Spectral analysis of inhomogeneous network models

Malhotra, N.

CHAPTER 4

Scale-free percolation: The graph Laplacian

This chapter is based on:
R.S. Hazra, N. Malhotra. *Spectral properties of the Laplacian of scale-free percolation models.* [*arxiv:2504.17552*], 2025.

Abstract

We consider scale free percolation on a discrete torus $\mathbf{V}_N$ of size N. Conditionally on an i.i.d. sequence of Pareto weights $(W_i)_{i\in\mathbf{V}_N}$ with tail exponent $\tau - 1 > 0$, we connect any two points i and j on the torus with probability

$$p_{ij} = \frac{W_i W_j}{\|i-j\|^\alpha} \wedge 1$$

for some parameter $\alpha > 0$. We focus on the (centered) Laplacian operator of this random graph and study its empirical spectral distribution. We explicitly identify the limiting distribution when $\alpha < 1$ and $\tau > 3$, in terms of the spectral distribution of some non-commutative unbounded operators.

§4.1 Introduction

In recent years, many random graph models have been proposed to model real-life networks. These models aim to capture three key properties that real-world networks exhibit: scale-free nature of the degree distribution, small-world property, and high clustering coefficients [van der Hofstad, 2024]. It is generally difficult to find random graph models which incorporate all three features. Classical random graph models typically fail to capture scale-freeness, small-world behaviour, and high clustering simultaneously. For instance, the Erdős-Rényi model only exhibits the small-world property, while models like Chung-Lu, Norros-Reittu, and preferential attachment models are scale-free (Chung and Lu [2002], Barabási and Albert [1999] and small-world but have clustering coefficients that vanish as the network grows. In contrast, regular lattices have high clustering but large typical distances. The Watts-Strogatz model (Watts and Strogatz [1998]) was an early attempt to create a network with high clustering and small-world features, but it does not produce scale-free degree distributions.

Scale-free percolation, introduced in Deijfen et al. [2013], blends ideas from long-range percolation (see e.g. Berger [2002]) with inhomogeneous random graphs such as the Norros–Reittu model. In this framework, vertices are positioned on the $\mathbb{Z}^d$ lattice, and each vertex x is independently assigned a random weight W_x. These weights follow a power-law distribution:

$$\mathbb{P}(W > w) = w^{-(\tau-1)}L(w),$$

where $\tau > 1$ and $L(w)$ is a slowly varying function at infinity.

Edges between pairs of vertices x and y are added independently, with a probability that increases with the product of their weights and decreases with their Euclidean distance. The edge probability is given by

$$p_{xy} = 1 - \exp\left(-\lambda \frac{W_x W_y}{\|x-y\|^\alpha}\right), \tag{4.1}$$

where $\lambda, \alpha > 0$ are model parameters and $\|\cdot\|$ denotes the Euclidean norm. This model has been proposed as a suitable representation for certain real-world systems, such as interbank networks, where both spatial structure and heavy-tailed connectivity distributions are relevant (Deprez et al. [2015]). Various properties of the model are now well known and we refer to the articles by Jorritsma et al. [2024], Cipriani and Salvi [2024], Cipriani et al. [2025], Heydenreich et al. [2017], Dalmau and Salvi [2021] for further references.

In recent times, there has been a lot of interest in the models which have connection probabilities similar to (4.1). Kernel-based spatial random graphs encompass a wide variety of classical random graph models where vertices are

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embedded in some metric space. In their simplest form (see Jorritsma et al. [2023] for a more complete exposition) they can be defined as follows. Let V be the vertex set of the graph and sample a collection of weights $(W_i)_{i\in V}$, which are independent and identically distributed (i.i.d.), serving as marks on the vertices. Conditionally on the weights, two vertices i and j are connected by an undirected edge with probability

$$\mathbb{P}\left(i \leftrightarrow j \mid W_i, W_j\right) = \kappa(W_i, W_j)\|i-j\|^{-\alpha} \wedge 1\,, \tag{4.2}$$

where κ is a symmetric kernel, $\|i-j\|$ denotes the distance between the two vertices in the underlying metric space and $\alpha > 0$ is a constant parameter. In a recent article, the present authors with A. Cipriani and M. Salvi (Cipriani et al. [2025]) proved the spectral properties of the adjacency matrix when $\alpha < d$ and the weights have a finite mean. In the above setting, the model was considered on a torus of side length N so that the adjacency operator as a matrix was easier to describe. In this article, we aim to extend this study to the case of a Laplacian matrix. Although our approach would extend to general kernel-based models, we shall stick to the product form kernel, that is, $\kappa(x,y) = xy$, so that the ideas can be clearly presented. It is one of the few cases where the limiting distribution can be explicitly described.

The Laplacian of a graph with N vertices is defined as $A_N - D_N$ where A_N is the adjacency matrix and D_N is the diagonal matrix where the i-th diagonal entry corresponds to the i-th degree. When the entries of the matrix are not restricted to 0 or 1, the matrix is also referred to as the *Markov matrix* (Bryc et al. [2006], Bordenave et al. [2014]). The graph Laplacian serves as the discrete analogue of the continuous Laplacian, essential in diffusion theory and network flow analysis. The Laplacian matrix has several key applications: The Kirchhoff Matrix-Tree Theorem relates the determinant of the Laplacian to the count of spanning trees in a graph (Chung [1997]), the multiplicity of the zero eigenvalue indicates the number of connected components (Chung [1997]), the second-smallest eigenvalue, known as the Fiedler value or the algebraic connectivity, measures the graph's connectivity; higher values signify stronger connectivity De Abreu [2007]. For a comprehensive overview of spectral methods in graph theory, refer to Chung's monograph Chung [1997] and Spielman's book Spielman [2012]. In modern machine learning, spectral techniques are pivotal in spectral clustering algorithms, where the techniques use the Laplacian eigenvalues and eigenvectors for dimensionality reduction before applying clustering algorithms like k-means (Abbe et al. [2020], Abbe [2017]). It is particularly effective for detecting clusters that are not linearly separable. Recent advancements integrate spectral clustering with graph neural networks to enhance graph pooling operations (Bianchi et al. [2020]). Spectral algorithms are also crucial for identifying communities within networks by analysing the spectral properties of the graph (Chung [1997]).

Bryc et al. [2006] established that for large symmetric matrices with independent and identically distributed entries, the empirical spectral distribution (ESD) of the corresponding Laplacian matrix converges to the free convolution of the semicircle law and the standard Gaussian distribution. In the context of sparse Erdős–Rényi graphs, Huang and Landon [2020] studied the local law of the ESD of the Laplacian matrix. They demonstrated that the Stieltjes transform of the ESD closely approximates that of the free convolution of the semicircle law and a standard Gaussian distribution down to the scale N^{-1}. Additionally, they showed that the gap statistics and averaged correlation functions align with those of the Gaussian Orthogonal Ensemble in the bulk. Ding and Jiang [2010] investigated the spectral distributions of adjacency and Laplacian matrices of random graphs, assuming that the variance of the entries of an $N \times N$ adjacency matrix depends only on N. They established the convergence of the ESD of these matrices under such conditions. These results of the Erdős-Rényi random graphs were extended to the inhomogeneous setting by Chakrabarty et al. [2021b]. In a recent work, Chatterjee and Hazra [2022] derived a combinatorial way to describe the limiting moments for a wide variety of random matrix models with a variance profile.

Our contribution

The empirical spectral distribution of the (centred) Laplacian of a graph that incorporates spatial distance has not been studied before. For example, we are not aware of a result that describes the spectral properties of the Laplacian for the long-range percolation model. Our main contribution is that we establish this result for the scale-free percolation model on the torus. We restrict ourselves to the dense regime, that is, $\alpha < 1$. We show that under mild assumptions on the weights (having finite variance), we establish the existence of the limiting distribution. It turns out to be a distribution that involves the Gaussian, the semicircle, and the Pareto distribution. In a symbolic (and rather informal) way, it is given by the spectral law of

$$W^{1/2}SW^{1/2} + m_1 W^{1/4}GW^{1/4},$$

where W is an unbounded operator with spectral law given by the Pareto distribution, S is a bounded compact operator whose spectral law is the semicircle law, and G is an unbounded operator whose law is given by the Gaussian distribution. Finally, m_1 is the first moment of W. The interaction between these operators comes from the fact that in the non-commutative space, $\{W, G\}$ is a commutative algebra, freely independent of S. Similar results have been established when the weights are bounded and degenerate, and no spatial distances are involved (Chatterjee and Hazra [2022] and Chakrabarty et al. [2021b]). The present work extends the results to settings that involve random heavy-tailed weights and spatial distances.

Outline of the article

In section 4.2 we explicitly describe the setup of the model and state our main results. In Theorem 4.2.1 we show the existence of the limiting spectral distribution, and in Theorem 4.2.5, we identify the measure and state some of its properties. In Section 4.3 we first introduce a Gaussianised version of the matrix, and this helps us to simplify the variance profile. We then truncate the weights and decouple the diagonal, which allows us to apply the moment method. In Section 4.4 we identify the limiting moments of the decoupled Laplacian and show that it does not depend on the spatial parameter $\alpha > 0$, which is crucial in the identification of the limiting measure of the original Laplacian. Finally, in Section 4.5 we identify the limiting measure using results from free probability. In Appendix 4.6 we provide references to some of the results we use in our proofs, which are collections of results from other articles and are stated here for completeness.

§4.2 Setup and main results

In this section we describe the setup of the model and also state the main results.

§4.2.1 Setup

(a) **V**ertex set: the vertex set is $\mathbf{V}_N := \{1, 2, \ldots, N\}$. The vertex set is equipped with torus the distance $\|i-j\|$, where

$$\|i-j\| = |i-j| \wedge (N - |i-j|).$$

(b) **Weights:** the weights $(W_i)_{i\in\mathbf{V}_N}$ are i.i.d. random variables sampled from a Pareto distribution W (whose law we denote by $\mathbf{P}$) with parameter $\tau - 1$, where $\tau > 1$. That is,

$$\mathbf{P}(W > t) = t^{-(\tau-1)}\mathbf{1}_{\{t\geq 1\}} + \mathbf{1}_{\{t<1\}}. \tag{4.3}$$

(c) **Long-range parameter:** $\alpha > 0$ is a parameter which controls the influence of the distance between vertices on their connection probability.

(d) **Connectivity function:** conditional on the weights, each pair of distinct vertices i and j is connected independently with probability $P^W(i \leftrightarrow j)$ given by

$$P^W(i \leftrightarrow j) := \mathbb{P}(i \leftrightarrow j \mid W_i, W_j) = \frac{W_i W_j}{\|i-j\|^{\alpha}} \wedge 1. \tag{4.4}$$

We will be using the short-hand notation $p_{ij} := \mathbb{P}(i \leftrightarrow j \mid W_i, W_j)$ for convenience. Note that the graph does not have self-loops.

In what follows, we denote by $\mathbb{P} = \mathbf{P} \otimes P^W$ the joint law of the weights and the edge variables. Note that $\mathbb{P}$ depends on N, but we will omit this dependence for simplicity. Let $\mathbb{E}, \mathbf{E}$, and E^W denote the expectation with respect to $\mathbb{P}, \mathbf{P}$, and P^W respectively.

The associated graph is connected, as nearest neighbours with respect to the torus distance are always linked. Let us denote the random graph generated by our choice of edge probabilities by $\mathbb{G}_N$. Let $\mathbb{A}_{\mathbb{G}_N}$ denote the adjacency matrix (operator) associated with this random graph, defined as

$$\mathbb{A}_{\mathbb{G}_N}(i,j) = \begin{cases} 1 & \text{if } i \leftrightarrow j, \\ 0 & \text{otherwise.} \end{cases}$$

Since the graph is finite and undirected, the adjacency matrix is always self-adjoint and has real eigenvalues. Let

$$\mathbb{D}_{\mathbb{G}_N} = \mathrm{Diag}(d_1, \cdots, d_N)$$

where d_i denotes the degree of the vertex i and in this case given by

$$d_i = \sum_{j \neq i} \mathbb{A}_{\mathbb{G}_N}(i,j).$$

We will consider the Laplacian of the matrix, which is denoted as follows:

$$\Delta_{\mathbb{G}_N} = \mathbb{A}_{\mathbb{G}_N} - \mathbb{D}_{\mathbb{G}_N}.$$

In general, when $\alpha < 1$, the eigenvalue distribution requires scaling in order to observe meaningful limiting behaviour. In Cipriani et al. [2025], it was shown that an appropriate scaling of the adjacency matrix, under which the convergence of the bulk eigenvalue distribution can be studied, is given by

$$c_N = \frac{1}{N} \sum_{i \neq j \in \mathbf{V}_N} \frac{1}{\|i-j\|^\alpha} \sim c_0 N^{1-\alpha}, \tag{4.5}$$

where c_0 is a positive constant. The scaled adjacency matrix is then defined as

$$\mathbf{A}_N := \frac{\mathbb{A}_{\mathbb{G}_N}}{\sqrt{c_N}}. \tag{4.6}$$

We define the corresponding (scaled) Laplacian as

$$\mathbf{\Delta}_N = \mathbf{A}_N - \mathbf{D}_N,$$

where $\mathbf{D}_N$ is given by $\mathbf{D}_N = \mathrm{Diag}(\mathbf{d}_1, \cdots, \mathbf{d}_N)$ with

$$\mathbf{d}_i = \sum_{k \neq i} \mathbf{A}_N(i,k).$$

The empirical measure that assigns a mass of $1/N$ to each eigenvalue of the $N \times N$ random matrix $\mathbf{M}_N$ is called the Empirical Spectral Distribution (ESD) of $\mathbf{M}_N$, denoted as

$$\mathrm{ESD}\left(\mathbf{M}_N\right) := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i},$$

where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ are the eigenvalues of $\mathbf{M}_N$. We are interested in the centred Laplacian matrix for the bulk distribution. So define

$$\mathbf{\Delta}_N^\circ = \mathbf{\Delta}_N - \mathbb{E}[\mathbf{\Delta}_N] \tag{4.7}$$

where $\mathbb{E}[\mathbf{\Delta}_N](i,j) = \mathbb{E}[\mathbf{\Delta}_N(i,j)]$. If we define $\mathbf{A}_N^\circ = \mathbf{A}_N - \mathbb{E}[\mathbf{A}_N]$ and $\mathbf{D}_N^\circ$ is the diagonal matrix $\mathrm{Diag}(\mathbf{d}_1^\circ, \ldots, \mathbf{d}_N^\circ)$ where $\mathbf{d}_i^\circ = \sum_{k \neq i} \mathbf{A}_N^\circ(i,k)$, then it is easy to see that

$$\mathbf{\Delta}_N^\circ = \mathbf{A}_N^\circ - \mathbf{D}_N^\circ.$$

In this article we will be interested in understanding the behaviour of $\mathrm{ESD}(\mathbf{\Delta}_N^\circ)$ as $N \to \infty$.

