Chapter 1

Introduction

In this thesis, we address two problems concerning the tautological rings of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. The first problem concerns expressing the Chern classes of the Hodge bundle within the tautological ring. Finding such expressions has been extensively studied in the literature, and several systematic methods have been developed [Mum83,JPPZ17,BGR19,MPS23]. In this context, in Chapter 2, we introduce a new systematic method of producing expressions of these Chern classes. The second problem we address is a proof of a conjecture by Costantini-Sauvaget-Schmitt which concerns two different expressions of the so-called *spin double ramification cycles* [CSS21]. These expressions can be regarded as a spin analogue of a conjecture [FP18, Sch18, Conjecture A], which was introduced by the authors of [JPPZ17], and proven in [HS21, BHP$^+$23]. The non-spin analogue of this conjecture played a key role in the computation of classes of strata of k-differentials. After proving the conjecture of Costantini-Sauvaget-Schmitt, we explain in the appendix how these expressions can be used to compute all spin classes of strata of k-differentials.

The two main chapters of this thesis, Chapters 2 and 3, are based on the papers [PS25, HPS25].

In Section 1.1, we discuss the general motivation for the study of the tautological rings and the tautological relations. In Section 1.2, we present the main results of Chapter 2. In Section 1.3, we introduce the problem of computing spin classes of strata of k-differentials, and how it can be tackled with the theory of spin *double ramification cycles*. The remainder of Chapter 3 is devoted to proving Conjecture 2.5 of [CSS21].

Introduction

Dans cette thèse, nous étudions deux problèmes concernant les anneaux tautologiques de l'espace de modules de courbes. Le premier problème porte sur l'expression des classes de Chern du fibré de Hodge dans l'anneau tautologique. La recherche de telles expressions a fait l'objet de nométudes approfondies dans la littérature, et plusieurs méthodes systématiques ont été développées [Mum83, JPPZ17, BGR19, MPS23]. Dans ce cadre, au Chapitre 2, nous introduisons une nouvelle méthode systématique pour produire des expressions de ces classes de Chern.

Le second problème que nous abordons est la démonstration d'une conjecture de Costantini, Sauvaget, et Schmitt relative à deux expressions différentes des *cycle de double ramification spin*. Ces expressions peuvent être vues comme un analogue spin d'une conjecture introduite par Janda, Pixton, Pandharipande, Zvonkine [FP18, Sch18, Conjectrure A] et récemment démontrée [HS21, BHP$^+$23]. L'analogue non spin de cette conjecture a joué un rôle clé dans le calcul des classes des strates de différentielles. Après avoir établi la conjecture de Costantini–Sauvaget–Schmitt, nous expliquons en annexe comment ces expressions peuvent être utilisées pour calculer toutes classes spins de strates de différentielles.

Les deux principaux chapitres de cette thèse, les Chapitres 2 et 3, sont basés sur les articles [PS25, HPS25].

Dans la Section 1.1, nous discutons de la motivation générale pour l'étude des anneaux tautologiques et des relations tautologiques. Dans la Section 1.2, nous présentons les principaux résultats du Chapitre 2. Dans la Section 1.3, nous introduisons le problème du calcul des classes de spin des strates de k-différentielles, ainsi que la manière dont il peut être abordé grâce à la théorie des *cycles de double ramification spin*. Le reste du Chapitre 3 est consacré à la démonstration de la Conjecture 2.5 de [CSS21].

Prelude

1.1 Moduli spaces of curves

Definition 1.1.1. Let g, n be non-negative integers. A *pre-stable curve* of genus g with n marked points over an algebraically closed field k is a tuple

$$(C, \sigma_1, \dots, \sigma_n),$$

where C is a reduced, proper, connected 1-dimensional scheme over k of arithmetic genus g with at worst nodal singularities, and σ_i are distinct smooth points of C. We denote by $\widetilde{C}$ the normalization of C. The points of $\widetilde{C}$ that are either marked or preimages of nodes are called *special points*. A pre-stable curve is called *stable* if $2g - 2 + n > 0$, and the following additional conditions hold for all irreducible components of $\widetilde{C}$:

(i) every rational component E of $\widetilde{C}$ has at least 3 special points,

(ii) every elliptic component E' of $\widetilde{C}$ has at least 1 special point.

Definition 1.1.2. A *family of (pre-)stable curves* over a scheme S with n marked points consists of the data

$$(\pi \colon C \to S, \sigma_1, \dots, \sigma_n \colon S \to C),$$

where π is a proper, flat, finitely presented morphism, and the morphisms $\sigma_1, \dots, \sigma_n$ are pairwise disjoint sections of π, whose images lie in the smooth locus of π, and such that the geometric fibers $(C_s, \sigma_1(s), \dots, \sigma_n(s))$ are (pre-)stable curves of genus g with n marked points. When the number of marked points is clear from context, we abbreviate $(\pi \colon C \to S, \sigma_i \colon S \to C)$.

Definition 1.1.3. An *isomorphism* between two families of (pre-)stable curves of genus g with n marked points $(\pi' \colon C' \to S, \sigma_i' \colon S \to C')$, and $(\pi \colon C \to S, \sigma_i \colon S \to C)$ over S is the datum of an isomorphism

$$\phi \colon C' \to C$$

over S such that $\phi \circ \sigma_i' = \sigma_i$ for all $i \in \{1, \dots, n\}$. We denote by $\overline{\mathcal{M}}_{g,n}(S)$ (resp. $\mathcal{M}_{g,n}(S)$) the groupoid of n-marked stable (resp. smooth) curves of genus g over S. Given a morphism $p \colon S' \to S$, the *pullback family of* $(\pi \colon C \to S, \sigma_i \colon S \to C)$ *under* p is the n-marked stable curve of genus g

$$(\pi' \colon C \times_S S' \to S', (\sigma_i \circ p, \mathrm{id}_{S'}) : S' \to C \times_S S'),$$

where π' is the pullback of π, and $(\sigma_i \circ p, \mathrm{id}_{S'})$ are the sections $S' \to C \times_S S'$ of π' induced by the pair of morphisms $\sigma_i \circ p$ and $\mathrm{id}_{S'}$ for all $i \in \{1, \dots, n\}$.

We define $\overline{\mathcal{M}}_{g,n}$ to be the category whose objects are families of stable curves of genus g with n marked points $(\pi\colon C \to S, \sigma_i\colon S \to C)$, and morphisms between two families $(\pi\colon C \to S, \sigma_i\colon S \to C)$ and $(\pi'\colon C' \to S', \sigma_i'\colon S' \to C')$, are pairs (p', p) such that the inner square of

$$\begin{array}{ccc} C' & \xrightarrow{p'} & C \\ \sigma_i' \uparrow \downarrow \pi' & & \pi \downarrow \uparrow \sigma_i \\ S' & \xrightarrow[p]{} & S \end{array}$$

is cartesian, and commutes with the sections σ_i and σ_i'. In particular, $(\pi'\colon C' \to S', \sigma_i'\colon S' \to C')$ is the pullback family of $(\pi\colon C \to S, \sigma_i\colon S \to C)$ under p. The functor

$$F\colon \overline{\mathcal{M}}_{g,n} \to \mathbf{Sch}$$

which sends a family of stable curves $(\pi\colon C \to S, \sigma_i\colon S \to C)$ to S, and morphisms (p', p) to p, makes $\overline{\mathcal{M}}_{g,n}$ a category fibered in groupoids over **Sch**. In particular, the fibers of F over a scheme S are the groupoids $\overline{\mathcal{M}}_{g,n}(S)$. We denote by $\mathcal{M}_{g,n}$ the full subcategory of $\overline{\mathcal{M}}_{g,n}$ defined by replacing stable with smooth in the above.

Theorem 1.1.4. *Let $g, n \in \mathbb{Z}_{\geq 0}$ such that $2g - 2 + n > 0$. The category fibered in groupoids $(F, \overline{\mathcal{M}}_{g,n})$, or simply $\overline{\mathcal{M}}_{g,n}$, is an irreducible, proper, and smooth Deligne-Mumford (DM) stack over* $\operatorname{Spec}(\mathbb{Z})$ *of dimension $3g - 3 + n$. Furthermore, the boundary $\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a normal-crossings divisor.*

By the 2-Yoneda lemma on fibered categories, there is a *universal family*

$$\begin{array}{c} \overline{\mathcal{C}}_{g,n} \\ \sigma_i \uparrow \downarrow \pi \\ \overline{\mathcal{M}}_{g,n}, \end{array}$$

where $\pi \circ \sigma_i = \mathrm{id}_{\overline{\mathcal{M}}_{g,n}}$ for $i \in \{1, \dots, n\}$. To be more precise, the family $(\pi\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}, \sigma_i\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n})$ is the family corresponding to the identity $\mathrm{id}_{\overline{\mathcal{M}}_{g,n}}$ under the 2-Yoneda lemma. The pullback of $\operatorname{Spec}(\mathbb{C}) \to \overline{\mathcal{M}}_{g,n}$ and π, is an n-marked stable curve of genus g. The sections σ_i are defined on geometric points by identifying the i-th marked point of the fiber. More generally, the universal family has the property that any family of n-marked stable curves of genus g over a base scheme S arises as the pullback of the universal family π via a unique morphism $S \to \overline{\mathcal{M}}_{g,n}$.

