Tautological relations and double ramification cycles with spin parity

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Appendix A

Recursive computation of spin classes of strata

Let $k \in \mathbb{Z}_{\geq 1}$ be an odd integer, and let $a \in \mathbb{Z}^n$ be a vector of odd integers such that

$$|a| = k(2g - 2 + n).$$

In this appendix we explain how Theorem 3.5.14 allows one to reduce the computation of $[\overline{\mathcal{M}}_g(a,k)]^{\pm}$ to the computation of spin classes of strata of holomorphic 1-differentials. Then we investigate how the spin classes of strata of holomorphic 1-differentials can be computed based on a conjecture.

A.1 Reduction to holomorphic spin classes of strata

Assume for now that $a \notin k\mathbb{Z}^n_{>0}$, so that Theorem 3.5.14 holds. We observe that the trivial graph contributes as $[\overline{\mathcal{M}}_g(a,k)]^{\pm}$ in the LHS of Theorem 3.5.14, i.e., we have

$$[\overline{\mathcal{M}}_g(a,k)]^{\pm} = \mathsf{P}^{\pm}_g(a,k) - \Delta, \tag{A.1}$$

where Δ is the contribution of all non-trivial simple star graphs (Γ, I) in $\mathrm{SStar}^{\mathrm{odd}}_{g,n}(a,k)$.

Lemma A.1.1. *Let $k \in \mathbb{Z}_{\geq 1}$ be an odd integer and let $a \notin k\mathbb{Z}^n_{>0}$ be a vector of odd integers such that $|a| = k(2g-2+n)$. Then the class $[\overline{\mathcal{M}}_g(a,k)]^{\pm}$ is completely determined by the classes of the form:*

(i) $\mathsf{P}^{\pm}_g(a,k)$*, and*

(ii) the classes $[\overline{\mathcal{M}}_{g'}(a',1)]^{\pm}$*, where* $a' \in \mathbb{Z}^{n'}_{>0}$ *and such that either* $g' \leq g$ *and* $n' < n$ *or* $g' < g$ *and* $n' \leq n$*.*

Proof. Under these hypotheses Theorem 3.5.14 applies, so we can express $[\overline{\mathcal{M}}_g(a,k)]^{\pm}$ as $\mathsf{P}^{\pm}_g(a,k)$ and the contribution obtained in Δ as in Equation A.1.

We now analyze the contribution of Δ. First, suppose that (Γ, I) has central vertex v_0 such that $g(v_0) = g$. Then the outer vertices v should necessarily satisfy $g(v) = 0$ and, by definition of simple star graphs, the contribution arising from outer vertices of (Γ, I) should be of the form $[\overline{\mathcal{M}}_0(I(v),1)]^{\pm}$, where $I(v)$ is a vector of positive integers. However, since there are no non-zero holomorphic differentials on $\mathbb{P}^1$, this contribution vanishes. This shows that any non-trivial meromorphic contribution in Δ must arise from simple star graphs (Γ, I) whose central vertex has genus strictly smaller than g. Applying Theorem 3.5.14 iteratively on the meromorphic contribution will drop the genus by at least one each time until we arrive to the base case

$$[\overline{\mathcal{M}}_0(I(v_0), k)]^{\pm} = 1,$$

where $I(v_0) \notin k\mathbb{Z}^{n'}_{>0}$. In that way we may eliminate all meromorphic contributions appearing in the computation of $[\overline{\mathcal{M}}_g(a, k)]^{\pm}$.

Furthermore, to identify the holomorphic contribution in Δ we argue as follows; in the case that a simple star graph (Γ, I) has central vertex of genus 0 and one outer vertex of genus g then, by stability, the contribution from the outer vertex will be of the form $[\overline{\mathcal{M}}_g(a', 1)]^{\pm}$ where $a' \in \mathbb{Z}^{n'}$ is a vector of integers such that n' is strictly less than the number of markings. The case where $g' < g$ and $n' \leq n$ arises from simple star graphs with central vertex v_0 of genus $0 < g(v_0) \leq g - 1$. □

Suppose that $a \in k\mathbb{Z}^n_{>0}$ such that $|a| = k(2g - 2 + n)$. Then any point of $\mathcal{M}_g(a/k, 1)$ naturally lies in $\mathcal{M}_g(a, k)$. In particular, we have natural inclusions

$$\mathcal{M}_g(a/k, 1) \subset \mathcal{M}_g(a, k), \text{ and } \widetilde{\mathcal{H}}^1_g(a/k) \subset \widetilde{\mathcal{H}}^k_g(a)$$