§4.2.2 Main Results

The Lévy-Prokhorov distance $d_L : \mathcal{P}(\mathbb{R})^2 \to [0, +\infty)$ between two probability measures μ and ν on $\mathbb{R}$ is defined as

$$d_L(\mu,\nu) := \inf\left\{\varepsilon > 0 \mid \mu(A) \leq \nu\left(A^\varepsilon\right) + \varepsilon \text{ and } \nu(A) \leq \mu\left(A^\varepsilon\right) + \varepsilon \quad \forall\, A \in \mathcal{B}(\mathbb{R})\right\},$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$, and A^ε is the ε-neighbourhood of A. For a sequence of random probability measures $(\mu_N)_{N \geq 0}$, we say that

$$\lim_{N \to \infty} \mu_N = \mu_0 \text{ in } \mathbb{P}\text{-probability}$$

if, for every $\varepsilon > 0$,

$$\lim_{N \to \infty} \mathbb{P}(d_L(\mu_N, \mu_0) > \varepsilon) = 0.$$

Our first main result is existential and is as follows.

Theorem 4.2.1.
Consider the random graph $\mathbb{G}_N$ on $\mathbf{V}_N$ with connection probabilities given by (4.4) with parameters $\tau > 3$ and $0 < \alpha < 1$. Let $\mathrm{ESD}(\mathbf{\Delta}_N^\circ)$ be the empirical spectral distribution of $\mathbf{\Delta}_N^\circ$ defined in (4.7). Then there exists a deterministic measure ν_τ on $\mathbb{R}$ such that

$$\lim_{N \to \infty} \mathrm{ESD}(\mathbf{\Delta}_N^o) = \nu_\tau \qquad \text{in } \mathbb{P}\text{–probability}.$$

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The characterisation of ν_τ is achieved by results from the theory of free probability. For convenience, we state some technical definitions. We refer the readers to [Anderson et al., 2010, Chapter 5] for further details.

For the following definitions, we refer the reader to Mingo and Speicher [2017], and recall from Chapter 1 that a W^*-algebra is a C^*-algebra of bounded operators on a Hilbert space closed in the weak operator topology.

Definition 4.2.2.
Let $(\mathcal{A}, \varphi)$ be a W^*-probability space, where $\mathcal{A}$ is a W^*-algebra, and φ is a faithful, tracial state. *A densely defined operator T is said to be affiliated with $\mathcal{A}$ if for every bounded measurable function h, we have $h(T) \in \mathcal{A}$. The law (or distribution) $\mathcal{L}(T)$ of such an affiliated operator T is the unique probability measure on $\mathbb{R}$ satisfying*

$$\varphi(h(T)) = \int_{\mathbb{R}} h(x)\, d\mathcal{L}(T)(x).$$

For a collection of self-adjoint operators $T_1, \dots, T_n$, their joint distribution is described by specifying

$$\varphi(h_1(T_{i_1}) \dots h_k(T_{i_k})),$$

for all $k \geq 1$, all index sequences $i_1, \dots, i_k \in \{1, \dots, n\}$, and all bounded measurable functions $h_1, \dots, h_k : \mathbb{R} \to \mathbb{R}$.

Definition 4.2.3.
Let $(\mathcal{A}, \varphi)$ be a W^*-probability space, and suppose $a_1, a_2 \in \mathcal{A}$. Then a_1 and a_2 are said to be freely independent if

$$\varphi(p_1(a_{i_1}) \dots p_n(a_{i_n})) = 0,$$

for every $n \geq 1$, every sequence $i_1, \dots, i_n \in \{1, 2\}$ with $i_j \neq i_{j+1}$ for all $j = 1, \dots, n-1$, and all polynomials $p_1, \dots, p_n$ in one variable satisfying

$$\varphi(p_j(a_{i_j})) = 0, \quad \text{for all } j = 1, \dots, n.$$

Definition 4.2.4.
Let $a_1, \dots, a_k$ and $b_1, \dots, b_m$ be operators affiliated with $\mathcal{A}$. The families $(a_1, \dots, a_k)$ and $(b_1, \dots, b_m)$ are freely independent if and only if

$$p(h_1(a_1) \dots h_k(a_k)) \quad \text{and} \quad q(g_1(b_1) \dots g_m(b_m))$$

are freely independent for all bounded measurable functions $h_1, \dots, h_k$ and $g_1, \dots, g_m$, and for all polynomials p and q in k and m non-commutative variables, respectively.

We are now ready to state our second main result.

Theorem 4.2.5.
Under the assumptions of Theorem 4.2.1, the limiting measure ν_τ can be identified as

$$\nu_\tau = \mathcal{L}\left(T_W^{1/2} T_s T_W^{1/2} + \mathbf{E}[W] T_W^{1/4} T_g T_W^{1/4}\right).$$

Here, T_g and T_W are commuting self-adjoint operators affiliated with a W^*-probability space $(\mathcal{A}, \varphi)$ such that, for bounded measurable functions h_1, h_2 from $\mathbb{R}$ to itself,

$$\varphi\left(h_1\left(T_g\right) h_2\left(T_W\right)\right) = \left(\int_{-\infty}^{\infty} h_1(x)\phi(x)dx\right)\left(\int_1^{\infty} h_2(u)(\tau-1)u^{-\tau}du\right)$$

with ϕ the standard normal density. Furthermore, T_s has a standard semicircle law and is freely independent of (T_g, T_W).

In particular, when W is degenerate, say $W \equiv 1$, then ν_τ is given by the free additive convolution of semicircle and Gaussian law.

§4.2.3 Discussion and simulations

(a) We now briefly describe the main steps of the proof.

1. Gaussianisation: In the first step, we show that replacing the Bernoulli entries with Gaussian entries having the same mean and variance results in empirical spectral distributions that are close.
2. Simplification of the variance profile: In this step, we show that the variance profile can be simplified to $W_i W_j / \|i-j\|^\alpha$, effectively removing the truncation at 1.
3. Truncation: Here, we show that in the Gaussian matrix, the weights W_i can be replaced by the truncated weights $W_i^m = W_i \mathbf{1}_{W_i \le m}$.
4. Decoupling the diagonal: In this step, we show that the Laplacian can be viewed as the sum of two independent random matrices (conditionally on the weights). Thus, we replace the diagonal matrix D_N with an independent copy Y_N, which has the same variance profile.
5. Moment method: With truncated weights and decoupled matrices, we apply the moment method to show convergence of the empirical spectral distribution and identify the limiting moments. A key observation is that the limiting measure and the method are independent of α, so the results remain valid even when $\alpha = 0$.

6. Identification of the limiting measure: Finally, we first identify the limiting measure in the case of truncated weights. These are typically associated with bounded operators (except in the Gaussian case). We then use techniques from Bercovici and Voiculescu [1993] to remove the truncation and identify the limiting measure in the general case.

(b) We now present some simulations that illustrate how the proof outline aligns with a specific value of α. In Figure 4.1, we plot the eigenvalue distribution of the centred Laplacian matrix, with the parameter range $N = 6000$, $\alpha = 0.5$ and $\tau = 4.1$. A crucial step in the proof of Theorem 4.2.1 requires us to replace the Bernoulli entries with Gaussian entries with the same variance profile. Also in the Gaussian case, we can simplify the variance to the following form:

$$\frac{W_i W_j}{\|i - j\|^\alpha}$$

for any $(i, j)^{\text{th}}$ entry. We compare the two spectra in Figure 4.2. We also consider the Gaussianised Laplacian matrix with a decoupled diagonal, and in Section 4.5, we apply an idea used in Cipriani et al. [2025], where we take $\alpha = 0$. We also compare the spectrum of this matrix to the original centred Laplacian in Figure 4.2. We see that the spectra are quite similar.

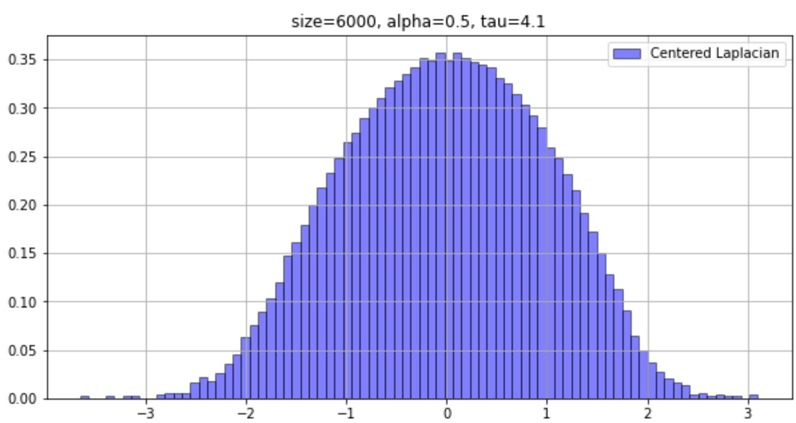

Figure 4.1: Spectrum of the centred Laplacian matrix .

(c) We remark that our results can be extended in two directions. Although we state and prove them for the case $d = 1$ and $\alpha < 1$, they naturally generalise to any dimension $d \geq 1$ and $\alpha < d$. In that case, the scaling

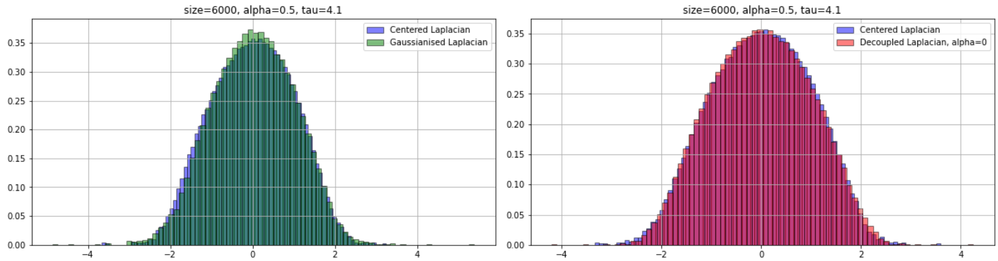

Figure 4.2: Comparing the spectrum of the centred Laplacian with the Gaussianised and the decoupled case.

constant requires an adjustment, with $c_N \sim c_0(d)\,N^{d-\alpha}$. For ease of presentation, we restrict ourselves to $d = 1$ in this work.

Another possible extension of our first result involves modifying the connection probabilities between vertices i and j to

$$p_{ij} = \frac{\kappa_\sigma(W_i, W_j)}{\|i-j\|^\alpha} \wedge 1,$$

where $\kappa_\sigma(x,y) = (x \vee y)(x \wedge y)^\sigma$. In this setting, we additionally assume $0 < \sigma < (\tau - 1)$. Such extensions have been studied in the context of adjacency matrices in Cipriani et al. [2025]. We strongly believe that in this case the limiting spectral distribution will exist, but it would be challenging to identify the limiting measure.

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§4.2.4 Notation

We will use the Landau notation o_N, O_N indicating in the subscript the variable under which we take the asymptotic (typically this variable will grow to infinity unless otherwise specified). Universal positive constants are denoted as c, c_1, ..., and their value may change with each occurrence. For an $N \times N$ matrix $\mathbf{A} = (a_{ij})_{i,j=1}^N$ we use $\mathrm{Tr}(\mathbf{A}) := \sum_{i=1}^N a_{ii}$ for the trace and $\mathrm{tr}(\mathbf{A}) := N^{-1}\,\mathrm{Tr}(\mathbf{A})$ for the normalised trace. When $n \in \mathbb{N}$ we write $[n] := \{1, 2, \ldots, n\}$. We denote the cardinality of a set A as $\#A$, and, with a slight abuse of notation, $\#\sigma$ also denotes the number of cycles in a permutation σ.

§4.3 Gaussianisation and setup for main proofs

To prove Theorem 4.2.1, we construct a Laplacian matrix with truncated weights and a simplified variance profile, with the diagonal decoupled from the adjacency matrix. We follow the ideas of Cipriani et al. [2025], albeit with a slightly modified approach, as follows:

(a) We begin by Gaussianising the matrix $\mathbf{\Delta}_N^o$ to obtain a matrix $\bar{\mathbf{\Delta}}_N$, using the ideas of Chatterjee [2005]. Since we have $\tau > 3$, the proof proceeds without the need to truncate the weight sequence $\{W_i\}_{i\in\mathbf{V}_N}$.

(b) We then tweak the entries of $\bar{\mathbf{\Delta}}_N$ further through a series of lemmas to obtain the Laplacian matrix $\widehat{\mathbf{\Delta}}_{N,g}$, whose corresponding adjacency has mean-zero Gaussian entries and a simplified variance profile.

(c) Next, we truncate the weights $\{W_i\}_{i\in\mathbf{V}_N}$ at $m \geq 1$, and construct the corresponding matrix $\mathbf{\Delta}_{N,g,m}$. We show that, in $\mathbb{P}$-probability, the Lévy distance vanishes in the iterated limit $m \to \infty$ and $N \to \infty$.

(d) We conclude by decoupling the diagonal of the matrix $\mathbf{\Delta}_{N,g,m}$ from the off-diagonal terms. This follows from classical results used in studying the spectrum of Laplacian matrices.

§4.3.1 Gaussianisation

Suppose $(G_{i,j})_{i>j}$ is a sequence of i.i.d. N$(0,1)$ random variables and independent of the sequence $(W_i)_{i\in\mathbf{V}_N}$. Define

$$\bar{\mathbf{A}}_N = \begin{cases} \frac{\sqrt{p_{ij}(1-p_{ij})}}{\sqrt{c_N}} G_{i\wedge j, i\vee j} + \frac{\mu_{ij}}{\sqrt{c_N}} & i \neq j \\ 0 & i = j, \end{cases}$$

where $\mu_{ij} = p_{ij} - \mathbb{E}[p_{ij}]$. Let $\bar{\mathbf{\Delta}}_N$ be the corresponding Laplacian of the matrix $\bar{\mathbf{A}}_N$. Let h be a 3 times differentiable function on $\mathbb{R}$ such that

$$\max_{0\leq k\leq 3} \sup_{x\in\mathbb{R}} |h^{(k)}(x)| < \infty\,,$$

where $h^{(k)}$ is the k-th derivative of h. Define the resolvent of the $N \times N$ matrix $\mathbf{M}_N$ as

$$R_{M_N}(z) = (\mathbf{M}_N - z\,\mathrm{I}_N)^{-1}\,, \qquad z \in \mathbb{C}^+,$$

where I_N is the $N \times N$ identity matrix and $\mathbb{C}^+$ is the upper-half complex plane. Further, define $H_z(\mathbf{M}_N) = \mathrm{S}_{M_N}(z) = \mathrm{tr}(R_{M_N}(z))$ for $z \in \mathbb{C}^+$.