Furthermore, there is also a *forgetful morphism*

$$p\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n},$$

which forgets the last marking and contracts any unstable component. The morphism p admits n, pairwise disjoint, sections $\sigma_i'\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$. On geometric points, these sections map an n-marked stable curve $(C, \sigma_1, \dots, \sigma_n)$ of genus g to $(C', \sigma_1, \dots, \sigma_n, \sigma_{n+1})$, where C' is obtained by gluing a rational tail at the i-th marked point of C, carrying the markings σ_i and σ_{n+1}.

Proposition 1.1.5. *[Knu83] There is an isomorphism*

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{C}}_{g,n}$$

of stacks over $\overline{\mathcal{M}}_{g,n}$.

In [Vis89], the author develops an intersection theory on a Deligne-Mumford stack in the spirit of Fulton and MacPherson which generalizes the intersection theory defined in [Ful98]. In particular, for $\overline{\mathcal{M}}_{g,n}$ we can define the Chow ring $A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. To simplify the notation, we write $A^*(\overline{\mathcal{M}}_{g,n})$, as we will always consider rational coefficients.

1.1.1 Stratification and the boundary strata

The fact that the boundary $\partial\overline{\mathcal{M}}_{g,n}$ is a normal-crossings divisor induces a stratification on $\overline{\mathcal{M}}_{g,n}$. Before we describe it, we give the following definition.

Definition 1.1.6. Let $g, n \in \mathbb{Z}_{\geq 0}$ such that $2g-2+n > 0$. A *pre-stable graph of type* (g, n) is the data of

$$\Gamma = (V, H, L, g\colon V \to \mathbb{Z}_{\geq 0}, i\colon H \to H, \phi\colon H \to V, \ell\colon L \to \{1, \dots, n\})$$

where V, H, and L are finite sets and:

(i) The function i is an involution of H.

(ii) Elements of H are called *half-edges* and the cycles of length 2 for i are called *edges*. We denote the set of cycles of length 2 by E.

(iii) The involution i has n fixed points, and L is the subset of such fixed points; elements of L are called *legs*. The function ℓ identifies L with the set $\{1, \dots, n\}$ of integers from 1 to n.

(iv) An element $v \in V$ is called a *vertex*, and the value $g(v)$ is called the *genus* of the vertex v. A half-edge h is *incident* to v if $\phi(h) = v$. We denote by $H(v)$ the set of half-edges incident to v, and by $n(v)$ the valency of the vertex v, i.e., the cardinality of $H(v)$.

(v) The genus of the graph, defined as

$$g(\Gamma) := h^1(\Gamma) + \sum_{v \in V} g(v), \text{ where } h^1(\Gamma) = |E| - |V| + 1,$$

is equal to g.

(vi) The graph is connected.

(vii) If for all vertices v we have $2g(v) - 2 + n(v) > 0$, the pre-stable graph is called *stable*.

We denote the set of pre-stable graphs (resp. stable graphs) by $\mathcal{G}^{\text{pst}}_{g,n}$ (resp. $\mathcal{G}_{g,n}$). For the sets H, E, and V, we will also use the notation $H(\Gamma)$, $E(\Gamma)$, and $V(\Gamma)$.

Definition 1.1.7. Let $(C, p_1, \dots, p_n)$ be a pre-stable curve of genus g with n marked points over an algebraically closed field. The *associated dual graph* of C, denoted by Γ_C, or simply Γ, is the pre-stable graph of type (g, n) obtained in the following way:

(i) The vertex set V is the set of irreducible components of the normalization $\widetilde{C}$ of C.

(ii) The function $g\colon V \to \mathbb{Z}_{\geq 0}$ assigns to each vertex the geometric genus of the corresponding irreducible component of $\widetilde{C}$.

(iii) The set of half-edges H consists of preimages of nodal points $q \in C$ via the normalization map $\nu\colon \widetilde{C} \to C$, and the marked points $\{p_1 \dots, p_n\}$.

(iv) The involution $i\colon H \to H$ is defined by exchanging the two preimages of every node and fixing each marked point. The map $\phi\colon H \to V$ sends each half-edge to the irreducible component of $\widetilde{C}$ containing it.

(v) The subset $L \subseteq H$ consists of the markings $\{p_1, \dots, p_n\}$, and the function $\ell\colon L \to \{1, \dots, n\}$ is given by $p_i \mapsto i$ for all $i = 1, \dots, n$.

Example 1.1.8. We outline the procedure described above in the following figure.

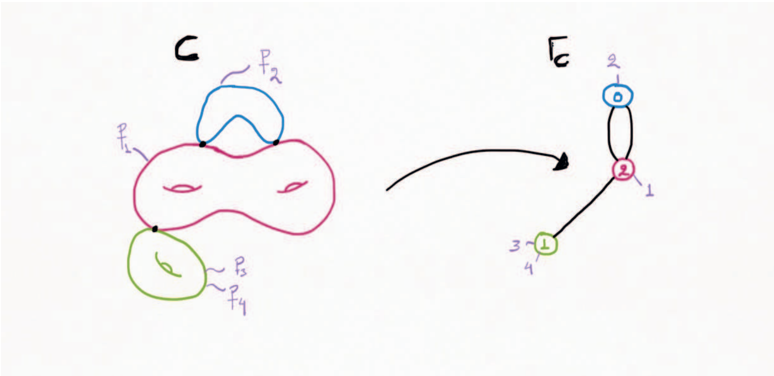

Figure 1.1: The graph Γ_C (on the right) associated to the geometric point (C, p_1, p_2, p_3, p_4) (on the left) of $\overline{\mathcal{M}}_{4,4}$

Definition 1.1.9. Let $(C, \sigma_1, \dots, \sigma_n)$ be a pre-stable curve over an algebraically closed field. Let $p \in C$ be a node and let C_p denote the partial normalization at p. Then:

(i) The node p is called *separating* if C_p is disconnected and *non-separating* otherwise.

(ii) The pre-stable curve $(C, \sigma_1, \dots, \sigma_n)$ is called of *compact type* if every node is separating.

A pre-stable graph Γ is called of *compact type* if removing any edge $e \in E(\Gamma)$ makes Γ disconnected. In particular, the associated dual graph of a pre-stable curve of compact type is a pre-stable graph of compact type. We denote by $\mathcal{M}^{ct}_{g,n}$ the restriction of $\overline{\mathcal{M}}_{g,n}$ to compact type curves.

If Γ is a stable graph of type (g, n), we define the spaces

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)} \text{ and } \mathcal{M}_\Gamma := \prod_{v \in V} \mathcal{M}_{g(v),n(v)}.$$

Given a geometric point of $\mathcal{M}_\Gamma$, say $(C_v)_{v \in V(\Gamma)}$, we can glue the curves C_v along the markings corresponding to the non-leg half edges of Γ to obtain a point in $\mathcal{M}_{g,n}$. The argument to extend the gluing process on families of stable curves is given in [ACG11, Chapter X, paragraph 7]. In that way, we have a gluing morphism[1]

$$\xi_\Gamma\colon \mathcal{M}_\Gamma \to \overline{\mathcal{M}}_{g,n}.$$

[1]We refer the reader to [ACG11, Chapter XII, paragraph 10] for generalities about gluing morphisms.

Its image is an irreducible, locally closed substack, and we have the following topological stratification by locally closed subsets

$$\overline{\mathcal{M}}_{g,n} = \coprod_{\Gamma \in \mathcal{G}_{g,n}} \xi_\Gamma \left(\mathcal{M}_\Gamma \right).$$

We call the closure of $\xi_\Gamma \left(\mathcal{M}_\Gamma \right)$ in $\overline{\mathcal{M}}_{g,n}$ the *boundary stratum* associated to Γ. Furthermore, we can extend the gluing morphism ξ_Γ on $\overline{\mathcal{M}}_\Gamma$; that is, there exists a gluing map

$$\zeta_\Gamma \colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n},$$

which is proper, finite of degree $|\mathrm{Aut}(\Gamma)|$, and a local complete intersection. The image of ζ_Γ is the closure of $\xi_\Gamma(\mathcal{M}_\Gamma)$. Essentially, the image of ζ_Γ parametrizes families of curves whose degenerations are forced by Γ. The class of the boundary stratum (Γ, ζ_Γ) is given by

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \zeta_{\Gamma *}(1).$$

1.1.2 Tautological rings and tautological relations

Definition 1.1.10. The system of tautological rings $\{R^*(\overline{\mathcal{M}}_{g,n})\}_{g,n}$ is the smallest set of $\mathbb{Q}$-subalgebras of the Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ satisfying:

1. The system is closed under the pushforward via all morphisms that forget markings, i.e., for all integers $m > n \geq 0$, and relevant g, we have

$$p_* : R^*(\overline{\mathcal{M}}_{g,m}) \to R^*(\overline{\mathcal{M}}_{g,n}).$$

