Sigma_{\overline{\mathcal{M}}_{1,2}}$ (the trivial subdivision in the case of the regular DR cycle). See Fig. 2 for an explanation of the coordinates x, y on the cone stack, where we now only draw the coarse space of the stacky cone

nondegeneracy (for the existence of a universal sheaf), and the formula given in Theorem B is not well-defined for degenerate θ.

The calculation of $\mathsf{logDR}_{g,A}$ plays a fundamental role in the log Gromov-Witten theory of toric varieties: by an elegant argument of Ranganathan and Urundolil Kumaran [74], the log Gromov-Witten theory of all nonsingular projective toric varieties (with log structure determined by the full toric boundary) can be reduced to $\mathsf{logDR}_{g,A}$.

After a discussion of Pixton's formula in the language of piecewise polynomials in Sect. 6, Theorem B is proven in Sect. 7.

1.8 Example in genus 1

We illustrate Theorem B for $\mathsf{logDR}_{1,(3,-3)}$ in Fig. 4 (giving a similar illustration of the regular DR cycle $\mathsf{DR}_{1,(3,-3)}$ for comparison). The general genus 1 computation is worked out in Sect. 8. A genus 2 calculation using the Sage package `logtaut` (part of `admcycles` [31]) is presented in Sect. 9.1.

1.9 Relations in $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$

The logarithmic double ramification cycle provides two nontrivial paths to tautological relations in $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$. The first path uses the choice of stability condition in Theorem B, and the second path is via Pixton's relations for the universal double ramification cycle [10].

Theorem C *The following two constructions yield relations in* $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$:

(i) *Let* θ *and* $\widehat{\theta}$ *be two small nondegenerate stability conditions of type* (g,n). *Then,*

$$\mathbf{P}^{g,\theta}_{g,A} = \mathbf{P}^{g,\widehat{\theta}}_{g,A} \in \mathsf{logR}^g(\overline{\mathcal{M}}_{g,n}).$$

(ii) *For every nondegenerate condition* θ *of type* (g,n),

$$\mathbf{P}^{h,\theta}_{g,A} = 0 \in \mathsf{logR}^h(\overline{\mathcal{M}}_{g,n})$$

for all $h > g$.

Theorem C is proven together with Theorem B in Sect. 7.

Higher double ramification cycles are discussed in Sect. 9. A third construction of relations in $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$ is presented in Sect. 9.2 via the *double-double ramification cycle*. The general study of $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$ will be taken up in a future paper.

1.10 Conventions

Throughout the paper, we work over the field $\mathbb{C}$ of complex numbers. All Chow groups are taken with $\mathbb{Q}$-coefficients. Furthermore, our monoids and log structures will always be fine and saturated (in the sense of [50, 67]), unless otherwise mentioned.

2 Logarithmic geometry

2.1 Overview

We recall here constructions from log geometry which will be needed later: tropicalization, Artin fans, and subdivisions. The discussion here is standard, but included for the convenience of the reader. Further details can be found in [1–3, 25, 68]. The main case of interest for us is the moduli space $\overline{\mathcal{M}}_{g,n}$ equipped with the divisorial log structure obtained from $\partial\overline{\mathcal{M}}_{g,n}$.

2.2 Artin fans and cone stacks

Definition 2 A *cone* is a pair (σ, N) consisting of a lattice N and a full dimensional strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$. The intersection $\sigma \cap N$ is an *integral structure* on σ. A *morphism* of cones

$$(\sigma, N) \to (\sigma', N')$$

is a homomorphism of lattices $N \to N'$ such that the induced map of vector spaces $N_{\mathbb{R}} \to N'_{\mathbb{R}}$ maps σ into σ'.

Let **RPC** be the category of cones. The category of cones is dual to the category of *fine and saturated sharp monoids*. A cone (σ, N) corresponds to the monoid

$$\overline{M}_\sigma = \mathsf{Hom}_{\mathrm{Mon}}(\sigma \cap N, \mathbb{N})\,.$$

Conversely, a monoid P corresponds to the cone

$$\left(\mathsf{Hom}_{\mathrm{Mon}}(P, \mathbb{R}_{\geq 0})\,,\ \mathsf{Hom}_{\mathrm{Gp}}(P^{\mathsf{gp}}, \mathbb{Z})\right).$$

A *face morphism* $(\sigma, N) \to (\sigma', N')$ is a map of cones that induces an isomorphism of σ with a (non-necessarily proper) face of σ'. We write $\mathbf{RPC}^f$ for the category of cones with face morphisms between them. Following [25], a *cone stack* is a category fibered in groupoids over $\mathbf{RPC}^f$.

An *Artin cone* is an algebraic stack of the form

$$\mathcal{A}_P = [\operatorname{Spec} \mathbb{C}[P] / \operatorname{Spec} \mathbb{C}[P^{\mathsf{gp}}]]$$

for a fine and saturated sharp monoid P. Artin cones carry natural logarithmic structures, obtained from the toric logarithmic structure on $\operatorname{Spec} \mathbb{C}[P]$ by descent.

A map of logarithmic stacks is *strict* if the induced map on log structures is an isomorphism.

Definition 3 ([25]) An Artin fan $\mathcal{A}$ is an algebraic stack with a logarithmic structure that has a strict étale cover by a disjoint union of Artin cones.

A cone (σ, N) determines an Artin cone by

$$\mathcal{A}_{(\sigma,N)} = \mathcal{A}_{\overline{M}_\sigma}\,.$$

Moreover, we have [1, 68]

$$\mathsf{Hom}_{\mathbf{RPC}}((\sigma, N), (\sigma', N')) = \mathsf{Hom}_{\mathrm{LogStacks}}(\mathcal{A}_{(\sigma,N)}, \mathcal{A}_{(\sigma',N')})\,.$$

In other words, the category of cones and the category of Artin cones are equivalent. In fact, this equivalence of categories extends to an equivalence of 2-categories between cone stacks and Artin fans [25, Theorem 3]. For a cone stack Σ, we will write $\mathcal{A}_\Sigma$ for the corresponding Artin fan.

Convention It is cumbersome to carry both the cones and the integral structures around everywhere in the notation. Thus, we will simply denote a cone by σ, which should be always understood as coming with an implicit integral structure.

2.3 Tropicalization

Let $X = (X, \mathsf{M}_X)$ be a log smooth algebraic stack with a logarithmic structure. To X, we can assign a combinatorial shadow of the *characteristic monoid*

$$\overline{\mathsf{M}}_X = \mathsf{M}_X / \mathcal{O}_X^*\,.$$

Definition 4 A *tropicalization* of X is an Artin fan $\mathcal{A}_X$ with a strict map

$$t_X : X \to \mathcal{A}_X$$

which has connected nonempty fibers.

We will write Σ_X for the cone stack corresponding to $\mathcal{A}_X$ and refer to Σ_X as a tropicalization for X as well. Tropicalizations are not unique.

A tropicalization is easy to construct when X is "sufficiently local". Local here is with respect to the log structure. The most local case as far as the characteristic monoid $\overline{\mathsf{M}}_X$ is concerned is when X is an *atomic* log scheme: X has a unique closed stratum, and for every point x in the closed stratum,

$$\Gamma(X, \overline{\mathsf{M}}_X) = \overline{\mathsf{M}}_{X,x} .$$

Atomic neighborhoods exist on a log smooth algebraic stack with a log structure smooth-locally. In the atomic case, Σ_X is simply the rational polyhedral cone

$$\sigma_{X,x} = (\mathrm{Hom}(\overline{\mathsf{M}}_{X,x}, \mathbb{R}_{\geq 0}), \mathrm{Hom}(\overline{\mathsf{M}}_{X,x}, \mathbb{N}))$$

for x a point in the closed stratum. A tropicalization map is constructed by taking a local chart $U_y \to \mathrm{Spec}\,\mathbb{C}[\overline{M}_{X,y}]$, and composing the natural map

$$U_y \to \mathcal{A}_{\overline{\mathsf{M}}_{X,y}}$$

with the map $\mathcal{A}_{\overline{\mathsf{M}}_{X,y}} \to \mathcal{A}_{\overline{\mathsf{M}}_{X,x}}$. The assumptions imply that the maps $U_y \to \mathcal{A}_{\overline{\mathsf{M}}_{X,x}}$ descend to X and have connected nonempty fibers.

In general, we can construct a tropicalization by presenting X as a (2-categorical) colimit

$$\varinjlim X_i \cong X \tag{20}$$

with each X_i atomic, and we define

$$\Sigma_X = \varinjlim \Sigma_{X_i} . \tag{21}$$

For a map $f : X_i \to X_j$ in the colimit (20), the associated map $\Sigma_{X_i} \to \Sigma_{X_j}$ is induced by the map

$$\overline{\mathsf{M}}_{X_j,x_j} \cong \Gamma(X_j, \overline{\mathsf{M}}_{X_j}) \xrightarrow{f^*} \Gamma(X_i, \overline{\mathsf{M}}_{X_i}) \cong \overline{\mathsf{M}}_{X_i,x_i}$$

of monoids. Equivalently,

$$\mathcal{A}_X = \varinjlim \mathcal{A}_{X_i} .$$

The corresponding tropicalization map $X \to \mathcal{A}_X$ is obtained by observing that the composed maps

$$X_i \to \mathcal{A}_{X_i} \to \mathcal{A}_X$$

descend to X. The colimit (21) is taken in the 2-category of stacks over rational polyhedral cones, and the corresponding colimit of Artin fans is taken in the 2-category of stacks with logarithmic structure.

Remark 5 The above construction in general depends on the choice of atomic cover of X and does not yield a canonical tropicalization. A canonical tropicalization $X \to \mathcal{A}_X$ is constructed in [1, 4]. The construction of the canonical Artin fan is however *not* functorial: a non-strict map $X \to Y$ does not in general induce a morphism $\mathcal{A}_X \to \mathcal{A}_Y$ between the canonical Artin fans of X and Y. However, given a tropicalization $Y \to \mathcal{A}_Y$, the substitute for functoriality of [1] says that we can find some tropicalization $X \to \mathcal{A}_X$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{A}_X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{A}_Y \end{array}$$

commutes. From the point of view of functoriality, working with the flexibility of an arbitrary tropicalization is more natural than just studying the canonical tropicalization.

Remark 6 Suppose X is a log smooth algebraic stack. Then every tropicalization Σ_X of X has the same cones as the canonical tropicalization, but with potentially different automorphism groups. In fact, both the cones of Σ_X and the cones of the canonical tropicalization are in bijection with the strata of X.

To understand the above constructions, there are two examples to keep in mind. The first is when X is a toric variety. Then, the canonical tropicalization Σ_X is the fan of X (though Σ_X is not embedded in the cocharacter lattice of the torus[19]).

The second example is when X is the log scheme associated to a nonsingular variety V with a normal crossings divisor D. When D is normal crossings in the Zariski topology, the canonical tropicalization Σ_X is the cone over the dual intersection complex of the irreducible components D_i of D. The construction is explained in [53]. Concretely, each stratum of X, which corresponds to a connected component of an intersection

$$D_{i_1} \cap D_{i_2} \cdots \cap D_{i_n}$$

of irreducible components, contributes the cone

$$(\mathbb{R}^n_{\geq 0}, \mathbb{Z}^n)$$

in Σ_X. These rational polyhedral cones are glued together in the obvious way: a cone becomes a face of every cone corresponding to a deeper stratum (further nonempty intersections of the D_i). In the Zariski normal crossings case, Σ_X does not have nontrivial stack structure.

[19] For example, if the toric variety has torus factors, the cocharacter lattice cannot be recovered from Σ_X.

In general, D may only be normal crossings étale locally: some D_i may self-intersect. Then, for integers $k > 1$, there can be strata which étale locally are the intersections of k distinct divisors $D_{i,1}, \ldots, D_{i,k}$ which are globally the k branches of a single divisor D_i coming together. Thus, intersections of the form

$$\bigcap_{j=1}^{n} D_{i_j,1} \cap \cdots \cap D_{i_j,k_j} \tag{22}$$

can also be strata of X.

Definition 7 [2, §6.2] The *monodromy group* G of a stratum of X is the image of the fundamental group of the stratum in the permutation group of the branches of the divisors cutting out the stratum.

For the stratum (22), the monodromy group G acts on

$$\mathbb{R}_{\geq 0}^{k_1+\cdots+k_n},$$

The interior – that is, the complement of its proper faces – of the corresponding stacky cone in Σ_X coincides with the stack quotient

$$[\mathbb{R}_{>0}^{k_1+\cdots+k_n}/G].$$

The details of the construction are explained in [2, §6.2].

Example 8 Let $X = (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$. Then the cone stack

$$\Sigma_{\overline{\mathcal{M}}_{g,n}} = \varinjlim_{\Gamma \in \mathcal{G}_{g,n}} \sigma_\Gamma$$

in equation (4) of the Introduction is a tropicalization for X. A tropicalization map

$$\overline{\mathcal{M}}_{g,n} \to \varinjlim_{\Gamma \in \mathcal{G}_{g,n}} \mathcal{A}_{\sigma_\Gamma}$$

is constructed in [25, Theorem 4]. The above tropicalization does *not* agree with the canonical Artin fan of [1, 4].

By the definition of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$, the automorphism group of the stacky cone σ_Γ is given by the group Aut_Γ of automorphisms of the graph Γ itself. However, Aut_Γ is different from the monodromy group G_Γ of the stratum $\mathcal{M}^\Gamma \subseteq \overline{\mathcal{M}}_{g,n}$ associated to Γ. Indeed, the branches of the boundary divisor of $\overline{\mathcal{M}}_{g,n}$ cutting out $\mathcal{M}^\Gamma$ precisely correspond to the edges $E(\Gamma)$. The monodromy group $G_\Gamma \subseteq \mathrm{Sym}(E(\Gamma))$ of the stratum $\mathcal{M}^\Gamma$ is the image of $\mathrm{Aut}_\Gamma \to \mathrm{Sym}(E(\Gamma))$ and fits into an exact sequence

$$\{\mathrm{id}_\Gamma\} \to \mathrm{Aut}_\Gamma^{\mathrm{loop}} \to \mathrm{Aut}_\Gamma \to G_\Gamma \to \{[\mathrm{id}_{E(\Gamma)}]\},$$

where $\mathrm{Aut}_\Gamma^{\mathrm{loop}} \subseteq \mathrm{Aut}_\Gamma$ is the subgroup of automorphisms of Γ acting as the identity on $E(\Gamma)$. The group of such automorphisms is the free abelian group generated by the involutions which interchange two half-edges forming a loop.

Since the automorphism groups Aut_Γ and G_Γ of the stacky cones in $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ and the cone stack associated to the canonical tropicalization of $\overline{\mathcal{M}}_{g,n}$ are distinct in general, we see that these two tropicalizations do not agree.

2.4 Subdivisions

Let X be a logarithmic algebraic stack. A choice of strict map from X to an Artin fan $\mathcal{A}_X$ provides access to one of the most important operations in the subject.

Definition 9 A *subdivision* (sometimes called a *proper subdivision* in the literature) of a rational polyhedral cone σ is a fan $F \subseteq \sigma$ whose support $\mathsf{Supp}(F)$ is equal to that of σ.

Definition 10 A *subdivision* of a cone stack Σ_X is a map $\widetilde{\Sigma}_X \to \Sigma_X$ from a cone stack $\widetilde{\Sigma}_X$ such that for every map $\sigma \to \Sigma_X$ from a rational polyhedral cone σ, the fiber product $\widetilde{\Sigma}_X \times_{\Sigma_X} \sigma \to \sigma$ is a subdivision.

Remark 11 Alternatively, a subdivision of a cone stack Σ_X can be defined by descent. The stack Σ_X is a combinatorial cone stack (in the sense of [25]): Σ_X has a presentation as a colimit

$$\Sigma_X = \varinjlim_{i \in \mathcal{C}} \Sigma_{X_i}$$

of a diagram of rationally polyhedral cones Σ_{X_i}, where all maps $\Sigma_{X_i} \to \Sigma_{X_j}$ in the colimit are isomorphisms to a (not necessarily proper) face. Every map from a cone $\sigma \to \Sigma_X$ necessarily factors through one of the Σ_{X_i}, and so subdivisions of Σ_X are the same as subdivisions of all Σ_{X_i} which respect all maps in the diagram. In particular, subdivisions respect identifications of faces and automorphisms in Σ_X.

A subdivision $\widetilde{\Sigma}_X \to \Sigma_X$ can be pulled back to X via the map from X to the Artin fan $\mathcal{A}_X$. Under the equivalence of categories between cone stacks and Artin fans, the map $\widetilde{\Sigma}_X \to \Sigma_X$ induces a proper, representable, and birational map of Artin fans $\widetilde{\mathcal{A}}_X \to \mathcal{A}_X$ which we can pull back to X:

$$\widetilde{X} = X \times_{\mathcal{A}_X} \widetilde{\mathcal{A}}_X \to X\,.$$

The map $\widetilde{X} \to X$ is proper and representable. When X is log smooth, the map $X \to \mathcal{A}_X$ is smooth, and thus $\widetilde{X} \to X$ is also birational.

Definition 12 A map $\widetilde{X} \to X$ of the form $X \times_{\mathcal{A}_X} \widetilde{\mathcal{A}}_X \to X$ is a *subdivision* or a *log modification* of X.

The functor of points of the log modification is characterized by the following result, which is essentially standard [51, 65]. As we could not find a precise reference covering the required level of generality, we include a proof for completeness.

Lemma 13 *Let $p : \widetilde{X} \to X$ be a log modification. Then, the following two properties hold*:

(i) *p is a representable monomorphism of logarithmic algebraic stacks,*
(ii) *a log map $S \to X$ lifts to $\widetilde{X}$ if and only if, étale locally on S, the map*[20] $\Sigma_S \to \Sigma_X$ *lifts through $\widetilde{\Sigma}_X$.*

Proof Since $\widetilde{X} = X \times_{\mathcal{A}_X} \widetilde{\mathcal{A}}_X$, it suffices to prove the properties for

$$q : \widetilde{\mathcal{A}}_X \to \mathcal{A}_X$$

instead of p. We fix a map $f : S \to \mathcal{A}_X$.

For property (i), we must check that any two lifts of f to maps $S \to \widetilde{\mathcal{A}}_X$ are equal. We can work étale locally on S. In particular, we may assume throughout that S and X are atomic, with global charts given by sharp fine saturated monoids Q and P respectively. Let

$$\sigma = \mathsf{Hom}(P, \mathbb{R}_{\geq 0}),$$
$$\tau = \mathsf{Hom}(Q, \mathbb{R}_{\geq 0}),$$

and let $N = \mathsf{Hom}(P, \mathbb{Z})$, $L = \mathsf{Hom}(Q, \mathbb{Z})$. Then, Σ_S is the single cone τ in the lattice L, and Σ_X is the single cone σ in the lattice N. The toric variety associated to σ in N is $\operatorname{Spec} k[P] = V(\sigma, N)$, the Artin fan is

$$\mathcal{A}_X = [\operatorname{Spec} k[P] / \operatorname{Spec} k[P^{\mathrm{gp}}]] = [V(\sigma, N)/T],$$

the global quotient of $V(\sigma, N)$ by its dense torus, and

$$\widetilde{\mathcal{A}}_X = [V(\widetilde{\sigma}_X, N)/T]$$

is the quotient of the toric variety defined by the fan $\widetilde{\Sigma}_X$ in N by its dense torus. Logarithmic maps $S \to \mathcal{A}_X$ are then very simple, with

$$\mathsf{Hom}(S, \mathcal{A}_X) = \mathsf{Hom}_{\mathrm{Mon}}(P, Q) = \mathsf{Hom}_{\mathrm{Cones}}(\Sigma_S, \Sigma_X).$$

Furthermore, $\widetilde{\mathcal{A}}_X$ has an open cover by the global quotients

$$\widetilde{\mathcal{A}}_i = [V(\sigma_i, N)/T],$$

where the σ_i are the cones of $\widetilde{\Sigma}_X$. If $\sigma_i < \sigma_j$ then $\widetilde{\mathcal{A}}_i \subset \widetilde{\mathcal{A}}_j$.

Suppose now that we are given two maps $g, h : S \to \widetilde{\mathcal{A}}_X$ lifting f. As we are working locally on S, we can assume that g factors through $\widetilde{\mathcal{A}}_i$ and h through $\widetilde{\mathcal{A}}_j$ for two cones σ_i, σ_j.

Since

$$\mathsf{Hom}(S, \widetilde{\mathcal{A}}_i) = \mathsf{Hom}(\Sigma_S, \sigma_i)$$

[20] The map $\Sigma_S \to \Sigma_X$ may only exist étale locally on S.

and similarly for $\widetilde{\mathcal{A}}_j$, the only way

$$q \circ g = q \circ h$$

can agree is if the corresponding maps $\Sigma_S \to \sigma_i \to \sigma$ and $\Sigma_S \to \sigma_j \to \sigma$ agree. This can only happen if the maps factor through $\sigma_i \cap \sigma_j$, which is a common face σ_k of both. But since $\sigma_k \to \sigma$ is a monomorphism, we must have $g = h$. Thus q is a monomorphism.

Once property (i) is established, to prove property (ii), we need only study the lifting of f étale locally on S. If

$$g : S \to \widetilde{\mathcal{A}}_X$$

is a lift f, then g will factor through one of the $\widetilde{\mathcal{A}}_i$, and so the map $\Sigma_S \to \Sigma_X$ will factor through $\widetilde{\Sigma}_X$. Conversely, suppose $\Sigma_S \to \Sigma_X$ factors through $\widetilde{\Sigma}_X$. The factorization determines maps $S \to \widetilde{\mathcal{A}}_i \to \widetilde{\mathcal{A}}_X$. □

Remark 14 When $X = (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$, the notions discussed here precisely coincide with the discussion of Sect. 1.3 of the Introduction. In particular, the definition of log Chow rings there also generalizes in a straightforward way to an arbitrary log smooth X:

$$\mathsf{logCH}(X) = \varinjlim_{X' \to X} \mathsf{CH}^*_{\mathrm{op}}(X') \,.$$

where $X' \to X$ ranges over all log modifications of X.

2.5 Piecewise linear functions

Definition 15 A *strict piecewise linear function* on a logarithmic algebraic stack X is a global section

$$\alpha \in H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}}) \otimes_{\mathbb{Z}} \mathbb{Q} \,.$$

For log smooth X, there is a natural interpretation of α as a function on any tropicalization Σ_X: for $x \in X$, restriction gives $\alpha_x \in \overline{\mathsf{M}}_{X,x}^{\mathsf{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ which induces an evaluation map

$$\mathsf{Hom}(\overline{\mathsf{M}}_{X,x}, \mathbb{R}_{\geq 0}) \to \mathbb{R} \,, \quad \varphi \mapsto \varphi(\alpha_x) \,,$$

where φ is extended in the obvious way from $\overline{\mathsf{M}}_{X,x}$ to $\overline{\mathsf{M}}_{X,x}^{\mathsf{gp}}$. Recall from Sect. 2.3 that the cone $\mathsf{Hom}(\overline{\mathsf{M}}_{X,x}, \mathbb{R}_{\geq 0})$ is a strict cover of the cone σ_x in Σ_X. Since the α_x come from a global section α, the above functions descend to a function on Σ_X. This function is strict piecewise linear, in the sense that it is continuous and linear on each cone of Σ_X. Furthermore, it takes rational values on the integral structure of Σ_X. We will refer to such a function as a strict piecewise linear function on Σ_X, and we will write sPL for the group of strict piecewise linear functions on a cone stack.

The construction above induces an isomorphism between the group of (combinatorial) strict PL functions on Σ_X and the group

$$H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}}) \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

of strict piecewise linear functions on X. In particular, the group $\mathsf{sPL}(\Sigma_X)$ does not depend on the choice of tropicalization of X.

Remark 16 In [43], the ring of strict piecewise polynomials on X was defined as the ring of global sections of the sheaf

$$\mathrm{Sym}^\bullet_{\mathbb{Q}} \overline{\mathsf{M}}_X^{\mathsf{gp}}\,.$$

In particular, the degree 1 part is given by

$$H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}}) \otimes_{\mathbb{Z}} \mathbb{Q}\,,$$

the same as the group of strict piecewise linear functions defined above. Locally on X the ring $\mathrm{Sym}^\bullet_{\mathbb{Q}} \overline{\mathsf{M}}_X^{\mathsf{gp}}$ is just the ring of polynomials in linear functions, and the same holds for the combinatorially-defined ring of piecewise polynomial functions, so the definition of [43] coincides with that of the present paper in all degrees.

If $\alpha \in H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}})$, we define the $\mathcal{O}_X^\times$-torsor $\mathcal{O}_X^\times(\alpha)$ to be the preimage of α under the natural map $\mathsf{M}_X^{\mathsf{gp}} \to \overline{\mathsf{M}}_X^{\mathsf{gp}}$. The preimage is an $\mathcal{O}_X^\times$-torsor because of the exact sequence

$$1 \to \mathcal{O}_X^\times \to \mathsf{M}_X^{\mathsf{gp}} \to \overline{\mathsf{M}}_X^{\mathsf{gp}} \to 0\,.$$

Definition 17 Let $\alpha \in H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}})$. We define $\mathcal{O}_X(\alpha)$ to be the line bundle obtained from $\mathcal{O}_X^\times(\alpha)$ by glueing in the *infinity* section.

