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Spectral analysis of inhomogeneous network models

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CHAPTER 1

Introduction

The broad goal of this thesis is to study the graph spectrum of various inhomogeneous random graph models, in particular, to characterise the eigenvalue distributions of random matrices associated with these random graphs. The first three chapters cover the graph *adjacency matrix*, whereas the fourth chapter is dedicated to the *Laplacian matrix*.

§1.1 Background

Throughout the history of mathematics, complex challenges have often driven the evolution of new areas of research. In the 20th century, there were major developments in several related disciplines, such as in the natural sciences (physics, chemistry, and biology), as well as in computer science, the social sciences, and medicine. Consequently, there was increasing interest in analysing data and describing phenomena observed through experimental methods, which in turn pushed the boundaries of mathematics. There was a need for precise mathematical frameworks to capture complex phenomena, giving rise to entirely new branches that are now fundamental in modern mathematics.

Typically, *complex systems* such as social networks, biological networks, and atomic nuclei, are difficult to analyse directly, even in the era of supercomputers and increasingly efficient algorithms. Mathematical models provide a reasonable approximation of such systems, and are built up over years of research. They often begin with deceptively simple “toy models”, and are subsequently generalised to more “realistic models” where the analysis can be challenging. Naturally, this also gives rise to several interesting questions in mathematics itself from a more abstract point of view. Moreover, while these branches of mathematics originate from distinct problems, as is the case for *random matrix theory* and *random graphs*, they cross paths frequently, yet continue to exist as independent research topics in their own right.

This chapter will serve as a preface to the material that will follow in the rest of the thesis. We dive into *spectral graph theory*, a topic that emerged in the 1950s and serves as the backbone of this thesis. We describe graphs and their matrices, namely the adjacency matrix and the Laplacian matrix, and give a brief overview of the relation between graph properties and the spectrum of their associated matrices.

Transitioning to the world of probability, we move on to *random graphs*, which were introduced in the mid 20th century. Over the years, a wide range of systems have been studied as complex networks, in particular biological and social networks. The explosive growth of these networks in the digital age and their increasing complexity underscore the need for robust mathematical models, which led to further development of the subject in the late 20th and early 21st century. Random graphs are graph-valued probabilistic objects and are essential in modelling real-world networks. We will present a brief overview of a toy model and various *graph regimes*, before proceeding with more general models.

We proceed with another kind of probabilistic object: a *random matrix*,

which is a matrix with random entries. This thesis focuses primarily on the eigenvalue distribution of random matrix models, that are associated with random graph models. It is important to note that although the main motivation comes from the study of random graph models, the essential tools of the trade come from random matrix theory, thereby also making the study of the spectrum relevant from a random matrix perspective.

The above naturally eases us into a more abstract theory of random variables. Abstraction is a fundamental aspect of mathematics, giving rise to areas such as *free probability* where one abstracts the notion of random variables and moves away from an underlying *sample space*. Despite this abstraction, a link with reality remains. Random matrix theory connects with free probability, and was born out of applications in statistics, operator algebras, and quantum physics.

With these notions well established, we proceed with a literature overview of *spectral theory for random graphs*, in particular for the Erdős-Rényi random graph. We conclude with an outline of the thesis and technical results that are used in later chapters, as well as a short discussion and concluding remarks.

§1.2 Spectral Graph Theory

Spectral graph theory is the study of the relation between geometric properties of graphs and the eigenvalues and eigenvectors of the associated graph matrices. Motivated by applications in quantum physics and chemistry, the theory is now used in various areas of mathematics, such as discrete mathematics and combinatorics, statistics, and probability, while also playing a crucial role in statistical physics and computer science. There are various references on the subject. We refer to Chung [1997] for an introduction, and to Spielman [2012] for a modern approach to the subject.

§1.2.1 Graphs and matrices

Graphs can be defined set-theoretically as a collection of two sets: a vertex set, and an edge set that indicates connections between the vertices. A *self-loop* is an edge from a vertex to itself. *Simple graphs* are graphs with no self-loops, and *at most* one edge between two vertices. Figure 1.1 illustrates a few special examples. For instance, a *tree* is a graph with no cycles: there is *exactly one* path from any vertex to any other vertex. On the other hand, a *clique* has an edge between every vertex. Figure 1.1 showcases *simple undirected graphs*, that is, the edges have no orientation (or direction). This thesis does not cover graphs that are directed, nor does it consider graphs that can have multiple

edges between two vertices.

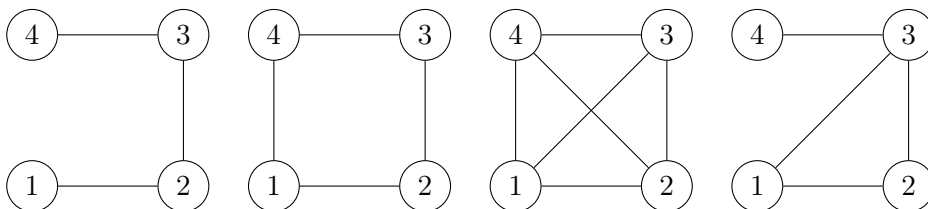


Figure 1.1: Some graphs on 4 vertices. The first three graphs are a tree, a cycle, a clique respectively.

A graph can be represented through its *adjacency matrix*. Let $G := (V, E)$ be the graph, with V being the vertex set and E the edge set. The adjacency matrix of G is defined as the matrix A with entries

$$A(i, j) := \begin{cases} 1, & \text{if } (i, j) \in E, \\ 0, & \text{if } (i, j) \notin E, \end{cases}$$

for all $i, j \in V$. For example, the cyclic graph in Figure 1.1 has the representation $G = (\{1, 2, 3, 4\}, \{(1, 2), (2, 3), (3, 4), (1, 4)\})$. The corresponding adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

We notice that A is symmetric. In fact, all undirected graphs have symmetric adjacency matrices, that is, $A(i, j) = A(j, i)$ for all $i, j \in V$. Moreover, A is zero on the diagonal, since G has no self-loops.

Another important graph matrix is the *graph Laplacian*. Let D denote the diagonal matrix with entries

$$D(i, j) = \begin{cases} \sum_{k \neq i, k \in V} A(i, k), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The *combinatorial graph Laplacian* L is defined as $L := A - D$. The *normalised Laplacian* is defined as $\mathcal{L} = I - D^{-1/2} A D^{-1/2}$, where $D^{-1/2}$ is the diagonal matrix defined as

$$D^{-1/2}(i, i) := \begin{cases} \frac{1}{\sqrt{D(i, i)}} & \text{if } D(i, i) \neq 0, \\ 0 & \text{if } D(i, i) = 0. \end{cases}$$

This thesis only covers the combinatorial graph Laplacian, which henceforth will be referred to as the Laplacian matrix. Note that if A is symmetric, then so are the graph Laplacians.

The Laplacian matrix gets its name from the fact that it can be viewed as the matrix form of the *discrete Laplacian operator*, which approximates the *continuous Laplacian operator* through a finite difference method (LeVeque [2007]). This can be illustrated by the discrete heat equation as follows: Let ϕ be a distribution across a graph $G = (V, E)$, with $\phi(i)$ being the temperature at a vertex $i \in V$. If $(i, j) \in E$, then the heat transfer between i and j is proportional to $\phi(i) - \phi(j)$. In particular, one obtains a matrix-vector ordinary differential equation of the form

$$\frac{d\phi}{dt} = kL\phi, \quad (1.1)$$

where $L = A - D$ is the graph Laplacian matrix and k is the thermal conductivity. This is analogous to the classical heat equation, hence the name “graph Laplacian”. The solution to (1.1) and its stability properties are obtained by analysing the eigenvalues of L .

§1.2.2 Spectral theory

Spectral theory traces its origins back to the works of David Hilbert in the early 20th century. He referred to the theory as *spectral analysis*. The name proved prophetic: a key result in the field, known as the *spectral theorem*, was later found to be useful in explaining atomic spectra in quantum mechanics.

Finite undirected graphs have adjacency (and Laplacian) matrices that are symmetric, which are diagonalisable and have real eigenvalues. For a finite graph G on N vertices, the spectrum of its adjacency matrix A (or its Laplacian matrix L) is the set of eigenvalues of A (or L , respectively). *Linear algebra* provides the necessary framework to study the eigenvalues and eigenvectors of graph matrices. For example, for a clique on N vertices, the spectrum of A is the eigenvalue $N - 1$ with multiplicity 1 and the eigenvalue -1 with multiplicity $N - 1$. The Laplacian, on the other hand, has eigenvalues 0 (once) and N (with multiplicity $N - 1$). Another powerful result, the *Perron-Frobenius theorem*, states that if G is connected, then the following is true (Spielman [2012, Theorem 4.5.1]): For the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ of the adjacency matrix of G , we have

- $\lambda_1 > \lambda_2$, and $\lambda_1 \geq -\lambda_N$, and
- λ_1 has a strictly positive eigenvector.

Although linear algebra provides a solid foundation, we outgrow it as N grows to infinity, and lean on spectral theory for our analysis. A *locally finite graph* $G = (V, E)$, that is, a graph where every vertex has finite degree, can be identified with a linear operator A on the Hilbert space $(\ell^2(V), \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the canonical inner product $\langle \phi, \psi \rangle := \sum_{i \in V} \overline{\psi(i)} \phi(i)$, and A acts on the canonical basis $(\delta_i)_{i \in V}$ as

$$A\delta_i = \sum_{j: (i,j) \in E} \delta_j.$$

We call the operator A the *adjacency operator*, and the Laplacian operator is defined similarly on $\ell^2(V)$. Note that in the finite-dimensional setting, we get back the graph matrices, and so we use the same notation for the matrix and the operator.

For an infinite graph, the spectrum of the adjacency operator is the set

$$\text{spec}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\},$$

and we define $\text{spec}(L)$ in a similar fashion. The spectrum of an operator on a finite-dimensional space is the set of eigenvalues. However, it consists of more components in the infinite-dimensional setting, and may have no eigenvalues or no point spectrum. In some instances, for a locally finite undirected graph, the operators A and L are self-adjoint, and $\text{spec}(A)$ and $\text{spec}(L)$ are contained in $\mathbb{R}$.