Lemma 4.3.1 (Gaussianisation of $\mathbf{\Delta}_N$).
Consider $\bar{\mathbf{\Delta}}_N$ and $\mathbf{\Delta}_N^o$ defined as above. Then for any h as above,

$$\lim_{N\to\infty} \left|\mathbb{E}[h(\Re H_z(\bar{\mathbf{\Delta}}_N))] - \mathbb{E}[h(\Re H_z(\mathbf{\Delta}_N^o))]\right| = 0\,,$$

and

$$\lim_{N\to\infty} \left|\mathbb{E}[h(\Im H_z(\bar{\mathbf{\Delta}}_N))] - \mathbb{E}[h(\Im H_z(\mathbf{\Delta}_N^o))]\right| = 0\,.$$

The proof is very similar to the one presented in Chatterjee and Hazra [2022] and is modified along the lines of Cipriani et al. [2025]. It uses the classical result of Chatterjee [2005], and we only give a brief sketch by showing the estimates of the error probabilities in this setting. In Cipriani et al. [2025], the Gaussianisation was done with truncated weights, but here we will not need that.

Proof. Following the proof of Cipriani et al. [2025] for the Laplacian, we define, conditional on the weights $(W_i)_{i\in\mathbf{V}_N}$, a sequence of independent random variables. Let $\mathbf{X}_b = (X_{ij}^b)_{1\le i<j\le N}$ be a vector with $X_{ij}^b \sim \mathrm{Ber}(p_{ij}) - \mathbb{E}[p_{ij}]$. Similarly, take another vector $\mathbf{X}_g = (X_{ij}^g)_{1\le i<j\le N}$ with $X_{ij}^g \sim \mathrm{N}\left(\mu_{ij}\,, p_{ij}(1-p_{ij})\right)$.

Let $n = N(N-1)/2$ and $\mathbf{x} = (x_{ij})_{1\le i<j\le N} \in \mathbb{R}^n$. Define $\mathrm{R}(\mathbf{x})$ to be the matrix-valued differentiable function given by

$$\mathrm{R}(\mathbf{x}) \coloneqq (\mathbf{M}_N(\mathbf{x}) - z\,\mathrm{I}_N)^{-1},$$

where $\mathbf{M}_N(\cdot)$ is the matrix-valued differentiable function that maps a vector in $\mathbb{R}^n$ to the space of $N\times N$ Hermitian matrices, given by

$$\mathbf{M}_N(\mathbf{x})_{ij} = \begin{cases} c_N^{-1/2} x_{ij} & \text{if } i<j,\\ c_N^{-1/2} x_{ji} & \text{if } i>j,\\ -c_N^{-1/2}\sum_{k\neq i} x_{ik} & \text{if } i=j. \end{cases}$$

Then, we see that $\mathbf{\Delta}_N^o = \mathbf{M}_N(\mathbf{X}_b)$ and $\bar{\mathbf{\Delta}}_N = \mathbf{M}_N(\mathbf{X}_g)$. Note that

$$E^W[X_{ij}^b] = E^W[X_{ij}^g] = \mu_{ij},$$

and

$$E^W[(X_{ij}^b)^2] = E^W[(X_{ij}^g)^2] = p_{ij} + \mathbb{E}[p_{ij}]^2 - 2p_{ij}\mathbb{E}[p_{ij}].$$

Consequently, using [Chatterjee, 2005, Theorem 1.1] we have that

$$\begin{aligned} &\left|\mathbb{E}[h(\Re H_z(\bar{\mathbf{\Delta}}_N))] - \mathbb{E}[h(\Re H_z(\mathbf{\Delta}_N^o))]\right| \\ &= \left|\mathbf{E}\left[E^W[h(\Re H_z(\bar{\mathbf{\Delta}}_N)) - h(\Re H_z(\mathbf{\Delta}_N^o))]\right]\right| \\ &\le C_1(h)\lambda_2(H) \sum_{1\le i<j\le N} \mathbb{E}[(X_{ij}^b)^2 \mathbf{1}_{|X_{ij}^b|>K_N}] + \mathbb{E}[(X_{ij}^g)^2 \mathbf{1}_{|X_{ij}^g|>K_N}] \qquad (4.8) \\ &+ C_2(h)\lambda_3(H) \sum_{1\le i<j\le N} \mathbb{E}[(X_{ij}^b)^3 \mathbf{1}_{|X_{ij}^b|\le K_N}] + \mathbb{E}[(X_{ij}^g)^3 \mathbf{1}_{|X_{ij}^g|\le K_N}] \qquad (4.9) \end{aligned}$$

where $\lambda_2(H) \leq C_2(\Im z)\frac{1}{Nc_N}$ and $\lambda_3(H) \leq C_3(\Im z)\frac{1}{Nc_N^{3/2}}$.

We first deal with the terms in (4.8). Note that since $p_{ij} \leq 1$, we have $|X_{ij}^b| \leq 1$, and as a consequence, for any $K_N \geq 2$, the first term in (4.8) is zero. For the Gaussian term, applying the Cauchy-Schwarz inequality followed by the second-moment Markov inequality yields

$$\mathbb{E}[(X_{ij}^g)^2 \mathbf{1}_{|X_{ij}^g|>K_N}] \leq \mathbb{E}[(X_{ij}^g)^4]^{1/2}\mathbb{P}(X_{ij}^g > K_N)^{1/2} \leq K_N^{-1}\mathbb{E}[(X_{ij}^g)^4]^{1/2}\mathbb{E}[(X_{ij}^g)^2]^{1/2}.$$

Since $\mathbb{E}[(X_{ij}^g)^2] = \mathbf{E}[p_{ij}+\mathbb{E}[p_{ij}]^2-2p_{ij}\mathbb{E}[p_{ij}]] \leq \mathbf{E}[p_{ij}]$, and similarly, $\mathbb{E}[(X_{ij}^g)^4] \leq \mathbf{E}[p_{ij}^2]$, we have

$$\begin{aligned}
&\lambda_2(H) \sum_{1\leq i<j\leq N} \mathbb{E}[(X_{ij}^g)^2 \mathbf{1}_{|X_{ij}^g|>K_N}] \\
&\leq \frac{\lambda_2(H)}{K_N} \sum_{1\leq i<j\leq N} \frac{\mathbf{E}[W_i^2]^{1/2}\mathbf{E}[W_j^2]^{1/2}}{\|i-j\|^{\alpha}} \frac{\mathbf{E}[W_i]^{1/2}\mathbf{E}[W_j]^{1/2}}{\|i-j\|^{\alpha/2}} \\
&\leq \frac{\lambda_2(H)}{K_N}\mathbf{E}[W_1]\mathbf{E}[W_1^2]N^{2-\frac{3\alpha}{2}} \\
&\leq \frac{\tilde{c_2}\mathbf{E}[W_1]\mathbf{E}[W_1^2]N^{2-\frac{3\alpha}{2}}}{K_N N^{2-\alpha}} = \mathrm{O}_N(N^{-\alpha/2}K_N^{-1}),
\end{aligned}$$

where the last equality follows as $\tau > 3$ and $\tilde{c_2}$ is a constant depending on $\Im(z)$ only. For the term containing the third moments, we see that

$$\begin{aligned}
&\lambda_3(H) \sum_{1\leq i<j\leq N} \mathbb{E}[(X_{ij}^b)^3 \mathbf{1}_{|X_{ij}^b|\leq K_N}] + \mathbb{E}[(X_{ij}^g)^3 \mathbf{1}_{|X_{ij}^g|\leq K_N}] \\
&\leq \lambda_3(H)K_N \sum_{1\leq i\leq j\leq N} \mathbb{E}[(X_{ij}^b)^2] + \mathbb{E}[(X_{ij}^g)^2] \\
&\leq \lambda_3(H)K_N 2\mathbf{E}[W_1]^2 \sum_{1\leq i\leq j\leq N} \frac{1}{\|i-j\|^{\alpha}} \\
&\leq \frac{c_3(\Im z)}{Nc_N^{3/2}}K_N\mathbf{E}[W_1]^2 N c_N \leq \tilde{c}_3 K_N c_N^{-1/2}.
\end{aligned}$$

Here $\tilde{c}_3$ is a constant depending on $\Im(z)$. Choosing any $2 \leq K_N \ll c_N^{1/2}$, both terms go to zero. This completes the proof of the Gaussianisation. □

§4.3.2 Simplification of the variance profile

We now proceed with a series of lemmas to simplify the variance profile of our Gaussianised matrix. First, we construct a new matrix $\mathbf{\Delta}_{N,g}$ as the Laplacian corresponding to the matrix $\mathbf{A}_{N,g}$, defined as follows:

Suppose $(G_{i,j})_{i>j}$ is a sequence of i.i.d. $\mathrm{N}(0,1)$ random variables as before, and independent of the sequence $(W_i)_{i\in\mathbf{V}_N}$. Define

$$\mathbf{A}_{N,g} = \begin{cases} \frac{\sqrt{p_{ij}(1-p_{ij})}}{\sqrt{c_N}} G_{i\wedge j, i\vee j} & i \neq j \\ 0 & i = j. \end{cases}$$

We now have the following result.

Lemma 4.3.2.
Let $\bar{\mathbf{\Delta}}_N$ and $\mathbf{\Delta}_{N,g}$ be as defined above. Then,

$$\lim_{N\to\infty} \mathbb{P}(d_L(\mathrm{ESD}(\bar{\mathbf{\Delta}}_N), \mathrm{ESD}(\mathbf{\Delta}_{N,g})) > \varepsilon) = 0.$$

Proof. The proof follows using Proposition 4.6.1. Taking expectation on the d_L distance, we have

$$\begin{aligned}
\mathbb{E}\left[d_L^3(\mathrm{ESD}(\mathbf{\Delta}_{N,g}, \mathrm{ESD}(\bar{\mathbf{\Delta}}_N)\right] &\leq \frac{1}{N}\mathbb{E}\,\mathrm{Tr}\left((\mathbf{\Delta}_{N,g} - \bar{\mathbf{\Delta}}_N)^2\right) \\
&= \frac{1}{N}\mathbb{E}\left[\sum_{1\leq i,j\leq N} \left(\mathbf{\Delta}_{N,g}(i,j) - \bar{\mathbf{\Delta}}_N(i,j)\right)^2\right] \\
&= \frac{1}{N}\sum_{1\leq i\neq j\leq N} \mathbb{E}\left[(\mathbf{A}_{N,g}(i,j) - \bar{\mathbf{A}}_N(i,j))^2\right] \\
&\quad + \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left(\sum_{k\neq i} \mathbf{A}_{N,g}(i,k) - \bar{\mathbf{A}}_N(i,k)\right)^2\right].
\end{aligned}$$

We deal with the last two terms separately. The first term is bounded above by

$$\frac{1}{Nc_N}\sum_{i\neq j}\mathbb{E}[\mu_{ij}^2] \leq \frac{1}{Nc_N}\sum_{i\neq j}\frac{\mathbf{E}[W_1^2]^2}{\|i-j\|^{2\alpha}} \approx \frac{N^{2-2\alpha}}{N^{2-\alpha}} = N^{-\alpha} \to 0.$$

Next, we have that

$$\sum_{k\neq i} \mathbf{A}_{N,g}(i,k) - \bar{\mathbf{A}}_N(i,k) \leq \frac{1}{\sqrt{c_N}}\sum_{k\neq i} p_{ik} = c_N^{-1/2}.$$

This makes the second term of the order $\mathrm{o}_N(c_N)$. We conclude the proof using Markov's inequality. $\square$

Define for $i \neq j$

$$\tilde{\mathbf{A}}_{N,g}(i,j) = \frac{\sqrt{p_{ij}}}{\sqrt{c_N}} G_{i\wedge j, i\vee j}$$

and put zero on the diagonal. Here $(G_{i,j})_{i\geq j}$ are the i.i.d. $\mathrm{N}(0,1)$ random variables used in the previous result. Let $\tilde{\mathbf{\Delta}}_{N,g}$ be analogously defined. The next lemma shows that $\mathbf{\Delta}_{N,g}$ and $\tilde{\mathbf{\Delta}}_{N,g}$ have asymptotically the same spectrum.

Lemma 4.3.3.

$$\lim_{N\to\infty} \mathbb{P}\left(d_L(\mathrm{ESD}(\tilde{\boldsymbol{\Delta}}_{N,g}), \mathrm{ESD}(\boldsymbol{\Delta}_{N,g})) > \varepsilon\right) = 0.$$

Proof. Again using Proposition 4.6.1, we have that

$$\begin{aligned}
&\mathbb{E}\left[d_L^3(\mathrm{ESD}(\tilde{\boldsymbol{\Delta}}_{N,g}), \mathrm{ESD}(\boldsymbol{\Delta}_{N,g}))\right] \\
&\leq \frac{1}{N}\mathbb{E}\,\mathrm{Tr}\left((\boldsymbol{\Delta}_{N,g} - \tilde{\boldsymbol{\Delta}}_{N,g})^2\right) \\
&= \frac{1}{N}\mathbb{E}\left[\sum_{1\leq i,j\leq N}\left(\boldsymbol{\Delta}_{N,g}(i,j) - \tilde{\boldsymbol{\Delta}}_{N,g}(i,j)\right)^2\right] \\
&= \frac{1}{N}\sum_{1\leq i\neq j\leq N}\mathbb{E}\left[(\mathbf{A}_{N,g}(i,j) - \tilde{\mathbf{A}}_{N,g})^2\right] \\
&\qquad + \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left(\sum_{k\neq i}\mathbf{A}_{N,g}(i,k) - \tilde{\mathbf{A}}_{N,g}(i,k)\right)^2\right]
\end{aligned}$$

Dealing with the last two terms separately as before, we proceed by bounding the first term by

$$\frac{1}{Nc_N}\sum_{i\neq j}\frac{\mathbf{E}[W_1^2]^2}{\|i-j\|^{2\alpha}} \approx \frac{N^{2-2\alpha}}{N^{2-\alpha}} = N^{-\alpha} \to 0.$$

Expanding the square in the second term, we have

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left(\sum_{k\neq i}\mathbf{A}_{N,g}(i,k) - \tilde{\mathbf{A}}_{N,g}(i,k)\right)^2\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{k\neq i}\mathbb{E}\left[\left(\mathbf{A}_{N,g}(i,k) - \tilde{\mathbf{A}}_{N,g}(i,k)\right)^2\right] \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\sum_{k\neq i}\sum_{\ell\neq i,k}\mathbb{E}\left[\left(\mathbf{A}_{N,g}(i,k) - \tilde{\mathbf{A}}_{N,g}(i,k)\right)\left(\mathbf{A}_{N,g}(i,\ell) - \tilde{\mathbf{A}}_{N,g}(i,\ell)\right)\right].
\end{aligned}$$

Again, the first term in above sum is of the order $N^{-\alpha}$ and the expectation in the second term is zero. Indeed, using the independence between $(W_i)_{i\in\mathbf{V}_N}$ and $G_{i,j}$ we have for $k\neq\ell$,

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathbf{A}_{N,g}(i,k) - \tilde{\mathbf{A}}_{N,g}(i,k)\right)\left(\mathbf{A}_{N,g}(i,\ell) - \tilde{\mathbf{A}}_{N,g}(i,\ell)\right)\right] \\
&= \mathbb{E}\left[(\sqrt{p_{ik}(1-p_{ik})} - \sqrt{p_{ik}})(\sqrt{p_{i\ell}(1-p_{i\ell})} - \sqrt{p_{i\ell}})\right]\mathbb{E}[G_{i,k}G_{i,\ell}] = 0.
\end{aligned}$$

This completes the proof of the lemma. □

We conclude this subsection with one final simplification. For any $i \neq j$, let

$$r_{ij} = \frac{W_i W_j}{\|i-j\|^\alpha},$$

and let $r_{ii} = 0$. Define the matrix $\widehat{\mathbf{A}}_{N,g}$ as follows: for $i \neq j$,

$$\widehat{\mathbf{A}}_{N,g}(i,j) = \frac{\sqrt{r_{i\wedge j, i\vee j}}}{\sqrt{c_N}} G_{i\wedge j, i\vee j}$$

and put 0 on the diagonal. Define Laplacian matrix $\widehat{\boldsymbol{\Delta}}_{N,g}$ accordingly with $\widehat{\mathbf{A}}_{N,g}$.