If $\alpha \leq \alpha'$, in the sense that

$$\alpha' - \alpha \in H^0(X, \overline{\mathsf{M}}_X) \subset H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}})\,,$$

then we have a natural map of line bundles

$$\mathcal{O}_X(\alpha) \to \mathcal{O}_X(\alpha')\,.$$

More generally, if $\mathcal{L}$ is a line bundle on X, we define $\mathcal{L}(\alpha) = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha)$.

Remark 18 The induced homomorphism

$$H^0(X, \overline{\mathsf{M}}_X^{\mathsf{gp}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathsf{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathsf{CH}^1(X)$$

coincides with the degree 1 part of the homomorphism

$$\Phi\colon \mathsf{sPP}(\Sigma_X) \to \mathsf{CH}(X)$$

from (10).

3 Logarithmic curves

3.1 Definitions

Up to now, we have been discussing *absolute* logarithmic geometry. We now turn our attention to the simplest relative situation: *logarithmic curves*.

Definition 19 Let S be a log scheme. A *logarithmic curve* $C \to S$ over S is a log smooth, proper, connected, integral, and vertical morphism with reduced geometric fibers which are of pure dimension 1.

Briefly, the adjective *integral* concerns the flatness of $C \to S$, and the adjective *vertical* means that we do not put additional log structure on the marked points of the family.

Remark 20 A warning to the reader about the notation: given a logarithmic curve $\pi : C \to S$ with sections $x_1, \ldots, x_n : S \to C$, later constructions (such as Definition 37) will use the notation

$$\omega^{\log} = \omega_\pi(\sum_{i=1}^{n} x_i)$$

for the relative log canonical sheaf, which is *not* the same as the sheaf of log differentials of C/S. Indeed, since we insist on the log-structure being vertical, the sheaf of log differentials is ω_π.

We do not formally review the relevant terminology, and refer the reader to [52] for a discussion. We only will use the most basic consequences of Definition 19: the map of underlying schemes of every log curve is a prestable curve, and the log structures on C and S give us access to a family of tropical curves.

More precisely, over every geometric point $s \in S$, the characteristic monoid $\overline{\mathsf{M}}_{C_s}$ of the fiber C_s satisfies:

- at the generic point c of an irreducible component, $\overline{\mathsf{M}}_{C_s,c} = \overline{\mathsf{M}}_{S,s}$,
- at a node q, there exists a unique (nonzero) element $\ell_q \in \overline{\mathsf{M}}_{S,s}$ such that

$$\overline{\mathsf{M}}_{C_s,q} = \overline{\mathsf{M}}_{S,s} \oplus_{\mathbb{N}} \mathbb{N}^2,$$

where the homomorphism $\mathbb{N} \to \mathbb{N}^2$ is the diagonal, and the homomorphism $\mathbb{N} \to \overline{\mathsf{M}}_{S,s}$ sends 1 to ℓ_q.

Following [25], we encode the above data as a *tropical curve over* S.

Definition 21 Let M be a sharp monoid. A *tropical curve* metrized by M is the data of a connected graph Γ together with elements $\ell(e) \in M - \{0\}$ for each edge of e. We call $\ell(e)$ the *length* of e.

For every log curve C/S and geometric point $s \in S$, we obtain a *tropical curve metrized by* $\overline{\mathsf{M}}_{S,s}$ consisting of the dual graph Γ_s of C_s with each edge e of Γ_s corresponding to a node $q \in C_s$ assigned the *length*

$$\ell(e) = \ell_q \in \overline{\mathsf{M}}_{S,s}\,.$$

Since the above data comes from a log map $C \to S$, it is severely constrained. First of all, the tropical curve $(\Gamma_s, \ell(e))$ is étale locally constant along strata of S. Furthermore, for each étale specialization $\zeta : t \rightsquigarrow s$, there is an induced map

$$p_\zeta^\sharp \colon \overline{\mathsf{M}}_{S,s} \to \overline{\mathsf{M}}_{S,t}$$

and an induced *edge contraction* map

$$f_\zeta : \Gamma_s \to \Gamma_t$$

which is compatible with $p_\zeta^\sharp$. If an edge e of Γ_s has length mapping to 0 then the edge is contracted, otherwise it is sent to an edge with length $p_{s,t}^\sharp(\ell(e))$.

Remark 22 When S is log smooth, the connection of Definition 21 with the constructions in Sect. 2.3 is as follows. The strata of S are in bijection with the cones of any tropicalization Σ_S. For a geometric point $s \in S$ in a given stratum O_s, the corresponding stacky cone σ_s in Σ_S has a strict étale cover by the rational polyhedral cone $\mathsf{Hom}(\overline{\mathsf{M}}_{S,s}, \mathbb{R}_{\geq 0})$. The étale specializations $\zeta : t \rightsquigarrow s$ correspond to gluing relations and automorphisms that produce σ_s out of the cover $\mathsf{Hom}(\overline{\mathsf{M}}_{S,s}, \mathbb{R}_{\geq 0})$.

The graphs Γ also appear naturally in two ways: as the dual graphs of the various fibers of $C \to S$ and as the fibers of the induced map of cone stacks $\Sigma_C \to \Sigma_S$. The latter fibers have the same combinatorial type in the interior of each cone of Σ_S, but their edge lengths vary. The parameter $\ell(e) \in \overline{\mathsf{M}}_{S,s}$ associated to an edge of a dual graph induces by evaluation a homomorphism

$$\ell(e) \colon \mathsf{Hom}(\overline{\mathsf{M}}_{S,s}, \mathbb{R}_{\geq 0}) \to \mathbb{R}_{\geq 0}\,.$$

The compatibility of the $\ell(e)$ with étale specializations means that these homomorphisms descend to Σ_S, and under this interpretation, they measure the length of the edge in the fibers of $\Sigma_C \to \Sigma_S$ as x moves in Σ_S.

3.2 Subdivisions of log curves

Let $C \to S$ be a logarithmic curve. As C is a logarithmic scheme, we may perform subdivisions on C, but the composition of a subdivision $C' \to C$ with $C \to S$ may no longer be a logarithmic curve (as $C' \to S$ may fail to be integral or have nonreduced fibers[21]).

[21] In good cases, when S is log smooth, $C' \to C$ is essentially a blow-up, and integrality is equivalent to flatness. Then, the pathology can be rephrased as *even though $C \to S$ is flat, the blow-up C' may fail to be flat over S*. Moreover, even if $C' \to S$ is flat, the fibers need not be reduced.

The subdivisions $C' \to C$ which are logarithmic curves over S are special. In terms of the tropicalizations $(\Gamma_s, \ell(e) \in \overline{\mathsf{M}}_{S,s}, f_\zeta)$, the subdivisons $C' \to C$ which are logarithmic curves over S are exactly the subdivisions of the following form:

- (Subdivision of each fiber) For each geometric point $s \in S$, a subdivision of each edge e_s of Γ_s into a union of edges $e(i)_s$ of length $\ell(e(i)_s) \in \overline{\mathsf{M}}_{S,s} - \{0\}$, with

$$\ell(e_s) = \sum \ell(e(i)_s)\,.$$

- (Compatibility with specialization) For each étale specialization $\zeta : t \rightsquigarrow s$ and edge e_t mapping to e_s under $E(\Gamma_t) \to E(\Gamma_s)$, a bijection φ between the edges $e(i)_s$ for which $\ell(e(i)_s)$ does not map to 0 under the map $p_\zeta^\sharp : \overline{\mathsf{M}}_{S,s} \to \overline{\mathsf{M}}_{S,t}$ and the edges $e(j)_t$ subdividing e_t, so that $p_\zeta^\sharp(\ell(e(i)_s)) = \ell(\varphi(e(i)_s)) \in \overline{\mathsf{M}}_{S,t}$.

Even though $C' \to S$ is not a log curve for every subdivision $C' \to C$, we have:

(i) $C \times_S S' \to S'$ is a log curve for every subdivision $S' \to S$ (as the definition of a log curve is stable under base change with respect to any log map $S' \to S$).
(ii) Subdivisions by log curves are in a sense cofinal among subdivisions of the total space C: for any subdivision $C' \to C$, there exists a log alteration (a log alteration is a composition of a log modification with a root stack along the log structure) $S' \to S$ and a further subdivision $C'' \to C' \times_S S'$ such that $C'' \to S'$ is a log curve.

We will not use (ii) in the paper. The result follows by the formulation of the weak semistable reduction theorem in [61].

A special class of subdivisions of log curves $C \to S$ will be central in what follows.

Definition 23 (Quasi-stable models) A quasi-stable model of a log curve $C \to S$ is a subdivision $\widehat{C} \to C$ such that $\widehat{C} \to S$ is a log curve and, fiberwise, the exceptional locus of $\widehat{C} \to C$ consists of isolated rational curves (every connected component of the exceptional locus is a nonsingular rational curve).

In particular, on the level of the dual graphs Γ_s, a quasi-stable model has at most one exceptional vertex on each edge. We have chosen the terminology quasi-stable model because, when the underlying family $C \to S$ is a family of stable curves, the curve $\widehat{C} \to S$ is called quasi-stable in the literature.

3.3 Abel-Jacobi theory on a tropical curve

The main notions of classical Abel-Jacobi theory have tropical analogues, originating in [11, 12, 60]. We present here a logarithmic approach to the part of the theory that we need for the present work.

Fix a tropical curve Γ metrized by a sharp monoid M. We briefly recall the definitions of divisors and flows on Γ from Sect. 1.7.2. A *divisor* on a tropical curve

Γ is an element of $\mathbb{Z}^{V(\Gamma)}$, while a *flow* on Γ is a function $f\colon \vec{E}(\Gamma) \to \mathbb{Z}$ satisfying $f(\tilde{e}) = -f(\vec{e})$ for all edges $e \in E(\Gamma)$. As written in equation (13), a flow on Γ induces a divisor by taking outgoing slopes,

$$\operatorname{div}\colon \mathsf{Flow}(\Gamma) \to \mathbb{Z}^{V(\Gamma)}.$$

Flows and divisors are essentially combinatorial notions: they depend only on the graph Γ and not the metric structure of Γ. On the other hand, the analogue of a rational function on a curve is a *piecewise linear* function, which requires the metric structure.

Definition 24 A *strict piecewise linear* function on Γ is a function

$$F\colon V(\Gamma) \to M^{\mathsf{gp}}$$

satisfying the following property: for every directed edge $\vec{e}$ from vertex u to vertex v, there is an integer $s(\vec{e}) \in \mathbb{Z}$, the *slope of F along $\vec{e}$*, such that

$$F(v) - F(u) = s(\vec{e})\ell(e). \tag{23}$$

We will use the abbreviation sPL for strict piecewise linear, and we write $\mathsf{sPL}(\Gamma)$ for the group of sPL functions on Γ.

By taking slopes, we obtain a homomorphism

$$\mathsf{sPL}(\Gamma) \to \mathsf{Flow}(\Gamma).$$

Composing with the homomorphism div yields a homomorphism

$$\mathsf{sPL}(\Gamma) \to \mathsf{Div}(\Gamma)$$

which we also denote by div. Explicitly, if $s(\vec{e})$ is the slope of F along the oriented edge $\vec{e}$, we have

$$\operatorname{div}(F) = \left(\sum_{\vec{e}\to v} s(\vec{e})\right)_{v\in V(\Gamma)} \in \mathbb{Z}^{V(\Gamma)}.$$

We say a divisor is *principal* if it is the divisor of some sPL function. Two divisors are *linearly equivalent* if their difference is principal. Whether a divisor is principal depends strongly on the monoid M (via the edge lengths), and *not* just on the underlying graph Γ.

If two sPL functions have the same slopes, then they differ by addition of a global constant. The next result determines when an (acyclic) flow arises as the slopes of a sPL function. To state the result, we require a slight generalization of the formula (14).

Definition 25 We define the pairing

$$\langle -, - \rangle_\ell\colon \mathsf{Flow}(\Gamma) \times \mathsf{Flow}(\Gamma) \to \mathsf{M}^{\mathsf{gp}}$$

by the formula

$$\langle f, g\rangle_\ell = \frac{1}{2} \sum_{\vec{e}\in\vec{E}(\Gamma)} \ell(e) f(\vec{e}) g(\vec{e}). \tag{24}$$

The prefactor $\frac{1}{2}$ is placed to count every edge contribution only once.

Every cycle $\gamma \in H_1(\Gamma)$ induces a flow by identifying $H_1(\Gamma)$ with the kernel of div as in (13). Hence, we sometimes view the pairing as being between flows and cycles.

Lemma 26 *Let f be a flow on Γ. Then f arises as the slopes of a sPL function if and only if, for every cycle γ in Γ, we have*

$$\langle \gamma, f\rangle_\ell = 0.$$

Proof The statement is well-known. That the pairing vanishes on the slopes of a sPL function, we leave to the reader. For the converse, start by choosing any value for F at a particular vertex. By moving around the graph, the condition (23) determines the value of F at every other vertex. The method can fail if two different ways of reaching a vertex were to yield different answers, which is precisely precluded by the vanishing of the pairing. □

In particular, if a flow f arises as the slopes of a sPL function, then f must be acyclic: starting at a vertex v in the cycle, the values $F(w)$ of a sPL function F must strictly increase when traversing edges with positive slope, but must return to $F(v)$ when traversing the whole cycle. The order here on M^{gp} is induced by the inclusion $M \subset M^{\mathsf{gp}}$: given $x, y \in M^{\mathsf{gp}}$, we say $x \leq y \in M^{\mathsf{gp}}$ if $y - x \in M \subset M^{\mathsf{gp}}$.

Remark 27 In the tropical literature, a tropical curve is typically a metric graph, and a divisor is a finite sum

$$\sum a_p p$$

with $a_p \in \mathbb{Z}$ and $p \in \Gamma$. For us, a tropical curve is a finite set of vertices and edges, with edges metrized by a monoid M. When the monoid M is $\mathbb{R}_{\geq 0}$, our tropical curves can be regarded as metric graphs. But for general monoids M, we do not have metric graphs. Moreover, the notion of a point of an edge of Γ does not have a clear meaning. Therefore, we choose to use finite formal sums of vertices in our definition of divisors. If an edge can be subdivided over M into two edges, it is reasonable to consider the new vertex of the subdivision a point of Γ, but we prefer to consider it as a divisor on the subdivision rather than on the original curve.

3.4 Algebraizing tropical Abel-Jacobi theory

Logarithmic geometry allows us to lift the combinatorics of tropical Abel-Jacobi theory to algebraic geometry.

A *strict piecewise linear* function on a log curve $C \to S$ is a global section of $\overline{\mathsf{M}}_C^{\mathsf{gp}}$. Over a geometric point s of S, such a sPL function

$$\alpha \in H^0(C, \overline{\mathsf{M}}_C^{\mathsf{gp}})$$

induces a combinatorial sPL function (in the sense of Definition 24) on the tropical curve, by sending a vertex v of Γ_s to the value of α at the generic point of the irreducible component of C_s corresponding to v.

When S is a geometric point, this gives a bijection between combinatorial sPL functions and $H^0(C, \overline{\mathsf{M}}_C^{\mathsf{gp}})$. On the other hand, suppose that we are given, for each geometric point s of S, a combinatorial sPL function

$$F_s \colon V(\Gamma_S) \to \overline{\mathsf{M}}_{S,s}^{\mathsf{gp}}\,.$$

Suppose, moreover, that the F_s are compatible with étale specialization: whenever $\zeta : t \rightsquigarrow s$ is an étale specialization inducing

$$p_\zeta^{\#} \colon \overline{\mathsf{M}}_{S,s} \to \overline{\mathsf{M}}_{S,t}$$

and

$$f_\zeta \colon \Gamma_s \to \Gamma_t\,,$$

we have

$$p_\zeta^{\#} F_s(v) = F_t(f_\zeta(v))\,.$$

Then there is a unique global section

$$\alpha \in H^0(C, \overline{\mathsf{M}}_C^{\mathsf{gp}})$$

inducing the combinatorial sPL functions F_s.

The geometric connection of the two constructions is the following: The sPL function α induces a line bundle $\mathcal{O}(\alpha)$ on C, as in Definition 17. Then, the multidegree of $\mathcal{O}(\alpha)$ on C_s is opposite to the tropical divisor associated to F_s:

$$\underline{\deg}\mathcal{O}(\alpha)|_{C_s} = -\mathsf{div}(F_s) \tag{25}$$

This equality is in particular responsible for our convention to glue the infinity rather than the zero section to the torsor $\mathcal{O}_C^{\times}(\alpha)$.

4 Stability conditions

4.1 Definitions

The definitions related to stability of line bundles here are adapted from [59]. Let

$$\pi : C \to S$$

be a logarithmic curve.

Definition 28 A *stability condition θ of degree d for π* consists of a function

$$\theta : V(\Gamma_s) \to \mathbb{Q}$$

for each geometric point $s \in S$, which satisfies:

(i) For $s \in S$, we have $\sum_{v \in V(\Gamma_s)} \theta(v) = d$.
(ii) For every étale specialization $\zeta : t \rightsquigarrow s$, the stability condition θ respects the induced map $\Gamma_s \to \Gamma_t$, in the sense that

$$\theta(w) = \sum_{v_i} \theta(v_i)$$

whenever $v_1, \ldots, v_n$ are the vertices of Γ_s mapping to $w \in \Gamma_t$.

For the universal family over the moduli space of stable curves,

$$\pi : C \to S = \overline{\mathcal{M}}_{g,n} ,$$

a stability condition of the form of Definition 28 is exactly a stability condition in the sense of Sect. 1.6.

If $T \to S$ is a logarithmic map, and $\widehat{C} \to C \times_S T = C_T$ is a quasi-stable model for C_T/T, then we can lift a degree-d stability condition θ for C/S canonically to a degree-d stability condition $\widehat{\theta}$ for $\widehat{C}/T$ by setting the value of $\widehat{\theta}$ to zero on every exceptional vertex of the dual graph of every fiber $\widehat{C}_t$ over every geometric point $t \in T$ (and setting the value of $\widehat{\theta}$ to equal the value of θ on all other vertices).

Definition 29 Let $T \to S$ be a logarithmic map. A line bundle $\mathcal{L}$ on a quasi-stable model $\widehat{C} \to C_T$ is *admissible* if for any geometric point $t \in T$ and exceptional component E of $\widehat{C}_t \to (C_T)_t$, the degree of $\mathcal{L}$ on E is 1.

The line bundle $\mathcal{L}$ tropicalizes to a divisor on the tropicalization of $\widehat{C} \to C_T$ in each fiber: for each geometric point $t \in T$, the divisor is given by the multidegree

$$\underline{\deg}(\mathcal{L}|_{\widehat{C}_t}) \in \mathbb{Z}^{V(\widehat{\Gamma}_t)} = \mathrm{Div}(\widehat{\Gamma}_t) .$$

A line bundle $\mathcal{L}$ is then admissible if and only if the associated tropical divisor has degree 1 on every exceptional vertex.

Definition 30 Let $T \to S$ be a logarithmic map, and let $\mathcal{L}$ be an admissible line bundle on a quasi-stable model $\widehat{C} \to C_T$ of C_T/T. Let θ be a stability condition for C/S with lift $\widehat{\theta}$ to $\widehat{C}/T$. Then, $\mathcal{L}$ is *θ-semistable* if, for every geometric point $t \in T$, the associated tropical divisor $\underline{\deg}(\mathcal{L}|_{\widehat{C}_t})$ on the graph $\widehat{\Gamma}_t$ of $\widehat{C}_t/t$ satisfies the condition:

$$\widehat{\theta}(G) - \frac{\mathrm{val}(G)}{2} \leq \underline{\deg}(\mathcal{L}|_{\widehat{C}_t})(G) \leq \widehat{\theta}(G) + \frac{\mathrm{val}(G)}{2}$$

for every subset $G \subseteq V(\widehat{\Gamma}_t)$, where

$$\widehat{\theta}(G) = \sum_{v \in G} \widehat{\theta}(v) , \quad \underline{\deg}(\mathcal{L}|_{\widehat{C}_t})(G) = \sum_{v \in G} \underline{\deg}(\mathcal{L}|_{\widehat{C}_t})(v)$$

and $\mathrm{val}(G)$ is the valence of G, defined as the number of edges connecting G and its complement.

An admissible line bundle $\mathcal{L}$ on $\widehat{C}/T$ is *θ-stable* if the inequalities of Definition 30 are strict except possibly when G or its complement are unions of exceptional vertices.

In Sect. 1.6, we cast the stability condition in terms of sub-curves which correspond here exactly to subsets of the vertices.

Definition 31 A stability condition θ for C/S is

- *nondegenerate* if, for every logarithmic map $T \to S$ and quasi-stable model $\widehat{C} \to C_T$, every θ-semistable line bundle $\mathcal{L}$ on $\widehat{C}/T$ is θ-stable,
- *small* if the trivial bundle is θ-semistable on C_T.

The category fibered in groupoids over $\mathbf{LogSch}/S$ which assigns to $T \to S$ the groupoid consisting of pairs of

(i) a quasi-stable model $\widehat{C} \to C_T$,
(ii) an admissible line bundle $\mathcal{L}$ on $\widehat{C} \to T$

forms an algebraic stack.[22] Imposing the condition that $\mathcal{L}$ is θ-semistable (for some nondegenerate θ) defines an open substack which we denote by $\mathbf{P}^\theta_\pi$, or $\mathbf{P}^\theta$ when $\pi : C \to S$ is understood. The relative inertia of $\mathbf{P}^\theta_\pi$ over S is given by $\mathbb{G}_m$. The associated rigidification over S, denoted by $\mathcal{P}^\theta_\pi$, is a proper algebraic space over S. Both $\mathbf{P}^\theta_\pi$ and $\mathcal{P}^\theta_\pi$ are log smooth over S, and their tropicalizations parametrize θ-semistable tropical divisors on the tropicalization $\Sigma_C \to \Sigma_S$ (see [59] for details).

Remark 32 According to the definition of θ-semistability, the trivial bundle is *not* semistable on quasi-stable models $\widehat{C} \to C_T$ that have exceptional components, as its degree fails to be 1 on them. Instability is forced here as otherwise there would be no chance of getting a separated space $\mathcal{P}^\theta_\pi$: over a discrete valuation ring T with generic point η, both $(C_T, \mathcal{O}_{C_T})$ and $(\widehat{C}, \mathcal{O}_{\widehat{C}})$ would be limits of $(C_\eta, \mathcal{O}_{C_\eta})$.

4.2 Nondegenerate stability conditions

4.2.1 Moduli of θ-stable bundles

Let $\pi : C \to S$ be a logarithmic curve together with a section x_1, where S is an algebraic stack with log structure. Let

$$\mathcal{L} \to C$$

[22] It is easy to see that $\mathbf{P}^\theta_\pi$ is a stack in the strict étale topology. We justify that this stack is algebraic. The Picard stack is algebraic [7, Theorem 7.3], and admissibility is an open condition, so it suffices to check that the stack of subdivisions is algebraic. But to specify a subdivision of a log curve along an edge e is to specify a monoid element that is strictly between 0 and the length of e. The space of monoid elements is representable by $[\mathbb{A}^1/\mathbb{G}_m]$, and strict inequalities between monoid elements cut out closed substacks.

be a line bundle on C of degree[23] d. Let θ be a nondegenerate stability condition of degree d for C/S. We present here a generalization of the construction of $\overline{\mathcal{M}}^{\theta}_{g,A}$ in Sect. 1.7.2. The main ideas are already contained in [6], but in a setting less general than we require.

Definition 33 Let $S^{\theta}_{\mathcal{L}}$ be the fibered category over $\mathbf{LogSch}/S$ with objects tuples

$$(T/S, \widehat{C} \to C_T, \alpha) \tag{26}$$

where T is a log scheme over S, $\widehat{C} \to C_T$ is a quasi-stable model of C_T/T, and α is a sPL function on $\widehat{C}/T$ vanishing on the component through which x_1 passes, and for which $\mathcal{L}_T(\alpha)$ is θ-stable.

Standard arguments show that $S^{\theta}_{\mathcal{L}}$ is a stack in groupoids for the strict étale topology: a quasi-stable model of C_T is associated to a subdivision of Σ_{C_T}, and α is a strict piecewise linear function on Σ_{C_T} (both of these are by definition étale local on T).

We write $C_{S^{\theta}_{\mathcal{L}}} \to S^{\theta}_{\mathcal{L}}$ for the pullback of C under the map $S^{\theta}_{\mathcal{L}} \to S$. Furthermore, we denote its universal quasi-stable model (written as $\widehat{C}$ in Definition 33 when taking $T = S^{\theta}_{\mathcal{L}}$) by

$$C^{\theta} \to C_{S^{\theta}_{\mathcal{L}}}$$

and the universal sPL function on C^{θ} by α^{θ}.

Remark 34 Let $\mathcal{P}^{\theta}_{\pi}$ be the moduli space of θ-stable line bundles over S trivialized along the section x_1 discussed at the end of Sect. 4.1. There is a natural Abel-Jacobi map

$$\mathsf{aj}\colon S^{\theta}_{\mathcal{L}} \to \mathcal{P}^{\theta}_{\pi} \tag{27}$$

sending $(T/S, \widehat{C} \to C_T, \alpha)$ to the rigidification of the line bundle $\mathcal{L}_T(\alpha)$ on $\widehat{C}$ along the section x_1.