2. The system is closed under the pushforward via all gluing morphisms

$$\zeta_{\delta *} \colon R^*(\overline{\mathcal{M}}_{g_1,n_1 \cup \{*\}}) \otimes_{\mathbb{Q}} R^*(\overline{\mathcal{M}}_{g_2,n_2 \cup \{\star\}}) \to R^*(\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}), \qquad \text{(type-tree)}$$
$$\zeta_{irr *} \colon R^*(\overline{\mathcal{M}}_{g-1,n+2}) \to R^*(\overline{\mathcal{M}}_{g,n}), \qquad \text{(type-loop)}$$

where ζ_δ denotes the gluing morphism parametrizing curves with one separating edge and two components of genus g_1 and g_2 with $g_1 + g_2 = g$, and ζ_{irr} denotes the gluing morphism parametrizing curves with one non-separating edge. Finally, given a Zariski-open substack $U \subseteq \overline{\mathcal{M}}_{g,n}$ (e.g., $\mathcal{M}_{g,n}$ and $\mathcal{M}^{\mathrm{ct}}_{g,n}$), the tautological rings of U are defined as the image of $R^*(\overline{\mathcal{M}}_{g,n})$ via the restriction morphism

$$A^*(\overline{\mathcal{M}}_{g,n}) \to A^*(U).$$

Definition 1.1.11. Let ω_π denote the relative dualizing sheaf of the universal curve $\overline{\mathcal{C}}_{g,n} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}$. We define the *Hodge bundle* $\overline{\mathcal{H}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ to be

$$\overline{\mathcal{H}}_{g,n} := \pi_* \left(\omega_\pi \right),$$

and we define the *lambda classes*

$$\lambda_i := c_i(\overline{\mathcal{H}}_{g,n}).$$

We write $c(\overline{\mathcal{H}}_{g,n})$ for the total Chern class of $\overline{\mathcal{H}}_{g,n}$. In particular, the fiber over a point in $\overline{\mathcal{M}}_{g,n}$ is given by $H^0(C, \omega_C)$. We denote by $\mathcal{H}_{g,n}$ the restriction of $\overline{\mathcal{H}}_{g,n}$ over $\mathcal{M}_{g,n}$.

Definition 1.1.12. For all $i \in \{1, \dots, n\}$, we define the *i-th cotangent line bundle* on $\overline{\mathcal{M}}_{g,n}$ as

$$\mathbb{L}_i := \sigma_i^* \omega_\pi,$$

where σ_i is the section corresponding to the i-th marking and ω_π is as above. Furthermore, we define

(i) the *psi-classes* $\psi_i := c_1(\mathbb{L}_i)$ for all $i \in \{1, \dots, n\}$, and

(ii) the *kappa classes* $\kappa_m := \pi_*(\psi_{n+1}^{m+1})$.

It can be shown that all ψ, κ, and λ classes lie in the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ [ACG11, Chapter XVII, paragraph 4]. However, it is not true in general that all classes are tautological (see [GP03, FP15]).

Proposition 1.1.13. *Let $g \geq 2$. The* tautological ring $R^*(\mathcal{M}_g)$*, is the $\mathbb{Q}$-subalgebra of $A^*(\mathcal{M}_g)$ generated by κ classes.*

Definition 1.1.14. Let Γ be a stable graph of type (g, n), and for all $v \in V(\Gamma)$ let $\theta_v \in R^*\left(\overline{\mathcal{M}}_{g(v),n(v)}\right)$ be an arbitrary monomial in ψ and κ classes of $\overline{\mathcal{M}}_{g(v),n(v)}$. We define

$$\theta := \mathrm{ep}\left(\bigotimes_{v \in V(\Gamma)} \theta_v\right) \in A^*(\overline{\mathcal{M}}_\Gamma),$$

where

$$\mathrm{ep}\colon \bigotimes_{v \in V(\Gamma)} A^*(\overline{\mathcal{M}}_{g(v),n(v)}) \to A^*(\overline{\mathcal{M}}_\Gamma)$$

is the exterior product morphism. The pair $[\Gamma, \theta]$ is called a *decorated strata class*. The *degree* of a decorated strata class $[\Gamma, \theta]$ is given by

$$|E(\Gamma)| + \sum_{v \in V(\Gamma)} \deg \theta_v,$$

where $\deg \theta_v$ denotes the degree of θ_v in the graded ring $R^*(\overline{\mathcal{M}}_{g(v),n(v)})$. We denote the $\mathbb{Q}$-linear space of decorated strata classes as $\mathcal{S}_{g,n}$.

Theorem 1.1.15. *[GP03, Proposition 11] The tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ is additively generated by classes of the form*

$$\zeta_{\Gamma *}(\theta)$$

where Γ and θ form a decorated strata class.

Since the dimension of $\overline{\mathcal{M}}_{g(v),n(v)}$ is finite for all $v \in V(\Gamma)$, there are finitely many non-zero choices of the classes θ_v for each component of $\overline{\mathcal{M}}_\Gamma$. Therefore, since there are finitely many stable graphs of type (g, n) up to isomorphism, we obtain that $R^*(\overline{\mathcal{M}}_{g,n})$ is finitely generated. Furthermore, in [GP03, Appendix A.4] the authors define an intersection product between decorated strata classes $[\Gamma, \theta]$ and $[\Gamma', \theta']$, which endows $\mathcal{S}_{g,n}$ with a graded algebra structure, and such that the $\mathbb{Q}$-linear map

$$\mathcal{S}_{g,n} \to R^*(\overline{\mathcal{M}}_{g,n})$$
$$[\Gamma, \theta] \mapsto \zeta_{\Gamma *}(\theta)$$

is a graded ring homomorphism. In particular, Theorem 1.1.15 shows that this ring morphism is surjective. The kernel of this morphism is called the *space of tautological relations*. The natural question to ask is:

"*Is there an explicit description of the tautological relations?*"

In cases such as $\overline{\mathcal{M}}_{0,n}$ [Kee92], $\overline{\mathcal{M}}_g$, for $g \leq 3$ [Mum83, Fab90a], and $\mathcal{M}_g$, for $g \leq 9$ [Fab90b, Iza95, PV15, CL24], the whole Chow ring has been determined in terms of generators and relations, and has been proven to be tautological. However, the problem of determining the full set of relations for arbitrary relevant (g, n) is still open. Mumford, in his seminal paper [Mum83], was the first to systematically produce relations in $R^*(\overline{\mathcal{M}}_g)$ for all $g \geq 2$. There he proved, among other relations between κ classes, the relation

$$c(\overline{\mathcal{H}}_g)c(\overline{\mathcal{H}}_g^\vee) = 1, \tag{1.1}$$

where $\overline{\mathcal{H}}_g^\vee$ denotes the dual of the Hodge bundle. In particular, a direct consequence of Equation 1.1 is the relation

$$\lambda_g^2 = 0 \text{ when } g > 0.$$

Additionally, Mumford in *loc.cit.* proved a precursor to the result of Graber-Pandharipande discussed earlier on the generators of $R^*(\overline{\mathcal{M}}_{g,n})$. Specifically, the tautological ring on $\mathcal{M}_g$ is generated by κ_i, for $i \in \{1, \ldots, g-2\}$. Mumford posed a conjecture, proven by Madsen and Weiss [MW07], which states that the stable cohomology of $\mathcal{M}_g$ is freely generated by the kappa classes:

$$\lim_{g \to \infty} H^*(\mathcal{M}_g) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots].$$

It was Carel Faber, after the work of Mumford, in an instructive and remarkably influential paper, who formulated a set of conjectures aimed at determining the structure of the tautological ring $R^*(\mathcal{M}_g)$ (see [Fab99]). Among these conjectures, Faber and Don Zagier formulated a conjecturally complete set of relations among κ classes for $R^*(\mathcal{M}_g)$ (FZ-relations). Furthermore, in 2012, Aaron Pixton conjectured a new set of relations in $R^*(\overline{\mathcal{M}}_{g,n})$ [Pix12], the so-called *Pixton-Faber-Zagier* relations (PFZ-relations). These relations generalize and extend the FZ relations. That is, when we set $n = 0$ and restrict the PFZ-relations to the interior $\mathcal{M}_g$, we obtain the FZ-relations. In [PPZ15] the authors provided a geometric realization of the PFZ-relations in the cohomology of $\overline{\mathcal{M}}_{g,n}$ by an analysis on the Witten's 3-spin class and using the Givental-Teleman reconstruction theorem. Janda gave a geometric realization of the PFZ-relations in the Chow ring of $\overline{\mathcal{M}}_{g,n}$ using similar techniques on the moduli of stable maps to $\mathbb{P}^1$ and its equivariant Gromov-Witten theory [Jan17]. The PFZ-relations constitute the largest known set of relations and are conjectured to contain all possible relations in the tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$.

1.2 Hodge relations

In this section, we present the results of [PS25] where we introduce a new family of relations, and which we refer to as *Hodge relations*.