Lemma 4.3.4.

$$\lim_{N\to\infty} \mathbb{P}\left(d_L(\mathrm{ESD}(\tilde{\boldsymbol{\Delta}}_{N,g}), \mathrm{ESD}(\widehat{\boldsymbol{\Delta}}_{N,g})) > \varepsilon\right) = 0.$$

Proof. For any $1 \leq i \neq j \leq N$, define the set $\mathcal{C}_{ij} = \{r_{ij} < 1\}$. Let $(X_{i,j})_{i\geq j}$ be defined as follows

$$X_{ij} = \frac{\sqrt{r_{ij}}}{\sqrt{c_N}} G'_{ij},$$

where $(G'_{ij})_{i\geq j}$ be a sequence of independent $\mathrm{N}(0,1)$ random variables, independent of the previously defined (G_{ij}) and $(W_i)_{i\in\mathbf{V}_N}$. Define a symmetric matrix $L_{N,g}$ as follows: for $1 \leq i < j \leq N$,

$$L_{N,g}(i,j) = \tilde{\mathbf{A}}_{N,g}(i,j)\mathbf{1}_{\mathcal{C}_{ij}} + X_{ij}\mathbf{1}_{\mathcal{C}_{ij}^c}.$$

We put zero on the diagonal and consider the $\boldsymbol{\Delta}_L$ as the Laplacian matrix corresponding to $L_{N,g}$. Note that $L_{N,g}$ has the same distribution as $\widehat{\mathbf{A}}_{N,g}$ and hence the $\boldsymbol{\Delta}_L$ has the same distribution as $\widehat{\boldsymbol{\Delta}}_{N,g}$.

By Proposition 4.6.1, we again have

$$\begin{aligned}
\mathbb{E}\left[d^3\left(\mathrm{ESD}(\boldsymbol{\Delta}_L), \mathrm{ESD}(\tilde{\boldsymbol{\Delta}}_{N,g})\right)\right] &\leq \frac{1}{N}\mathbb{E}\left[\sum_{1\leq i,j\leq N}\left(\boldsymbol{\Delta}_L(i,j) - \tilde{\boldsymbol{\Delta}}_{N,g}(i,j)\right)^2\right] \\
&= \frac{1}{N}\sum_{1\leq i\neq j\leq N}\mathbb{E}\left[(L_{N,g}(i,j) - \tilde{\mathbf{A}}_{N,g}(i,j))^2\right] \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\left(\sum_{k\neq i} L_{N,g}(i,k) - \tilde{\mathbf{A}}_{N,g}(i,k)\right)^2\right].
\end{aligned}$$

Expanding terms on the right-hand side, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[d^3\left(\mathrm{ESD}(\mathbf{\Delta}_L), \mathrm{ESD}(\tilde{\mathbf{\Delta}}_{N,g})\right)\right] \\
&\leq \frac{4}{N}\sum_{1\leq i\neq j\leq N}\mathbb{E}\left[(L_{N,g}(i,j)-\tilde{\mathbf{A}}_{N,g}(i,j))^2\right] \\
&\leq +\frac{1}{N}\sum_{i=1}^{N}\sum_{k\neq i}\sum_{\ell\neq i,k}\mathbb{E}\left[\left(L_{N,g}(i,k)-\tilde{\mathbf{A}}_{N,g}(i,k)\right)\left(L_{N,g}(i,\ell)-\tilde{\mathbf{A}}_{N,g}(i,\ell)\right)\right]
\end{aligned}
$$

Again, we deal with the two sums separately. The first sum can be bounded above as follows:

$$
\begin{aligned}
&\frac{4}{N}\sum_{1\leq i\neq j\leq N}\mathbb{E}\left[(L_{N,g}(i,j)-\tilde{\mathbf{A}}_{N,g}(i,j))^2\right] \\
&\leq \frac{4}{N}\sum_{1\leq i\neq j\leq N}\mathbb{E}\left[(\tilde{\mathbf{A}}_{N,g}(i,j)-X_{ij})^2\mathbf{1}_{\mathcal{C}_{ij}}\right] \\
&\leq \frac{8}{N}\sum_{1\leq i\neq j\leq N}\mathbb{E}\left[\tilde{\mathbf{A}}_{N,g}(i,j)^2\mathbf{1}_{\mathcal{C}_{ij}}\right]+\mathbb{E}\left[X_{ij}^2\mathbf{1}_{\mathcal{C}_{ij}}\right] \\
&\leq \frac{1}{Nc_N}\sum_{i\neq j\in\mathbf{V}_N}^{N}\mathbf{E}[G^2_{i\wedge j,i\vee j}\mathbf{1}_{\mathcal{C}^c_{ij}}]+\mathbf{E}[X^2_{ij}\mathbf{1}_{\mathcal{C}^c_{ij}}] \\
&\leq \frac{1}{Nc_N}\sum_{i\neq j\in\mathbf{V}_N}^{N}\mathbf{P}(\mathcal{C}^c_{ij})+\mathbf{E}[X^4_{ij}]^{1/2}\mathbf{P}(\mathcal{C}^c_{ij})^{1/2} \\
&\leq \frac{1}{Nc_N}\sum_{i\neq j\in\mathbf{V}_N}^{N}\mathbf{P}(\mathcal{C}^c_{ij})+\frac{3\mathbf{E}[W_i^2W_j^2]^{1/2}}{\|i-j\|^{\alpha}}\mathbf{P}(\mathcal{C}^c_{ij})^{1/2} \\
&\leq C(N^{-\alpha(\tau-2)}+N^{-\frac{\alpha}{2}(\tau-1)})=\mathrm{o}_N(1),
\end{aligned}
$$

where we have used in the last line the following estimate:

$$
\mathbf{P}(\mathcal{C}^c_{ij})\leq\mathbf{P}\left(W_iW_j\geq\|i-j\|^{\alpha}\right)\leq\frac{c}{\|i-j\|^{\alpha(\tau-1)}}
$$

which follows from Lemma 4.6.2. For the second term note that

$$
\begin{aligned}
&\mathbb{E}\left[\left(L_{N,g}(i,k)-\tilde{\mathbf{A}}_{N,g}(i,k)\right)\left(L_{N,g}(i,\ell)-\tilde{\mathbf{A}}_{N,g}(i,\ell)\right)\right] \\
&=\frac{1}{c_N}\mathbf{E}[\sqrt{p_{ik}}\sqrt{p_{i\ell}}\mathbf{1}_{\mathcal{C}^c_{ij}}\mathbf{1}_{\mathcal{C}^c_{i\ell}}]\mathbb{E}[G_{ik}G_{i\ell}] \\
&\quad-\frac{1}{c_N}\mathbf{E}[\sqrt{p_{ik}}\sqrt{r_{i\ell}}\mathbf{1}_{\mathcal{C}^c_{ij}}\mathbf{1}_{\mathcal{C}^c_{i\ell}}]\mathbb{E}[G_{ik}G'_{i\ell}]-\frac{1}{c_N}\mathbf{E}[\sqrt{r_{ik}}\sqrt{p_{i\ell}}\mathbf{1}_{C^c_{ij}}\mathbf{1}_{C^c_{i\ell}}]\mathbb{E}[G'_{ik}G_{i\ell}] \\
&\quad+\frac{1}{c_N}\mathbf{E}[\sqrt{r_{ik}}\sqrt{r_{i\ell}}\mathbf{1}_{C^c_{ij}}\mathbf{1}_{C^c_{i\ell}}]\mathbb{E}[G'_{ik}G'_{i\ell}],
\end{aligned}
$$

and since $k \neq \ell$, all the above terms are zero. Thus the proof follows. □

§4.3.3 Truncation

Let $m > 1$ be a truncation threshold and define $W_i^m = W_i \mathbf{1}_{W_i \le m}$ for any $i \in \mathbf{V}_N$. For all $N \in \mathbb{N}$, we define a new random matrix as follows: Let

$$r_{ij}^m = \frac{W_i^m W_j^m}{\|i-j\|^\alpha} \qquad i \neq j \in \mathbf{V}_N\,,$$

and let $\mathbf{A}_{N,g,m}$ be defined for $i \neq j$ as

$$\mathbf{A}_{N,g,m}(i,j) = \frac{\sqrt{r_{ij}^m}}{\sqrt{c_N}} G_{i\wedge j, i\vee j},$$

and put 0 on the diagonal. Analogously define $\mathbf{\Delta}_{N,g,m}$.

Lemma 4.3.5 (Truncation).
For every $\delta > 0$ one has

$$\limsup_{m\to\infty} \lim_{N\to\infty} \mathbb{P}\left(d_L(\mathrm{ESD}(\mathbf{\Delta}_{N,g,m}), \mathrm{ESD}(\tilde{\mathbf{\Delta}}_{N,g})) > \delta\right) = 0\,.$$

Proof. The proof follows the same idea as the previous lemmas. Recall that

$$\widehat{\mathbf{A}}_{N,g}(i,j) = \frac{\sqrt{r_{ij}}}{\sqrt{c_N}} G_{i\wedge j, i\vee j}$$

for all $i \neq j$, with 0 on the diagonal, and $\widehat{\mathbf{\Delta}}_{N,g}$ is the corresponding Laplacian. Once again, we have

$$\begin{aligned}
&\mathbb{E}\left[d^3\left(\mathrm{ESD}(\mathbf{\Delta}_{N,g,m}), \mathrm{ESD}(\widehat{\mathbf{\Delta}}_{N,g})\right)\right] \\
&\le \frac{1}{N}\mathbb{E}\left[\sum_{1\le i,j\le N}\left(\mathbf{\Delta}_{N,g,m}(i,j) - \widehat{\mathbf{\Delta}}_{N,g}(i,j)\right)^2\right] \\
&= \frac{1}{N}\sum_{1\le i\neq j\le N}\mathbb{E}\left[(\mathbf{A}_{N,g,m}(i,j) - \widehat{\mathbf{A}}_{N,g}(i,j))^2\right] \\
&\qquad + \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left(\sum_{k\neq i}\mathbf{A}_{N,g,m}(i,k) - \widehat{\mathbf{A}}_{N,g}(i,k)\right)^2\right] \\
&\le \frac{4}{N}\sum_{1\le i\neq j\le N}\mathbb{E}\left[(\mathbf{A}_{N,g,m}(i,j) - \widehat{\mathbf{A}}_{N,g}(i,j))^2\right] \\
&+ \frac{1}{N}\sum_{i=1}^N\sum_{k\neq i}\sum_{\ell\neq i,k}\mathbb{E}\left[\left(\mathbf{A}_{N,g,m}(i,k) - \widehat{\mathbf{A}}_{N,g}(i,k)\right)\left(\mathbf{A}_{N,g,m}(i,\ell) - \widehat{\mathbf{A}}_{N,g}(i,\ell)\right)\right].
\end{aligned}$$

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The proof of Lemma 4.3.4 aids us by taking care of the second factor in the last line, which turns out to be equal to 0 by the independence of Gaussian terms. For the first term, the common Gaussian factor pulls out by independence, yielding the upper bound

$$
\begin{aligned}
&\frac{4}{Nc_N}\sum_{1\le i\neq j\le N}\frac{\mathbf{E}\left[\left(\sqrt{W_iW_j}-\sqrt{W_i^mW_j^m}\right)^2\right]}{\|i-j\|^\alpha}\\
&\le \frac{4}{Nc_N}\sum_{1\le i\neq j\le N}\frac{\mathbf{E}[W_iW_j-W_i^mW_j^m]}{\|i-j\|^\alpha},
\end{aligned}
$$

where the inequality follows by using the identity $(a-b)^2\le|a^2-b^2|$ for any $a,b\ge 1$. Adding and subtracting the term $W_iW_j^m$ inside the expectation gives us that

$$
\begin{aligned}
&\frac{4}{N}\sum_{1\le i\neq j\le N}\mathbb{E}\left[(\mathbf{A}_{N,g,m}(i,j)-\widehat{\mathbf{A}}_{N,g}(i,j))^2\right]\\
&\le \frac{4}{Nc_N}\sum_{1\le i\neq j\le N}\frac{\mathbf{E}[W_i]\mathbf{E}[W_j\mathbf{1}_{\{W_j>m\}}]+\mathbf{E}[W_j^m]\mathbf{E}[W_i\mathbf{1}_{\{W_i>m\}}]}{\|i-j\|^\alpha}\\
&\le \frac{C_\tau}{Nc_N}\sum_{1\le i\neq j\le N}\frac{m^{2-\tau}}{\|i-j\|^\alpha}=\mathrm{O}_m(m^{2-\tau}),
\end{aligned}
$$

where the last inequality follows from Lemma 4.6.3, with C_τ a $\tau-$dependent constant. Markov inequality concludes the proof. □

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§4.3.4 Decoupling

Since we now have bounded weights, the decoupling result follows from the arguments from [Bryc et al., 2006, Lemma 4.12]. See also the proof of [Chakrabarty et al., 2021b, Lemma 4.2] for the inhomogeneous extension.