If A (and L) are self-adjoint operators defined over the Hilbert space $\ell^2(V)$, then the spectral theorem guarantees that these operators are in some sense “diagonalisable”. For example, consider the adjacency A . If A is a self-adjoint matrix, then the spectral theorem yields a *spectral decomposition* for A of the form

$$A = U\Lambda U^*,$$

where Λ is the diagonal matrix of eigenvalues of A , which are real, and U is a unitary matrix with columns as the orthonormal eigenvectors of A . If A is a self-adjoint operator, then we can still have a spectral decomposition in terms of a *spectral measure* (Rudin [1991]). The spectral theorem now gives us a *projection-valued measure* Λ on the spectrum $\text{spec}(A) \subset \mathbb{R}$ such that

$$A = \int_{\text{spec}(A)} \lambda \, d\Lambda(\lambda).$$

This formulation will later allow us to associate a probability measure to a certain class of “nice” operators.

Let us go back to a finite simple graph G . The largest eigenvalue λ_1 of the adjacency A is bounded above by the maximal degree of G . Additionally, λ_1

plays another crucial role in the study of the spread of epidemics on a graph. Following the ideas of [Pastor-Satorras et al. \[2015\]](#), we consider the following illustration.

Consider a community with (finite) vertex set V of size N and edge set E . Individuals are either *infected* or *susceptible*. Once an individual recovers, it becomes susceptible again. Any susceptible individual $x \in V$ gets infected at a rate β , and any infected individual recovers at a rate δ . If $p_x(t)$ denotes the probability that x is infected at time t , then,

$$p_x(t + \Delta t) = p_x(t)(1 - \delta\Delta t) + (1 - p_x(t))\beta\Delta t \sum_{y \in V} A(x, y)p_y(t).$$

Using this reasoning, one can derive an ODE for the dynamics of the spread as

$$\frac{dp_x(t)}{dt} = -\delta p_x(t) + \beta \sum_{y \in V} A(x, y)(1 - p_x(t))p_y(t).$$

To simplify the ODE, we can perform a linearisation trick by taking $1 - p_x(t) \approx 1$. This step is justified heuristically when the $p_x(t)$ is small for any x , that is, the epidemic spread is in the early stage. Let $p(t) = (p_1(t), \dots, p_N(t))^T$ be the vector of probabilities. We obtain a system of linearised ODEs given by

$$\frac{dp(t)}{dt} = (\beta A - \delta I)p(t).$$

Spectral analysis of the solution tells us that the equilibrium state is stable if

$$\frac{\beta}{\delta} < \frac{1}{\lambda_1},$$

that is, the infection dies out below this threshold. Heuristically, a large λ_1 indicates that nodes with many connections aid the spread of the disease.

The adjacency spectra have further applications. For instance, if the graph G has d_{max} as the maximal degree and d_{av} as the average degree, and λ_1 is the largest eigenvalue of the adjacency of G , then, by [Spielman \[2012, Lemma 4.2.1\]](#), we have

$$d_{av} \leq \lambda_1 \leq d_{max}.$$

If λ_1 and λ_N are the extremal eigenvalues, and if G is connected, then $\lambda_1 = -\lambda_N$ if and only if G is bipartite ([Spielman \[2012, Proposition 4.5.3\]](#)). Moreover, if $\alpha(G)$ is the chromatic number of a k -regular graph G on N vertices, then

$$\alpha(G) \leq \frac{-N\lambda_N}{k - \lambda_N}.$$

The above inequality is known as the Hoffman bound ([Haemers \[2021\]](#)). These results show that eigenvalues of the adjacency matrix can be used to study various properties of the graph.

The spectrum of the Laplacian can provide further insight into the graph structure. When the entries of the matrix are not restricted to 0 or 1, the matrix is also referred to as the *Markov matrix* ([Bryc et al. \[2006\]](#), [Bordenave et al. \[2014\]](#)). The graph Laplacian is essential in diffusion theory and network flow analysis, as it can be seen as the negative of the infinitesimal generator of a continuous-time random walk associated with a graph, and its spectral properties are useful in the analysis of mixing times and relaxation times of the random walk. It has several other key applications. The Kirchhoff Matrix-Tree Theorem relates the determinant of the Laplacian to the count of spanning trees in a graph ([Chung \[1997\]](#)), and the multiplicity of the zero eigenvalue indicates the number of connected components ([Chung \[1997\]](#)). The second-smallest eigenvalue, known as the Fiedler value or the algebraic connectivity, measures the graph's connectivity; higher values signify stronger connectivity [De Abreu \[2007\]](#).

In modern machine learning, spectral techniques are pivotal in spectral clustering algorithms, where the techniques use the Laplacian eigenvalues and eigenvectors for dimensionality reduction before applying algorithms like k -means clustering (see [Abbe et al. \[2020\]](#), [Abbe \[2017\]](#)). These algorithms are particularly effective for detecting clusters that are not linearly separable. Recent advancements integrate spectral clustering with graph neural networks to enhance graph pooling operations ([Bianchi et al. \[2020\]](#)). Spectral algorithms are also crucial for identifying communities within networks by analysing the spectral properties of the graph ([Chung \[1997\]](#)).

The normalised graph Laplacian, just like the graph Laplacian, is the negative of the infinitesimal generator of another continuous-time random walk associated with the graph. It has applications in studying the so-called Cheeger constant, as well as the diameter of the graph, but we will not be studying this matrix in this thesis.

§1.3 Random Graphs

How likely is it that you and another individual have a mutual friend? Will a disease spread in a community rapidly, or will it be restricted to isolated groups? How does one model social networks? Where are you likely to be if you walk randomly on the streets of Amsterdam? How likely are oil particles to percolate through a rock?

These questions only begin to scratch the surface of *random graph theory*. Random graphs first appeared in the context of sociology in the early 1900s. They reappeared in the context of mathematical biology, before the pioneering works of Paul Erdős and Alfred Rényi in 1959, which laid the foundation of the most elementary random graph model: The Erdős-Rényi random graph (ERRG). Thereafter, the interest in the topic grew rapidly, fuelled by the boom of computer science and the increasing interest in modelling complex networks.

§1.3.1 Erdős-Rényi random graphs

There are two models typically referred to as the Erdős-Rényi random graph. The first, introduced in Erdős and Rényi [1959], is a simple graph chosen *uniformly at random* from the set of all graphs on N vertices and m edges, and is parametrised by the tuple (N, m) . The second model, also called the *Gilbert-Erdős-Rényi* model, was introduced in Gilbert [1959] as a *percolation model* on the complete graph K_N on N vertices, where edges are kept with probability p and discarded with probability $1 - p$, for some $p \in [0, 1]$, and is parametrised by the tuple (N, p) . The two models are quite close. The latter will be the model used throughout this thesis and will be denoted as $\text{ER}_N(p)$ and abbreviated as ERRG. There are various texts on random graphs, and in particular on the ERRG. We refer to the monographs van der Hofstad [2017], van der Hofstad [2024] for an exposition of the topic.

In the setting where $p := \lambda/N$, the random graph is usually classified into three regimes:

- *Subcritical regime:* When $\lambda < 1$, the graph consists of small components that are tree-like. In particular, the graph is a *forest*, with the size of the largest component of the order $\text{O}_{N,\mathbb{P}}(\log N)$ (where O_N is the Landau notation, and the additional subscript $\mathbb{P}$ indicates the statement holds with high probability).
- *Critical regime:* When $\lambda = 1$, the graph exhibits a so-called *phase transition*. The largest component in the graph is now $\text{O}_{N,\mathbb{P}}(N^{2/3})$. This regime is the most delicate of the three, and we refer to Janson et al. [1993], Aldous [1997] for further analysis.
- *Supercritical regime:* When $\lambda > 1$, the graph has a *unique giant component* of size $\text{O}_N(N)$ with high probability, and other components of size $\text{O}_N(\log N)$. We have a further sub-regime in this regime:
 - *Connectivity regime:* If $p \gg \frac{\log N}{N}$, then with high probability the graph is *connected*, that is, there is only one component.

1. Introduction

In Figure 1.2, we see graph realisations for the three regimes.

Another classification for random graph models is *sparsity*. In particular, for this thesis, we say that the graph is *sparse* when the average degree of the graph is bounded. On the other hand, we say that the graph is *dense* if the average degree grows with N . For $\text{ER}_N(p)$, we have the following:

- *Dense regime*: $p := \varepsilon_N$ such that $\varepsilon_N \rightarrow 0$ and $N\varepsilon_N \rightarrow \infty$. In the literature, the dense regime is characterised by $\varepsilon_N \equiv \text{constant}$, but this regime will not be covered in this thesis, and hence we abuse terminology.
- *Sparse regime*: $p := \varepsilon_N$ such that $\varepsilon_N \rightarrow 0$ and $N\varepsilon_N \rightarrow \lambda \in (0, \infty)$.

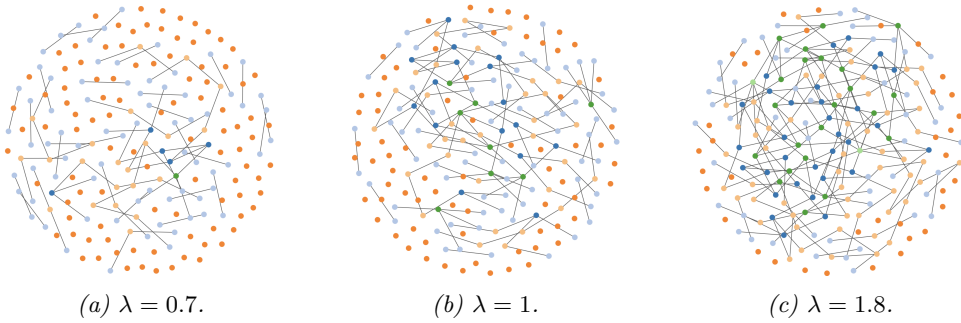


Figure 1.2: Realisations of $\text{ER}_N(p)$ for three regimes, with $p = \lambda/N$, and $N = 200$. Simulated on <https://www.networkpages.nl>.

Local weak convergence

The theory of local weak convergence builds on Aldous and Lyons [2007], Benjamini and Schramm [2001]. Since random graphs are essentially graph-valued random variables, this theory describes a framework to analyse the “limits” of sparse random graphs, by providing a natural topology to understand convergence. Consequently, any graph parameter that is continuous with respect to the local topology converges to the graph parameter of the limiting object, akin to how several functionals of random walks converge to functionals of Brownian motion in the appropriate topology. Local weak convergence is a remarkable tool, since in many instances the limit is easier to analyse than the prelimit. Before looking at formal details, we give a heuristic description:

Consider a uniformly chosen vertex of the random graph, say o_N , where N is the number of vertices. If the graph has no underlying geometry, as is the case for $\text{ER}_N(\cdot)$, we say that edges have length one. Fix a positive radius $r \in \mathbb{R}_+$, and from the vertex o_N , observe the graph up to the radius r . So, if $r \in [0, 1)$, observe only o_N and nothing more, if $r \in [1, 2)$, observe the immediate neighbours of

o_N , and so on. The local weak limit is in some sense what the graph “looks like” up to any finite radius r .