1.2.1 The projectivized Hodge bundle

Our starting point is the work of Sauvaget on the study of the projectivized Hodge bundle $\mathbb{P}\overline{\mathcal{H}}_{g,n}$ and its intersection theory [Sau19]. The space $\mathbb{P}\overline{\mathcal{H}}_{g,n}$ parametrizes pairs (C, ω) where C is a stable curve and ω is a non-zero holomorphic differential defined up to scaling. This space is of dimension $4g - 4 + n$. We denote by

$$p\colon \mathbb{P}\overline{\mathcal{H}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

the morphism which forgets the differential. General intersection theory provides a description of the Chow ring of $\mathbb{P}\overline{\mathcal{H}}_{g,n}$.[2] Pulling back classes via p defines an injective ring homomorphism

$$p^*\colon A^*(\overline{\mathcal{M}}_{g,n}) \to A^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$$

(see [EH16, Theorem 5.9]), allowing us to view $A^*(\overline{\mathcal{M}}_{g,n})$ as a subalgebra of $A^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$. Let $\xi := c_1(\mathcal{O}(1))$ denote the first Chern class of the dual of the tautological bundle of $\mathbb{P}\overline{\mathcal{H}}_{g,n}$. This class is a root of the monic polynomial

$$\Lambda_g^*(t) := t^g + t^{g-1}p^*\lambda_1 + \cdots + p^*\lambda_g,$$

that is, $\Lambda_g^*(\xi) = 0$ in $A^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$. In fact, any element $x \in A_k(\mathbb{P}\overline{\mathcal{H}}_{g,n})$ can be uniquely expressed at the level of Chow groups as $x = \sum_{i=0}^{g-1} \xi^i p^* \alpha_i$ for some classes $\alpha_i \in A_{k-g+i}(\overline{\mathcal{M}}_{g,n})$. As shown in [EH16, Theorem 9.6], extending p^* to a ring morphism $A^*(\overline{\mathcal{M}}_{g,n})[\xi] \to A^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$ by setting $p^*\xi := \xi$ yields an isomorphism of graded rings

$$A^*(\mathbb{P}\overline{\mathcal{H}}_{g,n}) \cong A^*(\overline{\mathcal{M}}_{g,n})[\xi]/\Lambda_g(\xi),$$

where $\Lambda_g(\xi) = \xi^g + \xi^{g-1}\lambda_1 + \cdots + \lambda_g$.

Definition 1.2.1. The *tautological ring* $R^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$ of $\mathbb{P}\overline{\mathcal{H}}_{g,n}$ is the subring of $A^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$ generated by ξ and the pullback of $R^*(\overline{\mathcal{M}}_{g,n})$ under p. In particular, we have

$$R^*(\mathbb{P}\overline{\mathcal{H}}_{g,n}) \cong R^*(\overline{\mathcal{M}}_{g,n})[\xi]/\Lambda_g(\xi).$$

Let $Z \in \mathbb{Z}^n_{\geq 0}$ be a vector of non-negative integers. The *size of* Z is defined as $|Z| := \sum_{i=1}^n z_i$. We define the locus

$$\mathcal{H}_g(Z) := \{(C, \omega) \in \mathcal{H}_{g,n} \mid \mathrm{ord}_{p_i}(\omega) = z_i\} \subset \mathcal{H}_{g,n},$$

and we denote by $\mathbb{P}\overline{\mathcal{H}}_g(Z)$ the Zariski closure of $\mathbb{P}\mathcal{H}_g(Z)$ inside $\mathbb{P}\overline{\mathcal{H}}_{g,n}$. In [Sau19] the author established a formula that computes the classes $[\mathbb{P}\overline{\mathcal{H}}_g(Z)]$ by induction on the size of Z. Below we will briefly sketch this induction, however, we refer the reader to [Sau19, Section 5.4] and results therein for a detailed analysis. Let $Z = (z_1 + 1, z_2, \dots)$ be a vector of integers. The formula of Sauvaget has shape

$$[\mathbb{P}\overline{\mathcal{H}}_g(z_1 + 1, z_2, \dots)] = (\xi + z_1\psi_1)[\mathbb{P}\overline{\mathcal{H}}_g(z_1, z_2, \dots)] + \Delta,$$

[2] In fact, the results stated below for $\mathbb{P}\overline{\mathcal{H}}_{g,n}$ hold more generally for projectivizations of vector bundles over smooth spaces.

where Δ is a sum of boundary terms, i.e., classes in $A^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$ supported in the boundary of $\mathbb{P}\overline{\mathcal{H}}_{g,n}$. These boundary terms consist of strata of meromorphic differentials with residue conditions in either lower genus or lower number of markings (see Section 2.2.2 for more details of the term Δ). Assuming that all strata of meromorphic differentials whose vector of zeroes has size smaller than $|Z|$ are computable, then using the identity above one can compute $[\mathbb{P}\overline{\mathcal{H}}_g(Z)]$. Besides providing an effective method for computing the classes $[\mathbb{P}\overline{\mathcal{H}}_g(Z)]$, this formula is used to prove that they lie in the tautological ring $R^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$ [Sau19, Theorem 2]. Finally, the shape of the boundary terms appearing in Δ allows one to inductively compute intersections of classes with $\mathbb{P}\mathcal{H}_g(Z)$ (e.g., [Sau18, PS25]), which makes it a very powerful tool.

1.2.2 Statements of results of Chapter 2

Our first result in [PS25] is an improvement on the approach of Sauvaget, achieved by lifting the inductive formula presented above from the quotient ring $R^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$ to the polynomial ring $R^*(\overline{\mathcal{M}}_{g,n})[\xi]$. More precisely, given a vector of positive integers $Z \in \mathbb{Z}^n_{>0}$, we define an inductive process that produces an element $\alpha(g, Z) \in R^*(\overline{\mathcal{M}}_{g,n})[\xi]$, whose image in $R^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$ via the quotient map

$$f : R^*(\overline{\mathcal{M}}_{g,n})[\xi] \to R^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$$

precisely recovers $[\mathbb{P}\overline{\mathcal{H}}_{g,n}(Z)]$. While our algorithm coincides with that of [Sau19] upon projection to $R^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$, it does not trivially lift to an algorithm in $R^*(\overline{\mathcal{M}}_{g,n})[\xi]$. The difficulty arises from the fact that the inductive algorithm producing $\alpha(g, Z)$ involves differentials on (possibly) disconnected curves, which are encoded via the combinatorial data of *level graphs*. However, since the edges of the dual graphs cannot be ordered canonically, we need to prove that our algorithm is invariant under simultaneous permutations of the markings corresponding to edges, of the entries of Z, and of permutations of connected components of the disconnected curves (see Lemma 2.2.8). Unlike the situation in [Sau19], where such invariance holds automatically, establishing it here is essential as $\alpha(g, Z)$ is not defined through an intrinsic geometric construction.

Definition 1.2.2. If $i \in \mathbb{N}$, then we denote by $R^*_i(\overline{\mathcal{M}}_{g,n})$ the linear subspace of $R^*(\overline{\mathcal{M}}_{g,n})$ spanned by push-forwards of polynomials in ψ and κ-classes along gluing morphisms of type-tree and at most i times the morphism of type-loop[3].

By our construction, the output of our algorithm, i.e., the class $\alpha(g, Z)$, is a polynomial in $R^*(\overline{\mathcal{M}}_{g,n})[\xi]$ of degree $|Z|$. Additionally, we prove a structural result on the output of our algorithm, that is, the coefficient of ξ^j in $\alpha(g, Z)$ satisfies

$$\operatorname{coef}(\xi^j) \in R^{|Z|-j}_j(\overline{\mathcal{M}}_{g,n}).$$

Lifting the algorithm to $R^*(\overline{\mathcal{M}}_{g,n})[\xi]$ provides a significant advantage which we put into practice. In particular, if we deliberately consider an empty condition (e.g., $|Z| > 2g - 2$), the image of $\alpha(g, Z)$ vanishes on $R^*(\mathbb{P}\overline{\mathcal{H}}_{g,n})$[4]. As a result, the remainder of the Euclidean division in $R^*(\overline{\mathcal{M}}_{g,n})[\xi]$ of the class $\alpha(g, Z)$ by the monic polynomial $\Lambda_g(\xi)$ must vanish, so that

$$\alpha(g, Z) = q_{\alpha(g,Z)}\Lambda_g(\xi).$$

[3] See Definition 1.1.10 for the definition of type-tree and type-loop morphisms

[4] Indeed, in this case, the locus $\mathbb{P}\mathcal{H}_g(Z)$ is empty, as there is no holomorphic differential with sum of order of zeroes greater than $2g - 2$.

Since this equality holds in $R^*(\overline{\mathcal{M}}_{g,n})[\xi]$, we can derive relations in $R^*(\overline{\mathcal{M}}_{g,n})$ by equating the coefficients on both sides.

Remark 1.2.3. If $q_{\alpha(g,Z)} = 0$, then the equation above yields no non-trivial relations. Therefore, to obtain non-trivial relations, it is necessary to choose Z such that $q_{\alpha(g,Z)} \neq 0$.

