Tautological relations and double ramification cycles with spin parity

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Summary

Moduli spaces arise as solutions to classification problems (*moduli problems*): given a type of geometric object (such as triangles or lines through the origin in the plane), one seeks to parametrize all such objects up to isomorphism. The resulting parameter space, when it exists, is called a moduli space. Put naively, a moduli space is a "mathematical map" of the totality of the geometric objects we are interested in. Just as in a map of the world, where every point on the map corresponds to a certain point in the world, every point in a moduli space corresponds to a particular isomorphism class among these objects that we wish to classify. We endow this map with a geometric structure, and by studying it we gain an overview of all the geometric objects we are interested in.

As mathematicians, we encounter moduli spaces without explicitly referring to them as such. Some examples are the projective space $\mathbb{P}^n$, parametrizing lines through the origin in $\mathbb{C}^{n+1}$, and the more intricate Grassmannian $\mathsf{Gr}(k, V)$, which parametrizes k-dimensional subspaces of a fixed n-dimensional vector space V. In this thesis, we are interested in *complex algebraic curves*: one-dimensional objects defined by polynomial equations over $\mathbb{C}$. Smooth algebraic curves are classified by a topological invariant called the *genus*. Complex algebraic curves and Riemann surfaces are two sides of the same coin. From this viewpoint, the genus represents the "number of holes" of a complex algebraic curve when viewed as a topological surface.

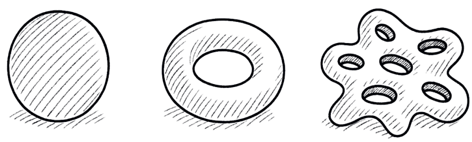

Figure A.1: From left to right, a genus 0, a genus 1, and a genus 6 complex algebraic curve.

The moduli space of smooth algebraic curves of genus g, denoted by $\mathcal{M}_g$, was already known to Riemann, who computed its dimension and coined the term "moduli". The moduli space $\mathcal{M}_g$ is not compact because families of smooth curves can degenerate into nodal ones, for example by pinching a cycle to produce a node. To capture these limits, Deligne and Mumford constructed a compactification, denoted by $\overline{\mathcal{M}}_g$, which includes singular *stable curves*, that is, nodal complex algebraic curves with finite automorphism group. Sometimes, to answer enumerative questions or to rigidify families of rational or elliptic curves, we introduce *markings* on stable curves, that is, smooth distinct points of stable curves. In this way, we obtain the *moduli space of stable curves of genus g with n markings*, denoted by $\overline{\mathcal{M}}_{g,n}$.

From the viewpoint of *intersection theory*, we are interested in the *Chow ring* of a moduli space. The Chow ring of an algebraic variety X, denoted by $A^*(X)$, encodes the geometric properties of its subvarieties, essentially by capturing how they intersect with one another. This framework allows us to "add" and "multiply" sub-varieties in order to study their interactions within the ambient space. As Mumford put it in his seminal paper [Mum83, Section 4], "Whenever a variety or topological space is defined by some universal property, one expects that by virtue of its defining property, it possesses certain cohomology (or Chow) classes called *tautological classes*". In the case of the Grassmannian, for example, these tautological classes completely determine the structure of $A^*(\mathsf{Gr}(k, V))$ in terms of generators and relations. The *tautological ring* of a moduli space X is the subring generated by its tautological classes. Unlike the situation for the Grassmanian, the tautological ring of $\overline{\mathcal{M}}_{g,n}$ does not, in general, coincide with the entire Chow ring. The central aim of this study is to completely determine the tautological ring of $\overline{\mathcal{M}}_{g,n}$ in terms of generators and relations. A set of generators, known as *decorated strata classes*, was found in complete generality by Graber-Pandharipande, while an explicit set of relations was conjectured by Aaron Pixton around 15 years ago but still remains open.

This thesis addresses two problems concerning the tautological ring of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. The first concerns the expression of the Hodge classes, or λ-classes, within the tautological ring. The *Hodge bundle*

$$\overline{\mathcal{H}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

is a vector bundle of rank g over $\overline{\mathcal{M}}_{g,n}$ whose fiber over an n-marked stable curve $(C, p_1, \dots, p_n)$ of genus g is the space of sections $H^0(C, \omega_C)$ of its canonical bundle ω_C. The class λ_i is defined to be the i-th Chern class of this vector bundle. The λ-classes have proven to be of fundamental importance in various contexts and have been studied quite thoroughly. Mumford was the first to produce explicit expressions of the λ-classes within the tautological ring. Since then, no other systematic method has been found to express *all* λ-classes within the tautological ring. To find such expressions, in Chapter 2, together with Adrien Sauvaget, we employed an auxiliary moduli space, the *projectivised Hodge bundle* $\mathbb{P}\overline{\mathcal{H}}_{g,n}$, building on the detailed analysis done by Sauvaget [Sau19]. In our paper [PS25], we used these results to develop a systematic approach for expressing the λ-classes in terms of decorated strata classes of a specific form.