We now state the above formally. A *rooted graph* $(\mathbb{G}, o)$ is a graph $\mathbb{G}$ with a specified root o . Let $\mathcal{G}^*$ denote the set of *locally finite connected rooted graphs* up to equivalence $\equiv$, where $\equiv$ denotes graph isomorphism. Given $r \in \mathbb{N}$, let $[\mathbb{G}, o]_r$ denote the finite rooted subgraph obtained from $(\mathbb{G}, o)$ by keeping vertices that are up to a distance r from o , including edges. We say that a sequence $(\mathbb{G}_N, o_N)_{N \in \mathbb{N}}$ converges *locally* to $(\mathbb{G}, o)$ if for each $r \in \mathbb{N}$ there exists an $n_r \in \mathbb{N}$ such that for all $N > n_r$, we have

$$[\mathbb{G}_N, o_N]_r \equiv [\mathbb{G}, o]_r.$$

If we define d_{LW} as

$$d_{LW} : (\mathbb{G}, o), (\mathbb{G}', o') \mapsto 1 / \sup\{r \in \mathbb{N} : [\mathbb{G}, o]_r \equiv [\mathbb{G}', o']_r\},$$

then $(\mathcal{G}^*, d_{LW})$ becomes a complete separable metric space. We can endow this space with its Borel σ -algebra, and consider the complete separable metric space of probability measures $\mathcal{P}(\mathcal{G}^*)$ on $\mathcal{G}^*$.

Definition 1.3.1 (Local weak convergence).

Let $(\mathbb{G}_N)_{N \geq 1}$ denote a sequence of (possibly disconnected) random graphs. If o_N is a uniformly chosen vertex (restricted to the connected component of $\mathbb{G}_N$), then we say that $(\mathbb{G}_N, o_N)$ converges *locally weakly* to $(\mathbb{G}, o)$ having law $\mathcal{L} \in \mathcal{P}(\mathcal{G}^*)$ if, for any bounded and continuous function $h : \mathcal{G}^* \rightarrow \mathbb{R}$, we have

$$\mathbb{E}[h(\mathbb{G}_N, o_N)] \rightarrow \mathbb{E}_{\mathcal{L}}[h(\mathbb{G}, o)]$$

as $N \rightarrow \infty$, where $\mathbb{E}$ is with respect to the law of the random graph and the root o_N .

As an example, consider the graph $\text{ER}_N(p)$, with $p = \lambda/N$ for a fixed λ . This graph converges locally weakly to a *Galton-Watson tree* (or a branching process) with offspring distribution $\text{Poi}(\lambda)$, that is, a process starting with a single vertex, giving birth to progeny that are distributed as $\text{Poi}(\lambda)$, and repeating this for each offspring.

Local weak convergence provides a powerful framework for the analysis of graph properties that are *local*, that is, continuous with respect to the local topology (see [van der Hofstad \[2024\]](#), [Salez \[2011\]](#)), as well as dynamics on the graph (see for example [Avena et al. \[2024\]](#) for *interacting particle systems*, and [Hupkes et al. \[2023\]](#) for a discussion on PDEs). Later, we will see how local weak convergence can be related to the spectrum of random graphs via the Stieltjes transform.

Inhomogeneous Erdős-Rényi random graphs

ERRG serve as the basis for many mathematical theories in random graphs. Real-world networks are highly inhomogeneous and have a far more complex structure. Various attempts have been made to generalise them to other kinds of random graph models. One of the successful extensions is the inhomogeneous Erdős-Rényi random graph model introduced by [Bollobás et al. \[2007\]](#). This graph has N vertices labelled by $[N] = 1, \dots, N$, and edges are present independently with probability p_{ij} given by $p_{ij} = \frac{f(x_i, x_j)}{N} \wedge 1$, where f is a symmetric kernel on a state space $S \times S$, and x_i are certain attributes associated with vertex i belonging to S . If f is bounded, then the graph is a sparse random graph. In this thesis, we study a variant of the above inhomogeneous random graph, namely, the vertex set remains the same, but the connection probabilities are given by

$$p_{ij} = \varepsilon_N f(w_i, w_j) \wedge 1,$$

where ε_N is a tuning parameter, (w_i) is a sequence of deterministic weights, and f is a symmetric bounded function on $[0, \infty)^2$. The weights can signify a property of vertex i . They can also be taken random, but are not considered to be so in this thesis for this model. Note that when $N\varepsilon_N \rightarrow \infty$, the average degree is unbounded, and when $N\varepsilon_N = O(1)$, the average degree is bounded. In the sparse case, the properties of the connected components were studied in [Bollobás et al. \[2007\]](#), which focused on the properties of the connected components and their relationship with the branching process. It was shown that the largest component of the graph has a size of order N if the operator norm of the kernel operator corresponding to f is strictly greater than 1 (see also [\[van der Hofstad, 2024, Theorem 3.9\]](#)). In the subcritical case, the sizes of the largest connected components can exhibit different behaviour compared to the ERRG. The study of the largest connected components in various inhomogeneous random graphs has attracted a lot of attention (see, for example, [Bhamidi et al. \[2010\]](#), [Broutin et al. \[2021\]](#), [Bet et al. \[2023\]](#), [van der Hofstad \[2013\]](#), [Devroye and Fraiman \[2014\]](#)). We abbreviate the inhomogeneous Erdős-Rényi random graph as IER.

§1.3.2 Kernel-based random graphs

In recent years, many random graph models have been proposed in an attempt to model real-life networks. These models aim to capture three key properties that real-world networks exhibit: scale-free nature of the degree distribution, small-world property, and high clustering coefficients [\[van der Hofstad, 2024\]](#). It is generally difficult to find random graph models that incorporate all three

features. Classical random graph models typically fail to capture scale-freeness, small-world behaviour, and high clustering simultaneously. For instance, the Erdős-Rényi model only exhibits the small-world property, while models like Chung-Lu, Norros-Reittu, and preferential attachment models are scale-free (Chung and Lu [2002], Barabási and Albert [1999]) and small-world, but have clustering coefficients that vanish as the network grows. In contrast, regular lattices have high clustering but large typical distances. The Watts-Strogatz model (Watts and Strogatz [1998]) was an early attempt to create a network with high clustering and small-world features, but it does not produce scale-free degree distributions.

Scale-free percolation, introduced in Deijfen et al. [2013], blends ideas from long-range percolation (see e.g. Berger [2002]) with inhomogeneous random graphs, such as the Norros-Reittu model. In this framework, vertices are positioned on $\mathbb{Z}^d$, and each vertex x is independently assigned a random weight W_x . These weights follow a power-law distribution:

$$\mathbb{P}(W > w) = w^{-(\tau-1)}L(w),$$

where $\tau > 1$ and $L(w)$ is a slowly varying function at infinity.

Edges between pairs of vertices x and y are added independently, with a probability that increases with the product of their weights and decreases with their Euclidean distance. The edge probability is given by

$$p_{xy} = 1 - \exp\left(-\lambda \frac{W_x W_y}{\|x - y\|^\alpha}\right), \quad (1.2)$$

where $\lambda, \alpha > 0$ are model parameters and $\|\cdot\|$ denotes the Euclidean norm. This model has been proposed as a suitable representation for certain real-world systems, such as interbank networks, where both spatial structure and heavy-tailed connectivity distributions are relevant (Deprez et al. [2015]). Various properties of the model are now well known, and we refer to the articles by Jorritsma et al. [2024], Cipriani and Salvi [2024], Cipriani et al. [2025], Heydenreich et al. [2017], Dalmau and Salvi [2021], van der Hofstad et al. [2024] for further references.

In recent times, there has been a lot of interest in models that have connection probabilities similar to (1.2). Kernel-based spatial random graphs encompass a wide variety of classical random graph models where vertices are embedded in some metric space. In their simplest form (see Jorritsma et al. [2023] for a more complete exposition) they can be defined as follows: Let V be the vertex set of the graph and, sample a collection of weights $(W_i)_{i \in V}$ that are independent and identically distributed (i.i.d.), serving as marks on the vertices. Conditionally on the weights, two vertices i and j are connected by an

undirected edge with probability

$$\mathbb{P}(i \leftrightarrow j \mid W_i, W_j) = \kappa(W_i, W_j) \|i - j\|^{-\alpha} \wedge 1,$$

where κ is a symmetric kernel, $\|i - j\|$ denotes the distance between vertices i and j in the underlying metric space and $\alpha > 0$ is a constant.

Common choices for κ include:

$$\begin{aligned} \kappa_{\text{triv}}(w, v) &\equiv 1, & \kappa_{\text{strong}}(w, v) &= w \vee v, \\ \kappa_{\text{prod}}(w, v) &= w v, & \kappa_{\text{pa}}(w, v) &= (w \vee v)(w \wedge v)^{\sigma_{\text{pa}}}. \end{aligned}$$

In the above, $\sigma_{\text{pa}} = \alpha(\tau - 1)/d - 1$, where $\tau - 1$ is the exponent of the tail distribution of the weights, so that the kernel κ_{pa} mimics the form that appears in preferential attachment models [Jorritsma et al., 2023]. While these models are well-studied from a random graph perspective, there is minimal literature on their spectral properties.

§1.4 Random Matrix Theory and Free Probability

§1.4.1 Random Matrices

Random matrices are matrix-valued random variables where each entry of the matrix is a classical random variable. They are of significant interest, not only from the point of view of modern probability and statistical physics, but also because they connect to various areas. First appearing in 1928 in the work of Wishart (Wishart [1928]) in the context of statistics and multivariate data analysis, the topic was further researched from a spectral analysis point of view in the pioneering work of Wigner (Wigner [1955]). There are now several connections with other areas. For instance, a connection with *number theory* was established when eigenvalues of certain random matrices were used to model the distribution of zeroes of the Riemann zeta function (Montgomery [1973]). There are also connections with dynamical systems, in particular with Painlevé's ordinary differential equations (Tracy and Widom [1994]), as well as with the Dyson Brownian motion (Dyson [1962]). There are several applications in numerical linear algebra, computer science, and statistics (see Johnstone [2001], or the textbook Tropp [2015]).

Quantum mechanics tells us that energy levels of large nuclei correspond to the eigenvalues of some Hermitian operator. Wigner chose to model this operator by using *Wigner matrix ensembles*, wherein he ignored all physical aspects of the system except symmetry. The reason to do so was the observation that gaps in energy levels of large nuclei followed similar patterns regardless of the material chosen. Systems with time-reversal symmetry were modelled by

using real symmetric random matrices with Gaussian entries, known as the *Gaussian orthogonal ensemble* (GOE), and those without were modelled by using complex Hermitian matrices with complex Gaussian entries, known as the *Gaussian unitary ensemble* (GUE).