Lemma 4.3.6.
Let $(Z_i : i\ge 1)$ be a family of i.i.d. standard normal random variables, independent of $(G_{i,j} : 1\le i\le j)$. Define a diagonal matrix Y_N of order N by

$$Y_N(i,i)=Z_i\sqrt{\frac{\sum_{k\neq i}r_{ik}^m}{c_N}},\quad 1\le i\le N.$$

and let

$$\boldsymbol{\Delta}_{N,g,c}=\mathbf{A}_{N,g,m}+Y_N. \tag{4.10}$$

Then for every $m > 1$, and for any $k \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\left[(\mathbf{\Delta}_{N,g,c})^{2k} - (\mathbf{\Delta}_{N,g,m})^{2k}\right]\right) = 0.$$

and

$$\lim_{N\to\infty} \frac{1}{N^2}\mathbb{E}\left(\mathrm{Tr}^2\left[(\mathbf{\Delta}_{N,g,c})^{k}\right] - \mathrm{Tr}^2\left[(\mathbf{\Delta}_{N,g,m})^{k}\right]\right) = 0.$$

§4.4 Moment method: Existence and uniqueness of the limit

We begin by stating a key proposition that describes the limit of the empirical spectral distribution of $\mathbf{\Delta}_{N,g,c}$. The majority of this section will be devoted to the proof of this proposition, and so, we defer the proof of the proposition to page 180.

Proposition 4.4.1.
Let $\mathrm{ESD}(\mathbf{\Delta}_{N,g,c})$ *be the empirical spectral distribution of* $\mathbf{\Delta}_{N,g,c}$ *defined in* (4.10). *Then there exists a deterministic measure* ν_τ *on* $\mathbb{R}$ *such that*

$$\lim_{N\to\infty} \mathrm{ESD}(\mathbf{\Delta}_{N,g,c}) = \nu_{\tau,m} \qquad \text{in } \mathbb{P}\text{–probability}.$$

We now use Proposition 4.4.1 and tools from Appendix 4.6 and Section 4.3 to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Combining Proposition 4.4.1 with Lemma 4.3.6 gives us that

$$\lim_{N\to\infty} \mathrm{ESD}(\mathbf{\Delta}_{N,g,m}) = \nu_{\tau,m} \qquad \text{in } \mathbb{P}\text{–probability}. \tag{4.11}$$

To show the existence of the limit $\nu_\tau := \lim_{m\to\infty} \nu_{\tau,m}$, we wish to apply Lemma 4.6.5. Equation (4.11) satisfies Condition (1) of Lemma 4.6.5. Moreover, Condition (2) can be easily verified by Lemma 4.3.5. Thus, there exists a unique limit ν_τ such that

$$\lim_{N\to\infty} \mathrm{ESD}(\tilde{\mathbf{\Delta}}_{N,g}) = \nu_\tau \qquad \text{in } \mathbb{P}\text{–probability}. \tag{4.12}$$

Combining equation (4.12) with Lemma 4.3.4, and subsequently with Lemma 4.3.3 and Lemma 4.3.2 yields

$$\lim_{N\to\infty} \mathrm{ESD}(\bar{\mathbf{\Delta}}_{N}) = \nu_\tau \qquad \text{in } \mathbb{P}\text{–probability}. \tag{4.13}$$

We now wish to show that the limiting empirical spectral distribution for $\mathbf{\Delta}_N^\circ$ is ν_τ in $\mathbb{P}$–probability. To this end, note that for any h satisfying conditions of

Chapter Four

Lemma 4.3.1, and H_z as in subsection 4.3.1, we have by the means of Lemma 4.3.1 that

$$\lim_{N\to\infty} h\left(\Re(H_z(\mathbf{\Delta}_N^\circ))\right) = h\left(\Re\, \mathrm{S}_{\nu_\tau}(z)\right).$$

The above characterises convergence in law. However, since ν_τ is a deterministic measure, the above convergence holds in $\mathbb{P}$–probability, and analogously for $\Im(H_z(\mathbf{\Delta}_N^\circ))$. This gives us that

$$\lim_{N\to\infty} \mathrm{S}_{\mathrm{ESD}(\mathbf{\Delta}_N^\circ)}(z) = \mathrm{S}_{\nu_\tau}(z) \qquad \text{in } \mathbb{P}\text{–probability}.$$

Since convergence of Stieltjes transforms characterises weak convergence, we obtain

$$\lim_{N\to\infty} \mathrm{ESD}(\mathbf{\Delta}_N^\circ) = \nu_\tau \qquad \text{in } \mathbb{P}\text{–probability},$$

completing the proof. □

We now provide the proof of Proposition 4.4.1. We borrow the main ideas of Chatterjee and Hazra [2022, Section 5.2.1, 5.2.2], and adapt them to our setting using the results of Cipriani et al. [2025, Section 4.4].

Proof of Proposition 4.4.1. The proof of the moment method is valid when the weights are bounded, and so for notational convenience, in this proof we will drop the dependence on m from $\{r_{ij}^m\}_{i,j\in\mathbf{V}_N}$. Thus, for the remainder of the proof, we have that

$$r_{ij} = \frac{W_i^m W_j^m}{\|i-j\|^\alpha}.$$

We apply the method of moments to show the convergence to the law $\nu_{\tau,m}$. The proof is split up into three parts as follows:

(a) For any $k \geq 1$, we compute the expected moment

$$\mathbb{E}\int_1^\infty x^k\, \mathrm{ESD}(\mathbf{\Delta}_{N,g,c})(\mathrm{d}\, x),$$

and show that as $N \to \infty$, the above quantity converges to a value $0 < M_k < \infty$ for k even, and 0 otherwise.

(b) We then show concentration by proving (under the law $\mathbb{P}$) that

$$\mathrm{Var}\left(\int_1^\infty x^k\, \mathrm{ESD}(\mathbf{\Delta}_{N,g,c})(\mathrm{d}\, x)\right) \to 0 \quad \text{as } N \to \infty.$$

(c) Lastly, we show that the sequence $\{M_k\}_{k\geq 1}$ uniquely determines a limiting measure.

Step 1. We begin by considering that k is even. By using the expansion for $(a+b)^k$, it is easy to see that

$$\begin{aligned}\mathbb{E}\int_1^\infty x^k \,\mathrm{ESD}(\boldsymbol{\Delta}_{N,g,c})(\mathrm{d}\,x) &= \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\left(\boldsymbol{\Delta}_{N,g,c}^k\right)\right] \\ &= \frac{1}{N}\sum_{\substack{m_1,\dots,m_k,\\ n_1,\dots,n_k}} \mathbb{E}\left[\mathrm{Tr}\left(\mathbf{A}_{N,g,m}^{m_1} Y_N^{n_1}\dots \mathbf{A}_{N,g,m}^{m_k} Y_N^{n_k}\right)\right],\end{aligned}$$

where $\mathbf{A}_{N,g,m}$ and Y_N are as in Lemma 4.3.6, and $\{m_i, n_i\}_{1\le i\le k}$ are such that $\sum_{i=1}^k m_i + n_i = k$.

Let $M(p)$ and $N(p)$ be defined as

$$M(p) = \sum_{i=1}^p m_i, \quad N(p) = \sum_{i=1}^p n_i$$

for any $1 \le p \le k$. To expand the trace term, we sum over all $\mathbf{i} = (i_1, \dots, i_{M(k)+N(k)+1}) \in [N]^{M(k)+N(k)+1}$, where $[p] := \{1, 2, \dots, p\}$, and we identify $i_{M(k)+N(k)+1} \equiv i_1$. Then, from Chatterjee and Hazra [2022, Eq. 5.2.2], we have

$$\begin{aligned}&\frac{1}{N}\mathrm{Tr}\left(\mathbf{A}_{N,g,m}^{m_1} Y_N^{n_1}\dots \mathbf{A}_{N,g,m}^{m_k} Y_N^{n_k}\right) \\ &= \frac{1}{N}\sum_{i_1,\dots,i_{M(k)}=1}^{N} \prod_{j=1}^{M(k)} G_{i_j\wedge i_{j+1}, i_j\vee i_{j+1}} \prod_{j=1}^{M(k)} \frac{\sqrt{r_{i_j i_{j+1}}}}{\sqrt{c_N}} \prod_{j=1}^{k}\left(\frac{1}{c_N}\sum_{t=1}^N r_{i_{1+M(j)}t}\right)^{\frac{n_j}{2}} \prod_{j=1}^k Z_{i_{1+M(j)}}^{n_j},\end{aligned} \tag{4.14}$$

where also in (4.14) we identify $i_{M(k)+1} \equiv i_1$. Taking expectation in (4.14), we have that

$$\begin{aligned}&\mathbb{E}\left[\mathrm{tr}\left(\mathbf{A}_{N,g,m}^{m_1} Y_N^{n_1}\dots \mathbf{A}_{N,g,m}^{m_k} Y_N^{n_k}\right)\right] \\ &= \frac{1}{N}\sum_{i_1,\dots,i_{M(k)}} \mathbb{E}\left[\prod_{j=1}^{M(k)} G_{i_j\wedge i_{j+1}, i_j\vee i_{j+1}}\right] \\ &\qquad \times \mathbb{E}\left[\prod_{j=1}^{M(k)} \frac{\sqrt{r_{i_j i_{j+1}}}}{\sqrt{c_N}} \prod_{j=1}^{k}\left(\frac{1}{c_N}\sum_{t=1}^N r_{i_{1+M(j)}t}\right)^{\frac{n_j}{2}}\right] \mathbb{E}\left[\prod_{j=1}^k Z_{i_{1+M(j)}}^{n_j}\right].\end{aligned} \tag{4.15}$$

It is well known that the expectation over a product of independent Gaussian random variables is simplified using the Wick's formula (see Lemma 4.6.7). In particular, if one were to partition the tuple $\{1, \dots, K\}$ for some non-negative integer K, the contributing partitions are typically non-crossing pair partitions (Nica and Speicher [2006]).

We now introduce some notation from Cipriani et al. [2025]. For any fixed non-negative even integer K, let $\mathcal{P}_2(K)$ and $NC_2(K)$ be the set of all pair partitions and the set of all non-crossing pair partitions of $[K]$, respectively. Let $\gamma = (1, \dots, K) \in S_K$ be the right-shift permutation (modulo K), and for any π which is a pair-partition, we identify it as a permutation of $[K]$, and read $\gamma\pi$ as a composition of permutations. Further, for any $\pi \in \mathcal{P}_2(K)$, let Cat_π denote the set

$$\mathrm{Cat}_\pi := \mathrm{Cat}_\pi(K, N) = \{\mathbf{i} \in [N]^K : i_r = i_{\gamma\pi(r)} \text{ for all } r \in [K]\}.$$

Let $C(K, N) = \mathrm{Cat}_\pi^c$, the complement of Cat_π, wherein we have $i_r = i_{\pi(r)}$ for any r. By Wick's formula for the Gaussian terms $\{G_{i,j}\}$, since the the sum over tuples $\mathbf{i}$ would be reduced to the sum over pair partitions $\pi \in \mathcal{P}_2(K)$ and the associated tuples $\mathbf{i} \in \mathrm{Cat}_\pi \cup C(K, N)$, we can write

$$\sum_{\mathbf{i}\in[K]^N} = \sum_{\pi\in\mathcal{P}_2(K)} \sum_{\mathbf{i}\in C(K,N)} + \sum_{\pi\in NC_2(K)} \sum_{\mathbf{i}\in\mathrm{Cat}_\pi} + \sum_{\pi\in\mathcal{P}_2(K)\setminus NC_2(K)} \sum_{\mathbf{i}\in\mathrm{Cat}_\pi} . \tag{4.16}$$

To analyse further, we use a key tool in the proof which is the following fact (Cipriani et al. [2025, Claim 4.10]).

Fact 4.4.2.
Let K be an even non-negative integer. Then, we have the following to be true:

(a) *For any $\pi \in NC_2(K)$, we have*

$$\lim_{N\to\infty} \frac{1}{N c_N^{K/2}} \sum_{\mathbf{i}\in\mathrm{Cat}_\pi} \prod_{(r,s)\in\pi} \frac{1}{\|i_r - i_{r+1}\|^\alpha} = 1 .$$

(b) *For any pair partition π, if $\mathbf{i} \in C(K, N)$, then,*

$$\lim_{N\to\infty} \frac{1}{N c_N^{K/2}} \sum_{\mathbf{i}\in C(K,N)} \prod_{(r,s)\in\pi} \frac{1}{\|i_r - i_{r+1}\|^\alpha} = 0 .$$

(c) *For a partition $\pi \in \mathcal{P}_2(K) \setminus NC_2(K)$, we have*

$$\lim_{N\to\infty} \frac{1}{N c_N^{K/2}} \sum_{\mathbf{i}\in\mathrm{Cat}_\pi\cup C(K,N)} \prod_{(r,s)\in\pi} \frac{1}{\|i_r - i_{r+1}\|^\alpha} = 0 .$$

Let $\tilde{\pi} := \gamma\pi$ for any choice of π. From Chatterjee [2005, Eq. 5.2.5], we have that

$$\mathcal{E}(\tilde{\pi}) := \mathbb{E}\left[\prod_{j=1}^{k} Z_{i_{1+M(j)}}^{n_j}\right] = \prod_{u\in\tilde{\pi}} \mathbb{E}\left[\prod_{\substack{j\in[k]:\\ 1+M(j)\in u}} Z_{\ell_u}^{n_j}\right] < \infty, \tag{4.17}$$

where u is a block in $\tilde{\pi}$ and ℓ_u its representative element. Note that this does not depend on the choice of $\mathbf{i}$, and to obtain a non-zero contribution, we must have that for all $u \in \tilde{\pi}$,

$$\sum_{j\in[k]:1+M(j)\in u} n_j \equiv 0 \pmod 2. \tag{4.18}$$

Observe that $\mathbb{E}\left[\prod_{j=1}^{M(k)} G_{i_j \wedge i_{j+1}, i_j \vee i_{j+1}}\right]$ depends only on $\tilde{\pi}$ and not the choice of $\mathbf{i}$, and as a consequence, we can define

$$\Phi(\tilde{\pi}) := \mathbb{E}\left[\prod_{j=1}^{M(k)} G_{i_j \wedge i_{j+1}, i_j \vee i_{j+1}}\right] < \infty\,. \tag{4.19}$$

Next, note that the sum

$$\frac{1}{c_N}\sum_{t=1}^{N} r_{i_{1+M(j)}t} = \mathrm{O}_N(1) \tag{4.20}$$

by definition of c_N (and the weights are uniformly bounded). Finally, if we look at the terms

$$\mathbb{E}\left[\left(\prod_{j=1}^{M(k)} \frac{r_{i_j i_{j+1}}}{c_N}\right)^{1/2}\right], \tag{4.21}$$

we can again bound the weights above by m. Recall that Wick's formula on the Gaussian terms imposes the restriction on choices of $\mathbf{i}$. Using these facts, in combination with (4.17), (4.19), and (4.20), we have that (4.15) gets bounded by

$$(4.15) \leq \frac{C}{N c_N} \sum_{\pi \in \mathcal{P}_2(M(k))} \sum_{\mathbf{i} \in \mathrm{Cat}_\pi \cup C(M(k),N)} \Phi(\tilde{\pi})\mathcal{E}(\tilde{\pi}) \prod_{(r,s)\in\pi} \frac{m^2}{\|i_r - i_{r+1}\|^\alpha}\,. \tag{4.22}$$

If we split (4.22) as (4.16), then using Fact 4.4.2, we see that in the cases when $\pi \in \mathcal{P}_2(M(k))$ and $\mathbf{i} \in C(M(k), N)$, and when $\pi \in \mathcal{P}_2(M(k)) \setminus NC_2(M(k))$ for all $\mathbf{i}$, the contribution in the limit $N \to \infty$ is 0.