In [PS25], we consider the vector $Z_{g-1} = (1, 2, \ldots, 2) \in \mathbb{Z}^g$, for which $|Z_{g-1}| = 2g-1$, so the aforementioned reasoning applies. We denote by $\pi_n \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,1}$ the morphism that forgets the last n markings on $\overline{\mathcal{M}}_{g,n+1}$. We extend the pushforward π_{n*} on

$$\pi_{n*} \colon R^*(\overline{\mathcal{M}}_{g,n+1})[\xi] \to R^*(\overline{\mathcal{M}}_{g,1})[\xi]$$

by sending a polynomial $\sum_{i=1}^k \xi^i \alpha_i$ to $\sum_{i=1}^k \xi^i \pi_{n*}\alpha_i$, and we define

$$\alpha(g, 1, g-1) := \frac{1}{(g-1)!} \pi_{(g-1)*} \left(\alpha(g, Z_{g-1}) \right).$$

Since $f(\alpha(g, 1, g-1)) = 0$ in $R^*(\mathbb{P}\overline{\mathcal{H}}_{g,1})$, we obtain

$$\alpha(g, 1, g-1) = q_{g-1} \Lambda_g(\xi) \in R^*(\overline{\mathcal{M}}_{g,1})[\xi]. \tag{1.2}$$

Before pushing forward via π_{g-1}, the class $\alpha(g, Z_{g-1})$ has degree $2g-1$ as a polynomial in ξ. On the other hand, the coefficient of ξ^j in $\alpha(g, Z_{g-1})$ is of degree $2g-1-j$ in the graded ring $R^*(\overline{\mathcal{M}}_{g,g})$. As π_{g-1} has relative dimension $g-1$ we obtain

$$\pi_{(g-1)*} \left(\xi^j \cdot \mathrm{coef}(\xi^j) \right) = \xi^j \cdot \pi_{(g-1)*} \mathrm{coef}(\xi^j) = 0$$

for all $j > g$. In particular, the class $\alpha(g, 1, g-1)$ is of degree g as a polynomial in ξ, and the coefficient of ξ^g lies in $\mathbb{Q}$. Let $a_g \in \mathbb{Q}$ be the coefficient of ξ^g; in fact one checks that $a_g = q_{g-1}$ in Equation 1.2. In Chapter 2 we show that $a_g = 2^{g-1} g$ (see Proposition 2.3.3), and thus,

$$\alpha(g, 1, g-1) = 2^{g-1} g \Lambda_g(\xi). \tag{1.3}$$

Finally, by comparing the coefficients of the two polynomials in the equation above, we obtain expressions for the classes λ_j for $1 \leq j \leq g$. We refer to these relations as *Hodge relations*. In particular, we prove the following theorem.

Theorem 1.2.4. *[PS25, Theorem 1.2] For all $i \in \mathbb{N}$, the class λ_{g-i} lies in $R_i^*(\overline{\mathcal{M}}_{g,n})$*

Remark 1.2.5. In the case $i = 0$, we obtain an expression for λ_g in terms of decorated strata classes with *only* separating nodes. The property of λ_g being expressed via compact type decorated graphs follows as a corollary of the so-called *DR/DZ conjecture* (see [BGR19, Section 4.7] for this corollary), which is a conjecture about the equivalence of two hierarchies produced from a given semi-simple CohFT: the first one is the Double ramification (DR) hierarchy proposed in [BGR19], and other is the Dubrovin-Zhang (DZ) hierarchy proposed in [DZ01]. At the time of posting [PS25] on the arXiv, the DR/DZ conjecture was still open. Recently, it was proven in a series of papers by Blot, Lewanski, Sauvaget, and Shadrin [BLS24, BSS25].

Question 1.2.6. Are the relations generated by the Hodge relations contained in the PFZ-relations? If so, what part of the PFZ-relations do they produce?

1.3 spin classes of strata of k-differentials and spin double ramification cycles

Let $g, n \in \mathbb{Z}_{\geq 0}$ be such that $2g-2+n > 0$. Let $k \in \mathbb{Z}_{\geq 1}$, and $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that

$$|a| := \sum_{i=1}^{n} a_i = k(2g-2+n). \tag{1.4}$$

We define the *stratum of k-differentials of type a* to be the closed substack of $\mathcal{M}_{g,n}$

$$\mathcal{M}_g(a,k) := \left\{ (C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n} \mid \mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \cong \omega_{\log}^{k} \right\},$$

where $\omega_{\log} := \omega_C(p_1 + \cdots + p_n)$. We denote by $\overline{\mathcal{M}}_g(a,k)$ the Zariski closure in $\overline{\mathcal{M}}_{g,n}$, and by $[\overline{\mathcal{M}}_g(a,k)] \in A^*(\overline{\mathcal{M}}_{g,n})$ the class of this cycle with the reduced sub-scheme structure. We call the vector a *holomorphic* if $a \in k\mathbb{Z}^n_{>0}$, and *meromorphic* otherwise.

Question 1.3.1. Are the classes $[\overline{\mathcal{M}}_g(a,k)]$ tautological? Is there an explicit formula for this class?

Let $a \in \mathbb{Z}^n_{>0}$ such that $|a| = 2g-2+n$. Recall from the previous section the classes $[\mathbb{P}\overline{\mathcal{H}}_g(a)]$ of holomorphic differentials with zeroes of order $a_i - 1$ at the i-th marking[5]. Then we have

$$p_*[\mathbb{P}\overline{\mathcal{H}}_g(a)] = [\overline{\mathcal{M}}_g(a,1)].$$

Sauvaget's approach [Sau19], presented in the previous subsection, shows that $[\mathbb{P}\overline{\mathcal{H}}_g(a)]$ is tautological and computable, giving an affirmative answer to the Question above in the case $k=1$. As we will see in the next subsections the theory of double ramification cycles reduces the computation of the case $k > 1$ to that of $k = 1$.

1.3.1 Approach via double ramification cycles

In general, there has been considerable effort and numerous approaches to the question we posed at the start of this section (see for example [Mul17, FP18, Sch18, PPZ19, Sau19, Won24] to name only a few). In this chapter, we focus on the approach via the theory of *double ramification cycles*, initially proposed in the appendix of [FP18] by the authors of [JPPZ17]. We begin with the following observation. First, consider the 0-section

$$\sigma_0 \colon \mathcal{M}_{g,n} \to \mathcal{J}_{g,n},$$

where $\mathcal{J}_{g,n}$ is the universal Jacobian over $\mathcal{M}_{g,n}$, sending an n-marked stable curve $(C, p_1, \ldots, p_n)$ to $(C, p_1, \ldots, p_n, \mathcal{O}_C)$. Then consider the *Abel-Jacobi section*

$$\begin{aligned} \sigma_{a,k} \colon \mathcal{M}_{g,n} &\to \mathcal{J}_{g,n} \\ (C, p_1, \ldots, p_n) &\mapsto (C, \omega_{\log}^{\otimes k}\left(-\sum_{i=1}^{n} a_i p_i\right)). \end{aligned}$$

[5]Here we adopt a slightly different notation than the one introduced in the previous subsection. This is because in this section we consider *log*-differentials instead of ordinary ones

These two maps allow us to realize geometrically the locus $\mathcal{M}_g(a,k)$ as the fiber product of the diagram

$$\begin{array}{ccc} & & \mathcal{M}_{g,n} \\ & & \downarrow{\scriptstyle \sigma_{a,k}} \\ \mathcal{M}_{g,n} & \xrightarrow[\sigma_0]{} & \mathcal{J}_{g,n}. \end{array}$$

We refer to the schematic intersection of these two sections, i.e., the fiber product $\mathcal{M}_{g,n} \times_{\mathcal{J}_{g,n}} \mathcal{M}_{g,n}$, as the *double ramification locus*. The double ramification locus is identical to $\mathcal{M}_g(a,k)$ over $\mathcal{M}_{g,n}$, but its behavior along the boundary is more subtle. We denote by $\overline{\mathcal{J}}_{g,n}$ the universal semi-abelian jacobian over $\overline{\mathcal{M}}_{g,n}$ parametrizing multi-degree zero line bundles on stable curves. The 0-section $\sigma_0\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}_{g,n}$ extends nicely to a regular closed immersion of codimension g. On the other hand, the Abel-Jacobi map does not extend over the whole boundary $\partial\overline{\mathcal{M}}_{g,n}$, as $\omega_{\log}^{\otimes k}\left(-\sum_{i=1}^n a_i p_i\right)$ fails to have multi-degree 0 on nodal stable curves in general. In particular, $\sigma_{a,k}$ only extends over the locus of irreducible stable curves. The problem of extending the Abel-Jacobi map has been extensively studied from various perspectives (see [HKP18, MW20, Hol21, HS21, AP21]). In [Hol21], Holmes constructs a birational model $\rho\colon \overline{\mathcal{M}}_g^a \to \overline{\mathcal{M}}_{g,n}$ over which $\sigma_{a,k}$ extends, and in which the schematic intersection of the Abel-Jacobi map and the 0-section, denoted by DRL, is proper over $\overline{\mathcal{M}}_{g,n}$ via the morphism ρ. Moreover, Holmes proves that this model is universal with respect to this property. The *double ramification cycle* on $\overline{\mathcal{M}}_{g,n}$ is then defined as

$$\mathsf{DR}_g(a,k) := \rho_*\left(\sigma_0^![\sigma_{a,k}]\right),$$

where $[\sigma_{a,k}]$ denotes the image of the section $\sigma_{a,k}$. We will delve deeper into the definition of $\overline{\mathcal{M}}_g^a$ in Section 3.2. The image of DRL in $\overline{\mathcal{M}}_{g,n}$ via ρ captures more information than just $\overline{\mathcal{M}}_g(a,k)$, as it has irreducible components supported entirely in the boundary. In [FP18, Sch18], the authors proposed two conjectural expressions for the class $\mathsf{DR}_g(a,k)$. They showed that these expressions, together with the approach via the projectivized Hodge bundle, are sufficient to compute the strata $[\overline{\mathcal{M}}_g(a,k)]$ for all a and k.

1.3.2 The moduli space of twisted canonical divisors

In [FP18], the authors define *the moduli space of k-twisted canonical divisors* for $k=1$, which was later generalized to higher k in [Sch18]. More precisely, these works define a proper space

$$\widetilde{\mathcal{H}}_g^k(a) \subset \overline{\mathcal{M}}_{g,n}$$

that contains $\mathcal{M}_g(a,k)$ as an open subset. Thus, we also have $\overline{\mathcal{M}}_g(a,k) \subset \widetilde{\mathcal{H}}_g^k(a)$. Moreover, for all $k \geq 1$, the space $\widetilde{\mathcal{H}}_g^k(a)$ has *pure dimension* $2g-3+n$ when $a \notin k\mathbb{Z}_{>0}^n$ [Sch18, Theorem 1.1]. As proven in [HS21], if $a \notin k\mathbb{Z}_{>0}^n$, the space $\widetilde{\mathcal{H}}_g^k(a)$ is the image of DRL under the morphism ρ.