Spectral analysis of random matrices is a broad subject, with a vast literature focusing on the distribution of eigenvalues of the matrix, the largest eigenvalue (or more generally, the k largest eigenvalues for some $k \in \mathbb{N}$), and the eigenvectors. One of the key statistics that we focus on in this thesis is the empirical spectral distribution, defined below.

Definition 1.4.1 (Empirical Spectral Distribution).

The empirical measure that assigns mass $1/N$ to each eigenvalue of random matrix $\mathbf{M}_N$ is called the Empirical Spectral Distribution (ESD), and is defined as

$$\text{ESD}(\mathbf{M}_N)(\cdot) = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}(\cdot),$$

where $\lambda_i := \lambda_i(\mathbf{M}_N)$ is the i -th eigenvalue of $\mathbf{M}_N$ and, for any x , $\delta_x(\cdot)$ is the Dirac delta mass at the value x .

Notice that since the eigenvalues are random, $\text{ESD}(\mathbf{M}_N)$ is a *random measure*. The *bulk distribution* of eigenvalues refers to the distribution of the non-extremal eigenvalues of the random matrix. The ESD is a central object of interest in studying the bulk of the eigenvalue distribution, so it would be heresy not to ask about its limiting behaviour as $N \rightarrow \infty$. The work of [Wigner \[1958\]](#) showed that for the GOE and GUE models, as well as for a large class of other random matrix models, under appropriate scaling of the entries the $\text{ESD}(\mathbf{M}_N)$ converges *weakly almost-surely* to a (deterministic) measure μ_{sc} , where μ_{sc} is the semicircle law with density

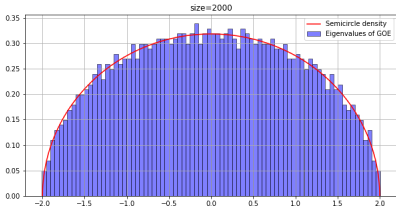
$$\mu_{sc}(\mathrm{d}x) := \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}_{|x| \leq 2} \mathrm{d}x.$$

For instance, consider the symmetric matrix $\mathbf{A}_N$ with entries

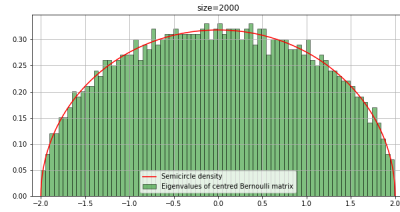
$$\mathbf{A}_N(i, j) \stackrel{d}{=} \frac{1}{\sqrt{p_N(1 - p_N)}} \text{Ber}(\varepsilon_N),$$

such that $N\varepsilon_N \rightarrow \infty$. This is the adjacency matrix of the Erdős-Rényi random graph $\text{ER}_N(\varepsilon_N)$. It is known that, as $N \rightarrow \infty$, $\text{ESD}(\mathbf{A}_N)$ converges to μ_{sc} in probability (see for example [Jung and Lee \[2018\]](#), [Tran et al. \[2013\]](#)). The fact that different empirical distributions converge to the same limit sparked the idea of *universality*, and we will later see how μ_{sc} is a universal limit in this area.

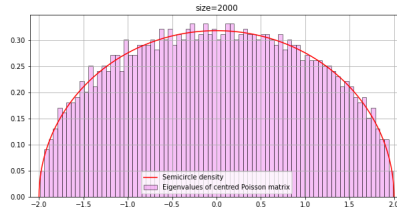
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(a) GOE ensemble, with entries distributed as $N(0, 1)$.



(b) Symmetric and centred Bernoulli matrix, with entries distributed as $\text{Ber}(0.5) - 0.5$.



(c) Symmetric and centred Poisson matrix, with entries distributed as $\text{Poi}(1) - 1$.

Figure 1.3: Eigenvalue distributions of some random matrix models.

The above has been generalised beyond the original Wigner matrices. In particular, some works consider matrix entries that are not i.i.d. and have a variance profile that is not constant. The study of the bulk blends flavours from various areas of mathematics. In favourable scenarios, the problem can be analysed within the framework of universality classes. Typically, in these cases, the matrix may have entries drawn from any distribution, but one can implement *Gaussianisation*, that is, replace them with Gaussian entries with the same mean and variance profile, without affecting the ESD too much (in probability). This technique follows ideas of Chatterjee [2005].

Of course, it would be naïve to assume that every random matrix model can be Gaussianised. There are numerous concrete examples, particularly in random graph theory, where this step fails. In such cases, one falls back on explicitly working with the ESD. One approach is through the *method of moments* (see Bordenave [2019]), which computes the moments of the ESD and find the limiting moments. By the spectral theorem, for any random matrix $\mathbf{M}_N$,

$$\int_{\mathbb{R}} x^k \text{ESD}(\mathbf{M}_N)(dx) = \frac{1}{N} \sum_{i=1}^k \lambda_i^k = \frac{1}{N} \text{Tr}(\mathbf{M}_N^k).$$

The “standard procedure” is to begin by computing expected moments and see

if they concentrate, that is, the target is to show that

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}} x^k \text{ESD}(\mathbf{M}_N)(dx) = M_k \quad \text{in } \mathbb{P}\text{-probability},$$

where $\mathbb{P}$ is the underlying law of the matrix entries. To guarantee the existence of a limiting measure, we have to check if the moments satisfy one of *Carleman's conditions*, that is, the moments uniquely determine a limiting measure if

$$\sum_{k \geq 0} M_{2k}^{-1/2k} = \infty, \quad \text{or equivalently,} \quad \limsup_{k \rightarrow \infty} M_{2k}^{1/2k} < \infty.$$

This approach has a strong combinatorial flavour due to computation of the combinatorial expression $\mathbb{E}[\text{tr}(\mathbf{M}_N^k)]$, where $\text{tr} := N^{-1} \text{Tr}$ is the normalised trace. Naturally, there are examples where the moments do not exist, and this approach then fails. This brings us to another classical approach, namely, the *Stieltjes transform approach*, where one translates the measure-theoretic problem into an analytic problem on the upper-half complex plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \Im(z) > 0\}$ (see [Bordenave \[2019\]](#), [Mingo and Speicher \[2017\]](#), [Anderson et al. \[2010\]](#)).

For any complex number $z \in \mathbb{C}^+$, we define the *resolvent* of a random matrix $\mathbf{M}_N$ as

$$\mathbf{R}_{\mathbf{M}_N}(z) := (\mathbf{M}_N - z \mathbf{I}_N)^{-1},$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix. For any measure μ and $z \in \mathbb{C}^+$, its *Stieltjes transform* is defined as

$$S_\mu(z) := \int_{\mathbb{R}} \frac{1}{x - z} \mu(dx).$$

So, we have that

$$S_{\text{ESD}(\mathbf{M}_N)}(z) = \int_{\mathbb{R}} \frac{1}{x - z} \text{ESD}(\mathbf{M}_N)(dx) = \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z} = \text{tr}(\mathbf{R}_{\mathbf{M}_N}(z)),$$

where $\text{tr} := N^{-1} \text{Tr}$ is the normalised trace operator. The Stieltjes transform allows us to work with analytic tools from complex analysis and functional analysis to deduce properties of the measure itself. For instance, if the Stieltjes transform is uniformly bounded (in z), then the measure has an absolutely continuous component ([Sen and Virág \[2011\]](#)). In some cases, we can also derive the exact density of the measure from its Stieltjes transform by using an inversion formula ([Bai and Silverstein \[2010\]](#)). Notable works show that one can bound the distance between two measures in terms of the distance between their respective Stieltjes transforms ([Bai and Silverstein \[2010\]](#), [Augeri \[2025\]](#)). In particular

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Augeri [2025] defines a distance d_S compatible with the weak topology on the space $\mathcal{P}(\mathbb{R})$ of probability measures on the real line as follows:

$$d_S(\mu, \nu) := \sup\{|S_\mu(z) - S_\nu(z)| : \Im(z) \geq 2, z \in \mathbb{C}^+\}, \quad \mu, \nu \in \mathcal{P}(\mathbb{R}).$$

If $d_{KL}(\cdot, \cdot)$ denotes the Kolmogorov-Smirnov distance and $\mathcal{W}^p(\cdot, \cdot)$ the L^p -Wasserstein distance for any $p \geq 1$, then

$$d_S(\mu, \nu) \leq d_{KL}(\mu, \nu) \wedge \mathcal{W}^p(\mu, \nu), \quad \mu, \nu \in \mathcal{P}(\mathbb{R}).$$

If a measure μ is compactly supported in $[-R, R]$ for some $R > 0$ with moments $\{M_k\}_{k \geq 1}$, then the Stieltjes transform can be related to the moments as:

$$S_\mu(z) = - \sum_{k \geq 0} \frac{M_k}{z^{k+1}}, \quad (1.3)$$

where the Laurent series on the right-hand side converges when $|z| > R$.

§1.4.2 An illustration

Let us next see a heuristic for the two methods of analysis. We begin with the moment method, following ideas from Speicher [2024].

Consider an i.i.d. sequence $\{G_{i,j} : N \geq i > j\}$ of random variables distributed as $N^{-1/2}N(0, 1)$, where $N(0, 1)$ is the standard Gaussian random variable with mean 0 and variance 1. Take an $N \times N$ Wigner matrix G , with entries $G(i, j) = G_{i \wedge j, i \vee j}$, where $G_{ii} = 0$. By trace expansion, we have

$$\mathbb{E}[\text{tr}(G^k)] = \frac{1}{N^{1+\frac{k}{2}}} \sum_{i_1, \dots, i_k=1}^N \mathbb{E}[G(i_1, i_2)G(i_2, i_3) \dots G(i_k, i_1)].$$

To compute this sum, we use the following well-known result (see for example Speicher [2024]).

Lemma 1.4.2 (Wick's formula).

Let $(X_1, \dots, X_n)$ be a real Gaussian vector and $\mathcal{P}_2(k)$ the set of pair partitions of $[k]$. Then, for any $1 \leq k \leq n$,

$$\mathbb{E}[X_{i_1} \dots X_{i_k}] = \sum_{\pi \in \mathcal{P}_2(k)} \prod_{(r,s) \in \pi} \mathbb{E}[X_{i_r} X_{i_s}], \quad (1.4)$$

where $(r, s) \in \pi$ indicates a pair (r, s) that is in the pair partition π .

