Spectral analysis of inhomogeneous network models

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Summary

This thesis comprises five chapters, three of which contain the core mathematical content. We study the limiting spectral distributions of random graph models with vertex inhomogeneity. In particular, we focus on the adjacency matrix of the inhomogeneous Erdős–Rényi random graph in the sparse regime, as well as the adjacency and Laplacian matrices for random graph models that incorporate spatial structure.

Random graph models provide a mathematical framework for understanding complex networks observed in fields such as physics, biology, computer science, and the social sciences. The classical Erdős–Rényi model, in which edges are added independently with equal probability, serves as a foundational model that continues to yield deep insights. Spectral graph theory plays a key role in this context, connecting the eigenvalues and eigenvectors of the adjacency and Laplacian matrices of graphs to structural and geometric properties of graphs. For instance, the Perron–Frobenius theorem ensures a unique largest eigenvalue for the adjacency matrix of a connected graph, with a corresponding positive eigenvector. More broadly, the spectrum gives information about the graph connectivity, subgraph counts, the chromatic number, and other topological features. Laplacian eigenvalues are central in the study of diffusion, mixing times of random walks, and spectral clustering algorithms. Notably, the Kirchhoff Matrix–Tree Theorem relates the determinant of the combinatorial Laplacian to the count of the spanning trees of the graph. These connections make spectral analysis a powerful tool for studying the geometry of complex networks. Chapter 1 provides a detailed introduction to spectral graph theory, random graphs, and random matrices.

The spectra of the adjacency and Laplacian matrices are well understood in the dense Erdős-Rényi random graph model. In the sparse case, three main analytical techniques are used: (i) characterising the limiting spectrum via local weak limits such as Galton–Watson trees; (ii) using combinatorial methods and special symmetric partitions to compute the moments of the limiting spectral measure; (iii) deriving the Stieltjes transform of the limiting measure using a fixed-point equation in an appropriate Banach space. In Chapter 2, we extend the Erdős–Rényi random graph model by incorporating deterministic vertex weights to introduce inhomogeneity, where now edges are added independently with a probability proportional to a function of the vertex weights. We study this model in the sparse setting, where the connectivity function is bounded. We analyse the empirical spectral distribution of the adjacency matrix using the moment method and the Stieltjes transform, and describe the limiting distribution through homomorphism densities, symmetric partitions, and a fixed-point equation.

Real-world networks often exhibit spatial structure in addition to vertex inhomogeneity. In Chapter 3, we consider a kernel-based random graph model on a discrete torus, where the vertices are equipped with random weights that follow a power-law distribution, and connection probabilities between two vertices depend directly on a function of the two weights and that is inversely proportional to the torus distance between the two vertices. Using the method of moments, we study the adjacency matrix of this model and show the existence and uniqueness of a limiting spectral measure. We further analyse the measure through its prelimit to show that it is absolutely continuous and non-degenerate. We characterise the Stieltjes transform of this measure through a fixed-point equation. When the kernel is rank-one, that is, it has a product structure, we identify the limiting measure explicitly as a free multiplicative convolution between the semicircle law and the Pareto law using tools from free probability.

In Chapter 4, we focus on the centred Laplacian matrix of the rank-one model, which is known as the Scale-Free percolation model. Using the method of moments, we show the existence of a unique limiting spectral measure. We further identify the measure in terms of the spectral distribution of some non-commutative unbounded operators, again using techniques from free probability theory.

In Chapter 5, we present simulations and provide a brief discussion of examples that fall outside the restrictions assumed in the previous chapters.