Schmitt proved that $\widetilde{\mathcal{H}}^1_g(a/k)$ is the only contribution in codimension $g - 1$ on the irreducible components of $\widetilde{\mathcal{H}}^k_g(a)$ [Sch18, Theorem 1.1]. We will use the notation

$$\mathcal{M}^{\mathrm{c}}_g(a, k) := \mathcal{M}_g(a, k) \setminus \mathcal{M}_g(a/k, 1).$$

Moreover, by [Ati71, Mum71], we obtain a decomposition

$$\mathcal{M}^{\mathrm{c}}_g(a, k) = \mathcal{M}^{\mathrm{c}}_g(a, k)^{+} \coprod \mathcal{M}^{\mathrm{c}}_g(a, k)^{-},$$

and we define $[\overline{\mathcal{M}}^{\mathrm{c}}_g(a, k)]^{\pm} = [\overline{\mathcal{M}}^{\mathrm{c}}_g(a, k)^{+}] - [\overline{\mathcal{M}}^{\mathrm{c}}_g(a, k)^{-}]$.

Proposition A.1.2. *Let $k \in \mathbb{Z}_{\geq 1}$ be an odd integer, and let $a \in k\mathbb{Z}^n_{>0}$ be a vector of odd integers. Then the following equality holds*

$$\mathsf{P}^{\pm}_g(a, k) = -a_1\psi_1[\overline{\mathcal{M}}_g(a/k, 1)]^{\pm} + [\overline{\mathcal{M}}^{\mathrm{c}}_g(a, k)]^{\pm} + \sum_{(\Gamma, I) \in \mathrm{SStar}^{\mathrm{odd}}_{g,1}(a,k)} \frac{\prod_{e \in E(\Gamma)} I(e)}{k^{|V_{\mathrm{out}}|}|\mathrm{Aut}(\Gamma, I)|} \zeta_{\Gamma *}[\overline{\mathcal{M}}_{\Gamma, I}]^{\pm}.$$

in $A^*(\overline{\mathcal{M}}_{g,n})$, where $\mathrm{SStar}^{\mathrm{odd}}_{g,1}(a, k)$ denotes the subset of $\mathrm{SStar}^{\mathrm{odd}}_g(a, k)$ such that the marking p_1 is incident on the central vertex.

Proof. The proof mirrors the arguments of the proof of the non-spin analogue of the proposition above, proven in [Sau20, Lemma 3.3]. In particular, one needs Theorem 3.5.14 and the polynomiality of $\mathsf{P}^{\pm}_g(a, k)$ in the entries of a (see [Pix23, Spe24]) to use the same arguments as in [Sau20, Lemma 3.3]. □

Corollary A.1.3. *Let $k \in \mathbb{Z}_{\geq 1}$ be an odd integer, and let $a \in \mathbb{Z}^n$ be a vector of odd integers such that $|a| = k(2g-2+n)$. Assume that $[\overline{\mathcal{M}}_{g'}(a', 1)]^{\pm}$ is tautological and computable for all $g' \leq g$ and the relevant $a' \in \mathbb{Z}^{n'}_{>0}$. Then $[\overline{\mathcal{M}}_g(a,k)]^{\pm}$ is tautological and computable.*

Proof. If $a \notin k\mathbb{Z}^n_{>0}$ then, by Lemma A.1.1, we can reduce the computation in spin classes of strata of holomorphic 1-differentials, which by assumption of the lemma are tautological and computable and thus, $\overline{\mathcal{M}}_g(a,k)$ is also computable. Now, the tautological nature follows from the assumption of the lemma together with the fact that $\mathsf{P}^{\pm}_g(a,k)$ is also tautological which conculdes the result for the meromorphic case.

On the other hand, if $a \in k\mathbb{Z}^n_{>0}$, then the proposition above implies that the only unknown term, i.e. $[\overline{\mathcal{M}}^{c}(a,k)]^{\pm}$, is computable by $\mathsf{P}^{\pm}_g(a,k)$ and classes of the form $[\overline{\mathcal{M}}_{g'}(a',1)]^{\pm}$ which proves the result. □

A.2 Conjectural expression for holomorphic spin classes of strata

The results of the previous section reduce the computation of $[\overline{\mathcal{M}}_g(a,k)]^{\pm}$ to the computation of spin classes of strata of holomorphic 1-differentials. In order to compute these classes we propose an adaptation of the method of Sauvaget [Sau19] described in Subsection 1.2.1, and which gives computes the strata of 1-differentials in the non-spin case.