We are now in the setting where we take $\pi \in NC_2(M(k))$ and $\tilde{\pi} := \gamma\pi$, and $\mathbf{i} \in \mathrm{Cat}_\pi$. First, note that $\tilde{\pi}$ is a partition of $[M(k)]$. We remark that if $M(k) \equiv 1 \pmod 2$ then $NC_2(M(k)) = \emptyset$, and so, $M(k)$ must be even.

Next, we focus on analysing the product $\prod_{j=1}^{M(k)} \sqrt{r_{i_j i_{j+1}}^m}$ appearing in (4.15). We wish to show that this depends only on π, and not on the choice of $\mathbf{i}$. We follow the idea of Cipriani et al. [2025], wherein one constructs a graph associated to a chosen partition π, and any tuple $\mathbf{i} \in \mathrm{Cat}_\pi$ is equivalent to a tuple $\tilde{\mathbf{i}}$ with as many distinct indices as the number of vertices in the constructed graph. First, note that the coordinates are pairwise distinct (we take $r_{ii} = 0$ for all i). Next, we construct a preliminary graph from the closed walk $i_1 \to i_2 \to \dots i_{M(k)} \to i_1$. Lastly, we collapse vertices and edges that are matched in Cat_π, and we denote the resulting graph as $G_{\tilde{\pi}}$, since it does not depend on the choice of $\mathbf{i}$ but rather the choice of π itself. The resulting graph $G_{\tilde{\pi}}$ is the graph associated to the partition π, and we refer the reader to Definition 4.6.8 for a formal description. For clarity, consider the following example:

Let $M(k) = 4$, and let $\pi = \{\{1,2\},\{3,4\}\}$. Then, $\tilde{\pi} = \{\{1,3\},\{2\},\{4\}\}$. For any $\mathbf{i} \in \mathrm{Cat}_\pi$, we see that $i_1 = i_3$, and i_2, i_4 are independent indices. Now, $G_{\tilde{\pi}}$ is a graph on 3 vertices, which are labelled as $\{\{1,3\}\}$, $\{2\}$ and $\{4\}$, and so its corresponding tuple $\tilde{\mathbf{i}}$ is exactly the same as $\mathbf{i}$.

We then have, from Chatterjee and Hazra [2022, Eq. 5.2.12], that

$$\prod_{j=1}^{M(k)} \sqrt{r^m_{i_j i_{j+1}}} = \prod_{e \in E_{\tilde{\pi}}} r_e^{t_e/2}, \tag{4.23}$$

where $E_{\tilde{\pi}}$ is the edge set of $G_{\tilde{\pi}}$ and t_e is the number of times an edge e is traversed in the closed walk on $G_{\tilde{\pi}}$. Also observe

$$\Phi(\tilde{\pi}) = \mathbb{E}\left[\prod_{e \in E_{\tilde{\pi}}} G_e^{t_e}\right].$$

Consequently, we must have that t_e to be even for all e, since the Gaussian terms are independent and mean 0. We claim that $t_e = 2$ for all $e \in E_{\tilde{\pi}}$. Indeed, if for all e, $t_e \geq 2$ with at least one e' such that $t_{e'} > 2$, then, $\sum_{e \in E_{\tilde{\pi}}} t_e > 2|E_{\tilde{\pi}}|$. Since $G_{\tilde{\pi}}$ is connected, $|E_{\tilde{\pi}}| \geq |V_{\tilde{\pi}}| - 1 = M(k)/2$, where $V_{\tilde{\pi}}$ is the vertex set. Thus, $\sum_{e \in E_{\tilde{\pi}}} t_e > M(k)$. But, $\sum_e t_e = M(k)$, gives a contradiction. We conclude that $t_e = 2$ for all $e \in E_{\tilde{\pi}}$.

A similar contradiction arises when we assume that there exists a self-loop in $G_{\tilde{\pi}}$. Thus $G_{\tilde{\pi}}$ is a tree on $\frac{M(k)}{2} + 1$ vertices with each edge traversed twice in the closed walk. As a consequence, every Gaussian term in $\Phi(\tilde{\pi})$ appears exactly twice, and so, $\Phi(\tilde{\pi}) = 1$.

Let b_s be the s^{th} block of $\tilde{\pi}$ and let ℓ_s its representative element. Define

$$\gamma_s := \#\left\{1 \leq j \leq k : 1 + M(j) \in b_s\right\},$$

and

$$\{s_1, s_2, \dots, s_{\gamma_s}\} = \{1 \leq j \leq k : 1 + M(j) \in b_s\}.$$

We then have

$$\prod_{j=1}^{k}\left(\frac{1}{c_N}\sum_{t=1}^{N} r_{i_{1+M(j)t}}\right)^{\frac{n_j}{2}} = \prod_{s=1}^{\frac{M(k)}{2}+1}\left(\frac{1}{c_N}\sum_{t=1}^{N} r_{\ell_s t}\right)^{\sum_{j=1}^{\gamma_s} n_{s_j}/2}. \tag{4.24}$$

Note that

$$\sum_{j=1}^{\gamma_s} n_{s_j} = \sum_{j\in[k]:1+M(j)\in b_s} n_j.$$

Let us define $\tilde{n}_s := \sum_{j=1}^{\gamma_s} n_{s_j}/2$. Then,

$$\sum_{s:b_s\in\tilde{\pi}} \tilde{n}_s = \frac{N(k)}{2}. \tag{4.25}$$

Using Chatterjee and Hazra [2022, Eq. 5.2.16], we obtain

$$\begin{aligned}
&\frac{1}{N{c_N}^{\frac{M(k)}{2}}}\sum_{\mathbf{i}\in\mathrm{Cat}_\pi}\prod_{j=1}^{M(k)}\sqrt{r_{i_j i_{j+1}}}\prod_{j=1}^{k}\left(\frac{1}{c_N}\sum_{t=1}^{N} r_{i_{1+M(j)t}}\right)^{\frac{n_j}{2}} \\
&= \frac{1}{N c_N^{\frac{M(k)+N(k)}{2}}}\sum_{\substack{\ell_1\neq\ldots\neq\ell_{M(k)/2+1},\\ p_{(s,1)},\ldots,p_{(s,\tilde{n}_s)}:s\in\left[\frac{M(k)}{2}+1\right]}}\prod_{(u,v)\in E_{\tilde{\pi}}} r_{\ell_u\ell_v}\prod_{s=1}^{\frac{M(k)}{2}+1}\prod_{t=1}^{\tilde{n}_s} r_{\ell_s p_{(s,t)}},
\end{aligned} \tag{4.26}$$

where for any two blocks b_{s_1} and b_{s_2}, $\{p(s_1,1),p(s_1,2),\ldots\}$ and $\{p(s_2,1),p(s_2,2),\ldots\}$ are non-intersecting sets of indices $\{p_1,p_2,\ldots,p_{\tilde{n}_{s_1}}\}$ and $\{p'_1,p'_2,\ldots,p'_{\tilde{n}_{s_2}}\}$. Note that for $(u,v)\in E_{\tilde{\pi}}$, $r_{\ell_u\ell_v}=r_{uv}$ as before, but we rewrite in terms of representative elements to indicate common factors with the terms $r_{\ell_s p_{(s,t)}}$. Taking expectation in (4.26) gives us

$$\begin{aligned}
&\mathbb{E}[(4.26)] \\
&= \frac{1}{N c_N^{\frac{M(k)+N(k)}{2}}}\sum_{\ell_1\neq\ldots\neq\ell_{\frac{M(k)}{2}+1}}\mathbb{E}\left[\prod_{(u,v)\in E_{\tilde{\pi}}}\frac{W_{\ell_u}^m W_{\ell_v}^m}{\|\ell_u-\ell_v\|^\alpha}\right. \\
&\qquad\left.\times\sum_{\substack{p_{(s,1)},\ldots,p_{(s,\tilde{n}_s)}:\\ s\in\left[\frac{M(k)}{2}+1\right]}}\prod_{s=1}^{\frac{M(k)}{2}+1}\prod_{t=1}^{\tilde{n}_s}\frac{W_{\ell_s}^m W_{p_{(s,t)}}^m}{\|\ell_s-p_{(s,t)}\|^\alpha}\right].
\end{aligned} \tag{4.27}$$

The vertex set $V_{\tilde{\pi}}$ of the graph $G_{\tilde{\pi}}$ yields $M(k)/2+1$ distinct indices, due to the tree structure. Using Fact 4.4.2, we see that the factor of

$$\sum_{\ell_1,\ldots,\ell_{\frac{M(k)}{2}+1}} \prod_{(u,v)\in E_{\tilde{\pi}}} \frac{1}{\|\ell_u-\ell_v\|^{\alpha}}$$

is of the order of $\mathrm{O}_N\left(c_N^{\frac{M(k)}{2}}\right)$ since the weights are uniformly bounded in the range $[1,m]$. For the second summand in (4.27), the index ℓ_s already appears in the graph $G_{\tilde{\pi}}$, and for any s, we have $\tilde{n}_s$ many distinct indices from the sequence $\{p_{s,t}\}$, and summing over all s yields $N(k)/2$ many distinct indices due to (4.25). The second summation is therefore of the order of $\mathrm{O}_N\left(c_N^{\frac{N(k)}{2}}\right)$.

We claim that as $N\to\infty$, (4.27) converges to the limit

$$\mathbb{E}\left[\prod_{(u,v)\in E_{\tilde{\pi}}} W^m_{\ell_u}W^m_{\ell_v}\prod_{s=1}^{\frac{M(k)}{2}+1}\prod_{t=1}^{\tilde{n}_s} W^m_{\ell_s}W_{p_{(s,t)}}\right].$$

First, note that the weights are bounded, and so, (4.27) is bounded above and below. Next, we note that with the scaling of $Nc_N^{M(k)/2}$, we have

$$\lim_{N\to\infty}\frac{1}{Nc_N^{M(k)/2}}\sum_{\ell_1\neq\ldots\neq\ell_{\frac{M(k)}{2}+1}}\mathbb{E}\left[\prod_{(u,v)\in E_{\tilde{\pi}}}\frac{W^m_{\ell_u}W^m_{\ell_v}}{\|\ell_u-\ell_v\|^{\alpha}}\right]=\mathbb{E}\left[\prod_{(u,v)\in E_{\tilde{\pi}}} W^m_{\ell_u}W^m_{\ell_v}\right],$$

which is the moments of the adjacency matrix of the model as in Cipriani et al. [2025]. Thus, combinatorially, the first summand in (4.27) corresponds with the graph $G_{\tilde{\pi}}$, as defined in Definition 4.5. Now, consider a modification of the graph as follows: For each vertex s in $G_{\tilde{\pi}}$, attach $\tilde{n}_s$ many independent leaves, and call the new graph $\tilde{G}_{\tilde{\pi}}$. We refer to Chatterjee and Hazra [2022] for a detailed description, and Figure 4.3 for a visual representation.

The second summand over the sequence $\{p_{s,t}\}$ for each s corresponds to the added leaves, since the only common index with the original graph is the index ℓ_s for each s. Keeping the index ℓ_s fixed (since it is summed out in the first summand involving the indices $\ell_1\neq\ldots\neq\ell_{\frac{M(k)}{2}+1}$) , we see that with the

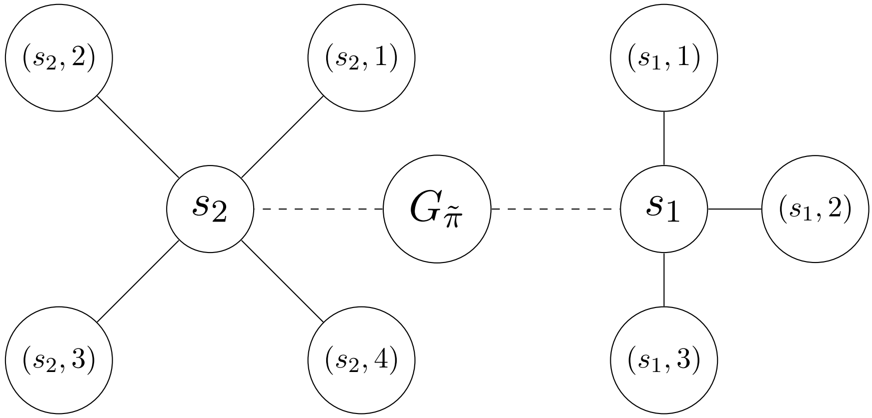

Figure 4.3: Modifying the graph $G_{\tilde{\pi}}$ to construct $\tilde{G}_{\tilde{\pi}}$. Here, we pick two vertices $s_1, s_2 \in V_{\tilde{\pi}}$, with $\tilde{n}_{s_1} = 3, \tilde{n}_{s_2} = 4$.

scaling $c_N^{N(k)/2}$ we have

$$\lim_{N\to\infty} \frac{1}{c_N^{N(k)/2}} \mathbb{E}\left[\sum_{p_{(s,1)},\ldots,p_{(s,\tilde{n}_s)}} \prod_{s=1}^{\frac{M(k)}{2}+1} \prod_{t=1}^{\tilde{n}_s} \frac{W_{\ell_s}^m W_{p_{(s,t)}}^m}{\|\ell_s - p_{(s,t)}\|^\alpha} \middle| W_{\ell_s}^m \right]$$
$$= \mathbb{E}\left[\prod_{s=1}^{\frac{M(k)}{2}+1} \prod_{t=1}^{\tilde{n}_s} W_{\ell_s}^m W_{p_{(s,t)}}^m \middle| W_{\ell_s}^m \right].$$

Due to the compact support of the weights, it is now easy to conclude that

$$\lim_{N\to\infty} (4.27) = \mathbb{E}\left[\prod_{(u,v)\in E_{\tilde{\pi}}} W_{\ell_u}^m W_{\ell_v}^m \prod_{s=1}^{\frac{M(k)}{2}+1} \prod_{t=1}^{\tilde{n}_s} W_{\ell_s}^m W_{p_{(s,t)}} \right] =: t(\tilde{G}_{\tilde{\pi}}, W^m) \tag{4.28}$$

where $W^m = (W_1^m, W_2^m, \ldots)$ and $\tilde{G}_{\tilde{\pi}}$ is the modified graph as described above and illustrated in Figure 4.3.