The definition of $S^{\theta}_{\mathcal{L}}$ is natural from the point of view of logarithmic geometry. While standard arguments using [63] show that $S^{\theta}_{\mathcal{L}}$ is a stack over $\mathbf{LogSch}/S$, what is not standard is whether $S^{\theta}_{\mathcal{L}}$ *can be represented by an algebraic stack with a log structure*. Theorem 35 gives an explicit description of $S^{\theta}_{\mathcal{L}}$ as a log modification of S, thus showing $S^{\theta}_{\mathcal{L}}$ is an algebraic stack with log structure.

Theorem 35 *The structure morphism $\rho : S^{\theta}_{\mathcal{L}} \to S$ is a log modification (obtained from a subdivision). In particular, ρ is proper, log étale, and relatively representable by logarithmic schemes. If S is log smooth then ρ is also birational.*

[23] Degree here means the degree of $\mathcal{L}$ on the fibers of π.

Remark 36 In Definition 33, the notation $S^\theta_{\mathcal{L}}$ is used to highlight the fact that $S^\theta_{\mathcal{L}}$ depends both on the stability condition θ and the line bundle $\mathcal{L}$. The proof of Theorem 35 will however show that $S^\theta_{\mathcal{L}}$ only depends on $\mathcal{L}$ through the associated multidegree $\underline{\deg}(\mathcal{L})$, which we can view as a tropical divisor on Σ_C. We could have thus equally written $S^\theta_{\underline{\deg}(\mathcal{L})}$.

Theorem 35 generalizes the construction of

$$\rho : \overline{\mathcal{M}}^\theta_{g,A} \to \overline{\mathcal{M}}_{g,n}$$

promised in (9). The following definition makes the latter claim precise.

Definition 37 Let $S = \overline{\mathcal{M}}_{g,n}$ with universal curve $\pi : C \to \overline{\mathcal{M}}_{g,n}$ carrying the standard log structure. Let

$$\mathcal{L} = (\omega_C^{\log})^{\otimes k}\left(-\sum_{i=1}^n a_i x_i\right)$$

be a line bundle on C with markings $x_1, \ldots, x_n$. For a nondegenerate stability condition θ for π, we define

$$\overline{\mathcal{M}}^\theta_{g,A} = (\overline{\mathcal{M}}_{g,n})^\theta_{\mathcal{L}}, \tag{28}$$

with Abel-Jacobi map

$$\mathsf{aj}\colon \overline{\mathcal{M}}^\theta_{g,A} \to \mathcal{P}^\theta_{g,n} \tag{29}$$

as in Remark 34.

4.2.2 Proof of Theorem 35

To check that a map $S' \to S$ is a log modification, it suffices to work locally in S. We may thus assume S has a strict map to an Artin fan $\mathcal{A}_S$. We present an explicit construction of the subdivision of the corresponding cone stack Σ_S which defines ρ. In the special case of Definition 37, where $S = \overline{\mathcal{M}}_{g,n}$, the subdivision reduces exactly to the construction in Sect. 1.7.2.

Let s be a geometric point of S and let $\sigma_s = \mathsf{Hom}(\overline{M}_{S,s}, \mathbb{R}_{\geq 0})$ be the corresponding cover of the stacky cone in Σ_S. By definition, a subdivision of Σ_S is a subdivision of the various cones σ_s as s ranges through S, which is compatible with all identifications of faces and automorphisms required in Σ_S. Let $\Gamma = \Gamma_s$ be the dual graph of C_s. We will write σ_Γ for σ_s to highlight the connection of the constructions below with Γ.

The logarithmic map $C \to S$ induces the structure of a tropical curve on Γ metrized by

$$\mathsf{M}_{\sigma_\Gamma} = \mathsf{M}_{S,s}$$

via the lengths $\ell(e) \in \overline{\mathsf{M}}_{S,s}$, as in Sect. 3. All together, the lengths $\ell(e)$ can be thought of as a homomorphism

$$\ell \colon \sigma_\Gamma \to \mathbb{R}_{\geq 0}^{E(\Gamma)},$$

so for $p \in \sigma_\Gamma = \mathsf{Hom}(\overline{\mathsf{M}}_{S,s}, \mathbb{R}_{\geq 0})$ we can write $(\ell(p))_e = \ell_e(p) = p(\ell(e))$. Let $\underline{\deg}(\mathcal{L})$ be the multidegree of $\mathcal{L}$ viewed as a tropical divisor on Γ. The cones of our subdivision in σ_Γ will be indexed by triples $(\widehat{\Gamma}, D, I)$ where

(i) $\widehat{\Gamma}$ is a quasi-stable graph with stabilization Γ,
(ii) D is a θ-stable divisor on $\widehat{\Gamma}$,
(iii) I an acyclic flow on $\widehat{\Gamma}$ satisfying $\mathrm{div}(I) = \underline{\deg}(\mathcal{L}) - D$.

We define the cone $\sigma_{\widehat{\Gamma}}$ by

$$\sigma_{\widehat{\Gamma}} = \sigma_\Gamma \times_{\mathbb{R}_{\geq 0}^{E(\Gamma)}} \mathbb{R}_{\geq 0}^{E(\widehat{\Gamma})}.$$

Concretely, a point of the cone $\sigma_{\widehat{\Gamma}}$ is given by

- a point $p \in \sigma_\Gamma$,
- for every edge e of Γ that is subdivided into edges e_1 and e_2 in $\widehat{\Gamma}$, a pair of nonnegative real numbers $\widehat{\ell}(e_1)$ and $\widehat{\ell}(e_2)$ such that $\ell(p)_e = \widehat{\ell}(e_1) + \widehat{\ell}(e_2)$.

The cone $\sigma_{\widehat{\Gamma}}$ has relative dimension over σ_Γ equal to the number of exceptional vertices of $\widehat{\Gamma}$.

In the universal case, when $S = \overline{\mathcal{M}}_{g,n}$, the map ℓ is an isomorphism, $\sigma_{\widehat{\Gamma}} = \mathbb{R}_{\geq 0}^{E(\widehat{\Gamma})}$, and the projection $\sigma_{\widehat{\Gamma}} \to \sigma_\Gamma$ is the map

$$\mathsf{pr} \colon \sigma_{\widehat{\Gamma}} \to \sigma_\Gamma, \quad \widehat{p} \mapsto p : e \mapsto \begin{cases} \widehat{p}(e_1) + \widehat{p}(e_2) & \text{if } \widehat{\Gamma} \text{ subdivides edge } e \text{ into } e_1, e_2, \\ \widehat{p}(e) & \text{otherwise,} \end{cases}$$

from Sect. 1.7.2.

We define a subcone $\tau_{\widehat{\Gamma},I} \subseteq \sigma_{\widehat{\Gamma}}$ by the condition

$$\tau_{\widehat{\Gamma},I} = \left\{ \widehat{p} \in \sigma_{\widehat{\Gamma}} \;\middle|\; \langle \gamma, I \rangle_{\widehat{p}} = 0 \text{ for all } \gamma \in H_1(\widehat{\Gamma}) \right\}, \tag{30}$$

where the pairing is (24) evaluated at the point $\widehat{p}$.

Lemma 38 *The map* $\mathsf{pr} : \sigma_{\widehat{\Gamma}} \to \sigma_\Gamma$ *induces an isomorphism from the cone* $\tau_{\widehat{\Gamma},I}$ *to the image*

$$\sigma_{\widehat{\Gamma},I} = \mathsf{pr}(\tau_{\widehat{\Gamma},I}) \subseteq \sigma.$$

Moreover, the collection of cones $\{\sigma_{\widehat{\Gamma},I}\}$ *together with their faces forms a fan* $\widetilde{\Sigma}_\Gamma^\theta$ *with support* σ_Γ.

Proof The claim is purely combinatorial and is proven by Abreu and Pacini [5, 6]. The second paper [6] works in a slightly more restricted setting (over the moduli stack

of stable 1-marked curves with a particular stability condition), though the arguments go through essentially unchanged in our more general setting. The heart of the proof is in the first paper [5], which works with stability conditions *more* general than ours.

We explain how to translate the notation of [6] to ours. The cones $C_{\Gamma,\mathcal{E},\varphi}$ of [6, Definition 3.4] correspond to our cones $\tau_{\widehat{\Gamma},I}$, and the cones $K_{\Gamma,\mathcal{E},\varphi}$ are our $\sigma_{\widehat{\Gamma},I}$. By [6, Proposition 3.7], the map

$$\mathsf{pr} : \tau_{\widehat{\Gamma},I} \to \sigma_{\widehat{\Gamma},I}$$

is an isomorphism. That these cones fit together to a fan which surjects onto σ_Γ is proven in [6, Theorem 3.9]. The content of the proof of surjectivity is [5, Theorem 5.6]: every degree d tropical divisor is linearly equivalent to a unique θ-stable divisor on a unique quasi-stable subdivision. □

To show that the construction of Lemma 38 defines a global subdivision $\widetilde{\Sigma}^\theta$ of Σ_S, we need furthermore to check that the cones $\sigma_{\widehat{\Gamma},I}$ descend to Σ_S: the cones must fit together along all identifications of faces and automorphisms of the various cones σ_Γ which define Σ_S. The proof of [6, Theorem 3.9] again goes through unchanged in our setting to show that a cone $\sigma_{\widehat{\Gamma}',I'}$ is a face of $\sigma_{\widehat{\Gamma},I}$ whenever

- $\sigma_{\Gamma'}$ is a face of σ_Γ,
- $\widehat{\Gamma}'$ is obtained from $\widehat{\Gamma}$ by contracting some edges,
- $I'(e) = I(e)$ for all edges $e \in E(\widehat{\Gamma}') \subset E(\widehat{\Gamma})$.

Thus, the cones $\sigma_{\widehat{\Gamma},I}$ form a polyhedral complex and define a global subdivision of Σ_S. For the gluing, it is necessary that the data $\widehat{\Gamma}$, I respects the monodromy and identifications of faces of σ_Γ. But this is precisely implied by condition (ii) in the definition of a stability condition.

Remark 39 The intuition behind the construction of the cones $\sigma_{\widehat{\Gamma},I}$ is as follows. Given I on $\widehat{\Gamma}$ and p in the interior of σ_Γ, it is in general impossible to both

(i) assign lengths to the edges of $\widehat{\Gamma}$ summing to the lengths on edges of Γ associated to $\ell(p)$,
(ii) find a strict piecewise linear function on $\widehat{\Gamma}$ whose slopes are given by I.

Indeed, by Lemma 26, the existence of such a lifting imposes several conditions on the lengths of the edges of $\widehat{\Gamma}$. The cone $\sigma_{\widehat{\Gamma}}$ is the universal cone over which the pullback of Γ admits a quasi-stable model of combinatorial type $\widehat{\Gamma}$. The subcones $\tau_{\widehat{\Gamma},I}$ are cut out inside $\sigma_{\widehat{\Gamma}}$ precisely by the linear conditions necessary to lift I into a strict piecewise linear function. Thus, the content of Lemma 38 is that the locus of points $p \in \sigma$ which support a quasi-stable model $\widehat{\Gamma}$ of Γ with a sPL function with slopes I is a cone $\sigma_{\widehat{\Gamma},I}$, isomorphic to $\tau_{\widehat{\Gamma},I}$.

Furthermore, the cones for various choices $\widehat{\Gamma}$, I that come from the stability condition θ subdivide σ_Γ: each p in the interior of σ_Γ belongs to the interior of precisely one such cone. Points p in the boundary of σ_Γ similarly belong to the interior of cones $\sigma_{\widehat{\Gamma}',I'}$, where Γ' is the contraction of Γ lying over p. Thus, the property of supporting a quasi-stable model with strict piecewise linear function with slopes I is cut out by inequalities on the lengths of the edges of Γ.

We will show below in Lemma 40 that the log modification

$$\tilde{\rho} : \widetilde{S}^{\theta}_{\mathcal{L}} \to S$$

corresponding to the subdivision $\widetilde{\Sigma}^{\theta}$ of Σ_S is isomorphic over S to $S^{\theta}_{\mathcal{L}}$.

Let $f : T \to S$ be any logarithmic scheme over S, and let $t \in T$ be a geometric point mapping to $s \in S$. The log curve $C \to S$ pulls back to a log curve $C_T \to T$. The induced tropical curve around t has underlying graph $\Gamma = \Gamma_s$ metrized by the lengths $f^{\sharp}(\ell(e)) \in \overline{\mathsf{M}}_{T,t}$, where

$$f^{\sharp} : \overline{\mathsf{M}}_{S,s} \to \overline{\mathsf{M}}_{T,t}$$

is the induced homomorphism. Equivalently, the tropical curve is determined by the composition

$$\tau \longrightarrow \sigma_{\Gamma} \xrightarrow{\ \ell\ } \mathbb{R}^{E(\Gamma)}_{\geq 0},$$

where τ is the cone dual to $\overline{\mathsf{M}}_{T,t}$. As in Sect. 3.2, to give a quasi-stable model $\widehat{C} \to C_T$, we must specify, for every geometric point $t \in T$, the structure of a quasi-stable model of Γ metrized by $\overline{\mathsf{M}}_{T,t}$, compatibly with étale specializations. In other words, to give a quasi-stable model with specified dual graph $\widehat{\Gamma}$, we must lift the homomorphism $\tau \to \mathbb{R}^{E(\Gamma)}_{\geq 0}$ to a homomorphism

$$\tau \to \mathbb{R}^{E(\widehat{\Gamma})}_{\geq 0}$$

for the various cones τ dual to the $\overline{\mathsf{M}}_{T,t}$, compatibly with étale specializations. By construction, the cone $\sigma_{\widehat{\Gamma}}$ is the universal cone over which the morphism ℓ lifts. Thus, $\widehat{\Gamma}$ inherits the structure of a tropical curve over $\sigma_{\widehat{\Gamma}}$, and also over its subcones $\tau_{\widehat{\Gamma},I}$. Since the $\sigma_{\widehat{\Gamma},I}$ are isomorphic to $\tau_{\widehat{\Gamma},I}$, we see that $\widehat{\Gamma}$ admits the structure of a tropical curve over $\sigma_{\widehat{\Gamma},I}$. For any face $\sigma_{\Gamma'} < \sigma_{\Gamma}$, the tropical curve Γ' over $\sigma_{\Gamma'}$ induced by $C \to S$ is obtained from Γ by contracting some edges. If over $\sigma_{\Gamma'}$ an edge e of Γ is contracted, both corresponding edges of $\widehat{\Gamma}$ are contracted as well, since their lengths must add up to that of e. Thus $\widehat{\Gamma}$ naturally contracts to a quasi-stable model of Γ' over $\sigma_{\Gamma'}$. Therefore, the various $\widehat{\Gamma}$ glue and, as described in Sect. 3.2, determine a quasi-stable subdivision

$$\widehat{C} \to C \times_S \widetilde{S}^{\theta}_{\mathcal{L}}.$$

Just as the cone $\sigma_{\widehat{\Gamma}}$ tautologically carries a quasi-stable subdivision of $\Gamma \times_{\sigma_{\Gamma}} \sigma_{\widehat{\Gamma}}$, the subcone $\tau_{\widehat{\Gamma},I}$ is the universal cone on which the flow I can be realized as the slopes of a strict piecewise linear function unique up to pullback from the base, because it is defined as the set of points in $\sigma_{\widehat{\Gamma}}$ where the pairing (24) vanishes, and Lemma 26 tells us that this is exactly the condition for the sPL function to exist. Hence the curve $\widehat{C}$ carries a unique sPL function

$$\alpha : V(\widehat{\Gamma}) \to \overline{\mathsf{M}}^{\mathsf{gp}}_{\sigma_{\widehat{\Gamma},I}} \tag{31}$$

which vanishes along x_1 (see Definition 33).

For each vertex $v \in \widehat{\Gamma}$, the sPL function α assigns an element of $\overline{\mathsf{M}}^{\mathsf{gp}}_{\sigma_{\widehat{\Gamma},I}}$, or equivalently, a homomorphism

$$\alpha(v)\colon \sigma_{\widehat{\Gamma},I} \to \mathbb{R}\,.$$

By construction, the function α has the following explicit description. Denote by v_0 the vertex in $\widehat{\Gamma}$ corresponding to the component that contains the marking x_1. Then, for any other $v \in V(\widehat{\Gamma})$, we have

$$\alpha(v) = \sum_{\vec{e} \in \gamma_{v_0 \to v}} I(\vec{e}) \cdot \widehat{\ell}_e\,,$$

where $\gamma_{v_0 \to v}$ is *any* oriented path from $v_0 \to v$, $I(\vec{e})$ is the integer slope of the flow I along $\vec{e}$, and

$$\widehat{\ell}_e : \sigma_{\widehat{\Gamma},I} \to \mathbb{R}$$

is the homomorphism determined by the length of the edge e. We thus obtain equation (16) of Sect. 1.7.3.

The definition of $\alpha(v)$ does not depend on the choice of a path between v_0 and v, If γ and γ' are paths from $v_0 \to v$, $\gamma - \gamma'$ is an oriented cycle, and

$$\langle \gamma - \gamma', I \rangle_{\widehat{\ell}} = 0$$

on $\sigma_{\widehat{\Gamma},I}$ by definition.[24] On the other hand, α does depend on the choice of v_0.

We have defined a combinatorial sPL function on the tropical curve $\widehat{\Gamma}$. It is easy to see that these sPL functions over the various cones $\sigma_{\widehat{\Gamma},I}$ are compatible and so, as explained in Sect. 3.4, they give rise to a global sPL function α on $\widehat{C}$.

The role of α is to twist the line bundle $\mathcal{L}$ into a θ-stable bundle. By applying (25) and part (iii) of our defining conditions for the tuples $(\widehat{\Gamma}, D, I)$, we obtain

$$\underline{\deg}\mathcal{L}(\alpha) = \underline{\deg}\mathcal{L} - \mathsf{div}(\alpha) = \underline{\deg}\mathcal{L} - \mathsf{div} I = D\,.$$

Therefore, the curve $\widehat{C}$ and the sPL function α together induce a map of stacks

$$\varphi\colon \widetilde{S}^{\theta}_{\mathcal{L}} \to S^{\theta}_{\mathcal{L}}$$

over S.

Lemma 40 *The map of stacks $\varphi\colon \widetilde{S}^{\theta}_{\mathcal{L}} \to S^{\theta}_{\mathcal{L}}$ is an isomorphism.*

Proof We may check the isomorphism strict-étale locally. Let C/T be a nuclear log curve (in the sense of [46, Definition 3.39]; this just means that T is 'small enough', and is automatic if for example T is strictly henselian local) with a line bundle $\mathcal{L}$ of degree d. We must show the following two conditions are equivalent:

[24] A similar argument is used in the proof of Lemma 26.

(i) There exists a quasi-stable subdivision $\widehat{C} \to C$ and a sPL function α on $\widehat{C}/T$ such that $\mathcal{L}(\alpha)$ is θ-stable.
(ii) There exists a quasi-stable subdivision $\widehat{\Gamma} \to \Gamma$ and a flow I on $\widehat{\Gamma}$ such that the condition $\langle \gamma, I \rangle_\ell = 0 \in \overline{\mathsf{M}}_T^{\mathsf{gp}}$ holds for every $\gamma \in H_1(\widehat{\Gamma})$.

The condition (i) characterizes the functor of points of $S^\theta_{\mathcal{L}}$, and condition (ii) characterizes the functor of points of $\widetilde{S}^\theta_{\mathcal{L}}$ by Lemma 13. The correspondence between subdivisions of the graph and of the curve is clear. If we start with a sPL function α, we define the flow I to be the associated slopes. Conversely given a flow I, we know I can be realised as the slopes of a sPL function by Lemma 26. □

Remark 41 During the course of the proof of Lemma 38, we have noted that while the subdivision of [6] is not constructed in the level of generality that we require (since the results of [6] are for $S = \overline{\mathcal{M}}_{g,n}$ and a stability condition pulled back from $\overline{\mathcal{M}}_{g,1}$), a slight modification of their argument suffices to construct $\widetilde{S}^\theta_{\mathcal{L}}$ in general. In fact, more can be said. From [5], for their specific choice of stability condition, there is a cone stack $\Sigma_{\mathcal{P}^\theta_{g,n}}$ subdividing the universal tropical Jacobian $\mathrm{Tro}\mathcal{P}\mathrm{ic}_{g,n}$ (see for example [63]), and a *tropical Abel-Jacobi section*

$$\Sigma_{\overline{\mathcal{M}}_{g,n}} \to \mathrm{Tro}\mathcal{P}\mathrm{ic}_{g,n}\,.$$

The fan $\widetilde{\Sigma}^\theta_\Gamma$ of [6] (and of Lemma 38 for their particular θ) is nothing but the pullback of the subdivision

$$\Sigma^\theta_{\mathcal{P}_{g,n}} \to \mathrm{Tro}\mathcal{P}\mathrm{ic}_{g,n} \tag{32}$$

to the various cones σ of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$. In the proof of [5] that (32) is a subdivision, nothing about the specific θ is used, except nondegeneracy. Their proof essentially produces a subdivision (32) for any stability condition, as discussed in [59, Theorem 5.5]. By pulling-back the Abel-Jacobi section, we obtain the same subdivision $\widetilde{\Sigma}^\theta_{\overline{\mathcal{M}}_{g,n}}$ that we construct in Lemma 38 for the case $S = \overline{\mathcal{M}}_{g,n}$.

The choice of [6] to work with stability conditions coming from 1-pointed curves is not about the subdivision $\overline{\mathcal{M}}^\theta_{g,A}$, but rather about the proof that the Abel-Jacobi section extends to a map

$$\mathsf{aj} : \overline{\mathcal{M}}^\theta_{g,A} \to \mathcal{P}^\theta_{g,n}$$

The claim is proven in [6] by writing a map in formal local coordinates and checking explicitly the required extension to a well-defined global map. The argument uses an explicit description of formal local charts of $\mathcal{P}^\theta_{g,1}$ for their particular choice of stability condition. Our main observation here is that in the category of log schemes, regardless of stability condition, the functor of points of $\widetilde{S}^\theta_{\mathcal{L}}$ is easy to describe and is represented by $S^\theta_{\mathcal{L}}$. The latter has an evident Abel-Jacobi section. So the formal local analysis is *not* needed to extend the Abel-Jacobi section.

A more geometrical argument can also be given. The line bundle $\mathcal{L}$ always determines an Abel-Jacobi section S to the logarithmic Jacobian $\mathrm{Log}\mathcal{P}\mathrm{ic}$ and a natural

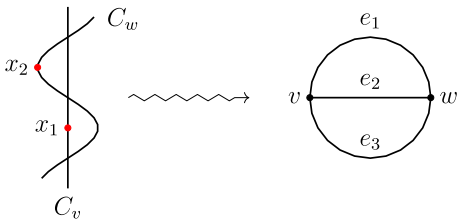
(logarithmic) fiber product diagram

$$\begin{array}{ccc} S^\theta_{\mathcal{L}} & \longrightarrow & S \\ \downarrow & & \downarrow \mathcal{L} \\ \mathcal{P}^\theta & \longrightarrow & \mathrm{Log}\mathcal{P}\mathrm{ic}, \end{array}$$

where the bottom map is a subdivision. Granting the theory of the logarithmic Jacobian, the above diagram shows that $S^\theta_{\mathcal{L}}$ is a subdivision and that the Abel-Jacobi section extends automatically to $S^\theta_{\mathcal{L}} \to \mathcal{P}^\theta$. The fiber product description readily leads to the definition of the functor of points of $S^\theta_{\mathcal{L}}$. This perspective will be explained carefully in the sequel to [59].

4.3 A graph of genus 2

A simple example in genus 2 is rich enough to contain most of the phenomena discussed. Let C_0 denote the *dollar sign* curve, the union of two rational curves C_v and C_w joined at 3 nodes. The dual graph Γ of C_0 has two vertices v and w and three edges e_1, e_2, e_3:



We place two markings on C_0. The first marking is on the component corresponding to v, and the second is on the component corresponding to w.

Let (S, s) be a nonsingular 3-dimensional base of a versal deformation of C_0,

$$\pi : C \to S\,, \quad \pi^{-1}(s) = C_0\,,$$

together with sections $x_1, x_2 : S \to C$ meeting the fiber C_0 as above. Both C and S carry canonical log structures. The tropicalization of S is the cone

$$\Sigma_S = \mathbb{R}^3_{\geq 0}\,,$$

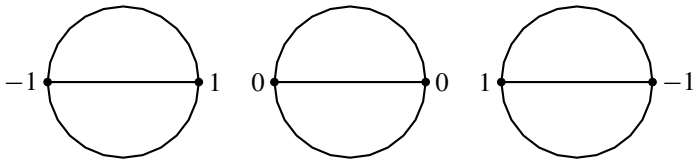
and the tropicalization of C/S is the fibration $\Sigma_C \to \Sigma_S$, which, over a point $(\delta_1, \delta_2, \delta_3) \in \Sigma_S$, assigns the graph Γ with edge e_i having length δ_i (with the understanding that when $\delta_i = 0$, the edge e_i is contracted in the fiber).