Theorem (2.1.2)**.** *The $(g-i)$-th Chern class of the Hodge bundle, λ_{g-i}, can be expressed as a linear combination of decorated strata classes of graphs with at most i loops. In particular, λ_g can be expressed with decorated strata classes supported only on compact type graphs.*

The second problem we address concerns two closely related concepts: the *double ramification cycle*, denoted by $\mathsf{DR}_g(a, k)$, and the *class of the stratum of k-differentials of type a*, denoted by $[\overline{\mathcal{M}}_g(a, k)]$. Both are elements of the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ defined via two numerical parameters: an integer $k \in \mathbb{Z}_{\geq 1}$, and a vector of integers $a := (a_i)_{i=1}^n \in \mathbb{Z}^n$ satisfying $|a| := \sum_{i=1}^n a_i = k(2g-2+n)$. The starting point of both constructions is the same: the *stratum of k-differentials of type a*

$$\mathcal{M}_g(a, k) := \left\{ (C, p_1, \dots, p_n) \in \mathcal{M}_{g,n} \mid \mathcal{O}_C\left(\sum_{i=1}^n a_i p_i\right) \cong \omega_{\log}^k \right\}. \tag{A.6}$$

For suitable choices (a, k), these are subspaces of $\mathcal{M}_{g,n}$ parametrizing smooth curves admitting a k-differential with zeroes and poles prescribed by the vector of integers a. The double ramification cycle is a class in $A^*(\overline{\mathcal{M}}_{g,n})$ associated to a compactification of $\mathcal{M}_g(a, k)$ inside $\overline{\mathcal{M}}_{g,n}$, while the class of a stratum of differentials is the class associated to the Zariski closure of $\mathcal{M}_g(a,k)$, denoted by $[\overline{\mathcal{M}}_g(a,k)]$. In recent years, the class $[\overline{\mathcal{M}}_g(a,k)]$ was expressed in terms of standard generators of the tautological ring via the theory of double ramification cycles by demonstrating two expressions of $\mathsf{DR}_g(a,k)$: the *star graph expression*, and *Pixton's formula*, namely

$$\text{star graph expression} = \mathsf{DR}_g(a,k) = \text{Pixton's formula}.$$

This approach was proposed in [FP18] for $k=1$ and generalized in [Sch18] for $k>1$. These expressions were established in a series of papers [JPPZ17, HS21, BHP$^+$23].

Before presenting the results obtained in Chapter 3, we introduce another central concept of this thesis. A *spin structure* on a smooth curve C of genus g over an algebraically closed field is a pair $(\mathcal{L},\phi)$, where $\mathcal{L}\to C$ is a line bundle of degree $g-1$ together with an isomorphism $\phi\colon \mathcal{L}^{\otimes 2}\to\omega_C$. The *parity* of a spin structure is defined as the parity of the dimension of its space of sections. Accordingly, we call a spin structure *even* when $h^0(C,\mathcal{L})\equiv 0 \mod 2$ and *odd* otherwise. Mumford showed that, given a family of smooth curves $C\to S$ and a line bundle $\mathcal{L}\to C$ of relative degree $g-1$ together with an isomorphism $\mathcal{L}^{\otimes 2}\to\omega_{C/S}$, the function

$$\begin{aligned} S &\to \mathbb{Z}/2\mathbb{Z} \\ s &\mapsto h^0(C_s,\mathcal{L}_s) \mod 2, \end{aligned}$$

which assigns to each point $s\in S$ the parity of the spin structure $\mathcal{L}_s$ on the fiber curve C_s is locally constant. In simpler terms, the parity of spin structures remains constant under deformations. As a corollary, the moduli space of spin structures on smooth curves, denoted by $\mathcal{M}_{g,n}^{1/2}$, splits into two components

$$\mathcal{M}_{g,n}^{1/2} = \mathcal{M}_{g,n}^{1/2,+}\sqcup\mathcal{M}_{g,n}^{1/2,-},$$

where the spaces with exponents $+,-$ parametrize even and odd spin structures, respectively. Cornalba constructed a compactification $\overline{\mathcal{M}}_{g,n}^{1/2}$ of $\mathcal{M}_{g,n}^{1/2}$ by introducing certain nodal degenerations and showed that this compactification respects the parity decomposition also along the boundary.