This expression already tells us that the odd moments are identically zero, since one cannot construct pair partitions for a tuple $[k]$ if k is odd. So, we only need to compute the even moments. Thus, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{E}[\text{tr}(G^{2k})] &= \frac{1}{N^{k+1}} \sum_{i_1, \dots, i_{2k}=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \prod_{(r,s) \in \pi} \mathbb{E}[G(i_r, i_{r+1})G(i_s, i_{s+1})] \\ &= \frac{1}{N^{k+1}} \sum_{i_1, \dots, i_{2k}=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \prod_{(r,s) \in \pi} \mathbf{1}_{(i_r, i_{r+1})=(i_s, i_{s+1})}. \end{aligned}$$

While there are two cases where the indicator is in force, namely $i_r = i_s$ or $i_r = i_{s+1}$, it turns out that the latter is the contributing factor in the limit. In particular, we get $i_r = i_{\pi(r)+1} = i_{\gamma\pi(r)}$, where $\gamma := (1, 2, \dots, 2k)$ is the shift by 1 modulo $2k$ permutation and, for any partition π , $\gamma\pi$ is read as a composition of two permutations by reading π as a permutation. Thus, we have that $\mathbf{i} := \{i_1, \dots, i_{2k}\}$ is constant on the cycles of $\gamma\pi$. We skip some technical steps, which involve the interchange of summands, and obtain the expression

$$\mathbb{E}[\text{tr}(G^{2k})] \sim \frac{1}{N^k} \sum_{\pi \in \mathcal{P}_2(2k)} N^{\#\gamma\pi},$$

where $\#\gamma\pi$ is the number of blocks in $\gamma\pi$, and $\sim$ means asymptotic. The contributing partitions are the *non-crossing pair partitions* $NC_2(2k)$, where we have that, for any $\pi \in NC_2(2k)$, $\#\gamma\pi = k + 1$, and, for $\pi \in \mathcal{P}_2(2k) \setminus NC_2(2k)$, $\#\gamma\pi \leq k$. Figure 1.4 illustrates some partitions of $\{1, 2, 3, 4\}$, with π_1 and π_2 as non-crossing pair partitions. This combinatorial approach yields that, for any even moments, we have

$$\lim_{N \rightarrow \infty} \mathbb{E}[\text{tr}(G^{2k})] = |NC_2(2k)| = C_k,$$

where C_k is the k -th Catalan number defined as

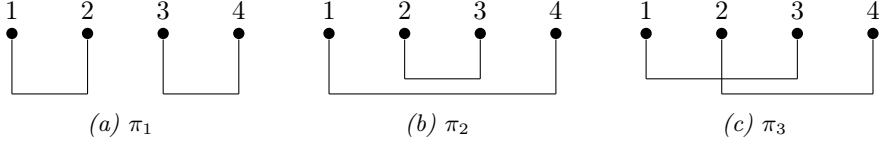
$$C_k = \frac{1}{k+1} \binom{2k}{k}.$$

The Catalan numbers are the even moments of the semicircle law μ_{sc} , which also has odd moments identically 0.

Let us proceed with the Stieltjes transform method. We begin by fixing $z \in \mathbb{C}^+$. Let μ_{sc} be the limiting measure of the ESD of G , which we a priori know is the semicircular law. To derive a recursive expression for $S_{\mu_{sc}}(z)$, one can use the moment relation (1.3), along with the following relation for Catalan numbers:

$$C_{k+1} = \sum_{i=0}^k C_i C_{k-i}$$

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Figure 1.4: Pair partitions of $\{1, 2, 3, 4\}$.

for any $k \in \mathbb{N}$. Manipulating the terms, we get

$$S_{\mu_{sc}}(z) = -\frac{1}{z + S_{\mu_{sc}}(z)}, \quad (1.5)$$

which is the unique analytic equation that characterises μ_{sc} (Bai and Silverstein [2010]). It is known that pointwise convergence of Stieltjes transforms (in z) implies weak convergence of measures (and vice versa). To that end, we analyse the resolvent matrix.

Let $r_{ij} := R_G(z)(i, j)$. It is well-known that $r_{ij} \in \mathbb{C}^+$ for all $i, j \in [N]$ (Bai and Silverstein [2010]). One approach to prove that $S_{\text{ESD}(G)}(z)$ converges to $S_{\mu_{sc}}(z)$ for each $z \in \mathbb{C}^+$ would be to use the resolvent identities (see Bordenave [2019]). However, there is a different approach that is used later on in the thesis. For any $z \in \mathbb{C}^+$, the following is a fact from complex analysis:

$$z = i \int_0^\infty e^{-itz^{-1}} dt.$$

So, we have for any $k \in [N]$

$$r_{kk} = i \int_0^\infty e^{-itr_{kk}^{-1}} dt.$$

From Bordenave [2019], we use the *Schur complement formula*, which gives us

$$r_{kk} = -\frac{1}{z + \sum_{i,j \neq k} \tilde{r}_{ij} G(i, k) G(j, k)},$$

where $\tilde{r}_{ij}$ is the (i, j) -th entry of the matrix $R_{G^{(k)}}(z) := (G^{(k)} - zI)^{-1}$, and $G^{(k)}$ is the matrix G with the k -th row and column deleted. We will not spell out details here, but rather give a heuristic of what the computation looks like.

Namely,

$$\begin{aligned}
 \mathbb{E}[r_{kk}] &= i\mathbb{E} \left[\int_0^\infty e^{itz} e^{it \sum_{i,j \neq k}^N \tilde{r}_{ij} G(i,k) G(j,k)} dt \right] \\
 &\approx i\mathbb{E} \left[\int_0^\infty e^{itz} e^{it \sum_{j \neq k}^N \tilde{r}_{jj} G(j,k)^2} dt \right] \\
 &\approx i \int_0^\infty e^{itz} \exp \left\{ it \mathbb{E} \left[\sum_{j \neq k}^N r_{jj} G(j,k)^2 \right] \right\} dt \\
 &\approx i \int_0^\infty \exp \{ it(z + \mathbb{E}[\text{tr}(\mathbf{R}_G(z))]) \} dt = -\frac{1}{z + \mathbb{E}[\text{tr}(\mathbf{R}_G(z))]},
 \end{aligned}$$

where each approximation requires justification, and becomes an equality in the limit $N \rightarrow \infty$. Summing over i on both sides, scaling by N and taking the limit $N \rightarrow \infty$ gives us (1.5) by the relation between the Stieltjes transform and the matrix resolvent. Using this approach has some advantages, particularly when dealing with random matrices with heavy-tailed entries ([Benaych-Georges et al. \[2014\]](#)).

Note that in both approaches, we only illustrate the convergence of the expected empirical measure. However, there are concentration results in both approaches that yield convergence in probability or almost surely ([Bordenave \[2019\]](#), [Speicher \[2024\]](#)).

Both approaches offer new insights into the problem as well as establish mysterious connections with numerous other areas, making random matrix theory an ideal playground for modern mathematics. The analytic approach has opened up the area of understanding local weak convergence, and many open problems were resolved in the last few years (see [Erdős and Yau \[2017\]](#) for more details).

§1.4.3 Free Probability

Consider two random variables X_1 and X_2 , where X_1 takes values -1 or 1 with probability $\frac{1}{2}$, and X_2 takes values 0 or 1 independently with probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Then, the distribution of the random variable $Y = X_1 X_2$ is the same as that of $\tilde{Y} = X_2 X_1$, and we write $X_1 X_2 \stackrel{d}{=} X_2 X_1$. Indeed, the probability that $X_1 X_2$ takes a value, for example 1 , is the same as the probability that $X_2 X_1$ takes that value, which in our example is $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. We say that X_1 and X_2 commute. We ask out of curiosity:

Are there instances when X_1 and X_2 do not commute? For instance, what if X_1 and X_2 are not real-valued, but are matrix-valued?

Classical probability studies random variables that commute, and a crucial concept in classical probability is that of *independence*: The outcome of X_1 does not affect X_2 and vice-versa. How do we abstract to a non-commutative setting? Does the concept of independence extend as well?

In the 1980s, the concept of *freeness*, or *free independence*, was studied by Dan Voiculescu in the context of operator algebras (Voiculescu [1985]). The generalisation of classic random variables to a non-commutative setting was through this very notion, which is a non-commutative analogue of (classical) independence. The combinatorial aspects are summarised in the classical text by Nica and Speicher [2006]. We now begin with some technical definitions.

Definition 1.4.3 (Non-commutative probability space).

A non-commutative probability space $(\mathcal{A}, \varphi)$ consists of a unital (associative) algebra $\mathcal{A}$ over $\mathbb{C}$ equipped with a linear functional $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ such that $\varphi(1) = 1$.

Let us fix an index set I . Elements of the space $(\mathcal{A}, \varphi)$ are called *non-commutative random variables*, and for any $a \in \mathcal{A}$, $\{\varphi(a^n)\}_{n \in \mathbb{N}}$ are the *moments* of a . The *joint distribution* of $a_1, \dots, a_k \in \mathcal{A}$ for any $k \in \mathbb{N}$ is the collection of mixed moments $\varphi(a_{i_1}, \dots, a_{i_\ell})$ for each $\ell \in \mathbb{N}$ and $i_1, \dots, i_\ell \in [k]$.

Definition 1.4.4 (Freeness).

Let $(\mathcal{A}_i)_{i \in I}$ be the unital subalgebras of $\mathcal{A}$. These are said to be *free* if, for any $k \in \mathbb{N}$, $\varphi(a_1 \dots a_k) = 0$ whenever:

- For $a_j \in \mathcal{A}_{i_j}$ with $i_j \in I$, $\varphi(a_j) = 0$ for all $j \in [k]$;
- $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{k-1} \neq i_k$.

Recall that in classical probability theory, one studies random variables over a (classical) probability space of the form $(\Omega, \mathcal{F}, \mathbb{P})$. The generalisation to the non-commutative setting deviates from the notion of an underlying event space and law, and is instead developed over the notion of a non-commutative algebra of random variables and their “expectations”. In fact, the functional φ is the non-commutative analogue of the classical notion of expectation. Similar to how classical random variables $(X_i)_{i \in I} \in (\Omega, \mathcal{F}, \mathbb{P})$ are said to be independent if the sigma-fields $(\mathcal{F}_i)_{i \in I}$ generated by them are independent, we say that random variables $(a_i)_{i \in I} \in (\mathcal{A}, \varphi)$ are said to be free if their generated unital subalgebras $(\mathcal{A}_i)_{i \in I}$ are free. This abstraction allows one to study a larger variety of objects, such as random matrices or random operators, as well as objects in other areas, notably in quantum mechanics.