For every nondegenerate degree 0 stability condition θ, we find that θ-stable divisors D must satisfy the inequalities

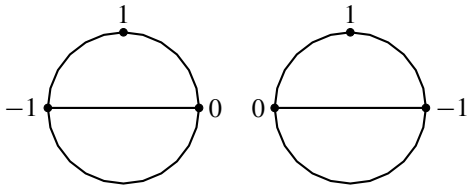
$$\theta(v) - \frac{3}{2} < D(v) < \theta(v) + \frac{3}{2}\,, \qquad \theta(w) - \frac{3}{2} < D(v) < \theta(w) + \frac{3}{2}\,.$$

Therefore, for a small stability condition (close enough to $\theta = 0$), θ-stable divisors must satisfy $-1 \leq D(v),\ D(w) \leq 1$. Picking such a θ, we find that the list of all admissible θ-stable divisors (up to isomorphism of the graph) is:

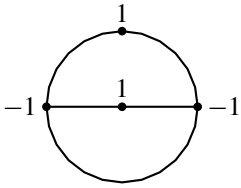
Depth 0:



Depth 1:



Depth 2:



Each graph of depth 1 and 2 stands for three graphs (after including those related by symmetry). There are 12 divisors in total on the list.

Consider the vector $A = (-3, 3)$ of Abel-Jacobi data for the double ramification cycle $\mathsf{DR}_{2,A}$. The associated tropical divisor $\underline{\deg}_{k=0,A}$ is given by $\underline{\deg}_{0,A} = 3v - 3w$, where in the following we denote divisors on graphs as $\mathbb{Z}$-linear combinations of their vertices. To determine the subdivision

$$\widetilde{\Sigma}_S^{\theta} \to \Sigma_S\,,$$

we must solve the tropical problem

$$\mathsf{div}(f) = \underline{\deg}_{0,A} - D\,,$$

where D is an admissible θ-semistable divisor on the above list, and f is a sPL function on Γ (or on $\widehat{\Gamma}$ in the depth 1 and 2 cases).

• Depth 0 cases:

For the divisor $D = 0v + 0w$, there is a unique ray of solutions in Σ_S. To build a strict piecewise linear function on Γ, we must choose 3 slopes $s_1, s_2, s_3 \in \mathbb{Z}$ on the edges of Γ. If we orient all three edges from w to v (with the convention that s_i is positive if the function increases from w to v), then, in order to have a sPL function on Γ, the condition

$$s_1\delta_1 = s_2\delta_2 = s_3\delta_3$$

on the lengths of the edges of Γ must be satisfied. In particular, the s_i must either all be 0, all positive, or all negative. If, in addition, we demand

$$\mathsf{div}(f) = (s_1 + s_2 + s_3)v + (-s_1 - s_2 - s_3)w = \underline{\deg}_{0,A} - D = 3v - 3w\,,$$

we find a unique possible solution

$$s_1 = s_2 = s_3 = 1$$

along the ray $\delta_1 = \delta_2 = \delta_3$, and no other solutions.

The divisor $D = -v + w$ can be analyzed similarly. The equation

$$\mathsf{div}(f) = \underline{\deg}_{0,A} - D = 4v - 4w$$

yields three possible solutions

$$(s_1, s_2, s_3) = (2, 1, 1)\,,\ (1, 2, 1)\,,\ (1, 1, 2)$$

along the three rays

$$\begin{aligned} 2\delta_1 = \delta_2 &= \delta_3\,, \\ \delta_1 = 2\delta_2 &= \delta_3\,, \\ \delta_1 = \delta_2 &= 2\delta_3\,. \end{aligned}$$

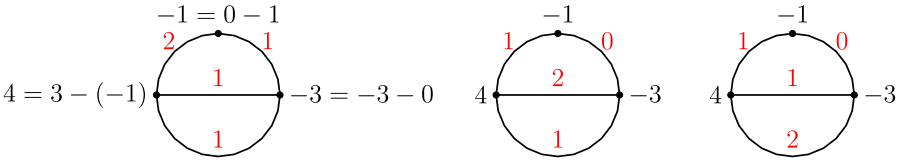
For the divisor $D = v - w$, there are no solutions at all to

$$\mathsf{div}(f) = \underline{\deg}_{0,A} - D = 2v - 2w\,.$$

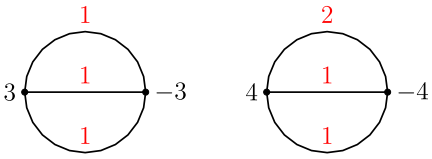
• Depth 1 cases:

The graph Γ is replaced here with a corresponding quasi-stable model $\widehat{\Gamma}$. Instead of studying the sPL function (as we did in the depth 0 cases), we simply specify the possible underlying slopes of such functions corresponding to acyclic flows I with associated divisor $\underline{\deg}_{0,A} - D$.

The first depth 1 divisor D on the list contributes three possible acyclic flows (with all edges oriented from right to left and slopes of I indicated above the edges):



The first of these flows, called $(\{2, 1\}, 1, 1)$ for brevity, has two specializations obtained by letting the exceptional vertex specialize to either of the original vertices v or w:



The flow $(\{2,1\},1,1)$ contributes a 2-dimensional cone of solutions to

$$\mathsf{div}(f) = \underline{\deg}_{0,A} - D$$

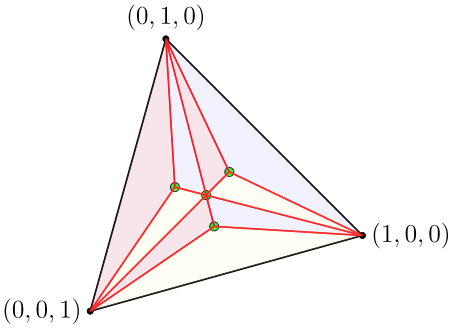
defined by the convex hull of the solutions of the two specializations (given by the rays $\delta_1 = \delta_2 = \delta_3$ and $2\delta_1 = \delta_2 = \delta_3$). Taken together, the depth 1 θ-stable divisors contribute 12 two dimensional cones: 9 from the three possible flows on the first divisor (and the those related by symmetry) and 3 from the second.

• Depth 2 cases:

A similar analysis can be applied to the three depth 2 divisors (which contain two exceptional vertices). Each divisor D contributes three 3-dimensional cones, for a total of 9.

• Full subdivision:

Altogether, the subdivision $\widetilde{\Sigma}^{\theta}_{S} \to \Sigma_S = \mathbb{R}^3_{\geq 0}$ has 4 new rays (corresponding to the depth 0 divisors), 12 new 2-dimensional cones (corresponding to depth 1 divisors), and 9 maximal cones (corresponding to the depth 2 divisors). The 2-dimensional intersection of the subdivision $\widetilde{\Sigma}^{\theta}_{S}$ with the hyperplane $\delta_1 + \delta_2 + \delta_3 = 1$ can be drawn as:



The vertices in green correspond to the depth 0 divisors, the red lines to the depth 1 divisors. The maximal cells which correspond to different flows realizing the same underlying divisor are shaded in the same color (Color figure online).

Remark 42 The geometrical meaning of the subdivision is as follows. There is the point $s \in S$ (of codimension 3) over which the curve $C \to S$ is a dollar sign curve. We blow-up S at s, which replaces s with $\mathbb{P}^2$. We then blow-up the 3 fixed points of $\mathbb{P}^2$ to obtain $S^{\theta}_{\mathcal{L}}$. The strata of $S^{\theta}_{\mathcal{L}}$ then correspond bijectively to the cones in the picture above. The Abel-Jacobi section at a point on the stratum corresponding to a cone lands in the part of the Jacobian parametrising line bundles whose multidegree is given exactly by the corresponding θ-stable multidegree D. The pullback of the Abel-Jacobi image to the normalization of the curve C is obtained by twisting the pullback of $\mathcal{L}$ to the normalization by multiples of the points lying over the nodes, where these multiples are determined by the flow f. The line bundle on C itself (equivalently, the glueing data for reassembling the line bundle on C from the line bundle on the normalization) varies as we move about in the stratum.

5 Theorem A

5.1 Universal constructions

Let $\mathfrak{M}_g$ be the stack of log curves, and let $\mathfrak{Pic}$ be the universal Picard stack of degree 0 line bundles on the universal curve over $\mathfrak{M}_g$, which we equip with the strict log structure. By the main result of [10], there is an operational class[25]

$$\mathsf{DR} \in \mathsf{CH}^g(\mathfrak{Pic}). \tag{33}$$

By [43], there is a natural lift of (33) to a logarithmic class

$$\mathsf{logDR} \in \mathsf{logCH}^g(\mathfrak{Pic}).$$

For a stack S, prestable curve C/S of genus g, and line bundle $\mathcal{L}$ on C of degree 0, we have a classifying map

$$\varphi_{\mathcal{L}} \colon S \to \mathfrak{Pic}.$$

The double ramification cycle is defined in [10] as an operational class on S by

$$\mathsf{DR}(\mathcal{L}) = \varphi_{\mathcal{L}}^* \mathsf{DR} \in \mathsf{CH}^g(S).$$

If S is a log smooth log stack, C/S a log curve of genus g, and $\mathcal{L}$ a line bundle on C of degree 0, we have a classifying map (of log stacks)

$$\varphi_{\mathcal{L}} \colon S \to \mathfrak{Pic},$$

and we can pull back logDR. Following [43], we define

$$\mathsf{logDR}(\mathcal{L}) = \varphi_{\mathcal{L}}^* \mathsf{logDR} \in \mathsf{logCH}^g(S).$$

Recall that $A = (a_1, \ldots, a_n)$ is a vector of integers summing to $k(2g-2+n)$. Let C/S be the universal curve $\mathcal{C} \to \overline{\mathcal{M}}_{g,n}$ over the moduli space of stable curves with markings $x_1, \ldots, x_n$, and let

$$\mathcal{L} = (\omega_C^{\log})^{\otimes k}\left(-\sum_{i=1}^{n} a_i x_i\right).$$

By [10] and [43], the above universal constructions are *both* compatible with the previous definitions of the double ramification cycle:

$$\mathsf{DR}(\mathcal{L}) = \mathsf{DR}_{g,A} \in \mathsf{CH}^g(\overline{\mathcal{M}}_{g,n}),$$
$$\mathsf{logDR}(\mathcal{L}) = \mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n}).$$

[25] In Sect. 1 and the reference [10], the more precise notation $\mathsf{CH}^*_{\mathsf{op}}$ is used for operational Chow. We will drop the subscript op to simplify the notation (unless needed for emphasis).

5.2 Almost twistability

Let $\widehat{S}$ be a log smooth log algebraic stack, let $\widehat{C}/\widehat{S}$ be a log curve of genus g, and let $\widehat{\mathcal{L}}$ be a line bundle on $\widehat{C}$ of degree 0. Let

$$\mathcal{J}_{\widehat{C}/\widehat{S}} \to \widehat{S}$$

be the multidegree $\underline{0}$ part of the relative Picard stack of $\widehat{C}/\widehat{S}$.

Definition 43 The pair $(\widehat{C}/\widehat{S}, \widehat{\mathcal{L}})$ is *almost twistable* if there exists a dense open $i : U \hookrightarrow \widehat{S}$ which satisfies the following two conditions:

(i) the line bundle $\widehat{\mathcal{L}}$ has multidegree $\underline{0}$ over U,

(ii) the map $U \xrightarrow{\varphi_{\mathcal{L}} \circ i} \mathcal{J}_{\widehat{C}/\widehat{S}}$ is a closed immersion (or, equivalently, the image of $\varphi_{\mathcal{L}} \circ i$ is closed in $\mathcal{J}_{\widehat{C}/\widehat{S}}$).

Definition 43 is the specialization of [43, Definition 4.10] obtained by setting the strict piecewise linear function α there to be zero. Indeed, under this specialization, Condition (i) of [43, Definition 4.11] that $\alpha = 0$ is a twisting function on U just says that $\widehat{\mathcal{L}}$ already has multidegree 0 there. On the other hand, the map $U \to \mathcal{J}_{\widehat{C}/\widehat{S}}$ is always a locally-closed immersion, and Condition (ii) of [43, Definition 4.11] is the valuative criterion for closedness. The motivation for Definition 43 lies in the following result.

Proposition 44 *Let $(\widehat{C}/\widehat{S}, \widehat{\mathcal{L}})$ be almost twistable. Let $\varphi_{\widehat{\mathcal{L}}} \colon \widehat{S} \to \mathfrak{Pic}$ be the map induced by $\mathcal{L}$. Then,*

$$\varphi_{\widehat{\mathcal{L}}}^* \mathsf{logDR} = \varphi_{\widehat{\mathcal{L}}}^* \mathsf{DR}$$

in $\mathsf{logCH}^g(\widehat{S})$.

Proof When $\widehat{S}$ is smooth, the claim is a special case of [43, Lemma 4.13]. However, the proof given there does not use smoothness as we work throughout with operational classes (the smoothness assumption is present in [43] only as a running assumption which simplifies some other parts of the paper). □

5.3 Category of twists

Let S be a log smooth log algebraic stack, let C/S be a log curve of genus g, and let $\mathcal{L}$ be a line bundle on C of degree 0. We consider the category $\mathbf{Twist}(S)$ of tuples

$$(T/S, \widehat{C} \to C_T, \alpha)$$

where T/S is a log scheme, $\widehat{C} \to C_T$ is a quasi-stable model of C_T/T, and α is a sPL function on $\widehat{C}$. Let

$$\mathbf{Twist}^{\underline{0}}(S, \mathcal{L}) \hookrightarrow \mathbf{Twist}(S)$$

be the open substack consisting of those objects where

$$\widehat{C} \xrightarrow{\sim} C_T$$

and $\mathcal{L}_T(\alpha)$ has multidegree $\underline{0}$ on C_T/T.

If T is a trait (the spectrum of a discrete valuation ring) with generic point $j : \eta \to T$, the log structure on T is *maximal* if the natural map

$$\mathsf{M}_T \to \mathcal{O}_T \times_{j_*\mathcal{O}_\eta} j_*\mathsf{M}_\eta$$

is an isomorphism. For example, if $\mathsf{M}_\eta = \mathcal{O}_\eta^*$, then $\overline{\mathsf{M}}_T(T) = \mathbb{N}$, and $\mathsf{sPL}(C)$ is canonically isomorphic to the group of Cartier divisors on C supported over the closed point of T. As explained in [56, §3.5], when checking the valuative criterion for properness in the logarithmic setting, test valuation rings should be restricted to those with maximal log structure.

Lemma 45 *Let T be a trait with generic point η and maximal log structure. Let C/T be a log curve with line bundles $\mathcal{F}$, $\mathcal{F}'$ on C, and $\alpha \in \overline{\mathsf{M}}_C^{\mathrm{gp}}(C_\eta)$ an sPL function, together with an isomorphism*

$$\psi : \mathcal{F}_\eta(\alpha) \to \mathcal{F}'_\eta .$$

Then there exists an sPL function $\bar{\alpha} \in \overline{\mathsf{M}}_C^{\mathrm{gp}}(C)$ on C restricting to α over η and such that ψ extends to an isomorphism $\mathcal{F}(\bar{\alpha}) \to \mathcal{F}'$.

Remark 46 The proof is an application of the separatedness of the logarithmic Picard stack of [63]. To a first approximation, log line bundles can be seen as equivalence classes of line bundles under the action of twisting by piecewise linear functions. As such, separatedness of the moduli of log line bundles means that, if two line bundles are equivalent modulo piecewise-linear functions over η, then they are so over the whole of T.

Proof of Lemma 45 The line bundles $\mathcal{F}$, $\mathcal{F}'$ naturally correspond to $\mathcal{O}_C^*$-torsors on C. By extending the structure group, we obtain two $\mathsf{M}_C^{\mathrm{gp}}$-torsors $\mathcal{F}^{\log}$, $\mathcal{F}'^{\log}$. By the exact sequence

$$\pi_*\overline{\mathsf{M}}_C^{\mathrm{gp}} \to R^1\pi_*\mathcal{O}_C \to R^1\pi_*M_C^{\mathrm{gp}}$$

these agree on the generic point of T, and hence they agree on T by [63, Theorem 4.10.1]. By another application of the same exact sequence we see that the group of line bundles on C whose associated $\mathsf{M}_C^{\mathrm{gp}}$ torsor is trivial is the image of $\overline{\mathsf{M}}_C^{\mathrm{gp}}(C)$, hence there exists $\beta \in \overline{\mathsf{M}}_C^{\mathrm{gp}}(C)$ and an isomorphism

$$\mathcal{F}(\beta) \xrightarrow{\sim} \mathcal{F}'.$$

Then, $\mathcal{O}_{C_\eta}(\alpha - \beta_\eta)$ is trivial, hence $\alpha - \beta_\eta$ is constant, so there exists $\gamma \in \overline{\mathsf{M}}_T^{\mathrm{gp}}(\eta)$ with

$$\alpha - \beta_\eta = \pi^*\gamma .$$

By maximality of the log structure, there exists $\bar{\gamma} \in \overline{\mathsf{M}}_T^{\mathsf{gp}}(T)$ restricting to γ. Setting $\bar{\alpha} = \beta + \pi^*\bar{\gamma}$ gives the result. □

Proposition 47 *Let $X \hookrightarrow \mathbf{Twist}(S)$ be an open substack containing $\mathbf{Twist}^{\underline{0}}(S, \mathcal{L})$ as a dense open. Write C_X for the pullback of C/S under $X \to S$, and write $\widehat{C}$ and α for the universal objects over X determined by the map $X \to \mathbf{Twist}(S)$. If X is separated, then $(\widehat{C}/X, \mathcal{L}(\alpha))$ is an almost twistable family.*

Proof Note that

$$\mathcal{J}_{\widehat{C}/X} = \mathcal{J}_{C_X/X},$$

since the curves differ by the insertion of rational bubbles. Let $U = \mathbf{Twist}^{\underline{0}}(S, \mathcal{L}) \hookrightarrow X$, so $\mathcal{L}(\alpha)$ defines a monomorphism

$$U \to \mathcal{J}_{C_X/X},$$

which we will show to be proper by the valuative criterion.

Let T be a trait with generic point η, and fix a map

$$t\colon T \to \mathcal{J}_{C_X/X}$$

such that η lands in the image of U. The map t corresponds to a line bundle $\mathcal{F}'$ on C_T/T of multidegree $\underline{0}$. Since $\mathcal{F}'_\eta$ lies in the image of $\mathbf{Twist}^{\underline{0}}(S, \mathcal{L})$, there exist an sPL function α on C_η and an isomorphism

$$\psi\colon \mathcal{L}_\eta(\alpha) \to \mathcal{F}'.$$

By Lemma 45, we can extend the sPL function α and isomorphism ψ over the whole of T, defining an object

$$(T/S, C_T \xrightarrow{\sim} C_T, \alpha)$$

of $\mathbf{Twist}^{\underline{0}}(S, \mathcal{L})$. The line bundle $\mathcal{L}(\alpha)$ induces a map $t'\colon T \to \mathcal{J}_{C_T/T}$, which agrees with t at η, and hence agrees with t by the separatedness of X. □

5.4 Proof of Theorem A

To prove Theorem A, we will apply Propositions 44 and 47 under the following specialization of the geometry:

- C/S is the universal curve $\mathcal{C} \to \overline{\mathcal{M}}_{g,n}$ over the moduli space of stable curves,
- $\mathcal{L}$ is the line bundle of total degree 0 on $\mathcal{C}/\overline{\mathcal{M}}_{g,n}$ defined by

$$\mathcal{L} = (\omega_C^{\log})^{\otimes k}\left(-\sum_{i=1}^n a_i x_i\right),$$

- $\widehat{S} = \overline{\mathcal{M}}_{g,A}^{\theta} \xrightarrow{\rho} \overline{\mathcal{M}}_{g,n}$ for a small nondegenerate stability condition θ,

- $\widehat{\mathcal{L}} = \mathcal{L}^\theta$ is the universal line bundle on the universal quasi-stable curve

$$\widehat{C} = \mathcal{C}^\theta \to \overline{\mathcal{M}}^\theta_{g,A},$$

- $\alpha = \alpha^\theta$ is the universal sPL function vanishing on x_1 on $\mathcal{C}^\theta$ and satisfying

$$\mathcal{L}^\theta = \mathcal{L}(\alpha^\theta).$$

Proposition 48 *For a small nondegenerate stability condition θ, the line bundle $\mathcal{L}^\theta$ on $\mathcal{C}^\theta \to \overline{\mathcal{M}}^\theta_{g,A}$ is almost twistable.*

Proof There is a natural open immersion

$$\overline{\mathcal{M}}^\theta_{g,A} \to \mathbf{Twist}(\overline{\mathcal{M}}_{g,n})$$

induced by the subdivision $\mathcal{C}^\theta$ and the sPL function α^θ, identifying $\overline{\mathcal{M}}^\theta_{g,A}$ with the open substack X of $(\widehat{C}, \alpha) \in \mathbf{Twist}(\overline{\mathcal{M}}_{g,n})$ where $\mathcal{L}|_{\widehat{C}}(\alpha)$ is θ-stable. We must verify the conditions required by Proposition 47 for the open set X:

- Since multidegree $\underline{0}$ line bundles on stable models are θ-stable, X contains $\mathbf{Twist}^{\underline{0}}(\overline{\mathcal{M}}_{g,n}, \mathcal{L})$ as a dense open.
- Separatedness of X follows from Theorem 35: since $X = \overline{\mathcal{M}}^\theta_{g,A} \xrightarrow{\rho} \overline{\mathcal{M}}_{g,n}$ is proper, X is also proper. The key input is the nondegeneracy of θ.

Since the conditions of Proposition 47 are satisfied, we conclude that $(\mathcal{C}^\theta/\overline{\mathcal{M}}^\theta_{g,A}, \mathcal{L}^\theta)$ is an almost twistable family. □

To complete the proof of Theorem A, the class

$$\varphi^*_{\mathcal{L}^\theta}\mathsf{DR} = \mathsf{DR}^{\mathsf{op}}_{g,\emptyset,\mathcal{L}^\theta} \in \mathsf{CH}^g_{\mathsf{op}}(\overline{\mathcal{M}}^\theta_{g,A})$$

must be shown to represent $\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$. We apply Proposition 44 to the family $(\mathcal{C}^\theta/\overline{\mathcal{M}}^\theta_{g,A}, \mathcal{L}^\theta)$ to conclude

$$\varphi^*_{\mathcal{L}^\theta}\mathsf{logDR} = \varphi^*_{\mathcal{L}^\theta}\mathsf{DR}$$

The last step is to prove that

$$\varphi^*_{\mathcal{L}^\theta}\mathsf{logDR} \in \mathsf{CH}^g_{\mathsf{op}}(\overline{\mathcal{M}}^\theta_{g,A})$$

represents $\mathsf{logDR}_{g,A} \in \mathsf{logCH}^g(\overline{\mathcal{M}}_{g,n})$.

The claim of the last step is true because of invariance properties of the logarithmic double ramification cycle [43]:

(i) If $f : C' \to C$ over S is a subdivision of log curves, then

$$\mathsf{logDR}(f^*\mathcal{L}) = \mathsf{logDR}(\mathcal{L}),$$

(ii) If β is a sPL function on C then

$$\mathsf{logDR}(\mathcal{L}(\beta)) = \mathsf{logDR}(\mathcal{L})\,,$$

(iii) If $f : S' \to S$ is a log modification, then $\mathsf{logCH}^*(S') = \mathsf{logCH}^*(S)$ and

$$\mathsf{logDR}(f^*\mathcal{L}) = \mathsf{logDR}(\mathcal{L})\,.$$

Invariances (i) and (ii) are proven in [43, Lemma 4.13] and [43, Theorem 4.18]. Invariance (iii) is by the construction of logDR.

We compute

$$\begin{aligned}\mathsf{logDR}(\mathcal{L}^\theta) &= \mathsf{logDR}(\mathcal{L}|_{C^\theta}) \\ &= \mathsf{logDR}(\mathcal{L}|_{C\times_{\overline{\mathcal{M}}_{g,n}}\overline{\mathcal{M}}^\theta_{g,A}}) \\ &= \mathsf{logDR}(\mathcal{L})\end{aligned}$$

by Invariances (ii), (i), and (iii) respectively.

6 Piecewise polynomials and strata classes

6.1 Overview

The $\mathbb{Q}$-algebras of piecewise polynomials and strict piecewise polynomials associated to $\overline{\mathcal{M}}_{g,n}$ were defined in Sect. 1.7.1 and used there to define the logarithmic tautological ring

$$\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n}) \subset \mathsf{logCH}^*(\overline{\mathcal{M}}_{g,n})\,.$$

In order to derive the formulas of Theorem B from Theorem A, an explicit correspondence between the normally decorated strata classes of [64] and strict piecewise polynomials[26] is required. The explicit correspondence is established in Proposition 57 of Sect. 6.2. In Sect. 6.3, we rewrite the formula from [10] for the universal double ramification cycle in the language of piecewise polynomials in preparation for the proof of Theorem B in Sect. 7.

6.2 Relating piecewise polynomials and graph sums

6.2.1 Normally decorated strata classes

Let (X, D) be a nonsingular algebraic stack equipped with a normal crossings divisor, viewed as a log stack. We begin by briefly recalling the definition of *normally decorated strata classes* for (X, D) from [64, §5.1].

[26] Warning to the reader: the piecewise polynomials of [64] are the strict piecewise polynomials here.

Let S be a codimension k stratum in X, and let B_S be the set of branches of D through S (so $|B_S| = k$). Let

$$\epsilon : \widetilde{S} \to X$$

be the normalization of the closure of S, let

$$p : P \to \widetilde{S}$$

be the G-torsor over $\widetilde{S}$ determined by the monodromy group G of Definition 7 acting on the set B_S, and let

$$j = \epsilon \circ p : P \to X$$

denote the composition. Since the map ϵ is unramified, we can define the normal bundle N_ϵ which splits when pulled back to P:

$$p^* N_\epsilon = \oplus_{b \in B_S} N_b \,.$$

The assignment $b \mapsto c_1(N_b)$ extends to a $\mathbb{Q}$-algebra homomorphism

$$c_1(N) \colon \mathbb{Q}[B_S] \to \mathsf{CH}^*(P)$$

where $\mathbb{Q}[B_S]$ is a polynomial ring in variables indexed by B_S. Given $F \in \mathbb{Q}[B_S]$, let

$$F(c_1(N)) \in \mathsf{CH}^*(P)$$

be the image under $c_1(N)$.