Returning to the main topic of the thesis, we observe that when $k\in\mathbb{Z}_{\geq 1}$ is *odd* and $a\in\mathbb{Z}^n$ is a vector of odd integers, every point $(C,p_1,\ldots,p_n)\in\mathcal{M}_g(a,k)$ naturally carries a spin structure. To be more precise,

$$(C,p_1,\ldots,p_n)\in\mathcal{M}_g(a,k)\Rightarrow \omega_{\log}^{\frac{1-k}{2}}\left(\sum_{i=1}^n \frac{a_i-1}{2}p_i\right) \text{ is a spin structure.}$$

Therefore, we obtain a decomposition

$$\mathcal{M}_g(a,k)=\mathcal{M}_g(a,k)^+\sqcup\mathcal{M}_g(a,k)^-$$

in terms of points whose associated spin structure is even or odd. Such a decomposition implies that

$$[\overline{\mathcal{M}}_g(a,k)] = [\overline{\mathcal{M}}_g(a,k)^+] + [\overline{\mathcal{M}}_g(a,k)^-]$$

in $A^*(\overline{\mathcal{M}}_{g,n})$. As stated above, the class $[\overline{\mathcal{M}}_g(a,k)]$ was computed in terms of standard generators of the tautological rings. However, what about the individual summands $[\overline{\mathcal{M}}_g(a,k)^+]$ and $[\overline{\mathcal{M}}_g(a,k)^-]$? Are they themselves tautological? If so, can we compute them in terms of the standard generators of the tautological ring? For this purpose, since the sum of these two terms has already been computed in terms of standard generators of the tautological ring, it suffices to do the same for their difference. We define the class of the *spin class of the stratum of k-differentials of type a* as:

$$[\overline{\mathcal{M}}_g(a,k)]^{\pm} := [\overline{\mathcal{M}}_g(a,k)^{+}] - [\overline{\mathcal{M}}_g(a,k)^{-}].$$

Towards the computation of $[\overline{\mathcal{M}}_g(a,k)]^{\pm}$, Costantini-Sauvaget-Schmitt defined a tautological class called the *spin double ramification cycle*, denoted by $\mathsf{DR}_g^{\pm}(a,k)$ by alternating the Pixton's formula. Motivated by the non-spin case described before, they proposed a conjecture [CSS21, Conjecture 2.5] in a similar fashion as in [FP18, Sch18]:

$$\text{spin version of Pixton's formula} := \mathsf{DR}_g^{\pm}(a,k) = \text{spin version of star graph expression.}$$

In Chapter 3, the main goal is to prove this conjecture. First, we give an overview of the theory of spin structures developed by Cornalba and establish certain computational results. Then, in Section 2, we provide an overview of the theory of double ramification cycles, and we study the geometry of this class. Combining the above and following the construction of the ordinary double ramification cycle $\mathsf{DR}_g(a,k)$ [MW20, Hol21], we give an *a priori* conjectural geometric realization of $\mathsf{DR}_g^{\pm}(a,k)$ in the following steps: first, we define a closed substack in a birational model $\rho_{\frac{1}{2}}\colon \overline{\mathcal{M}}_g^{1/2,a} \to \overline{\mathcal{M}}_{g,n}^{1/2}$, the *double ramification locus*, and consider its fundamental class. Then we push this class forward to $\overline{\mathcal{M}}_{g,n}^{1/2}$, intersect it with the *parity cycle*, that is, the difference between the even and the odd component, and we further push it along the morphism $\epsilon\colon \overline{\mathcal{M}}_{g,n}^{1/2} \to \overline{\mathcal{M}}_{g,n}$, which forgets the spin structure.

The space $\overline{\mathcal{M}}_g^{1/2,a}$ provides a much better-behaved framework to work with. At the end of Section 2, we discuss the relationship between the spaces involved in our definition with these in the definition of the usual double ramification cycle. This analysis allows us to obtain a complete description of the fundamental class of the double ramification locus in terms of its irreducible components and their multiplicities. In Section 3, we define an action of μ_2 on the double ramification locus, which we use to identify certain cancellations after we intersect with the parity cycle and push forward its fundamental class. Finally, building on the above, in Section 4, we show that our definition of $\mathsf{DR}_g^{\pm}(a,k)$ satisfies the spin version of star graph expression. In Section 5, we use the machinery developed in [BHP$^+$23] to show that our definition of $\mathsf{DR}_g^{\pm}(a,k)$ also satisfies the spin version of Pixton's formula. As a corollary, we obtain that our definition agrees with the one proposed by Costantini-Sauvaget-Schmitt, and conclude the proof of the aforementioned conjecture [CSS21, Conjecture 2.5].

Theorem (3.5.12)**.** *Let $k \in \mathbb{Z}_{\geq 1}$ be an odd integer, and let $a \in \mathbb{Z}^n$ be a vector of odd integers such that $|a| = k(2g-2+n)$. Then the class $\mathsf{DR}_g^{\pm}(a,k)$ is tautological and satisfies a spin version of Pixton's formula.*

Theorem (3.4.5)**.** *Under the assumptions of the theorem above, together with $a \notin k\mathbb{Z}_{>0}^n$ the spin double ramification cycle $\mathsf{DR}_g^{\pm}(a,k)$ also admits a star graph expression.*

In the Appendix, we show how this conjecture can be used to compute the class $[\overline{\mathcal{M}}_g(a,k)]^{\pm}$ in terms of standard generators of the tautological ring. Chapter 3 and the approach presented in the Appendix are parts of a joint work with Adrien Sauvaget and David Holmes.