Definition 1.3.2. Let $k \in \mathbb{Z}_{\geq 1}$, and let $a \in \mathbb{Z}^n$ such that $|a| = k(2g-2+n)$. A *k-twist compatible with a* on a pre-stable graph Γ is a function $I\colon H(\Gamma) \to \mathbb{Z}$ satisfying the following axioms:

(i) For all edges $e=(h,h') \in E(\Gamma)$, we have

$$I(h) + I(h') = 0.$$

(ii) For all legs ℓ_i, we have

$$I(\ell_i) = a_i.$$

(iii) For all vertices v, we have

$$\sum_{h \in H(v)} I(h) = k(2g(v) - 2 + n(v)).$$

(iv) If v, v' are vertices connected by two edges (h_1, h_1') and (h_2, h_2'), then

$$I(h_1) \geq 0 \Leftrightarrow I(h_2) \geq 0,$$

in which case we write $v \geq v'$.

(v) The relation on the set of vertices of Γ is transitive (i.e., it is a partial ordering).

We refer to pairs (Γ, I) as *k-twisted graphs*.

Instead of defining $\widetilde{\mathcal{H}}_g^k(a)$ as the image of DRL, we can properly define it following the original definitions in [FP18, Sch18]. Given a k-twisted graph (Γ, I), let $N_I \subseteq E(\Gamma)$ denote the set of edges $e = (h, h')$ of Γ such that $I(h), I(h') \neq 0$. Furthermore, given a geometric point $(C, p_1, \ldots, p_n)$ of $\overline{\mathcal{M}}_{g,n}$ and a k-twist on the dual graph of $(C, p_1, \ldots, p_n)$, we set

$$v_I \colon C_I \to C$$

to be the partial normalization of all nodes corresponding to edges in $N_I \subseteq E(\Gamma)$. Moreover, if $q_e \in N_I$ we denote by $q_h, q_{h'}$ the preimages of q_e in C_I via ν_I.

Definition 1.3.3. Let $(C, p_1, \ldots, p_n)$ be a geometric point of $\overline{\mathcal{M}}_{g,n}$. Let $k \in \mathbb{Z}_{\geq 1}$, and let $a \in \mathbb{Z}^n$ such that $|a| = k(2g - 2 + n)$. The divisor $\sum_{i=1}^n a_i p_i$ on $(C, p_1, \ldots, p_n)$ is called *k-twisted canonical* if there exists a k-twist on the dual graph of C such that

$$\mathcal{O}_{C_I}\left(\sum_{i=1}^n a_i p_i + \sum_{q_e \in N_I} I(h) q_h + I(h') q_{h'}\right) \cong \omega_{C_I, \log}^{\otimes k}.$$

Here $\omega_{C_I, \log}$ denotes the log-canonical bundle of C_I.

Definition 1.3.4. The moduli space of k-twisted canonical divisors is the subset of $\overline{\mathcal{M}}_{g,n}$ given by

$$\widetilde{\mathcal{H}}_g^k(a) := \left\{(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n} \mid \sum_{i=1}^n a_i p_i \text{ is a } k\text{-twisted canonical divisor}\right\}.$$

As shown in [FP18, Sch18] the space $\widetilde{\mathcal{H}}_g^k(a)$ is a closed substack of $\overline{\mathcal{M}}_{g,n}$. Moreover, we have that $\mathcal{M}_g(a, k) \subset \widetilde{\mathcal{H}}_g^k(a)$. Indeed, for a smooth curve $(C, p_1, \ldots, p_n)$ in $\mathcal{M}_{g,n}$, the divisor $\sum_{i=1}^n a_i p_i$ is a k-twisted canonical divisor if and only if $(C, p_1, \ldots, p_n) \in \mathcal{M}_g(a, k)$:

$$\widetilde{\mathcal{H}}_g^k(a) \cap \mathcal{M}_{g,n} = \mathcal{M}_g(a, k)$$

Therefore, since $\widetilde{\mathcal{H}}_g^k(a)$ is closed, we obtain that $\overline{\mathcal{M}}_g(a, k) \subset \widetilde{\mathcal{H}}_g^k(a)$.

Definition 1.3.5. We call a *simple star graph* a k-twisted graph (Γ, I) such that:

(i') The graph Γ is stable, and has a distinguished vertex v_0 (the *central* vertex) such that *all* edges attach v_0 to a vertex $v \in V_{\text{out}} := V(\Gamma) \setminus \{v_0\}$ (called an *outlying* vertex), and

(ii') the twist I is such that the value on the half edges h incident to vertices $v \neq v_0$ satisfies

$$I(h) > 0 \quad \text{and} \quad k | I(h).$$

We denote the set of simple star graphs by $\mathrm{SStar}_g(a, k)$. Finally, for an edge $e \in E(\Gamma)$ we define $I(e) := |I(h)| = |I(h')|$.

Remark 1.3.6. In [HS21], the authors impose an additional condition on the markings in their definition of simple star graphs: all markings ℓ_i with $a_i < 0$ must lie on the central vertex. In our setting, this condition is implicit in (ii') together with the fact that legs are regarded as half-edges.

If $a \notin k\mathbb{Z}^n_{>0}$, the irreducible components of $\widetilde{\mathcal{H}}^k_g(a)$ have a combinatorial description via simple star graphs [FP18, Sch18]. In particular, given a simple star graph (Γ, I), we define

$$\overline{\mathcal{M}}_{\Gamma,I} := \overline{\mathcal{M}}_{g(v_0)}(I(v_0), k) \times \prod_{v \in V_{\text{out}}} \overline{\mathcal{M}}_{g(v)}(I(v)/k, 1) \subset \overline{\mathcal{M}}_{\Gamma},$$

where $I(v)$ denotes the vector of integers with entries $(I(h))$ for all $h \in H(v)$. If $M \subset \widetilde{\mathcal{H}}^k_g(a)$ is an irreducible component, then the dual graph of the generic point is given by a star graph Γ, i.e., a stable graph satisfying condition (i') of the definition of simple star graphs above, and there exists a twist I such that the pair (Γ, I) is a simple star graph. Then M is an irreducible component of the closed subset $\zeta_\Gamma(\overline{\mathcal{M}}_{\Gamma,I})$ (see [Sch18, Section 3.1] or [FP18] for further details).

Definition 1.3.7. The *weighted fundamental class* of $\widetilde{\mathcal{H}}^k_g(a)$ is defined as the class

$$\mathsf{H}_g(a, k) = \sum_{(\Gamma,I) \in \mathrm{SStar}_g(a,k)} \frac{\prod_{e \in E(\Gamma)} I(e)}{k^{|V_{\text{out}}|} |\mathrm{Aut}(\Gamma)|} \zeta_{\Gamma*}[\overline{\mathcal{M}}_{\Gamma,I}] \tag{1.5}$$

in $A^g(\overline{\mathcal{M}}_{g,n})$.

Remark 1.3.8. The sum on the RHS of the equation above, may be viewed as a weighted sum over all irreducible components of $\widetilde{\mathcal{H}}^k_g(a)$. Essentially, the fundamental class of $\widetilde{\mathcal{H}}^k_g(a)$ with the reduced sub-scheme structure differs from the formula above by the coefficients appearing in front of each term. One of the main results of [HS21] is the fact that these numbers occur as the intersection theoretic multiplicity of the 0-section and the Abel-Jacobi map $\sigma_{a,k}$.

Theorem 1.3.9. *[HS21] Let $k \in \mathbb{Z}_{>0}$, and $a \notin k\mathbb{Z}^n_{>0}$ such that $|a| = k(2g - 2 + n)$. Then the following equality holds*

$$\mathsf{H}_g(a, k) = \mathsf{DR}_g(a, k)$$

in $A^g(\overline{\mathcal{M}}_{g,n})$.

1.3.3 Pixton's formula and the computation of strata

The second expression for the double ramification cycle $\mathsf{DR}_g(a,k)$, which in contrast to the star graph expression, holds for any k and $a \in \mathbb{Z}^n$ such that $|a| = k(2g-2+n)$, is given by the so-called *Pixton's class* $\mathsf{P}_g(a,k)$ (see [JPPZ17, Section 1.1] for its formal definition). This identity was established in complete generality in [BHP⁺23]. The special case $k=0$, which laid the foundation for the general result, was established in [JPPZ17].

Theorem 1.3.10. *[BHP⁺23] Let $k \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}^n$ such that $|a| = k(2g-2+n)$. Then the following equality holds*

$$\mathsf{DR}_g(a,k) = \mathsf{P}_g(a,k)$$

in $A^*(\overline{\mathcal{M}}_{g,n})$.