Definition 49 A *normally decorated strata* class of (X, D) is a class of the form

$$j_* F(c_1(N)) \in \mathsf{CH}^*(X)$$

where $F \in \mathbb{Q}[B_S]$ is a polynomial in the branches of D through a stratum S.

Remark 50 Normally decorated strata are defined using the monodromy torsors of [64], which are G-torsors over the normalizations of the strata of X for the monodromy group G. In other words, the monodromy torsors are connected to the automorphism group of the corresponding stacky cone in the *canonical* tropicalization of (X, D). It is possible to construct monodromy torsors that resolve the automorphism group of the stacky cone in an arbitrary tropicalization and develop a more flexible theory of normally decorated strata classes, but we will not pursue these directions here.

6.2.2 Piecewise polynomials

Let (X, D) be a nonsingular algebraic stack equipped with a normal crossings divisor, viewed as a log stack. For the rest of this section, we will work with the *canonical*

tropicalization $X \to \mathcal{A}_X$. The combinatorial definition of (strict) piecewise polynomials given in Sect. 1.7.1 for $\overline{\mathcal{M}}_{g,n}$ carries over unchanged to (X, D). See [43, 62, 64] for detailed foundations. We will describe the piecewise polynomial of (X, D) corresponding to a normally decorated strata class.

Let Σ_X denote the canonical fan of (X, D), and let σ be the stacky cone corresponding to a stratum $S \subset X$. The stacky cone σ has a strict cover by the cone $\tau = \mathbb{R}^{B_S}_{\geq 0}$, in the sense of [25, Definition 2.5]. In particular, the interior of σ is a quotient of the interior of τ by the monodromy group G. Let

$$\mathfrak{p} \colon \tau \to \sigma \quad \text{and} \quad \mathfrak{j} : \tau \to \Sigma_X$$

denote the quotient and the composition of the quotient with the inclusion. The set $\tau(1)$ of rays in τ is naturally identified with the set B_S of branches, so there is an identification of $\mathbb{Q}$-algebras

$$\mathsf{sPP}(\tau) = \mathbb{Q}[B_S]\,.$$

The monodromy group G acts naturally and compatibly on B_S, hence on τ, on $\mathbb{Q}[B_S]$, and on $\mathsf{sPP}(\tau)$.

Pullback of strict piecewise polynomials yields an injection

$$\mathfrak{i} : \mathsf{sPP}(\sigma) \to \mathsf{sPP}(\tau)\,. \tag{34}$$

The image of $\mathfrak{i}$ is contained in the G-invariant polynomials and contains those G-invariant polynomials which vanish on the boundary of τ; equivalently, the G-invariant polynomials which are divisible by

$$\delta_\tau = \prod_{b \in B_S} b\,.$$

We also define a map in the other direction

$$\mathfrak{j}_0 \colon \mathsf{sPP}(\tau) \to \mathsf{sPP}(\sigma)\,, \quad F \mapsto \delta_\tau \sum_{g \in G} g^* F. \tag{35}$$

As an illustration, take $\tau = \mathbb{R}^2_{\geq 0}$ with an action of $G = \mathbb{Z}/2\mathbb{Z}$ generated by $(x, y) \mapsto (y, x)$. For the projection $\mathfrak{p} : \tau \to \sigma = [\tau/G]$ we have $\mathsf{sPP}(\tau) = \mathbb{Q}[x, y]$ and $\mathsf{sPP}(\sigma) = \mathbb{Q}[x + y, x \cdot y]$. Then $\mathfrak{i} : \mathbb{Q}[x + y, x \cdot y] \to \mathbb{Q}[x, y]$ is the inclusion and

$$\mathfrak{j}_0 : \mathbb{Q}[x, y] \to \mathbb{Q}[x + y, x \cdot y],\ f(x, y) \mapsto xy \cdot (f(x, y) + f(y, x)),$$

so $\mathfrak{j}_0(1) = 2xy$. Geometrically, this example is related to the punctured Whitney umbrella.

6.2.3 Extension

Let σ be the stacky cone, with strict cover τ, corresponding to a codimension k stratum $S \subset X$. We describe how to canonically extend the polynomial function $\mathfrak{j}_0 F$

defined by (35) from σ to the whole fan Σ_X. We will construct a nontrivial extension of $\mathrm{j}_0 F$ over cones which contain σ. For cones not containing σ, the extension will be 0.

Let $\sigma' \supset \sigma$ be any stacky cone containing σ. As before, σ' is a quotient of a cone $\tau' = \mathbb{R}^{k'}_{\geq 0}$ for $k' \geq k$ with perhaps additional faces identified,

$$\mathfrak{p} : \tau' \to \sigma' .$$

The map $\mathfrak{p}$ sends some of the k-dimensional faces of τ' onto σ and the rest onto faces of σ' different from σ.

Let $\mathcal{F}_k$ be the set of the k-dimensional faces of the fiber product $\tau \times_{\sigma'} \tau'$. Explicitly, $\mathcal{F}_k$ is the set of pairs $(\widetilde{\tau}, \gamma)$ where $\widetilde{\tau}$ is a face of τ' and γ is an isomorphism fitting into the diagram

$$\begin{array}{ccc} \widetilde{\tau} & \xrightarrow{\subset} & \tau' \\ \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \mathfrak{p}} \\ \tau & \longrightarrow & \sigma'. \end{array}$$

Because the image of τ in σ' is generically the quotient $\sigma \cong \tau/G$, the group G acts freely on the isomorphisms

$$\gamma : \widetilde{\tau} \to \tau$$

commuting with the map $\tau \to \sigma'$, so $\mathcal{F}_k$ is a disjoint union of G-torsors.

The standard variables of the polynomials on τ' are in bijective correspondence to the set of rays $\tau'(1)$. Let

$$D^n(\tau') \subset \mathsf{sPP}(\tau')$$

be the $\mathbb{Q}$-linear subspace spanned by all monomials in at most n different variables. For example, if $|\tau'(1)| = 3$ with variables x_1, x_2, x_3, then

$$2x_1^9 x_2^3 + 7x_2x_3 \in D^2(\tau') \quad \text{and} \quad x_1x_2x_3 \notin D^2(\tau') .$$

Elements in $D^n(\tau')$ are determined by their values on all of the n-dimensional faces of τ'.

The subspaces $D^n(\tau')$ define an increasing filtration of $\mathsf{sPP}(\tau')$ as n increases. Pullback of piecewise polynomials by $\mathfrak{p}$ identifies $\mathsf{sPP}(\sigma')$ with a subring of $\mathsf{sPP}(\tau')$. Thus, we can restrict the filtration from $\mathsf{PP}(\tau')$ to obtain a filtration, denoted $D^n(\sigma')$, on $\mathsf{sPP}(\sigma')$.

Proposition 51 *Let $F \in \mathsf{sPP}(\tau)$. There is a unique polynomial*

$$\mathrm{j}^{\sigma'}_* F \in D^k(\sigma')$$

which agrees with $\mathrm{j}_0 F$ on σ and is 0 on all other k-dimensional faces of σ'.

Proof The uniqueness claim is immediate since the function is in D^k and its values on all of the k-dimensional faces of σ' are specified.

We will construct the desired function on τ' and then argue that it descends to σ'. For $(\widetilde{\tau}, \gamma) \in \mathcal{F}_k$, the polynomial $\delta_{\widetilde{\tau}}\gamma^* F \in \mathsf{sPP}(\widetilde{\tau})$ vanishes on the boundary of $\widetilde{\tau}$, and we write $\overline{\delta_{\widetilde{\tau}}\gamma^* F} \in \mathsf{sPP}(\tau')$ for the basic[27] extension. Define

$$\mathrm{j}_*^{\sigma'} F = \sum_{(\widetilde{\tau},\gamma)\in\mathcal{F}_k} \overline{\delta_{\widetilde{\tau}}\gamma^* F} \in \mathsf{sPP}(\tau').$$

Certainly, $\mathrm{j}_*^{\sigma'} F \in D^k(\tau')$.

We claim that $\mathrm{j}_*^{\sigma'} F$ lies in the image of $\mathsf{sPP}(\sigma') \hookrightarrow \mathsf{sPP}(\tau')$. Let G' be the monodromy group of σ' which acts on τ'. The group G' also acts on the set of k-dimensional faces of τ'. Since $\tau' \to \sigma'$ is G'-invariant, if a k-dimensional face $\kappa \subset \tau'$ is taken to another face κ' by $g' \in G'$, and the image of κ in σ' is σ, then the image of κ' is also σ. Furthermore, G' permutes all k-dimensional faces that do not surject to σ. Therefore, $\mathrm{j}_*^{\sigma'} F$ is invariant with respect to the G'-action. Furthermore, since $\mathrm{j}_*^{\sigma'} F$ vanishes on the boundary of every k-dimensional cone and is symmetric with respect to any additional identification of faces, $\mathrm{j}_*^{\sigma'} F$ lies in the image of $\mathsf{sPP}(\sigma') \hookrightarrow \mathsf{sPP}(\tau')$.

Finally, we must check that $\mathrm{j}_*^{\sigma'} F$ agrees with $\mathrm{j}_0 F$ on σ and is 0 on all other k-dimensional faces of σ'. The restriction of the function $\mathrm{j}_*^{\sigma'} F$ to σ is determined by the k-dimensional faces that surject onto σ. Choose such a face τ_0 of τ'. For any $(\widetilde{\tau}, \gamma) \in \mathcal{F}_k$ with $\widetilde{\tau} \neq \tau_0$, the function $\overline{\delta_{\widetilde{\tau}}\gamma^* F}$ vanishes on τ_0 by construction. Thus, since the isomorphisms $\gamma : \tau_0 \to \tau$ in $\mathcal{F}_k$ can be identified with elements of $g \in G$, we have

$$\mathrm{j}_*^{\sigma'} F|_{\tau_0} = \sum_{g\in G} \overline{\delta_\tau g^* F} = \mathfrak{p}_* F\,,$$

which is the required agreement. □

Definition 52 Given $F \in \mathsf{sPP}(\tau)$, we define $\mathrm{j}_* F \in \mathsf{sPP}(\Sigma_X)$ to be $\mathrm{j}_*^{\sigma'} F$ on those stacky cones σ' of Σ_X which contain σ as a face and 0 on the other stacky cones of Σ_X.

Example 53 For $(X, D) = (\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$, consider the cone $\sigma_{\Gamma_0} \subseteq \Sigma_X$ associated to the stable graph Γ_0 with one vertex of genus 0 and precisely g loops. Then $\tau = \mathbb{R}^g_{\geq 0}$ carries the strict piecewise polynomial $F = 1$. Then, the function[28]

$$\mathrm{j}_* F \in \mathsf{sPP}(\Sigma_X)$$

is given as follows:

[27] The *basic extension* of a polynomial on a face of $\mathbb{R}^n_{\geq 0}$ is by pullback via the canonical projection $\mathbb{R}^n_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ to the face.

[28] The function was explained to us by D. Ranganathan.

- on cones σ_Γ where Γ has a vertex of positive genus, the function $\mathrm{j}_* F$ vanishes (since these do not contain σ_{Γ_0}),
- on the other cones σ_Γ, where Γ has first Betti number equal to g, we have

$$\mathrm{j}_* F = g! \cdot \sum_{\substack{T \subseteq \Gamma \\ \text{spanning tree}}} \prod_{e \in E(\Gamma) \setminus E(T)} \ell_e \,.$$

Indeed, in this case the set $\mathcal{F}_g$ corresponds to the set of graph morphisms $\Gamma \to \Gamma_0$, modulo the equivalence relation identifying morphisms which differ by flipping some of the loops (see Example 8). Every such morphism precisely contracts a spanning tree $T \subseteq \Gamma$, and for a given tree T there are $g!$ such *equivalence classes of* morphisms. Indeed, these just correspond to the $g!$ bijections from the g edges f of Γ_0 to the noncontracted edges $e \in E(\Gamma) \setminus E(T)$ of Γ.

6.2.4 Commutation

The explicit correspondence of Sect. 6.2.5 requires the following basic commutation result.

Lemma 54 *Let S be a stratum of X of codimension k with branch set B_S and monodromy torsor*

$$p : P \to \widetilde{S} \,.$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Q}[B_S] & \xrightarrow{\mathrm{j}_*} & \mathsf{sPP}(\Sigma_X) \\ \downarrow{\scriptstyle c_1(N)} & & \downarrow{\scriptstyle \Phi} \\ \mathsf{CH}^*(P) & \xrightarrow{j_*} & \mathsf{CH}^*(X). \end{array} \tag{36}$$

Proof The argument is carried out in three steps.

Step I: The simple normal crossings case.

We first treat the case where (X, D) is a scheme with simple normal crossings and $\bar{S}$ is the intersection of the irreducible components D_b corresponding to the branches $b \in B_S$. The closure $\bar{S}$ is then normal, the monodromy group G is trivial, and $N_b = \mathcal{O}(D_b)|_{\bar{S}}$. The commutation of (36) is then easily verified:

$$\begin{aligned} j_* F(c_1(N)) &= j_* j^* F(D_b : b \in B_S) \\ &= [\bar{S}] \cdot F(D_b : b \in B_S) \\ &= \left(\prod_{b \in B_S} D_b \right) \cdot F(D_b : b \in B_S) \\ &= \Phi(\mathrm{j}_* F) \,. \end{aligned}$$

Step II: Reduction of the general case to $X = \mathcal{A}_{\sigma'}$ a stacky Artin cone.
We can move the issue to the Artin fan $\mathcal{A}_X$ of X. There is a Cartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{j} & X \\ \downarrow{\scriptstyle t_P} & & \downarrow{\scriptstyle t_X} \\ \mathcal{P} & \xrightarrow{\mathcal{A}_j} & \mathcal{A}_X \end{array}$$

where t_X is smooth, and $\mathcal{P}$ is the monodromy torsor over the normalization of the closure of the stratum $t_X(S) \subset \mathcal{A}_X$. Furthermore,

$$N_{\mathcal{P}/\mathcal{A}_X} = \oplus_{b \in B_S} \mathcal{N}_b$$

and $N_{P/X} = t_P^* N_{\mathcal{P}/\mathcal{A}_X}$ compatibly with the splittings of $N_{P/X}$ and $N_{\mathcal{P}/\mathcal{A}_X}$. Then,

$$j_*(F(c_1(N)) = j_*(t_P^* F(c_1(\mathcal{N})) = t_X^* \mathcal{A}_{j*} F(c_1(\mathcal{N})).$$

So, instead of proving the commutation of (36), we may instead prove the commutation of:

$$\begin{array}{ccc} \mathbb{Q}[B_S] & \xrightarrow{\mathrm{j}_*} & \mathsf{sPP}(\Sigma_X) \\ \downarrow{\scriptstyle c_1(N)} & & \downarrow{\scriptstyle \Phi} \\ \mathsf{CH}^*(\mathcal{P}) & \xrightarrow{\mathcal{A}_{j*}} & \mathsf{CH}^*(\mathcal{A}_X). \end{array}$$

Furthermore, for the Artin fan,

$$\Phi \colon \mathsf{sPP}(\Sigma_X) \to \mathsf{CH}^*(\mathcal{A}_X)$$

is an isomorphism [64], so we drop Φ from the notation.

Because the Chow ring of an Artin fan satisfies étale descent (see Lemma 55 below), we may prove the desired equalities étale locally on $X = \mathcal{A}_X$. Since passing to a Zariski open does not affect the monodromy or the normal bundle, we may further assume that

$$X = \mathcal{A}_{\sigma'},$$

where $\Sigma_X = \sigma'$ with $\sigma \subset \sigma'$.

Step III: Reduction to simple normal crossings.

A worked example for this step of the proof can be found in Remark 56.

Choose an étale cover $\mathcal{A}_{\tau'}$ of $\mathcal{A}_{\sigma'}$ with $\tau' = \mathbb{R}^{k'}_{\geq 0}$. The diagram commutes for $X = \mathcal{A}_{\tau'}$ by Step I. To prove commutativity for $X = \mathcal{A}_{\sigma'}$, we must keep track of how passing from $\mathcal{A}_{\sigma'}$ to $\mathcal{A}_{\tau'}$ affects the monodromy.

Consider the fiber diagram of stacks

$$\begin{array}{ccc} Q & \xrightarrow{h} & \mathcal{A}_{\tau'} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ P & \xrightarrow{j} & \mathcal{A}_{\sigma'} \end{array}$$

where $Q \widetilde{=} P \times_{\mathcal{A}_{\sigma'}} \mathcal{A}_{\tau'}$. Let $\widetilde{\tau}$ be a cone in τ' that maps isomorphically to $\sigma \subset \sigma'$. Geometrically, such a cone corresponds to a stratum $T_{\widetilde{\tau}} \subset \mathcal{A}_{\tau'}$ that maps to S, or, equivalently, to a connected component of the preimage of S in $\mathcal{A}_{\tau'}$. By definition, P is a G-torsor over the normalization $\widetilde{S}$ of the closure $\overline{S}$ of S. Since q is étale, the pullback of $\widetilde{S}$ to $\mathcal{A}_{\tau'}$ is the normalization of the union of the closures $\overline{T}_{\widetilde{\tau}}$ of the $T_{\widetilde{\tau}}$. Since $\mathcal{A}_{\tau'}$ is simple normal crossings, the latter is simply the disjoint union of the closures $\overline{T}_{\widetilde{\tau}}$. Furthermore, as $\mathcal{A}_{\tau'}$ has no monodromy, the monodromy torsor of each $\overline{T}_{\widetilde{\tau}}$ is $\overline{T}_{\widetilde{\tau}}$ itself, and the pullback of P to $\mathcal{A}_{\tau'}$ is a trivial G-torsor. We conclude that Q is a disjoint union

$$Q \cong \coprod_{(\widetilde{\tau},\gamma)\in\mathcal{F}_k} Q_{(\widetilde{\tau},\gamma)}$$

with isomorphic connected components $Q_{(\widetilde{\tau},\gamma)} \cong \overline{T}_{\widetilde{\tau}}$.

Let $B_{\widetilde{\tau}}$ be the set of branches of the divisor in $\mathcal{A}_{\tau'}$ cutting out $T_{\widetilde{\tau}}$. By construction, the group G acts on $B_{\widetilde{\tau}}$, and the set of G-equivariant bijections between B_S and $B_{\widetilde{\tau}}$ is a G-torsor, naturally identified with the torsor of pairs $(\widetilde{\tau},\gamma) \in \mathcal{F}_k$. Since q is étale, we find that

$$p^* N_{P/X} = N_{Q/\mathcal{A}_{\tau'}}\,.$$

Writing $p_{(\widetilde{\tau},\gamma)}$ for the restriction of p to $Q_{(\widetilde{\tau},\gamma)}$, we have that for each $b \in B_S$,

$$p^*_{(\widetilde{\tau},\gamma)} N_b = N_{(\widetilde{\tau},\gamma)(b)}$$

where we have written $(\widetilde{\tau},\gamma)(b)$ for the image of b under the bijection

$$B_S \to B_{\widetilde{\tau}}$$

corresponding to $(\widetilde{\tau},\gamma)$. Therefore, for any polynomial in $F \in \mathbb{Q}[B_S]$, an element $(\widetilde{\tau},\gamma) \in \mathcal{F}_k$ determines a polynomial $\gamma^* F \in \mathbb{Q}[B_{\widetilde{\tau}}]$, and the pullback of $j_* F(c_1(N_{P/X}))$ to $\mathcal{A}_{\tau'}$ is the sum

$$\sum_{(\widetilde{\tau},\gamma)\in\mathcal{F}_k} \gamma^* F(c_1(N_{\overline{T}_{\widetilde{\tau}}/\mathcal{A}_{\tau'}}))\,.$$

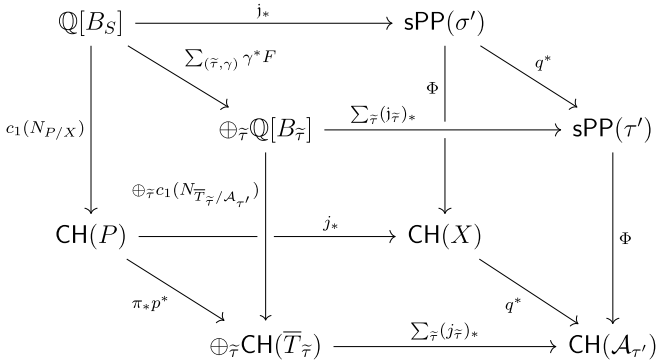
By Step I, the corresponding polynomial on $\mathcal{A}_{\tau'}$ is

$$\sum_{(\widetilde{\tau},\gamma)\in\mathcal{F}_k} \overline{\delta_{\widetilde{\tau}} \gamma^* F}\,,$$

which is precisely the polynomial on $\mathcal{A}_{\tau'}$ that descends to $\mathrm{j}_* F$ on X.

If we denote the projection $Q \to \coprod_{\widetilde{\tau}} \overline{T}_{\widetilde{\tau}}$ by π, the inclusion $\overline{T}_{\widetilde{\tau}} \to \mathcal{A}_{\tau'}$ by $j_{\widetilde{\tau}}$, and the homomorphism $\mathbb{Q}[B_{\widetilde{\tau}}] \to \mathsf{sPP}(\tau')$ by $(\mathrm{j}_{\widetilde{\tau}})_*$, we have proven that the following

diagram commutes:



Here, the (direct) sums are taken over the set of cones $\widetilde{\tau}$ in τ' that map isomorphically to $\sigma \subset \sigma'$. Since we know the commutation for the front face of the cube, and all arrows from the back face to the front face are injective, the commutation for the back face follows. □

Lemma 55 *Let $\mathcal{A}_X$ be an Artin fan, and write $(\mathcal{A}_X)_{et}$ for the small strict étale site of $\mathcal{A}_X$. Then the functor*

$$\mathsf{CH}^* \colon (\mathcal{A}_X)_{et}^{\mathsf{op}} \to \mathbb{Q}\mathbf{Alg}$$

is a sheaf.

Proof The result is a rephrasing of [64, Theorem 14] which shows that the Chow ring of the Artin fan coincides with the algebra of strict piecewise polynomials. The point is that this identification also holds for every strict étale cover of $\mathcal{A}_X$. More precisely, since for every strict étale map $U \to \mathcal{A}_X$, U is itself an Artin fan, [64, Theorem 14] shows that

$$\mathsf{CH}^*(U) = \mathsf{sPP}(U).$$

The result is proven in [64] for the case where $\mathcal{A}_X$ is the canonical Artin fan of a variety X, but all that is actually used is that the cone stack $\mathcal{A}_X$ can be stratified by a finite number of strata, each isomorphic to $B(\mathbb{G}_m^k \rtimes G)$ for nonnegative integers k and finite group G. Since sPP is by construction a strict étale sheaf, the result follows. □

Remark 56 To aid the reader in following the proof above, we work through the later steps in the special case where $X = \overline{\mathcal{M}}_{1,2}$ and $S = B\mathbb{Z}/2$ is the closed stratum corresponding to a cycle of two projective lines. Step II reduces us to the case where $X = \mathcal{A}_{\sigma'}$ is the Artin fan

$$\left[\frac{\operatorname{Spec} \mathbb{C}[\mathbb{N}^2]}{\operatorname{Spec} \mathbb{C}[\mathbb{Z}^2] \rtimes \mathbb{Z}/2}\right]$$

and $S = \widetilde{S}$ is the (stacky) closed point of X. We choose

$$\mathcal{A}_{\tau'} = \left[\frac{\operatorname{Spec}\mathbb{C}[\mathbb{N}^2]}{\operatorname{Spec}\mathbb{C}[\mathbb{Z}^2]}\right],$$

so that P is the closed point of $\mathcal{A}_{\tau'}$, and Q can be identified with the disjoint union of two copies of P:

$$\begin{array}{ccccc} Q = B\mathbb{G}_m^2 \sqcup B\mathbb{G}_m^2 & & \longrightarrow & & [\mathbb{A}^2/\mathbb{G}_m^2] = \mathcal{A}_{\tau'} \\ \downarrow & & & & \downarrow \\ P = B\mathbb{G}_m^2 & \longrightarrow & B\mathbb{G}_m^2 \rtimes \mathbb{Z}/2 & \longrightarrow & [\mathbb{A}^2/\mathbb{G}_m^2 \rtimes \mathbb{Z}/2] = \mathcal{A}_{\sigma'}. \end{array}$$

We find that $\sigma = \sigma'$ and $\tau = \tau'$ because we study a closed stratum. The diagram

$$\begin{array}{ccccc} \mathbb{Q}[B_S] & & \xrightarrow{\mathrm{j}_*} & & \mathsf{sPP}(\Sigma_X) \\ \downarrow{\scriptstyle c_1(N)} & & & & \downarrow{\scriptstyle \Phi} \\ \mathsf{CH}^*(P) & \xrightarrow{p_*} & \mathsf{CH}^*(\widetilde{S}) & \longrightarrow & \mathsf{CH}^*(X). \end{array}$$

specializes to

$$\begin{array}{ccccc} \mathbb{Q}[x,y] & & \xrightarrow{f(x,y)\mapsto xy(f(x,y)+f(y,x))} & & \mathbb{Q}[x,y]^{\mathbb{Z}/2} \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{Q}[x,y] & \xrightarrow{f(x,y)\mapsto f(x,y)+f(y,x)} & \mathbb{Q}[x,y]^{\mathbb{Z}/2} & \xrightarrow{g(x,y)\mapsto xyg(x,y)} & \mathbb{Q}[x,y]^{\mathbb{Z}/2} \end{array}$$

6.2.5 Explicit correspondence

We return to our main case of interest: strict piecewise polynomials on the stack of log curves $\mathfrak{M}_g$.