Let $a \in \mathbb{Z}^n_{>0}$ be a vector of odd integers. We define the sub-cone $\mathcal{SQ}_g(a) \subset \mathcal{H}_g(a)$ over $\mathcal{M}_{g,n}$ to be the cone of differentials admitting zeroes of order $a_i - 1$ at the i-th marking and non-marked zeroes of only even order. We denote by $\mathbb{P}\overline{\mathcal{SQ}}(a)$ the closure of the projectivization of $\mathcal{SQ}(a)$. By [Ati71, Mum71], we obtain a decomposition

$$\mathcal{SQ}_g(a) = \mathcal{SQ}_g(a)^{+} \coprod \mathcal{SQ}_g(a)^{-}.$$

Furthermore, we define

$$[\mathbb{P}\overline{\mathcal{SQ}}_g(a)]^{\pm} := [\mathbb{P}\overline{\mathcal{SQ}}(a)^{+}] - [\mathbb{P}\overline{\mathcal{SQ}}(a)^{-}],$$

where $\mathbb{P}\overline{\mathcal{SQ}}(a)^{+}$ and $\mathbb{P}\overline{\mathcal{SQ}}(a)^{-}$ denote the Zariski closures of $\mathbb{P}\mathcal{SQ}(a)^{+}$ and $\mathbb{P}\mathcal{SQ}(a)^{-}$ in $\mathbb{P}\overline{\mathcal{SQ}}(a)$. In the case where $|a| = 2g-2+n$, we have that

$$p_*[\mathbb{P}\overline{\mathcal{SQ}}_g(a)]^{\pm} = [\overline{\mathcal{M}}(a,1)]^{\pm}.$$

Therefore, establishing a recursive formula for $[\mathbb{P}\overline{\mathcal{SQ}}_g(a)]^{\pm}$ in the spirit of [Sau19] will allow us to argue inductively as we argued in Section 1.3. In [HPS25] we prove such a formula for $[\mathbb{P}\overline{\mathcal{SQ}}_g(a)]^{\pm}$, where $a \in \mathbb{Z}^n_{>0}$ is a vector of odd integers, and we show by induction that all classes of the form $p_*(\xi^b[\mathbb{P}\overline{\mathcal{SQ}}_g(a)]^{\pm})$ are tautological and computable. The base case for our induction turns out to be the classes

$$s_g^{\pm,b} := p_*(\xi^b[\mathbb{P}\overline{\mathcal{SQ}}_g(1,\ldots,1)]^{\pm}) \in A^b(\overline{\mathcal{M}}_{g,n}).$$

To express these classes, we consider the set $\mathrm{Tree}_{g,n} \subset \mathcal{G}_{g,n}$ of stable graphs with no loops and no genus 0 vertices. For all $t \in \mathbb{C}$, we denote

$$s_g^{\pm}(t) = \sum_{b \geq 0} t^b s_g^{\pm,b}, \text{ and} \tag{A.2}$$

$$L_g(t) = 2^{g-1}\Lambda(2t)\Lambda(-t) - 2^{2g-1}, \tag{A.3}$$

where $\Lambda(t) = 1 + t\lambda_1 + t^2\lambda_2 + \ldots$. The following conjecture expresses the classes $s_g^{\pm}(1)$ in terms of the $L_g(1)$-classes, which are tautological and computable because λ-classes are tautological and computable.

Conjecture A.2.1. For all $g, n \in \mathbb{Z}_{\geq 0}$ such that $2g - 2 + n > 0$, and for all $t \in \mathbb{C}$ we have

$$L_g(t) = \sum_{\Gamma \in \mathrm{Tree}_{g,n}} \frac{t^{|E(\Gamma)|}}{|\mathrm{Aut}(\Gamma)|} \zeta_{\Gamma *} \left(\bigotimes_{v \in V} s_{g(v)}(t) \right), \text{ and} \tag{A.4}$$

$$s_g(t) = \sum_{\Gamma \in \mathrm{Tree}_{g,n}} \frac{(-t)^{|E(\Gamma)|}}{|\mathrm{Aut}(\Gamma)|} \zeta_{\Gamma *} \left(\bigotimes_{v \in V} L_{g(v)}(t) \right). \tag{A.5}$$

The presence of the quadratic Hodge symbol in the LHS of formula A.4 is due to the identity

$$\Lambda(-1)\Lambda(1/2) = \epsilon_* \left([\pm] s_* \left(R^* \pi_* \mathcal{L} \right) \right)$$

that was used in the expression for the spin Hurwitz numbers and spin Gromov–Witten invariants of $\mathbb{P}^1$ [GKL21, GKLS22].