Inductive computation of strata

With these two expressions at hand, Farkas-Pandharipande proposed an inductive process that reduces the computation of classes of strata $[\overline{\mathcal{M}}_g(a,k)]$ to strata of either lower genus or fewer markings. The starting point is the observation that the contribution of the trivial simple star graph[6] in the LHS of Theorem 1.3.9 corresponds to $[\overline{\mathcal{M}}_g(a,k)]$ itself. Thus, when $a \notin k\mathbb{Z}^n_{>0}$, this observation, together with Theorem 1.3.9 and Theorem 1.3.10, yields the identity

$$[\overline{\mathcal{M}}_g(a,k)] = \mathsf{P}_g(a,k) - \Delta,$$

where Δ is the sum in the RHS of Equation 1.5 over non-trivial simple star graphs. A closer look at Δ shows that this is a class expressed in strata of k-differentials or 1-differentials of either lower genera g' or of lower number of markings n'. In particular, repeated applications of Theorem 1.3.9 to the contributions arising by the meromorphic strata reduce the aforementioned computation to the computation of holomorphic strata of 1-differentials $[\overline{\mathcal{M}}_{g'}(a',1)]$, where $a' \in \mathbb{Z}^{n'}_{>0}$ and either $g' \leq g$ and $n' < n$, or $g' < g$ and $n' \leq n$ (see also Lemma A.1.1).

At first glance, however, the theorems presented above do not seem sufficient to compute also $[\overline{\mathcal{M}}_{g'}(a',1)]$ for all holomorphic choices of a'. The classes $[\overline{\mathcal{M}}_{g'}(a',1)]$ were computed in the appendix of [FP18] via an induction on the set of pairs of non-negative integers (g,n) such that $2g-2+n > 0$. In particular, we endow the set of such pairs (g,n) with the partial ordering given by:

$$(g',n') < (g,n) \Leftrightarrow (g' < g) \text{ or } (g' = g \text{ and } n' \leq n).$$

The authors of [FP18] observed that if $a_+ = (a_1,\dots,a_n)$ is a vector of strictly positive integers such that $|a_+| = 2g-2+n$, then

$$[\overline{\mathcal{M}}_g\left((a_1,\dots,a_n+1,0),1\right)] = 0$$

as there is no differential admitting a single pole of order 1. Besides, Theorem 1.3.9 holds for the vector $(a_1,\dots,a_n+1,0)$. Applying this theorem and pushing forward the expression via

$$\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n},$$

[6] The *trivial simple star graph* is the graph with only one vertex.

one checks that the term $[\overline{\mathcal{M}}_g(a_+,1)]$ appears in the resulting expression [FP18, A.6.2]. A careful and intricate calculation then shows that the non-trivial base case is the computation of the stratum

$$[\overline{\mathcal{M}}_g((2g-1),1)]$$

according to the induction on genera or number of markings. That is, computing $[\overline{\mathcal{M}}_g(a_+,1)]$ can be reduced to the computation of the stratum $[\overline{\mathcal{M}}_g((2g-1),1)]$. Schmitt [Sch18] showed that for holomorphic strata of k-differentials, the case $\overline{\mathcal{M}}_g(a_+,1)$ is enough to determine also the computation of $\overline{\mathcal{M}}_g(ka_+,k)$. In total, for any k and $a \in \mathbb{Z}^n$, Theorem 1.3.9 and Theorem 1.3.10 reduces the computation of $[\overline{\mathcal{M}}_g(a,k)]$ to the computation of $\overline{\mathcal{M}}_g((2g-1),1)$, which, in turn, is computed by Sauvaget's algorithm [Sau19]. The tautological nature of $[\overline{\mathcal{M}}_g(a,k)]$ then follows from [Sau19, Theorem 3], the fact that $\mathsf{P}_g(a,k)$ is tautological, and the nature of the inductive process.

1.3.4 spin classes of strata of k-differentials

A *spin structure* on a smooth curve $(C,p_1,\ldots,p_n)$ is a line bundle $\mathcal{L} \to C$ together with an isomorphism $\phi\colon \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \omega_C$. The *parity* of a spin structure is defined as the parity of $h^0(C,\mathcal{L})$, and it is invariant under deformations of spin structures [Ati71, Mum71]. When $k \in \mathbb{Z}_{\geq 1}$ is *odd*, and $a \in \mathbb{Z}^n$ is a vector of *odd* integers such that $|a| = k(2g-2+n)$, any point of $\mathcal{M}_g(a,k)$ naturally carries a spin structure, namely

$$\omega_{\log}^{\frac{1-k}{2}}\left(\sum_{i=1}^{n}\frac{a_i-1}{2}\right).$$

In particular, this induces a decomposition

$$\mathcal{M}_g(a,k) = \mathcal{M}_g(a,k)^+ \coprod \mathcal{M}_g(a,k)^-$$

according to the parity of the associated spin structure. We denote by $\overline{\mathcal{M}}(a,k)^+$ and $\overline{\mathcal{M}}(a,k)^-$ the Zariski closures in $\overline{\mathcal{M}}_{g,n}$, and by $[\overline{\mathcal{M}}(a,k)^+]$ and $[\overline{\mathcal{M}}(a,k)^-]$ the classes of these cycles with the reduced sub-scheme structure

Question 1.3.11. Are the classes $[\overline{\mathcal{M}}_g(a,k)^+]$ and $[\overline{\mathcal{M}}_g(a,k)^-]$ tautological? Are there explicit formulas for these classes?

Based on the results presented in the previous subsection, we can compute the sum $[\overline{\mathcal{M}}_g(a,k)] = [\overline{\mathcal{M}}_g(a,k)^+]+[\overline{\mathcal{M}}_g(a,k)^-]$. Therefore, to compute each individual term, it suffices to compute their difference

$$[\overline{\mathcal{M}}_g(a,k)]^{\pm} := [\overline{\mathcal{M}}_g(a,k)^+] - [\overline{\mathcal{M}}_g(a,k)^-].$$

Towards the computation of this class, Costantini-Sauvaget-Schmitt [CSS21] proposed a conjecture in the spirit of [FP18]. To describe it, we begin by introducing the following two classes. First, for the remainder of this subsection, we assume that $k \in \mathbb{Z}_{\geq 1}$ is odd, and that $a \notin k\mathbb{Z}^n_{>0}$ is a vector of odd integers unless otherwise stated. We define

$$\mathsf{H}_g^{\pm}(a,k) := \sum_{(\Gamma,I)\in \mathrm{SStar}_g^{\mathrm{odd}}(a,k)} \frac{\prod_{e\in E(\Gamma)} I(e)}{k^{|V_{\mathrm{out}}|}|\mathrm{Aut}(\Gamma)|}\zeta_{\Gamma *}[\overline{\mathcal{M}}_{\Gamma,I}]^{\pm},$$

where $\mathrm{SStar}_g^{\mathrm{odd}}(a,k) \subset \mathrm{SStar}_g(a,k)$ denotes the subset of simple star graphs (Γ,I) such that I takes only odd values. Here, for every such simple star graph we write

$$[\overline{\mathcal{M}}_{\Gamma,I}]^{\pm} := [\overline{\mathcal{M}}_{g(v_0)}(I(v_0),k)]^{\pm} \times \prod_{v\in V_{out}} [\overline{\mathcal{M}}_{g(v)}(I(v),1)]^{\pm}.$$

Pixton's spin class

Definition 1.3.12. Let $r, k \in \mathbb{Z}_{\geq 0}$. An *$r/k$-weighting* on Γ is a function $w\colon H(\Gamma) \to \{0, \dots, r-1\}$ satisfying:

(i) For all edges $e = (h, h')$, we have

$$w(h) + w(h') \equiv 0 \mod r.$$

(ii) For all vertices v, we have

$$\sum_{h \in H(v)} w(h) \equiv k(2g(v) - 2 + n(v)) \mod r.$$

We say that this weighting is *compatible with the vector a* if for all $\ell_i \in L(\Gamma)$ we have $w(\ell_i) \equiv a_i$ mod r. In the case $k = 1$, we refer to the $r/1$-weightings as simply r-weightings.

If Γ is a stable graph, we denote by $W_\Gamma^{2r/k}(a)^{\mathrm{odd}}$ the set of odd $2r/k$-weightings, i.e., taking only odd values, compatible with a. For all positive integers c and r, we denote by $\mathsf{P}_g^{\pm,c,r}(a,k)$ the codimension c part of the class

$$\sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W_\Gamma^{2r/k}(a)^{\mathrm{odd}}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \zeta_{\Gamma *} \Bigg[\prod_{v \in V(\Gamma)} \exp(-\frac{k^2}{4}\kappa_1(v)) \prod_{i=1}^{n} \exp(\frac{a_i^2}{4}\psi_i) \cdot \prod_{e=(h,h')} \frac{1 - \exp(-\frac{w(h)w(h')}{4}(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}} \Bigg].$$

Similar arguments to those of Pixton [JPPZ17, Appendix] show that this class is polynomial in r for sufficiently large r. We define the *Pixton's spin class*, denoted by $\mathsf{P}_g^{\pm}(a,k)$, to be the constant term of $\mathsf{P}_g^{\pm,g,r}(a,k)$.

Conjecture 1.3.13. [CSS21, Conjecture 2.5] Let $k \in \mathbb{Z}_{\geq 1}$ be odd, and let $a \notin k\mathbb{Z}_{>0}^n$ be a vector of odd integers such that $|a| = k(2g - 2 + n)$. Then the following equality holds

$$\mathsf{H}_g^{\pm}(a,k) = \mathsf{P}_g^{\pm}(a,k)$$

in $A^*(\overline{\mathcal{M}}_{g,n})$.