• Let Γ be a graph corresponding to a cone σ_Γ. The rays of σ_Γ are identified with the edges $E(\Gamma)$ of Γ. The corresponding stratum $\mathfrak{M}^\Gamma \subseteq \mathfrak{M}_g$ is given by the immersion

$$\mathfrak{M}^\Gamma = \prod_{v\in V(\Gamma)} \mathfrak{M}^{sm}_{g(v),H(v)}/\operatorname{Aut}(\Gamma) \to \mathfrak{M}_g\,, \tag{37}$$

where $H(v)$ is the set of half-edges attached to the vertex v, see [9, Proposition 2.5]. Define

$$\mathfrak{M}_\Gamma = \prod_{v\in \mathrm{V}(\Gamma)} \mathfrak{M}_{g(v),H(v)}\,.$$

The normalization of the closure of the stratum is given by

$$\mathfrak{M}_\Gamma/\operatorname{Aut}_\Gamma \to \mathfrak{M}_g\,,$$

and the universal monodromy torsor is given by

$$\mathfrak{M}_\Gamma / \operatorname{Aut}_\Gamma^{\text{loop}} \to \mathfrak{M}_\Gamma / \operatorname{Aut}_\Gamma \,, \tag{38}$$

where as in Example 8 the group $\operatorname{Aut}_\Gamma^{\text{loop}} \subseteq \operatorname{Aut}_\Gamma$ is the subgroup of automorphisms of Γ acting as the identity on $E(\Gamma)$. By the same argument as in Example 8, we see that the monodromy group $G_\Gamma \subseteq \operatorname{Sym}(E(\Gamma))$ of the stratum $\mathfrak{M}^\Gamma$ is the quotient

$$G_\Gamma = \operatorname{Aut}_\Gamma / \operatorname{Aut}_\Gamma^{\text{loop}} \,.$$

Then (38) is precisely the G_Γ-monodromy torsor over the normalization of the strata closure for $\mathfrak{M}^\Gamma$.

For the calculations below, it will be more convenient to compose the monodromy torsor (38) with the additional étale cover $\mathfrak{M}_\Gamma \to \mathfrak{M}_\Gamma / \operatorname{Aut}_\Gamma^{\text{loop}}$ since that composition induces the familiar map

$$j_\Gamma \colon \mathfrak{M}_\Gamma \to \mathfrak{M}_g \,.$$

• Let $d \colon E(\Gamma) \to \mathbb{Z}_{\geq 0}$. Define a polynomial on the set of edges of Γ:

$$F = \prod_{e \in E(\Gamma)} e^{d_e} \in \mathbb{Q}[E(\Gamma)],$$

Then, $\mathrm{j}_* F$ is the sPP function on $\mathfrak{M}$ defined as follows: for a stable graph Γ', the value on the cone $\sigma_{\Gamma'}$ is given by the formula

$$\frac{1}{|\operatorname{Aut}_\Gamma^{\text{loop}}|} \sum_{f : \Gamma' \to \Gamma} \prod_{e \in E(\Gamma)} \ell_{f(e)}^{d_e + 1} \,.$$

As in Example 53, the graph morphisms $f : \Gamma' \to \Gamma$ precisely correspond to the elements $(\widetilde{\tau}, \gamma)$ that we sum over in the definition of $\mathrm{j}_*^{\sigma_{\Gamma'}}$. The factor $\frac{1}{|\operatorname{Aut}_\Gamma^{\text{loop}}|}$ arises since again we identify morphisms whose actions on vertices and edges agree. The equivalence classes of this relation are precisely the orbits of the free action of $\operatorname{Aut}_\Gamma^{\text{loop}}$ on the set of morphisms $\Gamma' \to \Gamma$.

The above formula automatically vanishes outside the star of the cone of Γ, since for any cone $\sigma_{\Gamma'}$ outside this star, there exist no graph morphisms $\Gamma' \to \Gamma$ and thus the above sum is empty. Define the class

$$\mathsf{m} = \prod_{e=(h,h') \in E(\Gamma)} (-\psi_h - \psi_{h'})^{d_e} \in \mathsf{CH}^*(\overline{\mathcal{M}}_\Gamma) \,.$$

Proposition 57 *The following explicit correspondence holds*:

$$\frac{1}{|\operatorname{Aut}_\Gamma^{\text{loop}}|} \cdot j_{\Gamma,*}(\mathsf{m}) = \Phi(\mathrm{j}_* F) \,.$$

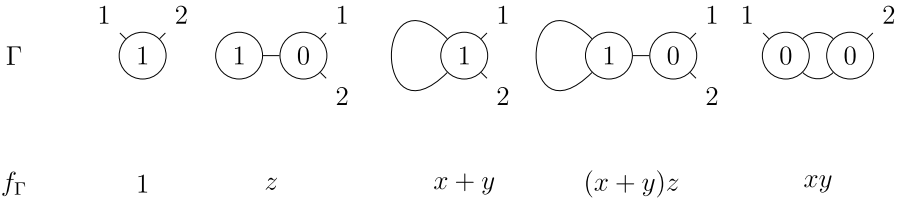


Fig. 5 A list of stable graphs Γ for $\overline{\mathcal{M}}_{1,2}$ and the corresponding strict piecewise polynomials f_Γ such that $\Phi(f_\Gamma)$ is the fundamental class of the associated stratum closure in $\overline{\mathcal{M}}_{1,2}$

Proof We apply Lemma 54 where $S = \mathfrak{M}^\Gamma$ is the stratum of $X = \mathfrak{M}_g$ defined by (37). Then, B_S is naturally in bijection with $E(\Gamma)$. We view F as an element of the top left corner of (36). We must show that

$$j_{\Gamma,*}(\mathsf{m}) = |\mathrm{Aut}_\Gamma^{\mathrm{loop}}| \cdot j_*(F(c_1(N)))\,.$$

Since $j\colon P = \mathfrak{M}_\Gamma/\mathrm{Aut}_\Gamma^{\mathrm{loop}} \to X = \mathfrak{M}_g$ fits in a diagram

$$\mathfrak{M}_\Gamma \xrightarrow{j'} \mathfrak{M}_\Gamma/\mathrm{Aut}_\Gamma^{\mathrm{loop}} \xrightarrow{j} \mathfrak{M}_g \qquad (j_\Gamma)$$

we need only show

$$\mathsf{m} = (j')^* F(c_1(N)) \in \mathsf{CH}(\mathfrak{M}_\Gamma)\,,$$

which is clear. □

Example 58 We write the correspondence of Proposition 57 explicitly in the case of the moduli space $\overline{\mathcal{M}}_{1,2}$. For more details of the computation (with illustrations), we refer the reader to [77, Chap. 2]. The strict piecewise polynomials on $\overline{\mathcal{M}}_{1,2}$ can be identified with the subring of the polynomial ring

$$\mathbb{Q}[x, y, z]$$

consisting of those polynomials which are symmetric under swapping x and y. Here, z is the strict piecewise linear function associated to the boundary divisor with a separating node, and x and y are the two branches of the divisor of curves with a non-separating node at its self-intersection point. Figure 5 gives the correspondence between strict piecewise polynomials and fundamental classes of strata closures.

6.3 Pixton's formula in terms of piecewise polynomials

6.3.1 Contributions

We translate here the formula of [10] for the universal double ramification cycle into the language of piecewise polynomials. The formula is written in the Chow cohomology of the universal Picard stack $\mathfrak{Pic}$ of degree 0 line bundles over the stack of

genus g curves $\mathfrak{M}_g$. The result will be used in Sect. 7 to prove our formula for the logarithmic double ramification cycle.

Let $\pi : C \to S$ be a log curve of genus g, and let $\mathcal{L}$ be a line bundle on C of degree 0. The double ramification cycle

$$\mathsf{DR} \in \mathsf{CH}^g(\mathfrak{Pic})$$

can be naturally expressed in terms of strict piecewise polynomials on S. We require the following two constructions:

- Let $\eta^{\mathfrak{Pic}} = \pi_*(c_1(\mathcal{L})^2) \in \mathsf{CH}^1(\mathfrak{Pic})$.
- For every positive integer r, let Cont^r be the strict piecewise power series on S defined on the cone associated to a graph Γ of genus g by:

$$\mathrm{Cont}^r_\Gamma = \sum_w r^{-h^1(\Gamma)} \prod_{e \in E(\Gamma)} \exp\left(\frac{\overline{w}(\vec{e})\overline{w}(\overleftarrow{e})}{2}\ell_e\right) \in \mathbb{Q}[[\ell_e : e \in E(\Gamma)]]\,. \quad (39)$$

The sum runs over *admissible weightings* mod r: flows w with values in $\mathbb{Z}/r\mathbb{Z}$ such that

$$\mathrm{div}(w) = \underline{\deg}(\mathcal{L}) \in (\mathbb{Z}/r\mathbb{Z})^{V(\Gamma)}\,.$$

Inside the exponential, $\overline{w}(\vec{e})$ and $\overline{w}(\overleftarrow{e})$ denote the unique representative of $w(\vec{e}) \in \mathbb{Z}/r\mathbb{Z}$ and $w(\overleftarrow{e}) \in \mathbb{Z}/r\mathbb{Z}$ in $\{0, \ldots, r-1\}$.

Remark 59 To make sense of the contribution formula (39), we must show that the collection of power series Cont^r_Γ yields a strict piecewise power series on the cone stack of S. The issue is about the compatibility of the formula on different cones Γ.

An étale specialization of geometric points of $\mathfrak{Pic}$,

$$\varphi\colon t \mapsto s\,,$$

yields a natural contraction map $\Gamma_s \to \Gamma_t$, an inclusion $E(\Gamma_t) \hookrightarrow E(\Gamma_s)$ of edges, and an injective map of rings

$$P_\varphi\colon \mathbb{Q}[[\ell_e\colon e \in E(\Gamma_t)]] \to \mathbb{Q}[[\ell_e\colon e \in E(\Gamma_s)]].$$

Suppose for every geometric point s in $\mathfrak{Pic}$, we are given a power series

$$f^s \in \mathbb{Q}[[\ell_e\colon e \in E(\Gamma_s)]]\,.$$

If, for every étale specialization $\varphi\colon t \mapsto s$, we have

$$P_\varphi(f^t) = f^s\big|_{\{\ell_e = 0 \,|\, e \notin E(\Gamma_t)\}}\,, \quad (40)$$

then the collection of power series $\{f_s\}$ glue to a global strict piecewise power series on the cone stack of S. Every truncation of the collection of power series $\{f^s\}$ can be viewed as a strictly piecewise polynomial on $\mathfrak{Pic}$.

If the f^s are compatible under graph contractions and graph automorphisms, then the compatibility (40) is satisfied over $\mathfrak{Pic}$. For the power series Cont^r of (39), these two compatibilities hold (see Lemma 60 below).

Lemma 60 *Formula* (39) *for* Cont^r *is compatible with respect to graph automorphisms and contractions.*

Proof The claim for automorphisms is clear by definition (since only the structure of the graph Γ is used).

Since a general contraction is a composition of contractions of single edges, we need only study the contraction of a single edge e,

$$\Gamma \to \Gamma' .$$

There are two cases:

(i) The edge e is not a loop. Weightings on the contracted graph Γ' are then naturally in bijection with weightings on Γ, and $h^1(\Gamma') = h^1(\Gamma)$.
(ii) The edge e is a loop. Then, $h^1(\Gamma') = h^1(\Gamma) - 1$. For a given weighting on Γ' there are exactly r lifts to weightings on Γ (obtained by assigning weights i and $r - i$ to half edges of e for $0 \le i < r$). To specialise Cont^r_Γ to Γ', we set $\ell_e = 0$. The result differs from the expression for $\mathrm{Cont}^r_{\Gamma'}$ in two ways: we sum over r additional weightings, and we divide by r; these cancel exactly.

Therefore, in both cases, compatibility (40) holds. □

Lemma 61 *For truncations of a fixed degree on quasi-compact opens of* S, Cont^r *is eventually polynomial in* r.

Proof For every graph Γ, the eventual polynomiality in r of Cont^r_Γ is obtained from the theory of Ehrhart polynomials, see [47, Appendix A]. Quasi-compactness is required to ensure that only finitely many graphs are considered. □

We define Cont to be the $r = 0$ value of the polynomial determined by Cont^r for large r. Every fixed degree truncation of Cont is a piecewise polynomial on the cone stack of $\mathfrak{Pic}$.

6.3.2 The universal DR cycle

With these preparations in place, we define

$$\mathbf{P}_g = \exp\Big(-\frac{1}{2}\eta^{\mathfrak{Pic}}\Big) \cdot \Phi(\mathrm{Cont}) \in \mathsf{CH}^*(\mathfrak{Pic}) .$$

On the other hand, a mixed degree operational Chow class

$$\mathsf{P}_{g,\emptyset,0} \in \mathsf{CH}^*(\mathfrak{Pic})$$

is defined in [10] whose codimension g part $\mathsf{P}^g_{g,\emptyset,0}$ computes DR.

Theorem 62 *We have the equality*

$$\mathbf{P}_g = \mathsf{P}_{g,\emptyset,0} \in \mathsf{CH}^*(\mathfrak{Pic}) .$$

Hence, we have $\mathsf{DR} = \mathbf{P}^g_g$, *where* $\mathbf{P}^g_g$ *denotes the codimension* g *part of* $\mathbf{P}_g$.

Proof By the definition of [10], $\mathsf{P}_{g,\emptyset,0}$ is given by the coefficient of r^0 of[29]

$$\exp\left(-\frac{1}{2}\eta^{\mathfrak{Pic}}\right)\sum_{\substack{\Gamma\in\mathsf{G}_{g,0,0}\\ w\in\mathsf{W}_{\Gamma,r}}}\frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\,j_{\Gamma *}\left[\prod_{e\in \mathrm{E}(\Gamma)}\frac{1-\exp\left(-\frac{w(\vec{e})w(\overleftarrow{e})}{2}(\psi_{\vec{e}}+\psi_{\overleftarrow{e}})\right)}{\psi_{\vec{e}}+\psi_{\overleftarrow{e}}}\right].$$

Following the notation of [10], the above sum is over the set of all possible graphs $\mathsf{G}_{g,0,0}$ for the universal Picard stack and over the set $\mathsf{W}_{\Gamma,r}$ of all possible admissible weightings mod r on Γ.

It suffices to show, for fixed r, that the class

$$\Phi(\mathrm{Cont}^r)=\Phi\left(\left\{\sum_w r^{-h^1(\Gamma)}\prod_e\exp\left(\frac{w(\vec{e})w(\overleftarrow{e})}{2}\ell_e\right)\right\}_\Gamma\right)$$

equals the class

$$\sum_{\substack{\Gamma\in\mathsf{G}_{g,0,0}\\ w\in\mathsf{W}_{\Gamma,r}}}\frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\,j_{\Gamma *}\left[\prod_{e\in \mathrm{E}(\Gamma)}\frac{1-\exp\left(-\frac{w(\vec{e})w(\overleftarrow{e})}{2}(\psi_{\vec{e}}+\psi_{\overleftarrow{e}})\right)}{\psi_{\vec{e}}+\psi_{\overleftarrow{e}}}\right].$$

Consider a graph $\Gamma\in\mathsf{G}_{g,0,0}$. Let $\mathrm{Cont}^r(\Gamma)$ be the part of Cont^r containing those monomials in which *all* the ℓ_e for $e\in E(\Gamma)$ appear with *positive* exponent.[30] We easily see that

$$\mathrm{Cont}^r=\sum_{\Gamma\in\mathsf{G}_{g,0,0}}\mathrm{Cont}^r(\Gamma)\,.$$

Indeed, on the cone associated to the stable graph Γ', for each monomial term $c_\alpha\ell^\alpha$ appearing in Cont^r let Γ be the graph obtained from Γ' by contracting all edges $e\in E(\Gamma')$ whose variable ℓ_e does not appear in the monomial ℓ^α. Then $c_\alpha\ell^\alpha$ appears in $\mathrm{Cont}^r(\Gamma)$ (and not in any other $\mathrm{Cont}^r(\Gamma'')$), and the above sum just expresses how the monomial terms of Cont^r are classified according to the possible contracted graphs Γ. We claim that the class $\Phi(\mathrm{Cont}^r(\Gamma))$ is exactly equal to

$$\sum_{w\in\mathsf{W}_{\Gamma,r}}\frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\,j_{\Gamma *}\left[\prod_{e\in \mathrm{E}(\Gamma)}\frac{1-\exp\left(-\frac{w(\vec{e})w(\overleftarrow{e})}{2}(\psi_{\vec{e}}+\psi_{\overleftarrow{e}})\right)}{\psi_{\vec{e}}+\psi_{\overleftarrow{e}}}\right].$$

[29] The half-edges h, h' appearing in the formula from [10] naturally correspond to the directed edges $\vec{e}$, $\overleftarrow{e}$, and under this correspondence, the notions of admissible weightings mod r likewise agree.

[30] More precisely, for a graph $\Gamma\in\mathsf{G}_{g,0,0}$, we can describe the piecewise polynomial $\mathrm{Cont}^r(\Gamma)$ as follows. On the cone associated to a stable graph Γ', $\mathrm{Cont}^r(\Gamma)$ is given by those monomial terms of $\mathrm{Cont}^r_{\Gamma'}$ of (39) for which the contraction of all edges of Γ' associated to variables ℓ_e which *do not* appear in the monomial is a graph isomorphic to Γ.

The equality follows from Proposition 57. Indeed, for the power series

$$F_\Gamma = |\mathrm{Aut}_\Gamma^{\mathrm{loop}}| \cdot \sum_{w \in \mathsf{W}_{\Gamma,r}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \left[\prod_{e \in \mathrm{E}(\Gamma)} \frac{1 - \exp\left(-\frac{w(\vec{e})w(\bar{e})}{2} e\right)}{e} \right] \in \mathbb{Q}[[E(\Gamma)]]$$

in formal variables e (for $e \in E(\Gamma)$), we can check monomial by monomial that $\mathrm{j}_* F_\Gamma = \mathrm{Cont}^r(\Gamma)$, in the notation from Proposition 57, whereas the ψ-monomial m_Γ on $\overline{\mathcal{M}}_\Gamma$ associated to F_Γ is precisely the term appearing in the pushforward $j_{\Gamma *}$ above. The intuition here is that replacing ℓ_e^d by $(-\psi_{\vec{e}} - \psi_{\bar{e}})^{d-1}$ everywhere in the expression $\exp(\frac{w(\vec{e})w(\bar{e})}{2}\ell_e)$ yields exactly

$$\frac{1 - \exp\left(-\frac{w(\vec{e})w(\bar{e})}{2}(\psi_{\vec{e}} + \psi_{\bar{e}})\right)}{\psi_{\vec{e}} + \psi_{\bar{e}}}.$$

The factor $|\mathrm{Aut}_\Gamma^{\mathrm{loop}}|$ in the formula of F_Γ precisely cancels the opposite factor appearing in Proposition 57. We see that the language of piecewise polynomials expresses Pixton's formula more efficiently. □

7 Theorems B and C

7.1 Families of curves

Let S be a log smooth log algebraic stack, let C/S be a log curve of genus g, and let $\mathcal{L}$ be a line bundle on C of degree 0. We assume, in addition, that we have:

(i) a log modification $\widehat{S} \to S$,
(ii) a quasi-stable model $\widehat{C} \to C \times_S \widehat{S}$,
(iii) $\alpha \in \overline{\mathsf{M}}_{\widehat{C}}^{\mathsf{gp}}(\widehat{C})$ a strict piecewise linear function (in the sense of Sect. 3.4).

For the proofs of Theorems B and C, the fundamental geometry is the following. Recall that $A = (a_1, \ldots, a_n)$ is a vector of integers summing to $k(2g-2+n)$. Let C/S be the universal curve $\mathcal{C} \to \overline{\mathcal{M}}_{g,n}$ over the moduli space of stable curves with markings $x_1, \ldots, x_n$, and let

$$\mathcal{L} = (\omega_C^{\log})^{\otimes k}\left(-\sum_{i=1}^n a_i x_i\right).$$

The additional data is given by:

(i) $\widehat{S} = \overline{\mathcal{M}}_{g,A}^\theta \xrightarrow{\rho} \overline{\mathcal{M}}_{g,n}$ for a small nondegenerate stability condition θ,
(ii) $\widehat{C} = \mathcal{C}^\theta \to \mathcal{C} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,A}^\theta$ is the universal quasi-stable curve,
(iii) $\alpha = \alpha^\theta$ is the universal sPL function on $\mathcal{C}^\theta$ and satisfies $\mathcal{L}^\theta = \mathcal{L}(\alpha^\theta)$.

7.2 Geometric construction of a logarithmic class on S

The line bundle $\widehat{\mathcal{L}} = \mathcal{L}(\alpha)$ on $\widehat{C}$ defines a map

$$\varphi_{\widehat{\mathcal{L}}} \colon \widehat{S} \to \mathfrak{Pic}\,,$$

and, in turn, a class

$$\mathsf{DR}(\widehat{\mathcal{L}}) = \varphi_{\widehat{\mathcal{L}}}^*(\mathsf{DR}) \in \mathsf{CH}^g(\widehat{S}) \subset \mathsf{logCH}^g(S)\,.$$

To emphasise that we are considering $\mathsf{DR}(\widehat{\mathcal{L}})$ as a logarithmic class on S, we use the notation

$$\mathsf{DR}^{\approx}_S(\widehat{S}, \widehat{\mathcal{L}}) = \mathsf{DR}(\widehat{\mathcal{L}}) \in \mathsf{logCH}^g(S)\,,$$

an approximation to the true logarithmic double ramification cycle

$$\mathsf{logDR}(\mathcal{L}) \in \mathsf{logCH}^g(S)$$

of [43].

By Proposition 44, a sufficient condition for the equality

$$\mathsf{DR}^{\approx}_S(\widehat{S}, \widehat{\mathcal{L}}) = \mathsf{logDR}(\mathcal{L})$$

is that $(\widehat{C}/\widehat{S}, \widehat{\mathcal{L}})$ is almost twistable.

In Sect. 7.4, we give a formula for $\mathsf{DR}^{\approx}_S(\widehat{S}, \widehat{\mathcal{L}})$ which applies regardless of whether the family is almost twistable. Applied with Theorem A, the formula yields Theorem B. Theorem C is obtained from Pixton's relations in the universal setting [10].

7.3 The functions $\mathfrak{P}$ and $\mathfrak{L}$

We take a (strict) geometric point $\widehat{s}$ of $\widehat{S}$ mapping to s, a strict geometric point of S. Over these points, we have a graph $\widehat{\Gamma}$ of $\widehat{C}$ contracting to a graph Γ of C, a map of cones

$$\mathrm{Hom}(\overline{\mathsf{M}}_{\widehat{S},\widehat{s}}, \mathbb{R}_{\geq 0}) = \widehat{\sigma} \to \sigma_\Gamma = \mathrm{Hom}(\overline{\mathsf{M}}_{S,s}, \mathbb{R}_{\geq 0})\,,$$

and a sPL function α on $\widehat{\Gamma}$.

Remark 63 In the geometric setting of Theorems B and C (as specified at the end of Sect. 7.1),

$$\sigma_\Gamma = \mathbb{R}_{\geq 0}^{E(\Gamma)}\,,$$

and the choice of the point $\widehat{s}$ determines a flow I. The cone $\widehat{\sigma}$ is then given by

$$\widehat{\sigma} = \sigma_{\widehat{\Gamma},I}$$

defined in (30). The sPL function α is uniquely determined by the conditions:

- $\alpha(x_1) = 0$,
- the slopes of α are equal to I,

see (31). In particular, α here exactly matches (16).

We resume the general case, continuing to view the sPL function $\alpha \in \overline{\mathsf{M}}_{\widehat{C}}(\widehat{C})$ as a function

$$V(\widehat{\Gamma}) \to \overline{\mathsf{M}}^{\mathrm{gp}}_{\widehat{S},\widehat{s}},$$

following Sect. 3.4. Let

$$\underline{\deg}(\widehat{\mathcal{L}}) : V(\widehat{\Gamma}) \to \mathbb{Z}$$

be the multidegree. We have a length function

$$\ell \colon E(\widehat{\Gamma}) \to \overline{\mathsf{M}}_{\widehat{S},\widehat{s}},$$

and seeing elements of $\overline{\mathsf{M}}_{\widehat{S},\widehat{s}}$ as strict piecewise linear functions on $\widehat{\sigma}$, we can define a power series on $\widehat{\sigma}$ by

$$\mathrm{Cont}^r_{\widehat{\sigma}} = \sum_w r^{-h^1(\widehat{\Gamma})} \prod_{e \in E(\widehat{\Gamma})} \exp\left(\frac{\overline{w}(\vec{e})\overline{w}(\overleftarrow{e})}{2}\ell_e\right) \in \mathbb{Q}[[\ell_e : e \in E(\widehat{\Gamma})]]\,. \tag{41}$$

The sum runs over *admissible weightings* mod r: flows w with values in $\mathbb{Z}/r\mathbb{Z}$ such that

$$\mathrm{div}(w) = \underline{\deg}(\widehat{\mathcal{L}}) \in (\mathbb{Z}/r\mathbb{Z})^{V(\widehat{\Gamma})}\,.$$

Inside the exponential, $\overline{w}(\vec{e}), \overline{w}(\overleftarrow{e}) \in \{0, \ldots, r-1\}$ denote the unique representative of $w(\vec{e}), w(\overleftarrow{e}) \in \mathbb{Z}/r\mathbb{Z}$ respectively.