Recovering classical probability is fairly straightforward, and we illustrate it for bounded random variables as follows: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability

space. Set $\mathcal{A} := L^\infty(\Omega, \mathbb{P})$ as the unital algebra of bounded measurable functions (“random variables”) $X : \Omega \rightarrow \mathbb{C}$, and set φ to be the unital linear functional on $\mathcal{A}$ as the “expectation” with respect to $\mathbb{P}$, that is,

$$\varphi(X) := \mathbb{E}[X], \quad X \in \mathcal{A}.$$

Note that $\varphi(1)$ corresponds to $\mathbb{P}(\Omega) = 1$.

Voiculescu, in Voiculescu [1985], studied the notion of freeness in the context of *Von-Neumann algebras* (also called W^* -algebras). In particular, if G is group, then saying that its subgroups $(G_i)_{i \in I}$ are free is equivalent to saying that the subalgebras $(\mathbb{C}G_i)_{i \in I}$ are free in the space $(\mathbb{C}G, \varphi_G)$ (Speicher [2011, Proposition 1.3]), where $\mathbb{C}G$ is the group algebra of G and $\varphi_G : \mathbb{C}G \rightarrow \mathbb{C}$ is a unital functional. Certain $*$ -algebras, in particular C^* -algebras, are of particular interest.

Definition 1.4.5 (C^* -algebras).

A C^* -algebra is a Banach algebra $\mathcal{A}$ over $\mathbb{C}$ such that it is a $*$ -algebra possessing the involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$ satisfying $\|xx^*\| = \|x\|^2$ for each $x \in \mathcal{A}$.

Any C^* -algebra is isomorphic to a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, for some Hilbert space $\mathcal{H}$.

Definition 1.4.6 (W^* -algebras).

A W^* -algebra, or a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, is a C^* -algebra that is closed under the weak operator topology, that is, if any net $A_\alpha \in \mathcal{A}$ converges to $A \in \mathcal{B}(\mathcal{H})$ in the weak operator topology, then $A \in \mathcal{A}$.

The respective non-commutative probability spaces are called a C^* -probability space and a W^* -probability space.

Definition 1.4.7 (Tracial, state, and faithful functionals).

Let $(\mathcal{A}, \varphi)$ be a C^* -probability space.

- If $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$, then φ is a state.
- We say φ is tracial if $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$.
- We say φ is faithful if for all $a \in \mathcal{A}$, $\varphi(a^*a) = 0$ implies $a = 0$.

Naturally, one would wonder if there are universality results in free probability, as is the case for classical probability. In particular, a natural question would be regarding a generalisation of the classical Central Limit Theorem, where sums of independent and identically distributed (i.i.d.) centred random variables having a finite variance converge to the *standard normal distribution* under appropriate scaling. In free probability, such a result does exist, but the limiting law is not the normal distribution; rather, it is the non-commutative analogue of the normal distribution (see Speicher [2011]).

Theorem 1.4.8 (Free Central Limit Theorem).

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $(a_i)_{i \in I} \in \mathcal{A}$ be a family of free random variables such that $\varphi(a_i) = 0$ and $\varphi(a_i^2) = 1$ for each $i \in I$. Further, assume that $(a_i)_{i \in I}$ are identically distributed, in the sense that $\varphi(a_i^r) = \varphi(a_j^r)$ for any $r \in \mathbb{N}$ and all $i, j \in I$. Then, if $S_n = \sum_{i=1}^n a_i$, then, for any $k \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} \varphi(n^{-k/2} S_n^k) = \varphi(s^k),$$

where s is the semicircle variable, or the semicircle element, with

$$\varphi(s^k) = \begin{cases} 0, & k \text{ is odd,} \\ C_m = \frac{1}{m+1} \binom{2m}{m}, & k = 2m \text{ for some } m \in \mathbb{N}. \end{cases}$$

In the literature, the limiting law is called the semicircle law, or the *Wigner semicircle law*, named after the theoretical physicist Eugene Wigner, whose pioneering work in the 1950s on the study of eigenvalue statistics of random matrices led to the foundation of *random matrix theory*.

Connections between random matrices and free probability were established in 1991 in the seminal work of Voiculescu ([Voiculescu \[1991\]](#)), where it was shown that random matrix models exhibit *asymptotic freeness*. This allows one to exploit tools from free probability to analyse various random matrix problems.

For any $N \in \mathbb{N}$, a $*$ -probability space of random $N \times N$ matrices is just $(M_N(L^{\infty-}(\Omega, \mathbb{P})), \text{tr} \otimes \mathbb{E})$, where $(\Omega, \mathbb{P})$ is a classical probability space, and

$$L^{\infty-}(\Omega, \mathbb{P}) := \bigcap_{1 \leq p < \infty} L^p((\Omega, \mathbb{P})),$$

and for any complex algebra $\mathcal{A}$, $M_N(\mathcal{A}) \cong M_N(\mathbb{C}) \otimes \mathcal{A}$ is the space of $N \times N$ matrices with entries drawn from $\mathcal{A}$. Moreover, $\mathbb{E}$ is the expectation with respect to the law $\mathbb{P}$. Recall that, for any (random) matrix $\mathbf{M}_N$, we have

$$\text{Tr}(\mathbf{M}_N) := \sum_{i=1}^N \mathbf{M}_N(i, i) = \sum_{i=1}^N \lambda_i(\mathbf{M}_N),$$

where $(\lambda_i(\mathbf{M}_N))_{i=1}^N$ are the *eigenvalues* of $\mathbf{M}_N$. The following result, which is an extension of the original work by Voiculescu, shows asymptotic freeness of Gaussian ensembles and deterministic matrices (see [Speicher \[2011, Theorem 6.14\]](#)).

Theorem 1.4.9 (Asymptotic freeness in matrix ensembles).

For $t \in \mathbb{N}$ and $N \in \mathbb{N}$, let $\mathbf{G}^{(1)}, \dots, \mathbf{G}^{(t)}$ be t independent $N \times N$ Gaussian unitary ensembles $\text{GUE}(N)$. Let $X_N \in M_N(\mathbb{C})$ be a deterministic $N \times N$ matrix such

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that $\sup_N \|X_N\| \leq C$ for some $C > 0$ (where $\|\cdot\|$ denotes the Hilbert-Schmidt norm) and $X_N \xrightarrow{d} x$ in the space $(\mathcal{A}, \varphi)$, that is,

$$\lim_{N \rightarrow \infty} \text{tr}(X_N^k) = \varphi(x^k)$$

for each $k \in \mathbb{N}$. Then

$$(G^{(1)}, \dots, G^{(t)}, X_N) \xrightarrow{d} (s_1, \dots, s_t, x),$$

where $s_1, \dots, s_t$ are semicircle elements in $(\mathcal{A}, \phi)$ and $s_1, \dots, s_t, x$ are free, that is, for all $m \in \mathbb{N}$, $q : m \rightarrow \mathbb{N}_0$, and $p : [m] \rightarrow [t]$,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\text{tr} \left(G^{(p(1))} X_N^{q(1)} \dots G^{(p(m))} X_N^{q(m)} \right) \right] = \varphi \left(s_1^{(p(1))} x^{q(1)} \dots s_t^{(p(m))} x^{q(m)} \right).$$

The above theorem shows that $(\{G^{(i)}\}_{1 \leq i \leq t}, X_N)$ are asymptotically free, allowing us to conclude results about sums and products of random matrices.

The remaining technical details are quoted from [Anderson et al. \[2010\]](#), [Hazra and Maulik \[2013\]](#).

Definition 1.4.10 (Affiliated operators).

A self-adjoint operator X is said to be affiliated to a W^* -algebra $\mathcal{A}$, if $f(X) \in \mathcal{A}$ for an bounded Borel function f on $\mathbb{R}$.

We call self-adjoint operators associated to $\mathcal{A}$ random elements of $\mathcal{A}$. For any affiliated random element X , the algebra generated by X is defined as $\mathcal{A}_X := \{f(X) : f \text{ bounded measurable}\}$. Naturally, $X_1, X_2 \in \mathcal{A}$ are free if $\mathcal{A}_{X_1}, \mathcal{A}_{X_2}$ are free, as in the following definition.

Definition 1.4.11 (Free operators).

Self-adjoint operators $(X_i)_{i \in I}$ affiliated with a W^* -algebra $\mathcal{A}$ are said to be free if and only if the algebras generated by $\{f(X_i) : f \text{ bounded measurable}\}_{i \in I}$ are free.

Definition 1.4.12 (Law of an operator).

For a self-adjoint operator (or a random element) X affiliated to a W^* -algebra $\mathcal{A}$, and the probability space $(\mathcal{A}, \varphi)$, the law of X is the unique probability measure μ_X on $\mathbb{R}$ satisfying

$$\varphi(f(X)) = \int_{\mathbb{R}} f(t) \mu_X(\mathrm{d}x)$$

for every bounded Borel function f on $\mathbb{R}$.

If Λ is the projection-valued spectral measure associated with X (which is guaranteed by the spectral theorem), with Λ_A denoting the measure evaluated at a set A , then

$$\mu_X(-\infty, x] = \varphi(\Lambda_{(-\infty, x]}(X)).$$

The following is quoted from [Anderson et al. \[2010, Proposition 5.3.34\]](#).

Proposition 1.4.13.

Let $\mu_1, \dots, \mu_p$ be probability measures on $\mathbb{R}$. Then there exists a W^ -probability space $(\mathcal{A}, \varphi)$ with φ a normal faithful tracial state, and self-adjoint operators $(X_i)_{1 \leq i \leq p}$ affiliated with $\mathcal{A}$, with laws $(\mu_i)_{1 \leq i \leq p}$ that are free.*

From [Anderson et al. \[2010, Property 5.3.34, Corollary 5.3.35\]](#), one can always construct a Hilbert space $\mathcal{H}$, a tracial state φ , and two free variables X_1 and X_2 with laws μ_1 and μ_2 , respectively, affiliated with the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$. Then, *free additive convolution* of μ_1 and μ_2 , denoted as $\mu_1 \boxplus \mu_2$, is the law of $X_1 + X_2$. Additionally, if either X_1 or X_2 is non-negative, then the *free multiplicative convolution* $\mu_1 \boxtimes \mu_2$ is the law of $X_1 X_2$. The extension of free convolutions to unbounded measures can be done in the context of finite von Neumann algebras. Assume that $\mathcal{A}$ is a finite von Neumann algebra with a normal faithful tracial state φ , that is, $(\mathcal{A}, \varphi)$ is a tracial W^* -probability space and $\mathcal{A}$ is acting on a Hilbert space $\mathcal{H}$. A closed, densely defined operator T on $\mathcal{H}$ is *affiliated* with $\mathcal{A}$ if its polar decomposition $T = uX$ has the property that $u \in \mathcal{A}$ and X is affiliated with $\mathcal{A}$. Let $\tilde{\mathcal{A}}$ denote the set of all operators on $\mathcal{H}$ that are affiliated with $\mathcal{A}$. Then, $\tilde{\mathcal{A}}$ is an algebra, that is, if $X, Y \in \tilde{\mathcal{A}}$, then $X + Y$ and XY are densely defined, closable, and their closures are in $\tilde{\mathcal{A}}$. See [Bercovici and Voiculescu \[1993\]](#) for further details.