We can therefore conclude that for all even k,

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[\mathrm{tr}(\mathbf{\Delta}_{N,g,c}^k) \right] = \sum_{\substack{m_1,\ldots,m_k,\\ n_1,\ldots,n_k}} \sum_{\pi \in NC_2(M(k))} \mathcal{E}(\tilde{\pi}) t(\tilde{G}_{\tilde{\pi}}, W^m). \tag{4.29}$$

Now, consider the case when k is odd. Due to (4.18), we have that $M(k)$ must be odd. Thus, π cannot be a pair partition, and in particular, $\pi \notin$

$NC_2(M(k))$. Consider the term $\Phi(\tilde{\pi})$ in (4.19), and notice that by Wick's formula, this term is identically 0 if $M(k)$ is odd. Since the other expectations in (4.15) are of order $\mathrm{O}_N(1)$, we conclude that the odd moments are 0 in expectation.

Step 2. We now wish to show the concentration of the moments. Define

$$P(\mathbf{i}) := \mathbb{E}\left[\prod_{j=1}^{M(k)} G_{i_j\wedge i_{j+1}, i_j\vee i_{j+1}} \prod_{j=1}^{M(k)} \frac{\sqrt{r_{i_j i_{j+1}}}}{\sqrt{c_N}} \prod_{j=1}^{k}\left(\frac{1}{c_N}\sum_{t=1}^{N} r_{i_{1+M(j)}t}\right)^{\frac{n_j}{2}} \prod_{j=1}^{k} Z_{i_{1+M(j)}}^{n_j}\right],$$

and

$$\begin{aligned}P(\mathbf{i},\mathbf{i}') &:= \mathbb{E}\left[\prod_{j=1}^{M(k)} G_{i_j\wedge i_{j+1}, i_j\vee i_{j+1}} \prod_{j=1}^{M(k)} \frac{\sqrt{r_{i_j i_{j+1}}}}{\sqrt{c_N}} \prod_{j=1}^{k}\left(\frac{1}{c_N}\sum_{t=1}^{N} r_{i_{1+M(j)}t}\right)^{\frac{n_j}{2}} \prod_{j=1}^{k} Z_{i_{1+M(j)}}^{n_j}\right.\\ &\times \left.\prod_{j=1}^{M(k)} G_{i'_j\wedge i'_{j+1}, i'_j\vee i'_{j+1}} \prod_{j=1}^{M(k)} \frac{\sqrt{r_{i'_j i'_{j+1}}}}{\sqrt{c_N}} \prod_{j=1}^{k}\left(\frac{1}{c_N}\sum_{t=1}^{N} r_{i'_{1+M(j)}t}\right)^{\frac{n_j}{2}} \prod_{j=1}^{k} Z_{i'_{1+M(j)}}^{n_j}\right].\end{aligned}$$

Then,

$$\begin{aligned}&\mathrm{Var}\left(\int_{\mathbb{R}} x^k \,\mathrm{ESD}(\boldsymbol{\Delta}_{N,g,c})(\mathrm{d}\,x)\right)\\ &= \frac{1}{N^2}\sum_{\substack{m_1,\dots,m_k,\\ n_1,\dots,n_k}} \sum_{\mathbf{i},\mathbf{i}':[M(k)]\to[N]} \left[P(\mathbf{i},\mathbf{i}') - P(\mathbf{i})P(\mathbf{i}')\right], \qquad (4.30)\end{aligned}$$

and we would like to show (4.30) $\to 0$. If $\mathbf{i}$ and $\mathbf{i}'$ have no common indices, then $P(\mathbf{i},\mathbf{i}') = P(\mathbf{i})P(\mathbf{i}')$ by independence. If there is *exactly* one common index, say $i_1 = i'_1$, then by independence of Gaussian terms, the factors $\mathbb{E}[G_{i_1,i_2}]$ and $\mathbb{E}[G_{i_1,i'_2}]$ would pull out, causing (4.30) to be identically 0. Thus, we have *at least* one matching of the form $(i_1,i_2) = (i'_1,i'_2)$.

Let us begin by taking k to be even. Consider exactly one matching, which we take to be $(i_1,i_2) = (i'_1,i'_2)$ without loss of generality. Let π,π' be partitions of $\{1,2,\dots,M(k)\},\{1',2',\dots,M(k)'\}$ respectively. Let $\sum^{(1)}$ denote the sum over index sets $\mathbf{i},\mathbf{i}'$ with exactly one matching. Then, we have by an extension of the previous argument

$$\begin{aligned}&\frac{1}{N^2}\sum_{\mathbf{i},\mathbf{i}':[M(k)]\to[N]}^{(1)} P(\mathbf{i},\mathbf{i}')\\ &\leq \frac{1}{N^2 c_N^{M(k)}}\sum_{\pi,\pi'}\Phi(\tilde{\pi})\mathcal{E}(\tilde{\pi})\Phi(\tilde{\pi}')\mathcal{E}(\tilde{\pi}')\sum_{\mathbf{i},\mathbf{i}'}\mathbb{E}\left[r_{i_1 i_2}\prod_{j=2}^{M(k)}\sqrt{r_{i_j i_{j+1}}}\prod_{j=2}^{M(k)}\sqrt{r_{i'_j i'_{j+1}}}\right].\end{aligned} \tag{4.31}$$

Expanding the expression for r_{ij} and using the fact that $W_i^m \leq m$ gives us that (4.31) is bounded above by

$$\frac{m^{2M(k)}}{N^2 c_N^{M(k)}}\sum_{\pi}\Phi(\tilde{\pi})\mathcal{E}(\tilde{\pi})\Phi(\tilde{\pi}')\mathcal{E}(\tilde{\pi}')\sum_{\mathbf{i},\mathbf{i}'}\frac{1}{\|i_1-i_2\|^{\alpha}}\prod_{j=1}^{M(k)}\frac{1}{\|i_j-i_{j+1}\|^{\alpha/2}}\frac{1}{\|i'_j-i'_{j+1}\|^{\alpha/2}}. \tag{4.32}$$

We are now precisely in the setting of Cipriani et al. [2025], and in particular, following the ideas from Cipriani et al. [2025, p24] and using Fact 4.4.2, we obtain that the right-hand side of (4.32) is of order $\mathrm{O}_N(c_N^{-1})$. For t matchings in $\mathbf{i},\mathbf{i}'$, the order is $\mathrm{O}(c_N^{-t})$, giving us that (4.26) is of order $\mathrm{O}(c_N^{-1})$ when k is even.

The argument for the case where k is odd is similar. Since the optimal order is achieved when we take $\mathbf{i}\setminus\{i_1,i_2\}\in\mathrm{Cat}_\pi$ and $\mathbf{i}'\setminus\{i'_1,i'_2\}\in\mathrm{Cat}_{\pi'}$, with $\pi,\pi'\in NC_2(M(k))$, one cannot construct these partitions with k being odd with the restriction from (4.18) imposing that $M(k)$ must be odd. Consequently, we have convergence in $\mathbb{P}$–probability of the moments of $\mathrm{ESD}(\mathbf{\Delta}_{N,g,c})$. Thus, we conclude that

$$\lim_{N\to\infty}\mathrm{tr}(\mathbf{\Delta}_{N,g,c}^k) = M_k \quad \text{in } \mathbb{P}\text{–probability}\,,$$

where

$$M_k = \begin{cases}\sum_{\mathcal{M}(k)}\sum_{\pi\in NC_2(M(k))} t(\tilde{G}_{\tilde{\pi}}, W^m)\mathcal{E}(\tilde{\pi})\,, & k \text{ even},\\ 0\,, & k \text{ odd}\,,\end{cases} \tag{4.33}$$

where $\mathcal{M}(k)$ is the multiset of all numbers $(m_1,\dots,m_k,n_1,\dots,n_k)$ that appear in the expansion $(a+b)^k$ for two non-commutative variables a and b.

Step 3. We are now left to show that these moments uniquely determine a limiting measure. This follows from Chatterjee and Hazra [2022, Section 5.2.2], but we show the bounds for the sake of completeness.

First, from Chatterjee and Hazra [2022, Section 5.2.2], we have that $\mathcal{E}(\tilde{\pi}) \leq 2^k k!$. Next, observe from (4.28) that $|t(\tilde{G}_{\tilde{\pi}}, W^m)| \leq (m^2)^{\frac{k}{2}} = m^2$, since $W_i \leq m$ for all i and $\tilde{G}_{\tilde{\pi}}$ is a graph on $\frac{k}{2}+1$ vertices with $\frac{k}{2}$ edges. Lastly, $|NC_2(M(k)| \leq |NC_2(k)| = C_k$, where C_k is the k^{th} Catalan number, and moreover, $|\mathcal{M}(k)| \leq 2^k$. Combining these, we have

$$\beta_k := |M_k| \leq 2^k.C_k.m^k.2^k k! = (4m)^k C_k k!\,.$$

Using Sterling's approximation, we have

$$\frac{1}{k}\beta_k^{\frac{1}{k}} \leq \frac{4m}{(k+1)^{\frac{1}{k}}}.\frac{4\mathrm{e}^{-\left(1+\frac{1}{k}\right)}}{\pi^{\frac{1}{k}}}\,,$$

where π here is now the usual constant, and subsequently, we have

$$\limsup_{k\to\infty} \frac{1}{2k}\beta_{2k}^{\frac{1}{2k}} < \infty\,. \tag{4.34}$$

Equation (4.34) is a well-known criteria to show that the moments uniquely determine the limiting measure (see Lin [2017, Theorem 1]). This completes the argument. $\square$

Chapter Four

§4.5 Identification of the limit

§4.5.1 Removing geometry

In Section 4.4, we show the existence of a unique limiting measure ν_τ such that

$$\lim_{N\to\infty} \mathrm{ESD}(\mathbf{\Delta}_N^\circ) = \nu_\tau \qquad \text{in } \mathbb{P}\text{–probability}\,.$$

We have also shown that ν_τ is the limiting measure for the ESD of the Laplacian matrix $\widehat{\mathbf{\Delta}}_{N,g}$. In particular, through the proof of Proposition 4.4.1, we show that the limit $\nu_{\tau,m}$ is independent of the choice of α, and consequently, ν_τ is α–independent. We then use the idea of substituting $\alpha = 0$ from Cipriani et al. [2025, Section 6] in the matrix $\widehat{\mathbf{\Delta}}_{N,g}$, to obtain the Laplacian matrix $\mathbf{\Delta}_{N,g}^\circ$, which corresponds to the adjacency matrix $\mathbf{A}_{N,g}^\circ$ with entries given by

$$\mathbf{A}_{N,g}^\circ(i,j) = \begin{cases} \frac{\sqrt{W_iW_j}}{\sqrt{N}} G_{i\wedge j, i\vee j}, & i \neq j \\ 0, & i = j. \end{cases}$$

Then, $\lim_{N\to\infty} \mathrm{ESD}(\mathbf{\Delta}_{N,g}^\circ) = \nu_\tau$ in $\mathbb{P}$–probability. Recall that for all $1 \leq i \leq N$, $W_i^m := W_i \mathbf{1}_{W_i \leq m}$ for any $m \geq 1$. We can now apply Lemmas 4.3.5 and 4.3.6 to contruct a matrix $\mathbf{\Delta}_{N,g,c}^\circ = \mathbf{A}_{N,g,m}^\circ + Y_N^\circ$ such that

$$\limsup_{m\to\infty} \lim_{N\to\infty} \mathbb{P}\left(d_L(\mathrm{ESD}(\mathbf{\Delta}_{N,g}^\circ), \mathrm{ESD}(\mathbf{\Delta}_{N,g,c}^\circ)) > \delta\right) = 0\,,$$

where

$$\mathbf{A}^{\circ}_{N,g,m}(i,j) = \begin{cases} \frac{\sqrt{W_i^m W_j^m}}{\sqrt{N}} G_{i\wedge j, i\vee j}, & i \neq j \\ 0, & i = j, \end{cases}$$

and Y_N° is a diagonal matrix with entries

$$Y_N^{\circ}(i,i) = Z_i \sqrt{\frac{\sum_{k\neq i} W_i^m W_k^m}{N}}\,.$$

By Proposition 4.4.1, we have that $\lim_{N\to\infty} \mathrm{ESD}(\mathbf{\Delta}^{\circ}_{N,g,c}) = \nu_{\tau,m}$ in $\mathbb{P}$–probability. Thus, we begin by identifying $\nu_{\tau,m}$. To that end, consider the matrix $\widehat{\mathbf{\Delta}}^{\circ}_{N,g,c} := \mathbf{A}_{N,g,c} + \widehat{Y}_N^{\circ}$, with $\mathbf{A}_{N,g,c}$ as before, and $\widehat{Y}_N^{\circ}$ a diagonal matrix with entries

$$\widehat{Y}_N^{\circ}(i,i) = Z_i \sqrt{W_1^m}\sqrt{\mathbf{E}[W_1^m]}\,.$$

We now have the following lemma.