Theorem 1.3.14 (Theorem 3.5.14)**.** *Conjecture 1.3.13 is true.*

Corollary 1.3.15 (Proposition A.1.3)**.** *Assume that the classes $[\overline{\mathcal{M}}_{g'}(a',1)]^{\pm}$ are tautological and computable for all $g' \leq g$ and suitable $a' \in \mathbb{Z}_{>0}$. Then the classes $[\overline{\mathcal{M}}_g(a,k)]^{\pm}$ are tautological and computable for all suitable $k \in \mathbb{Z}_{\geq 1}$ and $a \in \mathbb{Z}^n$.*

Remark 1.3.16. In [CSS21], the original formulation of the conjecture compares the star graph expression, i.e., $\mathsf{H}_g^{\pm}(a,k)$ to a class that appears different at first. However, one can show that the constant term of the Pixton's spin class $\mathsf{P}_g^{\pm}(a,k)$, as defined above, matches the codimension g part of the constant term of the polynomial defined in [CSS21, Section 2.3]. This equivalence follows from applying the exact same arguments as in [JPPZ17, Proposition 5] to reduce the contribution of the constant term defined in [CSS21] in simpler terms and then extract $2r^2$ from the expression.

The main result of this chapter is a proof of Conjecture 1.3.13. Following the approach via the theory of DR cycles, we will introduce a new class $\mathsf{DR}_g^{\pm}(a,k)$, and demonstrate an equality of this class with the LHS and RHS of the equation above.

1.3.5 Spin DR cycles

We work over the moduli space of spin structures, denoted by $\mathcal{M}_{g,n}^{1/2}$, i.e., the moduli space of tuples $(C, p_1, \ldots, p_n, \mathcal{L}, \phi)$, where $(C, p_1, \ldots, p_n)$ is a smooth curve of genus g with n marked points, $\mathcal{L}$ is a line bundle on C, and $\phi\colon \mathcal{L}^{\otimes 2} \to \omega_C$ is an isomorphism of sheaves on C. Various compactifications of this space have been constructed in the literature (see [Cor89,Jar00,Chi08,HO23]), each introducing different objects in the boundary. In Section 3.1, we recall the compactification $\overline{\mathcal{M}}_{g,n}^{1/2}$ constructed by Cornalba in [Cor89] where the objects in the boundary are certain pre-stable degenerations. This space is a smooth DM stack admitting a finite flat morphism of DM stacks

$$\epsilon\colon \overline{\mathcal{M}}_{g,n}^{1/2} \to \overline{\mathcal{M}}_{g,n}$$

of degree 2^{2g-1}, obtained by forgetting the spin structure. Cornalba proves that the parity of spin structures remains constant in degenerations also in the boundary, extending the results of [Ati71, Mum83]. We denote by $\overline{\mathcal{M}}_{g,n}^{1/2,+}$ (resp. $\overline{\mathcal{M}}_{g,n}^{1/2,-}$) the connected component of $\overline{\mathcal{M}}_{g,n}^{1/2}$ parametrizing spin structures whose parity is even (resp. odd). Moreover, we define the *parity cycle* as the difference

$$[\pm] := [\overline{\mathcal{M}}_{g,n}^{1/2,+}] - [\overline{\mathcal{M}}_{g,n}^{1/2,-}] \in A^0(\overline{\mathcal{M}}_{g,n}^{1/2}).$$

Let $\overline{\mathcal{C}}_{g,n}^{1/2} \to \overline{\mathcal{M}}_{g,n}^{1/2}$ denote the universal curve over $\overline{\mathcal{M}}_{g,n}^{1/2}$, and let $\mathcal{L} \to \overline{\mathcal{C}}_{g,n}^{1/2}$ denote the universal spin structure on the universal curve. Moreover, we define $\sigma_{a,k}^{1/2}$ to be the section $\mathcal{M}_{g,n}^{1/2} \to \mathcal{J}_{g,n}$ associated to the line bundle

$$\mathcal{L}_{a,k} := \mathcal{L} \otimes \omega_{\log}^{\frac{k-1}{2}} \left(-\sum_{i=1}^{n} \frac{a_i - 1}{2} \right)$$

on the universal curve $\mathcal{C}_{g,n}^{1/2}$. Similarly to the definition of the double ramification locus, we define the double ramification locus of $\mathcal{L}_{a,k}$ as the fiber product of $\sigma_{a,k}^{1/2}$ with the 0-section $\sigma_0\colon \mathcal{M}_{g,n}^{1/2} \to \mathcal{J}_{g,n}$. The universal jacobian admits an extension over $\overline{\mathcal{M}}_{g,n}^{1/2}$. However, as in the non-spin case, the section $\sigma_{a,k}^{1/2}$ does not extend over $\overline{\mathcal{M}}_{g,n}^{1/2}$ since $\mathcal{L}_{a,k}$ fails to have multi-degree 0 over the boundary. In Section 3.2, we construct a birational model

$$\rho_{\frac{1}{2}}\colon \overline{\mathcal{M}}_{g}^{1/2,a} \to \overline{\mathcal{M}}_{g,n}^{1/2}$$

in the spirit of [MW20, Hol21, HS21], over which $\sigma_{a,k}^{1/2}$ extends, and on which the schematic intersection of σ_0 and $\sigma_{a,k}^{1/2}$, denoted by $\mathrm{DRL}^{1/2}$, is proper over $\overline{\mathcal{M}}_{g,n}^{1/2}$. We define the *spin double ramification cycle* as

$$\mathsf{DR}_g^{\pm}(a,k) := 2\epsilon_* \left(\rho_{\frac{1}{2}*}(\sigma_0^![\sigma_{a,k}^{1/2}]) \cdot [\pm] \right).$$

In Section 3.4, we establish the equality

$$\mathsf{DR}_g^{\pm}(a,k) = \mathsf{H}_g^{\pm}(a,k), \qquad \text{(Theorem 3.4.5)}$$

and in Section 3.5, we prove

$$\mathsf{DR}_g^{\pm}(a,k) = \mathsf{P}_g^{\pm}(a,k). \qquad \text{(Theorem 3.5.12)}$$

Finally, in the appendix, we explain how Conjecture 1.3.13 (see also [CSS21, Conjecture 2.5] can be used to compute $[\overline{\mathcal{M}}_g(a,k)^{\pm}]$ for all values of a and k in analogy with the method developed in [FP18].

Organization of Chapter 3

In Section 3.1 we summarize the definition of *spin curves* and their moduli, as constructed by Cornalba in [Cor89]. We recall the description of the stratification of the boundary indexed by 2-*weighted* graphs and prove a vanishing statement for the classes of strata with at least one 0 weight (see Proposition 3.1.16). We could not find a proper proof of this "folklore" proposition in the literature, although the argument is essentially borrowed from Cornalba's original paper and used in the computations of [GKL21].

In Section 3.2, we develop the necessary framework to extend the Abel-Jacobi section, and we give proper definitions for $\overline{\mathcal{M}}_g^a$ and $\overline{\mathcal{M}}_g^{1/2,a}$ using logarithmic geometry. Additionally, we carry out a detailed algebraic analysis on the étale local picture of these spaces, which allows us to compute the lengths of the Artin local rings at generic points of the irreducible components of the corresponding double ramification loci. Similar computations were previously done in [HS21]. However, there the authors extract étale local charts from the minimal log structures over only *fine* log schemes. Since we would like to work over the category of *fine and saturated* log schemes in the next section, we analyze the relationship between $\overline{\mathcal{M}}_g^a$ and the space constructed in [HS21]. Upon completing these computations, we construct a morphism

$$\mathrm{DRL}^{1/2} \to \mathrm{DRL}$$

between the double ramification loci in $\overline{\mathcal{M}}_g^{1/2,a}$ and in $\overline{\mathcal{M}}_g^a$. In particular, we show that this map is a μ_2-gerbe, which allows us to reduce the length calculations of $\mathrm{DRL}^{1/2}$ to the ones we did for DRL.

In Section 3.3, we establish an action by μ_2, the group of square roots of unity, on the sub-locus of $\mathrm{DRL}^{1/2}$ consisting of curves with topological type given by simple star graphs (Γ, I), where I admits at least one even value. Using this action, we prove that, in such cases, the even and odd components of this sub-locus of $\mathrm{DRL}^{1/2}$ are isomorphic. As a corollary, we obtain a vanishing result similar to that presented in Proposition 3.1.16.

In Section 3.4, we examine the image of the even and the odd parts of the irreducible components of $\mathrm{DRL}^{1/2}$ under $\epsilon \circ \rho_{\frac{1}{2}}$, corresponding to simple star graphs (Γ, I), where I admits only odd values. As a result, we show that $\mathsf{DR}_g^{\pm}(a,k)$ can be expressed in terms of the classes $\zeta_\Gamma[\overline{\mathcal{M}}_{\Gamma,I}]^{\pm}$. Finally, the length calculations of irreducible components of $\mathrm{DRL}^{1/2}$ yield the star graph expression for $\mathsf{DR}_g^{\pm}(a,k)$, forming the first part of the proof of the Conjecture 1.3.13.

Finally, in Section 3.5, we recall one of the main theorems of [BHP+23] which we use in order to prove that $\mathsf{DR}_g^{\pm}(a,k)$ is equal to Pixton's spin class $\mathsf{P}_g^{\pm}(a,k)$ in $A^*(\overline{\mathcal{M}}_{g,n})$. This completes the proof of Conjecture 1.3.13.