§1.5 Spectral approach to random graphs

Spectral analysis of random graph models studies the limiting spectral distribution of the associated random matrices. The analysis follows a similar structure as in random matrix theory, where we begin with the ESD of the matrix of the finite graph and study its behaviour asymptotically as the size tends to infinity.

Results on the bulk distribution in random matrix theory and spectral theory of random graphs are *CLT-type*, that is, they have the same flavour as the free central limit theorem. In particular, if $\mathbf{M}_N$ is some Hermitian random matrix with entries having law $\mathbb{P}$, which could also be an adjacency or a Laplacian matrix, then the main question is as follows: Does there exist a (possibly random) measure μ_0 such that

$$\text{ESD} \left(\frac{\mathbf{M}_N - \mathbb{E}[\mathbf{M}_N]}{c_N} \right) \xrightarrow{*} \mu_0 ?$$

Here, $*$ denotes that this convergence could be (weakly, in the measure-theoretic sense) in distribution, in $\mathbb{P}$ –probability, or $\mathbb{P}$ –almost surely, and c_N is a scaling that is of the order of the variance of the entries, given by

$$c_N = \mathbb{E}[\text{Tr}(\mathbf{M}_N^2)] = \sum_{i,j=1}^N \mathbb{E}[\mathbf{M}_N(i,j)^2].$$

For random graph models, this scaling also turns out to be the expected degree of a uniformly chosen vertex.

§1.5.1 Revisiting the Erdős-Rényi random graph

In the case of the homogeneous ERRG(N, ε_N), it is known that in the dense case the empirical distribution converges to the semicircle law after an appropriate scaling (Jung and Lee [2018], Tran et al. [2013]). The Laplacian spectrum for the dense case was studied in Ding and Jiang [2010], Jiang [2012].

In the sparse case, the spectra converge to limiting measures that depend on the parameter $\lambda := \lim_{N \rightarrow \infty} N\varepsilon_N$. The behaviour is much more complicated in this setting. Various interesting properties for spectra of the adjacency matrix were predicted by Bauer and Golinelli [2001]. The existence of the limiting distribution was proved by Khorunzhy et al. [2004], who study both the adjacency and the Laplacian matrices, and also show some interesting properties of the moments and the limiting Stieltjes transform. The local geometric behaviour of sparse random graphs can be studied using the theory of local weak convergence (LWC), which builds on the works Aldous and Lyons [2007] and Benjamini and Schramm [2001]. LWC describes how a graph looks like around a uniformly chosen vertex in the limit as the size of the graph tends to infinity. For a detailed review of LWC and various other applications, see van der Hofstad [2024]. In a remarkable work by Bordenave and Lelarge [2010], where the authors study the adjacency and the Laplacian matrices, it was proved that if a graph with N vertices converges locally weakly to a Galton-Watson tree, then the Stieltjes transform of the empirical spectral distribution converges in L^1 to the Stieltjes transform of the spectral measure of the tree, and satisfies a recursive distributional equation. The example of a homogeneous ERRG was treated in [Bordenave and Lelarge, 2010, Example 2].

The limiting measure of the adjacency matrix of the sparse ERRG depends on λ and is still very non-explicit. It was proved by Bordenave et al. [2017], Arras and Bordenave [2023] that the measure has an absolutely continuous component if and only if $\lambda > 1$. The size of the atom at the origin was computed by Bordenave et al. [2011], and the nature of the atomic part of the measure was studied in Salez [2020], where it was shown that the set of atoms is dense

when $\lambda < 1$, and is linked with a countable dense ring $\mathbb{A}$ of *totally real algebraic integers*. The study of so-called extended states at the origin was initiated in [Coste and Salez \[2021\]](#), and it was shown that for $\lambda < e$ there were no extended states, while for $\lambda > e$, there are extended states.

All these results were conjectured in [Bauer and Golinelli \[2001\]](#). Most results on local limits show that properties are generally true for unimodular Galton-Watson trees. In the simulations of [Bauer and Golinelli \[2001\]](#), it is clear that when λ is slightly larger than 1, the limiting measure already starts taking the shape of the semicircle law. It was shown in [Jung and Lee \[2018\]](#) that, indeed, if $\lambda \rightarrow \infty$, then the limiting measure converges to the semicircle law. Some key questions still remain open for the sparse ERRG, such as the following:

- What are the explicit moments of μ_λ ?
- How “close” is the measure μ_λ to μ_{sc} ? Is there a way to quantify the distance between the two measures?

§1.5.2 Local weak convergence

The seminal work of [Bordenave and Lelarge \[2010\]](#) characterises the limiting spectral distribution for locally tree-like graphs. In particular, consider $\mathbf{A}_N$ and $\mathbf{\Delta}_N$ to be the scaled adjacency and Laplacian matrices, respectively, of a random graph model $\mathbb{G}_N$, such that the following hold:

- The sequence of random graphs $\{\mathbb{G}_N\}_{N \geq 1}$ has a weak limit $\mathbb{G}$.
- For a uniformly chosen root $o_N \in \mathbb{G}_N$, the degree sequence of the rooted graph $(\deg(\mathbb{G}_N, o_N))_{N \geq 1}$ is uniformly integrable.
- Let $\mathcal{G}^*$ denote the set of rooted isomorphism classes of rooted connected locally finite graphs, and let $U_2(\mathbb{G})$ be the distribution on $\mathcal{G}^* \times \mathcal{G}^*$ of the pair of rooted graphs $((\mathbb{G}, o_1), (\mathbb{G}, o_2))$, where o_1, o_2 are uniformly chosen roots of G . Then, $U_2(\mathbb{G}_N)$ converges weakly to $\mathbb{G} \otimes \mathbb{G}$, that is, to two independent and identical copies of $\mathbb{G}$.

Under the above conditions, there exists unique probability measures μ_λ and ν_λ on $\mathbb{R}$ such that $\lim_{N \rightarrow \infty} \text{ESD}(\mathbf{A}_N) = \mu_\lambda$ and $\lim_{N \rightarrow \infty} \text{ESD}(\mathbf{\Delta}_N) = \nu_\lambda$ weakly in probability. Furthermore, if $\mathbb{G}_N$ is the graph $\text{ER}_N(\varepsilon_N)$, and $\mathbf{A}_{\mathbb{G}_N}$ is the adjacency matrix of the graph, then $\mathbf{A}_N := \lambda^{-1/2} \mathbf{A}_{\mathbb{G}_N}$, and the measure μ_λ represents the expected spectral measure associated with the root of a Galton-Watson tree with offspring distribution $\text{Poi}(\lambda)$ and weights $1/\sqrt{\lambda}$. This result comes from the theory of local weak convergence (see [Benjamini and Schramm](#)

[2001], [van der Hofstad \[2024\]](#)), which is a powerful tool to study spectral measures associated with many sparse random graph models.

In particular, consider the adjacency matrix (though a similar result holds for the Laplacian matrix). Consider the space $\mathbb{H}$ of holomorphic functions $f : \mathbb{C}^+ \rightarrow \mathbb{C}^+$, equipped with the topology induced by uniform convergence on compact sets. Then, $\mathbb{H}$ is a complete separable metrizable compact space. The *resolvent* of the adjacency matrix is given as

$$R_{\mathbf{A}_N}(z) = (\mathbf{A}_N - zI)^{-1}$$

for each $z \in \mathbb{C}^+$. The map $z \mapsto R_{\mathbf{A}_N}(z)(i, i)$ is in $\mathbb{H}$, and the Stieltjes transform of $\text{ESD}(\mathbf{A}_N)$ is given by $\text{tr } R_{\mathbf{A}_N}(z)$, where $\text{tr} = N^{-1} \text{Tr}$ denotes the normalised trace operator. Let $\mathcal{G}^*$ denote the set of rooted isomorphism classes of rooted connected locally finite graphs. Assume that the random graph sequence $(\mathbb{G}_N)_{N \geq 1}$ has the random local limit $\mathbb{G} \in \mathcal{G}^*$, and assume further that $\mathbb{G}$ is a *Galton-Watson Tree* with degree distribution F_* , that is, a rooted random tree obtained from a Galton-Watson process with root having offspring distribution F_* and all children having a distribution F (which may or may not be the same as F_*).

Let $S_{\mathbf{A}_N}(z)$ denote the Stieltjes transform of the empirical measure $\text{ESD}(\mathbf{A}_N)$. It was shown in [[Bordenave and Lelarge, 2010](#), Theorem 2] that there exists a unique probability measure Q on $\mathbb{H}$ such that, for each $z \in \mathbb{C}^+$,

$$Y(z) \stackrel{d}{=} \left(z + \sum_{i=1}^P Y_i(z) \right)^{-1}$$

where P has distribution F and $Y, \{Y_i\}_{i \geq 1}$ are i.i.d. with law Q and independent of P . Moreover,

$$\lim_{N \rightarrow \infty} S_{\mathbf{A}_N}(z) = \mathbb{E}X(z) \text{ in } L^1,$$

where $X(z)$ is such that:

$$X(z) \stackrel{d}{=} - \left(z + \sum_{i=1}^{P_*} Y_i(z) \right)^{-1},$$

where $\{Y_i\}_{i \geq 1}$ are i.i.d. copies with law Q , and P_* is a random variable independent of $\{Y_i\}_{i \geq 1}$ having distribution F_* .

The analysis and expressions are similar for S_{Δ_N} , as illustrated in [Bordenave and Lelarge \[2010\]](#).