Lemma 4.5.1.
Let $\mathbf{\Delta}^{\circ}_{N,g,c}$ and $\widehat{\mathbf{\Delta}}^{\circ}_{N,g,c}$ be as defined above. Then,

$$\lim_{N\to\infty} \mathbb{P}\left(d_L(\mathrm{ESD}(\mathbf{\Delta}^{\circ}_{N,g,c}), \mathrm{ESD}(\widehat{\mathbf{\Delta}}^{\circ}_{N,g,c})) > \delta \right) = 0\,.$$

Proof. We apply Proposition 4.6.1 to obtain

$$\begin{aligned}
&\mathbb{E}\left[d_L(\mathrm{ESD}(\mathbf{\Delta}^{\circ}_{N,g,c}), \mathrm{ESD}(\widehat{\mathbf{\Delta}}^{\circ}_{N,g,c}))^3 \right] \\
&\leq \frac{1}{N}\mathbb{E}\sum_{i=1}^{N}\left(Y_N^{\circ}(i,i) - \widehat{Y}_N^{\circ}(i,i) \right)^2 \\
&\leq \frac{1}{N}\mathbb{E}[Z_1^2]\mathbf{E}[W_1^m]\sum_{i=1}^{N}\mathbf{E}\left[\left(\frac{\sqrt{\sum_{k\neq i} W_k^m}}{\sqrt{N}} - \sqrt{\mathbf{E}[W_1^m]} \right)^2 \right] \\
&\leq \frac{m}{N}\sum_{i=1}^{N}\mathbf{E}\left[\left| \frac{\sum_{k=1}^{N} W_k^m}{N} - \mathbf{E}[W_1^m] \right| \right]\,.
\end{aligned} \tag{4.35}$$

We have that $(W_i^m)_{i\in\mathbf{V}_N}$ is a bounded sequence of i.i.d. random variables, and in particular have finite variance. By the strong law of large numbers, we have that

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} W_k^m}{N} = \mathbf{E}[W_1^m] \quad \mathbb{P}\text{–almost surely}\,.$$

However, by the boundedness of the weights, we have that $N^{-1}\sum_{i=1}^{N} W_i^m$ is uniformly bounded by m, which is integrable (with respect to $\mathbb{E}$). By the dominated convergence theorem, we have convergence in L^1, and consequently, (4.35) goes to 0 as $N\to\infty$. We conclude with Markov's inequality. □

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We can now conclude that $\nu_{\tau,m}$ is the limiting measure of the ESD of the matrix $\widehat{\boldsymbol{\Delta}}^{\circ}_{N,g,c}$.

§4.5.2 Identification of the truncated measure

We have that

$$\lim_{N\to\infty} \mathrm{ESD}(\widehat{\boldsymbol{\Delta}}^{\circ}_{N,g,c}) = \nu_{\tau,m} \quad \text{in } \mathbb{P}\text{–probability}\,.$$

Notice that $\widehat{\boldsymbol{\Delta}}^{\circ}_{N,g,c}$ can be written as

$$\begin{aligned}\widehat{\boldsymbol{\Delta}}^{\circ}_{N,g,c} &= \mathbf{A}^{\circ}_{N,g,m} + \widehat{Y}^{\circ}_{N} \\ &= \mathrm{W}_m^{1/2}\left(\frac{1}{\sqrt{N}}\mathrm{G}\right)\mathrm{W}_m^{1/2} + \sqrt{\mathbf{E}[W_1^m]}\mathrm{W}_m^{1/4}\mathrm{Z}\mathrm{W}_m^{1/4}\,,\end{aligned}$$

where $\mathrm{W}_m = \mathrm{Diag}(W_1^m, \dots, W_N^m)$, G is a standard Wigner matrix with i. i. d $\mathrm{N}(0,1)$ entries above the diagonal and 0 on the diagonal, and Z is a diagonal matrix with i.i.d. $\mathrm{N}(0,1)$ entries.

First, we need to show that

$$\begin{aligned}&\lim_{N\to\infty} \mathrm{ESD}\left(\mathrm{W}_m^{1/2}\left(\frac{1}{\sqrt{N}}\mathrm{G}\right)\mathrm{W}_m^{1/2} + \sqrt{\mathbf{E}[W_1^m]}\mathrm{W}_m^{1/4}\left(\frac{1}{\sqrt{N}}\mathrm{Z}\right)\mathrm{W}_m^{1/4}\right) \\ &= \mathcal{L}\left(T_{W_m}^{1/2}T_sT_{W_m}^{1/2} + \sqrt{\mathbf{E}[W_m]}T_{W_m}^{1/4}T_gT_{W_m}^{1/4}\right) \text{ weakly in probability}\,.\end{aligned}$$

This easily follows by retracing the arguments in the proof of [Chakrabarty et al., 2021b, Theorem 1.3] and using the Lemma 4.6.6 presented in the appendix. This shows that

$$\nu_{\tau,m} = \mathcal{L}\left(T_{W_m}^{1/2}T_sT_{W_m}^{1/2} + \sqrt{\mathbf{E}[W_m]}T_{W_m}^{1/4}T_gT_{W_m}^{1/4}\right).$$

§4.5.3 Identification of the limiting measure

We now conclude with the proof of Theorem 4.2.5.

Proof of Theorem 4.2.5. Consider the measure μ_{W^m} and μ_W which are laws of $W^m = W\mathbf{1}_{W\le m}$ and W respectively. Also consider μ_g and μ_s to be the laws of the standard Gaussian and semicircle law, respectively. We have for all $t \in \mathbb{R}$,

$$|F_{\mu_{W^m}}(t) - F_{\mu_W}(t)| \le \varepsilon \tag{4.36}$$

for m large enough. Hence from [Bercovici and Voiculescu, 1993, Theorem 3.9] there exists a W^* probability space (A, φ) and self-adjoint operators T_{W^m}, T_W, T_g

and T_s affiliated to (A, φ) and projection $p \in A$ such that $pT_{W^m}p = pT_Wp$ and $\varphi(p) \geq 1-\varepsilon$. Also the spectral laws of T_{W^m}, T_W, T_g and T_s are given respectively by μ_{W^m}, μ_W, μ_g and μ_s respectively.

We can consider the commutative subalgebra generated by $\{T_{W^m}, T_g\}$. Then using [Bercovici and Voiculescu, 1993, Proposition 4.1], it is possible to generate random variable from $\{T_{W^m}, T_g\}$ that is free from T_s. Analogously, one can do the same for $\{T_W, T_g\}$.

Consider a self-adjoint polynomial Q_m of $\{T_{W^m}, T_g, T_s\}$ and let the law of this polynomial be given by ν_m. Similarly, let Q be the same self-adjoint polynomial of $\{T_W, T_g, T_s\}$ and ν be its law. Then using $pT_{W^m}p = pT_Wp$ and (4.36) and [Bercovici and Voiculescu, 1993, Corollary 4.5 and Theorem 3.9] we have that $d_\infty(\nu_m, \nu) \leq \varepsilon$ for all m large enough. Here d_∞ is the Kolmogorov distance. Picking $Q(x, y, z) = x^{1/2}yx^{1/2} + cx^{1/4}zx^{1/4}$ for some constant $c = \sqrt{\mathbf{E}[W]}$, completes the proof. □

§4.6 Appendix

In this section we collect some technical lemmas that are used in the proofs of our main results.

§4.6.1 Technical lemmas

For bounding the d_L distance between the ESDs of two matrices, we will need the following inequality, due to Hoffman and Wielandt (see Bai and Silverstein [2010, Corollary A.41]).

Proposition 4.6.1 (Hoffman-Wielandt inequality).
Let $\mathbf{A}$ and $\mathbf{B}$ be two $N \times N$ normal matrices and let $\mathrm{ESD}(\mathbf{A})$ and $\mathrm{ESD}(\mathbf{B})$ be their ESDs, respectively. Then,

$$d_L\left(\mathrm{ESD}(\mathbf{A}), \mathrm{ESD}(\mathbf{B})\right)^3 \leq \frac{1}{N} \operatorname{Tr}\left[(\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})^*\right]. \tag{4.37}$$

Here $\mathbf{A}^*$ denotes the conjugate transpose of $\mathbf{A}$. Moreover, if $\mathbf{A}$ and $\mathbf{B}$ are two Hermitian matrices of size $N \times N$, then

$$\sum_{i=1}^{N} (\lambda_i(\mathbf{A}) - \lambda_i(\mathbf{B}))^2 \leq \operatorname{Tr}[(\mathbf{A}-\mathbf{B})^2]. \tag{4.38}$$

The next two straightforward lemmas control the tail of the product of two Pareto random variables and the expectation of a truncated Pareto.

Lemma 4.6.2.
Let X and Y be two independent Pareto r.v.'s with parameters β_1 and β_2 respectively, with $\beta_1 \leq \beta_2$. There exist constants $c_1 = c_1(\beta_1, \beta_2) > 0$ and $c_2 = c_2(\beta_1) > 0$ such that

$$\mathbf{P}(XY > t) = \begin{cases} c_1 t^{-\beta_1} & \text{if } \beta_1 < \beta_2 \\ c_2 t^{-\beta_1} \log t & \text{if } \beta_1 = \beta_2. \end{cases}$$

Lemma 4.6.3.
Let X be a Pareto random variable with law $\mathbf{P}$ and parameter $\beta > 1$. For any $m > 0$ it holds

$$\mathbf{E}\left[X \mathbf{1}_{X \geq m}\right] = \frac{\beta}{(\beta - 1)} m^{1-\beta}.$$

We state one final auxiliary lemma related to the approximation of sums by integrals.

Lemma 4.6.4.
Let $\beta \in (0, 1]$. Then there exists a constant $c_1 = c_1(\beta) > 0$ such that

$$\frac{1}{N} \sum_{i \neq j \in \mathbf{V}_N} \frac{1}{\|i - j\|^\beta} \sim c_1 \max\{N^{1-\beta}, \log N\}. \tag{4.39}$$

If instead $\beta > 1$, there exists a constant $c_2 > 0$ such that

$$\frac{1}{N} \sum_{i \neq j \in \mathbf{V}_N} \frac{1}{\|i - j\|^\beta} \sim c_2 \,.$$

We end this section by quoting, for the reader's convenience, the following lemma from Chakrabarty et al. [2016, Fact 4.3].

Lemma 4.6.5.
Let (Σ, d) be a complete metric space, and let $(\Omega, \mathcal{A}, P)$ be a probability space. Suppose that $\big(X_{mn} : (m, n) \in \{1, 2, \ldots, \infty\}^2 \backslash \{\infty, \infty\}\big)$ is a family of random elements in Σ, that is, measurable maps from Ω to Σ, the latter being equipped with the Borel σ-field induced by d. Assume that

(1) for all fixed $1 \leq m < \infty$

$$\lim_{n \to \infty} d\left(X_{mn}, X_{m\infty}\right) = 0 \text{ in } P\text{-probability.}$$

(2) For all $\varepsilon > 0$,

$$\lim_{m \to \infty} \limsup_{n \to \infty} P\left(d\left(X_{mn}, X_{\infty n}\right) > \varepsilon\right) = 0.$$

Then, there exists a random element $X_{\infty\infty}$ of Σ such that

$$\lim_{m\to\infty} d\left(X_{m\infty}, X_{\infty\infty}\right) = 0 \text{ in } P\text{-probability} \tag{4.40}$$

and

$$\lim_{n\to\infty} d\left(X_{\infty n}, X_{\infty\infty}\right) = 0 \text{ in } P\text{-probability.}$$

Furthermore, if $X_{m\infty}$ is deterministic for all m, then so is $X_{\infty\infty}$, and (4.40) simplifies to

$$\lim_{m\to\infty} d\left(X_{m\infty}, X_{\infty\infty}\right) = 0. \tag{4.41}$$

Lemma 4.6.6 (Fact A.4 Chakrabarty et al. [2021b]).
Suppose that W_N is an $N \times N$ scaled standard Gaussian Wigner matrix, i.e., a symmetric matrix whose upper triangular entries are i.i.d. normal with mean zero and variance $1/N$. Let D_N^1 and D_N^2 be (possibly random) $N \times N$ symmetric matrices such that there exists a deterministic C satisfying

$$\sup_{N\geq 1, i=1,2} \left\|D_N^i\right\| \leq C < \infty$$

where $\|\cdot\|$ denotes the usual matrix norm (which is same as the largest singular value for a symmetric matrix). Furthermore, assume that there is a W^*-probability space $(\mathcal{A}, \varphi)$ in which there are self-adjoint elements d_1 and d_2 such that, for any polynomial p in two variables, it

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\left(p\left(D_N^1, D_N^2\right)\right) = \varphi\left(p\left(d_1, d_2\right)\right) \text{ a.s.}$$

Finally, suppose that $\left(D_N^1, D_N^2\right)$ is independent of W_N. Then there exists a self-adjoint element s in $\mathcal{A}$ (possibly after expansion) that has the standard semicircle distribution and is freely independent of (d_1, d_2), and is such that

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\left(p\left(W_N, D_N^1, D_N^2\right)\right) = \varphi\left(p\left(s, d_1, d_2\right)\right) \text{ a.s.}$$

for any polynomial p in three variables.

Lemma 4.6.7 (Wick's formula).
Let $(X_1, X_2, \ldots, X_n)$ be a real Gaussian vector, then, and $\mathcal{P}_2(k)$ the set of pair partitions of $[k]$. Then, for any $1 \leq k \leq n$,

$$\mathbb{E}[X_{i_1} \cdots X_{i_k}] = \sum_{\pi\in\mathcal{P}_2(k)} \prod_{(r,s)\in\pi} \mathbb{E}[X_{i_r} X_{i_s}]\,. \tag{4.42}$$

We borrow the following definition from Avena et al. [2023, Definition 2.3].

Definition 4.6.8 (Graph associated to a partition).
For a fixed $k \geq 1$, let γ denote the cyclic permutation $(1, 2, \ldots, k)$. For a partition π, we define $G_{\gamma\pi} = (\mathbf{V}_{\gamma\pi}, E_{\gamma\pi})$ as a rooted, labelled directed graph associated with any partition π of $[k]$, constructed as follows.

- *Initially consider the vertex set $\mathbf{V}_{\gamma\pi} = [k]$ and perform a closed walk on $[k]$ as $1 \to 2 \to 3 \to \cdots \to k \to 1$ and with each step of the walk, add an edge.*

- *Evaluate $\gamma\pi$, which will be of the form $\gamma\pi = \{V_1, V_2, \ldots, V_m\}$ for some $m \geq 1$ where $\{V_i\}_{1 \leq i \leq m}$ are disjoint blocks. Then, collapse vertices in $\mathbf{V}_{\gamma\pi}$ to a single vertex if they belong to the same block in $\gamma\pi$, and collapse the corresponding edges. Thus, $\mathbf{V}_{\gamma\pi} = \{V_1, \ldots, V_m\}$.*

- *Finally root and label the graph as follows.*

  – Root: *we always assume that the first element of the closed walk (in this case '1') is in V_1, and we fix the block V_1 as the root.*

  – Label: *each vertex V_i gets labelled with the elements belonging to the corresponding block in $\gamma\pi$.*

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