Definition 64 (The function $\mathfrak{P}$) As explained in Sect. 6.3, the formula (41) is polynomial in r for sufficiently large r. The function $\mathfrak{P}$, a strict piecewise power series on the cone complex of $\widehat{S}$, is defined by the the $r = 0$ specialization of the polynomial in r defined by formula (41).

Definition 65 (The function $\mathfrak{L}$) Following (17), we define a sPL function $\mathfrak{L}$ on the cone stack of $\widehat{S}$. On $\widehat{\sigma}$, the definition is

$$\mathfrak{L} = \sum_{v \in V(\widehat{\Gamma})} \underline{\deg}(\mathcal{L} \otimes \widehat{\mathcal{L}})(v) \cdot \alpha(v)\,, \tag{42}$$

where[31] $\deg(\mathcal{L} \otimes \widehat{\mathcal{L}})(v)$ is an integer and $\alpha(v) \in \overline{\mathsf{M}}^{\mathrm{gp}}_{\widehat{S},\widehat{s}}$ is viewed as a linear function on $\widehat{\sigma}$. We extend the expression (42) to a strict piecewise linear function on the cone stack of $\widehat{S}$ using the method of Remark 59.

[31] Over $\widehat{S}$, we follow the convention that $\mathcal{L}$ denotes the pullback of $\mathcal{L}$ from C to $\widehat{C}$.

7.4 The formula

Application of the map Φ of (11) to $\mathfrak{P}$ and $\mathfrak{L}$ yields classes

$$\Phi(\mathfrak{P}),\, \Phi(\mathfrak{L}) \in \mathsf{logCH}^*(S).$$

Let the logarithmic class $\eta \in \mathsf{logCH}^1(S)$ be the image of

$$\eta = \pi_*(c_1(\mathcal{L})^2) \in \mathsf{CH}^1(\widehat{S})$$

under the inclusion $\mathsf{CH}^*(\widehat{S}) \subset \mathsf{logCH}^*(S)$.

Definition 66 (The class $\mathbf{P}^{\widehat{S}}_{g,\widehat{\mathcal{L}}}$) The logarithmic class associated to $(\widehat{C}/\widehat{S}, \widehat{L})$ by Pixton's formula is defined by:

$$\mathbf{P}^{\widehat{S}}_{g,\widehat{\mathcal{L}}} = \exp\Big(-\frac{1}{2}(\eta + \Phi(\mathfrak{L}))\Big) \cdot \Phi(\mathfrak{P}) \in \mathsf{logCH}^*(S)\,.$$

Let $\mathbf{P}^{g,\widehat{S}}_{g,\widehat{\mathcal{L}}}$ be the codimension g part of $\mathbf{P}^{\widehat{S}}_{g,\widehat{\mathcal{L}}}$,

$$\mathbf{P}^{g,\widehat{S}}_{g,\widehat{\mathcal{L}}} \in \mathsf{logCH}^g(S)\,.$$

Following Sect. 6.3.2, the mixed degree operational Chow class

$$\mathsf{P}_{g,\emptyset,0} \in \mathsf{CH}^*(\mathfrak{Pic})$$

from [10] has codimension g part $\mathsf{P}^g_{g,\emptyset,0}$ which computes the universal DR class.

Theorem 67 *We have*

$$\varphi^*_{\widehat{\mathcal{L}}}(\mathsf{P}_{g,\emptyset,0}) = \mathbf{P}^{\widehat{S}}_{g,\widehat{\mathcal{L}}} \in \mathsf{logCH}^*(S)\,. \tag{43}$$

In particular, we have

$$\mathsf{DR}^{\approx}_S(\widehat{S}, \widehat{\mathcal{L}}) = \mathbf{P}^{g,\widehat{S}}_{g,\widehat{\mathcal{L}}} \in \mathsf{logCH}^g(S)\,. \tag{44}$$

Proof On $\widehat{S}$, we have the classes

$$\eta = \pi_*(c_1(\mathcal{L})^2) \quad \text{and} \quad \widehat{\eta} = \pi_*(c_1(\widehat{\mathcal{L}})^2)\,,$$

which are related by

$$\begin{aligned}
\widehat{\eta} &= \pi_*(c_1(\widehat{\mathcal{L}})^2) \\
&= \pi_*(c_1(f^*\mathcal{L})^2) + 2\pi_*(c_1(f^*\mathcal{L}) \cdot c_1(\mathcal{O}(\alpha))) + \pi_*(c_1(\mathcal{O}(\alpha))^2) \\
&= \eta + \pi_*(c_1(\mathcal{L} \otimes \widehat{\mathcal{L}}) \cdot c_1(\mathcal{O}(\alpha)))\,.
\end{aligned}$$

Next, for every line bundle $\mathcal{F}$ on $\widehat{C}$ we claim that

$$\pi_*(c_1(\mathcal{F}) \cdot c_1(\mathcal{O}(\alpha))) = \Phi\Big(\sum_{v \in \widehat{V}} (\underline{\deg}\mathcal{F})(v) \cdot \alpha(v) \Big), \tag{45}$$

To prove formula (45), we first reduce to the case where $c_1(\mathcal{O}_C(\alpha))$ is a vertical prime divisor D on $\widehat{C}$ lying over a prime divisor $\pi(D)$ on $\widehat{S}$. The left side yields $\pi(D)$ times the degree of F on the generic fiber of $D \to \pi(D)$, which is exactly what is computed by the right side.

Applying (45) yields $\widehat{\eta} = \eta + \Phi(\mathfrak{L})$. By Theorem 62, we have[32]

$$\begin{aligned}
\varphi^*_{\widehat{\mathcal{L}}}(\mathsf{P}_{g,\emptyset,0}) &= \varphi^*_{\widehat{\mathcal{L}}}\left(\exp\Big(-\frac{1}{2}\eta^{\mathfrak{Pic}} \Big) \cdot \Phi(\mathrm{Cont}) \right) \\
&= \exp\Big(-\frac{1}{2}\widehat{\eta} \Big) \cdot \varphi^*_{\widehat{\mathcal{L}}}\Phi(\mathrm{Cont}) \\
&= \exp\Big(-\frac{1}{2}(\eta + \Phi(\mathfrak{L})) \Big) \cdot \varphi^*_{\widehat{\mathcal{L}}}\Phi(\mathrm{Cont})
\end{aligned}$$

By definition, $\mathfrak{P}$ exactly matches the piecewise power series $\varphi^*_{\widehat{\mathcal{L}}}\Phi(\mathrm{Cont})$. Hence,

$$\varphi^*_{\widehat{\mathcal{L}}}(\mathsf{P}_{g,\emptyset,0}) = \exp\Big(-\frac{1}{2}(\eta + \Phi(\mathfrak{L})) \Big) \cdot \Phi(\mathfrak{P}) \in \mathsf{logCH}^*(S),$$

which proves (43). Equation (44) follows from the codimension g part of (43) together with [10]. □

7.5 Proofs of Theorems B and C

Proof of Theorem B The result follows immediately by specializing Theorem 67 to the fundamental geometry specified in Sect. 7.1. By Theorem A, the left side of equation (44) specializes to $\mathsf{logDR}_{g,A}$, and the right side of equation (44) specializes specializes to $\mathbf{P}^{g,\theta}_{g,A}$. We exactly obtain the formula of Theorem B. □

Proof of Theorem C Part (i) is a consequence of Theorem B, since both sides of the equality compute the cycle $\mathsf{logDR}_{g,A}$. Part (ii) follows from the vanishing

$$\mathsf{P}^h_{g,\emptyset,0} = 0 \in \mathsf{CH}^h(\mathfrak{Pic}) \quad \text{for } h > g \tag{46}$$

combined with equality (43) of Theorem 67. The vanishing (46) follows from the universal DR relations of [10, Theorem 8]. □

The proof of [10, Theorem 8] uses a vanishing result for Pixton's double ramification formula with target varieties on moduli spaces of stable maps (which was shown in [8] using a strategy that generalizes the earlier proof on the moduli space of stable curves given in [28]).

[32] Here, $\eta^{\mathfrak{Pic}}$ denotes the universal class on $\mathfrak{Pic}$ from Sect. 6.3.

8 Genus 1 calculations

8.1 Form of the answer

We apply Theorem B here to explicitly compute $\mathsf{logDR}_{g,A}$ in the case $g = 1$. The result, presented in Theorem 72, is of the form

$$\mathsf{logDR}_{1,A} = \mathsf{DR}_{1,A} + (\text{piecewise linear correction term}) \in \mathsf{logCH}^1(\overline{\mathcal{M}}_{1,n}),$$

where the term $\mathsf{DR}_{1,A}$ is the (pullback of the) standard double ramification cycle in $\mathsf{CH}^1(\overline{\mathcal{M}}_{1,n})$. The formula for $\mathsf{DR}_{1,A}$ is written in sPP language using Theorem 62. We have

$$\mathsf{DR}_{1,A} = -\frac{1}{2}\eta + \Phi(\mathrm{Cont}),$$

where η is the class from (18) and Cont is the strict piecewise linear function given by

$$\mathrm{Cont}_\Gamma = -\frac{1}{12}\left(\sum_{e \text{ a nonseparating edge}} \ell_e\right) - \frac{1}{2}\left(\sum_{e \text{ a separating edge}} w(e)^2 \ell_e\right), \tag{47}$$

where w is any flow on Γ satisfying $\mathrm{div}(w) = \underline{\deg}_{k,A}$. Here $\underline{\deg}_{k,A}$ is the multidegree of the line bundle $(\omega_C^{\log})^{\otimes k}(-\sum_i a_i x_i)$ as defined in Sect. 1.7.2. In the second sum, we have one term for each *undirected* separating edge; although evaluating $w(e)$ requires choosing an orientation for e, the value of $w(e)^2$ does not depend on this choice because $w(\vec{e}) = -w(\vec{e})$. While there can be different choices for the flow w in general, its value on a separating edge is uniquely determined by $\mathrm{div}(w)$ (as the sum of the degrees on one side of the edge), so the term $w(e)^2$ in (47) is well-defined.

8.2 θ-stability

Let θ be a small nondegenerate stability condition of type $(1, n)$ and degree 0. We can completely describe the θ-stable multidegrees D. A graph Γ of genus 1 has cycle number either 0 or 1. Nothing interesting happens when the cycle number is 0 (in which case the graph is a tree).

Lemma 68 *Let Γ be a tree. Let θ be a small nondegenerate stability condition of degree 0. Then, the only θ-stable multidegree D on Γ is $D = 0$.*

Proof Let e be any edge of Γ which splits Γ into subgraphs Γ_1 and Γ_2. Because there is only one edge connecting Γ_1 with Γ_2, there is only one integer satisfying the θ-stability inequality for $D(\Gamma_1)$. Because θ is small, this integer must be 0. Therefore, D has total degree 0 on each of the subgraphs Γ_1 and Γ_2. We easily conclude that D has degree 0 on every vertex. □

The situation is slightly more complicated for a graph Γ with cycle number 1. Such a graph has a unique cycle C_Γ along with trees glued to the vertices of C_Γ.

Lemma 69 *Let Γ be a connected graph with cycle number 1 and cycle C_Γ. Let θ be a small nondegenerate stability condition of degree 0. Let D be a θ-stable multidegree D on Γ. Then, $D(v) = 0$ for all v not on the cycle C_Γ. Moreover, as one goes around the cycle, the nonzero values of $D(v)$ alternate between 1 and -1.*

Proof The first claim follows by the same argument as for Lemma 68. For the second claim, it suffices to show that, for any subset of consecutive vertices around the cycle C_Γ, the sum of their degrees is in the set $\{-1, 0, 1\}$.

Consider the subgraph Γ_1 consisting of consecutive vertices around C_Γ together with the trees glued at those vertices. Because there are only two edges connecting Γ_1 with its complement, there are only two integers satisfying the θ-stability inequality for $D(\Gamma_1)$. Because θ is small, one of these integers must be 0, and then the other is ± 1. Therefore, $D(\Gamma_1) \in \{-1, 0, 1\}$, and since D has degree 0 outside the cycle, we obtain the desired property. □

The Lemmas 68 and 69 apply not only to a genus 1 stable graph Γ, but to any quasi-stable modification $\widehat{\Gamma}$ (we extend the stability condition to the quasi-stable model as explained after Definition 28).

8.3 The formula

The computation of logDR is particularly simple in genus 1 because the function $\mathfrak{P}$ of (19) in degree 1 is just the first term in (47): the strict piecewise linear function

$$-\frac{1}{12}\left(\sum_{e \text{ a nonseparating edge}} \ell_e\right).$$

The interesting part of formula (19) in genus 1 is $\mathfrak{L}$, which we will compute explicitly.

We begin with a general result (which holds in any genus) rewriting $\mathfrak{L}$ in a more convenient form.

Lemma 70 *Let $(\widehat{\Gamma}, D, I)$ be the data giving a cone $\sigma_{\widehat{\Gamma},I} \in \widetilde{\Sigma}^\theta_\Gamma$. Let J be any flow on $\widehat{\Gamma}$ satisfying $\operatorname{div}(J) = D$, and let $I_0 = I + J$ (so $\operatorname{div}(I_0) = \underline{\deg}_{k,A}$). Then on $\sigma_{\widehat{\Gamma},I}$, we have*

$$\mathfrak{L} = \sum_{e \in E(\widehat{\Gamma})} (I_0(\vec{e})^2 - J(\vec{e})^2) \cdot \widehat{\ell}_e\big|_{\widehat{\ell}=\widehat{\ell}(\ell)},$$

where $\widehat{\ell}(\ell)$ is the unique collection of edge lengths of the subdivision $\widehat{\Gamma}$ of Γ (depending on the lengths ℓ of Γ) such that there exists a strict piecewise linear function on $\widehat{\Gamma}$ with slopes given by the flow I (see Lemma 38), and where the sum runs over the unoriented edges of $\widehat{\Gamma}$: any orientation of e can be chosen (the summand is invariant).

Proof We use the same notation as in the definition of $\mathfrak{L}$ in (16) and (17), but we omit the final change of variables $|_{\widehat{\ell}=\widehat{\ell}(\ell)}$ for brevity of notation. We have

$$\mathfrak{L} = \sum_{v \in V(\widehat{\Gamma})} (D + \underline{\deg}_{k,A})(v)\alpha(v)$$

$$
\begin{aligned}
&= \sum_{v,\vec{e}\to v} (I_0(\vec{e}) + J(\vec{e}))\alpha(v) \\
&= \sum_{e=(v,v')\in E(\widehat{\Gamma})} \Big((I_0(v\to v') + J(v\to v'))\alpha(v') \\
&\quad + (I_0(v'\to v) + J(v'\to v))\alpha(v) \Big) \\
&= \sum_{e=(v,v')\in E(\widehat{\Gamma})} (I_0(v\to v') + J(v\to v'))(\alpha(v') - \alpha(v)) \\
&= \sum_{e\in E(\widehat{\Gamma})} (I_0(\vec{e}) + J(\vec{e}))I(\vec{e})\widehat{\ell}_e \\
&= \sum_{e\in E(\widehat{\Gamma})} (I_0(\vec{e}) + J(\vec{e}))(I_0(\vec{e}) - J(\vec{e}))\widehat{\ell}_e
\end{aligned}
$$

as desired. In the last two lines, any orientation of e can be chosen (the summand is invariant). □

Let Γ be a stable graph of genus 1, and let $(\widehat{\Gamma}, D, I)$ be the data giving a cone

$$\sigma_{\widehat{\Gamma},I} \in \widetilde{\Sigma}^{\theta}_{\Gamma}.$$

We will compute $\mathfrak{L}$ on $\sigma_{\widehat{\Gamma},I}$. If Γ is a tree, then $D = 0$ and $\widehat{\Gamma} = \Gamma$ by Lemma 68. Nothing interesting is happening: there is a unique flow I with $\operatorname{div}(I) = \underline{\deg}_{k,A}$, and the resulting function $\mathfrak{L}$ simply contributes part of the regular double ramification cycle formula.

Assume now that Γ has cycle number 1 with cycle C_Γ. We fix an orientation on C_Γ, and let C be the flow on Γ (or on $\widehat{\Gamma}$) with value 1 around that cycle and 0 elsewhere. In order to apply Lemma 70, we must pick a flow J with $\operatorname{div}(J) = D$. By Lemma 69, D must have a very specific form, and we can find such [33] a J supported on C_Γ that only takes on values 0 and 1 (when read in the chosen orientation around the cycle). We then have

$$
\begin{aligned}
\mathfrak{L} &= \sum_{e\in E(\widehat{\Gamma})} (I_0(e)^2 - J(e)^2)\cdot \widehat{\ell}_e\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \\
&= \sum_{e\in E(\widehat{\Gamma})} (I_0(e)^2 - J(e)C(e))\cdot \widehat{\ell}_e\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \\
&= \sum_{e\in E(\widehat{\Gamma})} (I_0(e)^2 - I_0(e)C(e))\cdot \widehat{\ell}_e\big|_{\widehat{\ell}=\widehat{\ell}(\ell)} \\
&= \sum_{e\in E(\Gamma)} (I_0(e)^2 - I_0(e)C(e))\cdot \ell_e\,,
\end{aligned}
$$

[33] If $D \neq 0$, it is easy to see that such a J is actually unique.

where the first equality is Lemma 70, the second equality follows from J only taking on values 0 and 1 (when oriented consistently with C around the cycle), the third equality follows from the relation between $I = I_0 - J$ and the variable substitution $\widehat{\ell} = \widehat{\ell}(\ell)$, and the fourth equality follows because I_0 and C are constant along the edge subdivided by $\widehat{\Gamma}$.

This final expression only depends on I_0. Since $I_0 = I + J$ and

$$C(e)^2 \geq C(e)J(e) \geq 0$$

for all e, we have the inequalities

$$\sum_{e \in E(\widehat{\Gamma})} I_0(e)C(e)\widehat{\ell}_e \geq \sum_{e \in E(\widehat{\Gamma})} I(e)C(e)\widehat{\ell}_e \geq \sum_{e \in E(\widehat{\Gamma})} (I_0(e) - C(e))C(e)\widehat{\ell}_e \,.$$

The middle term here vanishes after the variable substitution $\widehat{\ell} = \widehat{\ell}(\ell)$, while the first and third terms can be simplified because I_0 and C are constant along the edge subdivided by $\widehat{\Gamma}$. If we let $F = I_0$ denote the flow on Γ, we obtain the inequalities

$$\sum_{e \in E(\Gamma)} F(e)C(e)\ell_e \geq 0 \geq \sum_{e \in E(\Gamma)} (F(e) - C(e))C(e)\ell_e \,. \tag{48}$$

But for general ℓ_e, exactly one flow F with $\operatorname{div}(F) = \underline{\deg}_{k,A}$ satisfies these inequalities (since the set of such flows is an arithmetic progression with difference C). Therefore the inequalities (48) describe a (coarser) subdivision where $\mathfrak{L}$ can be defined. We summarize what we have computed in the following result.

Proposition 71 *Let Γ be a graph with cycle number 1. Let C be a flow on Γ with value 1 around the unique cycle and 0 elsewhere. Then*

$$\mathfrak{L} = \sum_{e \in E(\Gamma)} (F(e)^2 - F(e)C(e)) \cdot \ell_e,$$

where F is any flow on Γ satisfying $\operatorname{div}(F) = \underline{\deg}_{k,A}$ and the inequalities (48).

By putting the pieces of the formula for logDR together, we obtain a simple description of the piecewise linear function giving the difference between logDR and standard DR in genus 1.

Theorem 72 *In genus 1, we have*

$$\mathsf{logDR}_{1,A} = \mathsf{DR}_{1,A} - \frac{1}{2}\Phi(\mathfrak{L}'),$$

where $\mathfrak{L}'$ is the piecewise linear function defined as follows: $\mathfrak{L}'$ is zero on cones coming from a stable tree Γ, and otherwise $\mathfrak{L}'$ is given by

$$\mathfrak{L}' = \sum_{e \in E(C_\Gamma)} (F(e)^2 - F(e)C(e)) \cdot \ell_e$$

(for C and F as in Proposition 71).

Proof In genus 1 and degree 1, the logDR formula (19) yields

$$\mathsf{logDR}_{1,A} = -\frac{1}{2}\eta + \Phi([\mathfrak{P}]_{\deg 1}) - \frac{1}{2}\Phi(\mathfrak{L}).$$

We compare the above with the standard DR formula discussed in Sect. 8.1. The η terms are identical, and we have already observed that the degree 1 part of $\mathfrak{P}$ agrees with the nonseparating edge term in (47). Meanwhile, the part of $-\frac{1}{2}\Phi(\mathfrak{L})$ coming from separating edges in Proposition 71 agrees with the separating edge term in (47). An edge in Γ is nonseparating if and only if it is in the cycle C_Γ, so removing those terms from $\mathfrak{L}$ leaves precisely $\mathfrak{L}'$. □

Remark 73 After a little combinatorial manipulation, we obtain an alternative description of the piecewise linear function $\mathfrak{L}'$. Suppose F is a flow on Γ satisfying $\mathrm{div}(F) = \underline{\deg}_{k,A}$. Define a piecewise linear function on σ_Γ by

$$m_F = \min\left(\sum_{\substack{e\in E(\Gamma)\\ F(e)C(e)>0}} F(e)C(e)\ell_e, \sum_{\substack{e\in E(\Gamma)\\ F(e)C(e)<0}} -F(e)C(e)\ell_e \right).$$

The function m_F is zero if F is not acyclic (where not being acyclic means that we can find a cycle γ of directed edges in Γ such that $F(\vec{e}) \geq 0$ for all $\vec{e}$ in γ and with at least one of the inequalities being strict). It is then easily checked that on σ_Γ we have

$$\mathfrak{L}' = 2\sum_F m_F\,,$$

where the sum runs over all acyclic F.

Remark 74 The logarithmic genus 1 computation can be used to compute an interesting class in $\mathsf{CH}^2(\overline{\mathcal{M}}_{1,n})$ as follows. Take two genus 1 cycles with the same n,

$$\mathsf{logDR}_{1,A}\,,\ \mathsf{logDR}_{1,B} \in \mathsf{logCH}^1(\overline{\mathcal{M}}_{1,n})\,.$$

Individually, the cycles push forward to the standard cycles

$$\mathsf{DR}_{1,A}, \mathsf{DR}_{1,B} \in \mathsf{CH}^1(\overline{\mathcal{M}}_{1,n})$$

but the pushforward of their product,

$$\pi_*(\mathsf{logDR}_{1,A} \cdot \mathsf{logDR}_{1,B}) \in \mathsf{CH}^2(\overline{\mathcal{M}}_{1,n})\,,$$

is *not* equal to the product of the standard DR cycles. These *double-double ramification cycles* are discussed further in Sect. 9.1. Here, we simply sketch how to compute them in genus 1 using the above formulas for logDR.

By Theorem 72 and a straightforward argument for the vanishing of the cross terms, we have

$$\pi_*(\mathsf{logDR}_{1,A} \cdot \mathsf{logDR}_{1,B}) = \mathsf{DR}_{1,A} \cdot \mathsf{DR}_{1,B} + \frac{1}{4}\pi_*(\Phi(\mathfrak{L}'_A \cdot \mathfrak{L}'_B))\,.$$

By Remark 73, this second term is defined on a cone σ_Γ by the polynomial

$$\pi_* \left(\sum_{F_A, F_B} m_{F_A} m_{F_B} \right),$$

where F_A, F_B run over acyclic flows balancing A and B respectively and π_* is the pushforward from piecewise polynomials on a subdivision of a cone to polynomials on that cone. After working out the algebra of the pushforward and translating into standard tautological class notation, we obtain

$$\pi_*(\mathsf{logDR}_{1,A} \cdot \mathsf{logDR}_{1,B}) = \mathsf{DR}_{1,A} \cdot \mathsf{DR}_{1,B} - \sum_{\Gamma \text{ double edge graph}} \frac{1}{2} j_{\Gamma *} \left(\frac{A_\Gamma^2 B_\Gamma^2 - A_\Gamma^2 - B_\Gamma^2 + \gcd(A_\Gamma, B_\Gamma)^2}{12} \right),$$

where the sum runs over the isomorphism classes of stable graphs Γ with two vertices and a double edge connecting them and A_Γ, B_Γ are the absolute values of the degrees of $\underline{\deg}_{k,A}$, $\underline{\deg}_{k,B}$ respectively on one of the vertices (it does not matter which one since the total degree is 0).