§1.5.3 Further literature

Adjacency matrix

In recent years, there has been significant research on inhomogeneous Erdős–Rényi random graphs, which can be equivalently modelled by Wigner matrices with a variance profile. The limiting spectral distribution of the adjacency matrix of such graphs has been studied in [Chakrabarty et al. \[2021b\]](#), [Zhu \[2020\]](#), [Bose et al. \[2022\]](#), while local eigenvalue statistics have been analysed in [Dumitriu and Zhu \[2019\]](#), [Ajanki et al. \[2019\]](#). [Zhu and Zhu \[2024\]](#) studies the fluctuations of the linear eigenvalue statistics for a wide range of such inhomogeneous graphs. Additionally, various properties of the largest eigenvalue have been investigated in [Cheliotis and Louvaris \[2024\]](#), [Husson \[2022\]](#), [Chakrabarty et al. \[2022\]](#), [Ducatez et al. \[2024\]](#), [Dionigi et al. \[2023\]](#). One of the most significant properties of the limiting spectral measure for random graphs is its absolute continuity with respect to the Lebesgue measure, which is closely tied to the concept of mean quantum percolation [[Bordenave et al., 2017](#), [Anantharaman et al., 2021](#), [Arras and Bordenave, 2023](#)]. Quantum percolation investigates whether the limiting measure has a non-trivial absolutely continuous spectrum. Recently, it was shown in [Arras and Bordenave \[2023\]](#) that the adjacency operator of a supercritical Poisson Galton-Watson tree has a non-trivial absolutely continuous part when the average degree is sufficiently large. Additionally, [Bordenave et al. \[2017\]](#) demonstrated that supercritical bond percolation on $\mathbb{Z}^d$ has a non-trivial absolutely continuous part for $d = 2$. These results motivate similar questions for *kernel-based random graphs* and other percolation models. In [Bhamidi et al. \[2012\]](#) the spectra of the adjacency matrix of random trees are studied, including the *preferential attachment tree*. Spectral analysis of weighted adjacency matrices has also been used in hidden clique problems (see [Chatterjee et al. \[2025\]](#)).

Laplacian Matrix

[Bryc et al. \[2006\]](#) established that, for large symmetric matrices with i.i.d. entries, the empirical spectral distribution (ESD) of the corresponding Laplacian matrix converges to the free convolution of the semicircle law and the standard Gaussian distribution. In the context of sparse Erdős–Rényi random graphs, [Huang and Landon \[2020\]](#) studied the local law of the ESD of the Laplacian matrix. They demonstrated that the Stieltjes transform of the ESD closely approximates that of the free convolution of the semicircle law and a standard Gaussian distribution, down to scale N^{-1} . Additionally, they showed that the gap statistics and averaged correlation functions align with those of the Gaussian Orthogonal Ensemble in the bulk. [Ding and Jiang \[2010\]](#) investigated the

spectral distributions of adjacency and Laplacian matrices of random graphs, assuming that the variance of the entries depend only on N . They established the convergence of the ESD of these matrices under such conditions. The results for the Erdős-Rényi random graphs were extended to the inhomogeneous setting by [Chakrabarty et al. \[2021b\]](#). In a recent work, [Chatterjee and Hazra \[2022\]](#) derived a combinatorial way to describe the limiting moments for a wide variety of random matrix models with a variance profile.

§1.6 Outline of the thesis

The three main chapters of this thesis are based on three papers on spectral properties of inhomogeneous random graph models.

Chapter 2

In Chapter 2, we study the inhomogeneous Erdős-Rényi random graph model on N vertices in the sparse setting, where vertices have deterministic weights and edges are added between two vertices independently with a probability that is proportional to a function of their two weights, scaled by a factor of N . We take the vertex set $[N]$, and consider a sequence of *deterministic* weights $(w_i)_{i=1}^N$, such that if o_N is a uniform random variable on $[N]$, then there exists a limiting random variable W with law μ_w such that $w_{o_N} \xrightarrow{d} W$. We add edges independently with probability

$$p_{ij} := \varepsilon_N f(w_i, w_j), i, j \in [N],$$

where ε_N is a sparsity parameter such that $N\varepsilon_N \rightarrow \lambda \in (0, \infty)$, and f is a bounded continuous function.

We study the scaled adjacency matrix $\mathbf{A}_N$ of the random graph, with entries given by

$$\mathbf{A}_N(i, j) = \mathbf{A}_N(j, i) \stackrel{d}{=} \frac{1}{\sqrt{\lambda}} \text{Ber}(p_{ij}).$$

In Theorem 2.3.7, we find that there exists a deterministic non-degenerate limiting measure μ_λ such that $\lim_{N \rightarrow \infty} \text{ESD}(\mathbf{A}_N) = \mu_\lambda$ in probability, and the moments of μ_λ are given by

$$\int x^k \mu_\lambda(dx) = \begin{cases} 0, & k \text{ is odd,} \\ \sum_{l=2}^{k/2+1} \sum_{\substack{\pi \in SS(k): \\ |\gamma\pi|=l}} \lambda^{l-1-\frac{k}{2}} t(G_{\gamma\pi}, f, \mu_w), & k \text{ is even,} \end{cases}$$

where $SS(k)$ is the set of Simple Symmetric partitions of $[k]$, as in [Bose et al. \[2022\]](#), $G_{\gamma\pi}$ is a graph associated to a partition π that is described later, and

$t(\cdot, \cdot, \cdot)$ is a generalisation of the graph homomorphism density that appears in graphon theory in [Lovász and Szegedy \[2006\]](#). We further find that $\lim_{\lambda \rightarrow \infty} \mu_\lambda = \mu_f$, where μ_f is the measure in the dense regime that appears in [Chakrabarty et al. \[2021b\]](#), [Zhu \[2020\]](#), which extends the results of [Jung and Lee \[2018\]](#).

In Theorem 2.3.9, under the assumption that f is Lipschitz in one coordinate, we show that, in an appropriate Banach space $\mathcal{B}$, there exists a functional $\phi_z^* \in \mathcal{B}$ that is the unique solution to a fixed-point equation in $\mathcal{B}$, such that

$$S_{\mu_\lambda}(z) = i \int_0^\infty e^{-\lambda d_f(y)} \int_0^\infty e^{ivz} e^{\lambda \phi_z^*(y, \frac{v}{\lambda})} dv \mu_w(dy), \quad z \in \mathbb{C}^+,$$

where $d_f(y) = \int f(x, y) \mu_w(dx)$. This chapter is based on the paper [Avena et al. \[2023\]](#).

Chapter 3

In Chapter 3, we study a model with spatial geometry. We consider a kernel-based random graph model on a d -dimensional discrete torus $\mathbf{V}_N$, which serves as the vertex set of the random graph. Each vertex $i \in \mathbf{V}_N$ has a random weight W_i , where $(W_i)_{i \in \mathbf{V}_N}$ are i.i.d. random variables sampled from a Pareto distribution W (whose law is denoted by $\mathbf{P}$ and measure μ_W) with parameter $\tau - 1$, where $\tau > 1$, that is,

$$\mathbf{P}(W > t) = t^{-(\tau-1)} \mathbf{1}_{\{t \geq 1\}} + \mathbf{1}_{\{t < 1\}}.$$

Conditionally on the weights, edges are added independently with probability

$$p_{ij} := P^W(i \leftrightarrow j) = \frac{\kappa(W_i, W_j)}{\|i - j\|^\alpha} \wedge 1,$$

where $\|\cdot\|$ is the torus distance, $\alpha \in (0, d)$ is a parameter of choice, and κ is a kernel that has the form $\kappa(x, y) := (x \vee y)(x \wedge y)^\sigma$ for some $0 < \sigma < \tau - 1$, as in [Jorritsma et al. \[2023\]](#).

We consider the scaled adjacency matrix of this graph, which is a symmetric random matrix with entries

$$\mathbf{A}_N(i, j) = \mathbf{A}_N(j, i) \stackrel{d}{=} c_N^{-1/2} \text{Ber}(p_{ij}),$$

where $c_N = N^{1-\alpha}$. For $\tau > 2$, Theorems 3.2.1 and 3.2.3 show that there exists a deterministic non-degenerate limiting measure $\mu_{\sigma, \tau}$ with finite second moment such that

$$\lim_{N \rightarrow \infty} \text{ESD}(\mathbf{A}_N) = \mu_{\sigma, \tau}, \quad \text{in } \mathbb{P}\text{-probability},$$

where $\mathbb{P} = \mathbf{P} \otimes P^W$ is the joint law.

Theorem 3.2.4 shows that $\mu_{\sigma, \tau}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Theorem 3.2.5 shows that, when $\tau > 3$ and $\sigma < \tau - 2$,

in an appropriate Banach space $\mathcal{B}$ there exists a unique analytic solution $a^* \in \mathcal{B}$ to a fixed-equation in $\mathcal{B}$, such that

$$S_{\mu_{\sigma,\tau}}(z) = \int_1^\infty a^*(z, x) \mu_W(dx), \quad z \in \mathbb{C}^+.$$

When $\sigma = 1$, there is an explicit description of the measure. In particular, Theorem 3.2.2 tells us that $\mu_{1,\tau} = \mu_{sc} \boxtimes \mu_W$, with tail asymptotic $\mu_{1,\tau}(x, \infty) \sim C_\tau x^{-2(\tau-1)}$ as $x \rightarrow \infty$, for some τ -dependent constant $C_\tau < \infty$. Here, $\boxtimes$ is the *free multiplicative convolution* of measures. This chapter is based on the paper [Cipriani et al. \[2025\]](#).

Chapter 4

In Chapter 4, we take the model from Chapter 3 with $\sigma = 1$ and $\tau > 3$, that is, weights with finite variance. This model is called the scale-free percolation model. We begin with the scaled adjacency $\mathbf{A}_N$ as in Chapter 3, and define the corresponding Laplacian as $\Delta_N = \mathbf{A}_N - \mathbf{D}_N$. We study the centred Laplacian $\Delta_N^\circ := \Delta_N - \mathbb{E}[\Delta_N]$. Theorem 4.2.1 shows that there exists a deterministic limiting measure ν_τ such that

$$\lim_{N \rightarrow \infty} \text{ESD}(\Delta_N^\circ) = \nu_\tau \quad \text{in } \mathbb{P}\text{-probability}.$$

Theorem 4.2.5 identifies ν_τ in terms of the spectral distribution of some non-commutative operators. Heuristically, ν_τ has (in an operator sense) the law given by the spectral law of

$$W^{1/2} S W^{1/2} + m_1 W^{1/4} G W^{1/4},$$

where W is an unbounded operator with spectral law given by the Pareto distribution, S is a bounded compact operator whose spectral law is the semicircle law, and G is an unbounded operator whose law is given by the Gaussian distribution. We will see a more formal description of this later on in Chapter 4. This chapter is based on the paper [Hazra and Malhotra \[2025\]](#).

Chapter 5

In Chapter 5, we show some further simulations of the above models, and conclude with a short discussion on open problems.

§1.7 Concluding remarks

The thesis gives a spectral perspective to some inhomogeneous random graph problems. The results mainly describe properties of the bulk distribution. There are many other interesting features, and we hope that this thesis will form a baseline for future research.