9 Computational tools and higher double ramification cycles

9.1 A Sage implementation of logDR

A Sage package `logtaut` for computations with logarithmic double ramification cycles is being developed as part of the `admcycles` package [31]. The package is able to compute/manipulate the following mathematical structures discussed in our paper:

- the vector space of stability conditions θ of type (g, n) and degree d (with a description of a natural basis),
- cone stacks and their (strict) piecewise polynomial functions (together with natural operations such as pushforwards under subdivision maps),
- the particular cone stack $\Sigma_{\overline{\mathcal{M}}_{g,n}}$, the subdivision induced by a vector $A \in \mathbb{Z}^n$ together with a nondegenerate stability condition θ of degree $|A|$, and the natural map

$$\mathsf{sPP}(\Sigma_{\overline{\mathcal{M}}_{g,n}}) \to \mathsf{R}^*(\overline{\mathcal{M}}_{g,n})$$

  discussed in Sect. 6.2,
- the piecewise polynomial functions $\mathfrak{L}$ and $\mathfrak{P}$ of equation (19) determining the logarithmic double ramification cycle via Theorem B.

As an example of a calculation in `logtaut`, we can write an explicit relation in $\mathsf{logR}^2(\overline{\mathcal{M}}_{1,3})$ obtained by Theorem C (ii). Concretely, we can combine the vanishing

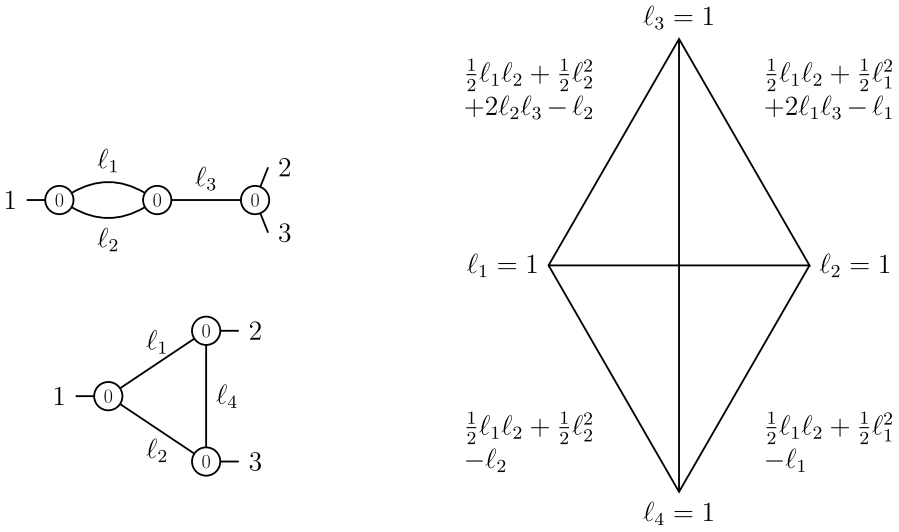


Fig. 6 The piecewise polynomial $\mathfrak{P}$, which is nonvanishing on precisely two (neighboring) maximal cones of $\Sigma_{\overline{\mathcal{M}}_{1,3}}$. We illustrate the stable graphs associated to these cones and the slice through them corresponding to all edge lengths summing to 1. The cones are subdivided at the ray $\ell_1 = \ell_2 = 1$, $\ell_3 = \ell_4 = 0$, and we give the piecewise polynomial $\mathfrak{P}$ on the resulting four chambers

from Theorem C with the classical vanishing of the formula $\mathsf{P}^h_{g,A}$ for the double ramification cycle in degree h higher than g.[34] Taking the difference, we obtain

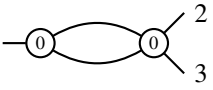
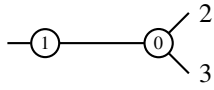
$$\Delta = \mathbf{P}^{2,\theta}_{1,(2,-1,-1)} - \mathsf{P}^2_{1,(2,-1,-1)} = 0 \in \mathsf{logR}^2(\overline{\mathcal{M}}_{1,3}),$$

for a small and nondegenerate stability condition θ.[35] After expressing $\mathsf{P}^2_{1,(2,-1,-1)}$ in terms of piecewise polynomials via Theorem 62, the advantage of the difference Δ is that the piecewise polynomial parts of $\mathbf{P}^{2,\theta}_{1,(2,-1,-1)}$ and $\mathsf{P}^2_{1,(2,-1,-1)}$ cancel on all but two maximal cones in $\Sigma_{\overline{\mathcal{M}}_{1,3}}$. With `logtaut`, we then compute that Δ is the codimension 2 part of the mixed-degree class

$$\left(1 + 2\psi_1 + \frac{1}{2}(\psi_2 + \psi_3)\right) \cdot \Phi(\mathfrak{P}) \tag{49}$$

for the piecewise polynomial function $\mathfrak{P}$ on $\Sigma_{\overline{\mathcal{M}}_{1,3}}$ illustrated in Fig. 6.

Using the methods of Sect. 6.2, we can translate the relation $\Delta = 0$ into the language of normally decorated strata classes: consider the graphs

$\Gamma =$  , $\delta_{2,3} =$  .

[34] The formula $\mathsf{P}^h_{g,A}$ was first presented in [47] and the claimed vanishing was proven in [28].

[35] The choice of the stability condition θ does not affect the relation Δ in this case.

Let $\pi : \widehat{\mathcal{M}} \to \overline{\mathcal{M}}_{1,3}$ be the blow-up of the stratum associated to Γ, and denote by E_Γ the exceptional divisor of π. After expressing $\Phi(\mathfrak{P})$ in terms of normally decorated strata classes, we obtain

$$\Delta = \frac{1}{2} \cdot \pi^* [\Gamma] - (2\psi_1 + \frac{1}{2}(\psi_2 + \psi_3) - 2\delta_{23}) \cdot E_\Gamma + \frac{1}{2} E_\Gamma^2 \in \mathsf{CH}^2(\widehat{\mathcal{M}}) . \quad (50)$$

It is interesting to note that the coefficient $(2\psi_1 + \frac{1}{2}(\psi_2 + \psi_3) - 2\delta_{23})$ appearing here is precisely the compact type part of the formula $\mathsf{P}^1_{1,(2,-1,-1)}$, giving the pullback of the theta divisor under the Abel-Jacobi map associated to $(2, -1, -1)$.

Due to the complexity of the objects involved (such as the subdivisions of fans and their rings of piecewise polynomials), the calculations in `logtaut` are mostly feasible for low genus and numbers of markings (around $2g + n \leq 7$). With future optimization, further cases could be covered.

9.2 Double-double ramification cycles

One of the initial motivations for the study of logDR was that of understanding the *double-double ramification cycle* which arises in the virtual localization formula for the Gromov-Witten theory of log toric surfaces [37] as a vertex term (replacing the quadratic Hodge integrals in the localization formula for the Gromov-Witten theory of toric surfaces [38]). The geometry of these higher double ramification cycles plays a central role in the study of stable maps to log toric targets [73, 74]. Using the formula of Theorem B and the Sage package `logtaut`, we can now compute these cycles in moderate genus (limited by computing capacity).

To motivate the higher double ramification cycles, we begin by recalling that the natural formula

$$\mathsf{DR}_{g,A} \cdot \mathsf{DR}_{g,B} \stackrel{?}{=} \mathsf{DR}_{g,A+B} \cdot \mathsf{DR}_{g,B}$$

holds on the locus of curves of compact type, but fails on $\overline{\mathcal{M}}_{g,n}$. However, by a result of [45], the analogue for logDR is always true:

$$\mathsf{logDR}_{g,A} \cdot \mathsf{logDR}_{g,B} = \mathsf{logDR}_{g,A+B} \cdot \mathsf{logDR}_{g,B} \in \mathsf{logCH}^{2g}(\overline{\mathcal{M}}_{g,n}) , \quad (51)$$

which is a special case of a $\mathrm{GL}_2(\mathbb{Z})$-invariance property for logDR. The *double-double ramification cycle* is defined by

$$\mathsf{DDR}_{g,A,B} = \pi_* \left(\mathsf{logDR}_{g,A} \cdot \mathsf{logDR}_{g,B} \right) \in \mathsf{CH}^{2g}(\overline{\mathcal{M}}_{g,n}),$$

and may be viewed as a "corrected" version of the product $\mathsf{DR}_{g,A} \cdot \mathsf{DR}_{g,B}$. Higher double ramification cycles are defined by pushforwards of products of more logarithmic double ramification cycles.

The double-double ramification cycle is trivial in genus 0:

$$\mathsf{DDR}_{0,A,B} = 1 \in \mathsf{logCH}^0(\overline{\mathcal{M}}_{0,n}) .$$

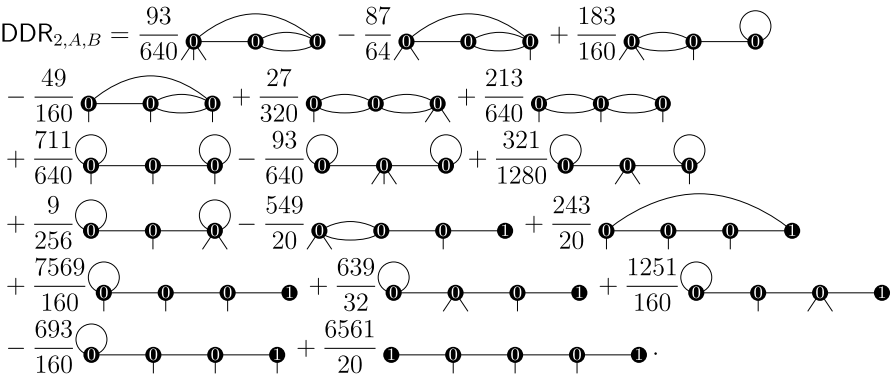
A full calculation in genus 1 of $\mathsf{DDR}_{1,A,B}$ was presented in Remark 74 using the formula for logDR of Theorem B. That $\mathsf{DDR}_{g,A,B}$ and the higher analogues are tautological classes on the moduli spaces of curves for all g follows also from results of [43, 62], but the formula of Theorem B provides the only known effective method of calculation. The package `logtaut` produces explicit formulas for $\mathsf{DDR}_{g,A,B}$ in the tautological ring.

We illustrate below the double-double cycle in case $g = 2$, $n = 3$, $A = [3, -3, 0]$, and $B = [0, 3, -3]$. Since the class

$$\mathsf{DDR}_{2,A,B} \in \mathsf{CH}^4(\overline{\mathcal{M}}_{2,3})$$

is invariant under permuting the markings (as can be seen by applying (51)), we can express the answer via `logtaut` as a sum over graphs[36] without specifying the ordering of the legs:

$\mathsf{DDR}_{2,A,B} =$



Two constructions of tautological relations in $\mathsf{logR}^*(\overline{\mathcal{M}}_{g,n})$ were given in Theorem C. The $\mathrm{GL}_2(\mathbb{Z})$-invariance property for the double-double ramification cycle yields a third construction: apply Theorem B to all four terms of (51).

Acknowledgements We are very grateful to T. Graber, L. Herr, D. Ranganathan, and J. Wise for discussions about the log Chow ring, log Gromov-Witten theory, and the possibility of a formula for the log double ramification cycle. We have also benefited from related conversations with A. Abreu, Y. Bae, L. Caporaso, C. Faber, N. Pagani, R. Schwarz, P. Spelier, and D. Zvonkine. We thank A. Szenes for organizing the *Helvetic Algebraic Geometry Seminar* in Geneva in August 2021 where the results of the paper were first presented. We thank the referees for many improvements.

Funding D.H. was supported by NWO grants VI.Vidi.193.006 and 613.009.103. S.M. was supported by ERC-2017-AdG-786580-MACI. R.P. was supported by SNF-200020-182181, SNF-200020-219369, ERC-2017-AdG-786580-MACI, and SwissMAP. A.P. was supported by NSF grant 1807079. J.S. was supported by SNF Early Postdoc Mobility grant 184245 held at the Max Planck Institute for Mathematics in Bonn, SNF-184613 at the University of Zürich, SNF-200020-219369, and SwissMAP. This project has received funding from the European Research Council (ERC) under the European Union Horizon 2020 research and innovation program (grant agreement No 786580).

[36] Following the conventions of `admcycles`, in the formula for $\mathsf{DDR}_{2,A,B}$, each stable graph Γ represents the sum of $3! = 6$ pushforwards of the fundamental class under gluing maps (associated to the 6 ways of adding labels to the 3 legs of the graph).

References

1. Abramovich, D., Wise, J.: Birational invariance in logarithmic Gromov-Witten theory. Compos. Math. **154**(3), 595–620 (2018)
2. Abramovich, D., Caporaso, L., Payne, S.: The tropicalization of the moduli space of curves. Ann. Sci. Éc. Norm. Supér. (4) **48**(4), 765–809 (2015)
3. Abramovich, D., Chen, Q., Marcus, S., Ulirsch, M., Wise, J.: Skeletons and fans of logarithmic structures. In: Nonarchimedean and Tropical Geometry, Simons Symp, pp. 287–336. Springer, Cham (2016)
4. Abramovich, D., Chen, Q., Marcus, S., Wise, J.: Boundedness of the space of stable logarithmic maps. J. Eur. Math. Soc. **19**(9), 2783–2809 (2017)
5. Abreu, A., Pacini, M.: The universal tropical Jacobian and the skeleton of the Esteves' universal Jacobian. Proc. Lond. Math. Soc. (3) **120**(3), 328–369 (2020)
6. Abreu, A., Pacini, M.: The resolution of the universal Abel map via tropical geometry and applications. Adv. Math. **378**, 107520 (2021)
7. Artin, M.: Algebraization of formal moduli. I. In: Global Analysis (Papers in Honor of K. Kodaira), pp. 21–71 (1969)
8. Bae, Y.: Tautological relations for stable maps to a target variety. Ark. Mat. **58**(1), 19–38 (2020)
9. Bae, Y., Schmitt, J., Skowera, J.: Chow rings of stacks of prestable curves I. Forum Math. Sigma **10** (2022)
10. Bae, Y., Holmes, D., Pandharipande, R., Schmitt, J., Schwarz, R.: Pixton's formula and Abel–Jacobi theory on the Picard stack. Acta Math. **230**(2), 205–319 (2023)
11. Baker, M., Faber, X.: Metric properties of the tropical Abel–Jacobi map. J. Algebraic Comb. **33**, 349–381 (2011)
12. Baker, M., Norine, S.: Riemann–Roch and Abel–Jacobi theory on a finite graph. Adv. Math. **215**(2), 766–788 (2007)
13. Barrott, L.J.: Logarithmic Chow theory (2019). arXiv:1810.03746
14. Bini, G., Fontanari, C., Viviani, F.: On the birational geometry of the universal Picard variety. Int. Math. Res. Not. **2012**(4), 740–780 (2012)
15. Blot, X., Lewanski, D., Shadrin, S.: On the strong DR/DZ conjecture (2024). arXiv:2405.12334
16. Brannetti, S., Melo, M., Viviani, F.: On the tropical Torelli map. Adv. Math. **226**(3), 2546–2586 (2011)
17. Bunnett, D., Joswig, M., Pfeifle, J.: Stacky fans and tropical moduli in polymake (2021). arXiv:2101.07316
18. Buryak, A., Rossi, P.: A generalisation of Witten's conjecture for the Pixton class and the noncommutative KdV hierarchy. J. Inst. Math. Jussieu **22**(6), 2987–3009 (2023)
19. Buryak, A., Shadrin, S., Spitz, L., Zvonkine, D.: Integrals of ψ-classes over double ramification cycles. Am. J. Math. **137**(3), 699–737 (2015)
20. Buryak, A., Guéré, J., Rossi, P.: DR/DZ equivalence conjecture and tautological relations. Geom. Topol. **23**(7), 3537–3600 (2019)
21. Caporaso, L.: A compactification of the universal Picard variety over the moduli space of stable curves. J. Am. Math. Soc. **7**(3), 589–660 (1994)
22. Caporaso, L., Christ, K.: Combinatorics of compactified universal Jacobians. Adv. Math. **346**, 1091–1136 (2019)
23. Casalaina-Martin, S., Kass, J.L., Viviani, F.: The singularities and birational geometry of the compactified universal Jacobian. Algebr. Geom. **4**(3), 353–393 (2017)
24. Cavalieri, R.: Hurwitz theory and the double ramification cycle. Jpn. J. Math. **11**(2), 305–331 (2016)
25. Cavalieri, R., Chan, M., Ulirsch, M., Wise, J.: A moduli stack of tropical curves. Forum Math. Sigma **8**, e23 (2020)
26. Cavalieri, R., Markwig, H., Ranganathan, D.: Pluricanonical cycles and tropical covers (2022). arXiv:2206.14034
27. Chan, M.: Combinatorics of the tropical Torelli map. Algebra Number Theory **6**(6), 1133–1169 (2012)
28. Clader, E., Janda, F.: Pixton's double ramification cycle relations. Geom. Topol. **22**(2), 1069–1108 (2018)
29. Clader, E., Grushevsky, S., Janda, F., Zakharov, D.: Powers of the theta divisor and relations in the tautological ring. Int. Math. Res. Not. **2018**(24), 7725–7754 (2018)
30. Costantini, M., Sauvaget, A., Schmitt, J.: Integrals of ψ-classes on twisted double ramification cycles and spaces of differentials (2021)

31. Delecroix, V., Schmitt, J., van Zelm, J.: Admcycles—a Sage package for calculations in the tautological ring of the moduli space of stable curves. J. Softw. Algebra Geom. **11**(1), 89–112 (2021)
32. Eliashberg, Y., Glvental, A., Hofer, H.: Introduction to Symplectic Field Theory, pp. 560–673. Birkhäuser, Basel (2010)
33. Esteves, E.: Compactifying the relative Jacobian over families of reduced curves. Trans. Am. Math. Soc. **353**(8), 3045–3095 (2001). (electronic)
34. Esteves, E., Pacini, M.: Semistable modifications of families of curves and compactified Jacobians. Ark. Mat. **54**(1), 55–83 (2016)
35. Fan, H., Wu, L., You, F.: Structures in genus-zero relative Gromov–Witten theory. J. Topol. **13**(1), 269–307 (2020)
36. Farkas, G., Pandharipande, R.: The moduli space of twisted canonical divisors. J. Inst. Math. Jussieu, 1–58 (2016). With an appendix by Janda, Pandharipande, Pixton, and Zvonkine
37. Graber, T.:. In preparation
38. Graber, T., Pandharipande, R.: Localization of virtual classes. Invent. Math. **135**(2), 487–518 (1999)
39. Graber, T., Pandharipande, R.: Constructions of nontautological classes on moduli spaces of curves. Mich. Math. J. **51**(1), 93–109 (2003)
40. Graber, T., Vakil, R.: Relative virtual localization and vanishing of tautological classes on moduli spaces of curves. Duke Math. J. **130**(1), 1–37 (2005)
41. Holmes, D.: Extending the double ramification cycle by resolving the Abel-Jacobi map. J. Inst. Math. Jussieu (2019). https://doi.org/10.1017/S1474748019000252
42. Holmes, D., Orecchia, G.: Logarithmic moduli of roots of line bundles on curves. Expo. Math. **41**(3), 577–602 (2023)
43. Holmes, D., Schwarz, R.: Logarithmic intersections of double ramification cycles. Algebr. Geom. **9**(5), 574–605 (2022)
44. Holmes, D., Kass, J., Pagani, N.: Extending the double ramification cycle using Jacobians. Eur. J. Math. **4**(3), 1087–1099 (2018). https://doi.org/10.1007/s40879-018-0256-7
45. Holmes, D., Pixton, A., Schmitt, J.: Multiplicativity of the double ramification cycle. Doc. Math. **24**, 545–562 (2019)
46. Holmes, D., Molcho, S., Orecchia, G., Poiret, T.: Models of Jacobians of curves. J. Reine Angew. Math. **2023**(801), 115–159 (2023)
47. Janda, F., Pandharipande, R., Pixton, A., Zvonkine, D.: Double ramification cycles on the moduli spaces of curves. Publ. Math. IHES **125**(1), 221–266 (2017)
48. Janda, F., Pandharipande, R., Pixton, A., Zvonkine, D.: Double ramification cycles with target varieties. J. Topol. **13**(4), 1725–1766 (2020)
49. Kass, J.L., Pagani, N.: The stability space of compactified universal Jacobians. Trans. Am. Math. Soc. **372**, 4851–4887 (2019)
50. Kato, K.: Logarithmic structures of Fontaine-Illusie. In: Algebraic Analysis, Geometry, and Number Theory (Baltimore, MD, 1988), pp. 191–224. Johns Hopkins University Press, Baltimore (1989)
51. Kato, K.: Toric singularities. Am. J. Math. **116**(5), 1073–1099 (1994)
52. Kato, F.: Log smooth deformation theory. Tohoku Math. J. **48**(3), 317–354 (1996)
53. Kempf, G., Knudsen, F.F., Mumford, D., Saint-Donat, B.: Toroidal Embeddings. I. Lecture Notes in Mathematics, vol. 339. Springer, Berlin (1973)
54. Li, J.: Stable morphisms to singular schemes and relative stable morphisms. J. Differ. Geom. **57**(3), 509–578 (2001)
55. Li, J.: A degeneration formula of GW-invariants. J. Differ. Geom. **60**(2), 199–293 (2002)
56. Marcus, S., Wise, J.: Logarithmic compactification of the Abel-Jacobi section. Proc. Lond. Math. Soc. (3) **121**(5), 1207–1250 (2020)
57. Melo, M.: Compactifications of the universal Jacobian over curves with marked points (2015). arXiv: 1509.06177
58. Melo, M.: Universal compactified Jacobians. Port. Math. **76**(2), 101–122 (2019)
59. Melo, M., Molcho, S., Ulirsch, M., Viviani, F.: Tropicalization of the universal Jacobian. Épijournal Géom. Algébr. **6**, 15 (2022)
60. Mikhalkin, G., Zharkov, I.: Tropical curves, their Jacobians and theta functions. Curves Abelian Var. **465**, 203–230 (2008)
61. Molcho, S.: Universal stacky semistable reduction. Isr. J. Math. **242**(1), 55–82 (2021)
62. Molcho, S., Ranganathan, D.: A case study of intersections on blowups of the moduli of curves. Algebra Number Theory **18**(10), 1767–1816 (2024)
63. Molcho, S., Wise, J.: The logarithmic Picard group and its tropicalization. Compos. Math. **158**(7), 1477–1562 (2022)

64. Molcho, S., Pandharipande, R., Schmitt, J.: The Hodge bundle, the universal 0-section, and the log Chow ring of the moduli space of curves. Compos. Math. **159**(2), 306–354 (2023)
65. Nakayama, C.: Logarithmic étale cohomology, II. Adv. Math. **314**, 663–725 (2017)
66. Oberdieck, G., Pixton, A.: Holomorphic anomaly equations and the Igusa cusp form conjecture. Invent. Math. **213**(2), 507–587 (2018)
67. Ogus, A.: Lectures on Logarithmic Algebraic Geometry. Cambridge Studies in Advanced Mathematics, vol. 178. Cambridge University Press, Cambridge (2018)
68. Olsson, M.C.: Log algebraic stacks and moduli of log schemes. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.), University of California, Berkeley (2001)
69. Pandharipande, R.: A compactification over $\overline{M}_g$ of the universal moduli space of slope-semistable vector bundles. J. Am. Math. Soc. **9**(2), 425–471 (1996)
70. Pandharipande, R.: A calculus for the moduli space of curves. In: Proceedings of Algebraic Geometry–Salt Lake City, pp. 459–488 (2015)
71. Payne, S.: Equivariant Chow cohomology of toric varieties. Math. Res. Lett. **13**(1), 29–41 (2006)
72. Pixton, A.: Generalized boundary strata classes. In: The Abel Symposium, pp. 285–293. Springer, Berlin (2017)
73. Ranganathan, D.: A note on cycles of curves in a product of pairs (2019). arXiv:1910.00239
74. Ranganathan, D., Kumaran, A.U.: Logarithmic Gromov–Witten theory and double ramification cycles. J. Reine Angew. Math. **2024**(809), 1–40 (2024)
75. Schmitt, J.: Dimension theory of the moduli space of twisted k-differentials. Doc. Math. **23**, 871–894 (2018)
76. Schmitt, J., van Zelm, J.: Intersections of loci of admissible covers with tautological classes. Sel. Math. New Ser. **26**(5), 79 (2020)
77. Schwarz, R.: Logarithmic approach to the double ramification cycle. PhD thesis, Universiteit Leiden (2023)
78. van Ittersum, J.-W., Oberdieck, G., Pixton, A.: Gromov-Witten theory of K3 surfaces and a Kaneko-Zagier equation for Jacobi forms. Sel. Math. New Ser. **27**(4), 64 (2021)
79. Vistoli, A.: Intersection theory on algebraic stacks and on their moduli spaces. Invent. Math. **97**(3), 613–670 (1989)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Authors and Affiliations

D. Holmes[1] · S. Molcho[2] · R. Pandharipande[3] · A. Pixton[4] · J. Schmitt[3]

✉ A. Pixton
pixton@umich.edu

D. Holmes
holmesdst@math.leidenuniv.nl

S. Molcho
samouil.molcho@uniroma1.it

R. Pandharipande
rahul@math.ethz.ch

J. Schmitt
johannes.schmitt@math.ethz.ch

1 Mathematisch Instituut, Universiteit Leiden, Leiden, The Netherlands

[2] Dipartimento di Matematica, Sapienza Università di Roma, Rome, Italy

[3] Departement Mathematik, ETH Zürich, Zürich, Switzerland

[4] Department of Mathematics, University of Michigan, Ann Arbor, USA