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Editors' Suggestion

Promoting Fluctuation Theorems into Covariant Forms

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The principle of covariance, a cornerstone of modern physics, asserts the equivalence of all inertial frames of reference. Fluctuation theorems, as extensions of the second law of thermodynamics, establish universal connections between irreversibility and fluctuation in terms of stochastic thermodynamic quantities. However, these relations typically assume that both the thermodynamic system and the heat bath are at rest with respect to the observer, thereby failing to satisfy the principle of covariance. In this Letter, by introducing covariant work and heat that incorporate both energy-related and momentum-related components, we promote fluctuation theorems into covariant forms applicable to moving thermodynamic systems and moving heat baths. We illustrate this framework with two examples: the work statistics of a relativistic stochastic field and the heat statistics of a relativistic Brownian motion. Although our Letter is carried out in the context of special relativity, the results can be extended to the nonrelativistic limit. Our Letter combines the principle of covariance and fluctuation theorems into a coherent framework and may have applications in the study of thermodynamics relevant to cosmic microwave background as well as the radiative heat transfer and noncontact friction between relatively moving bodies.

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Introduction—The second law of thermodynamics is a statement about irreversibility. Many macroscopic phenomena occur in one direction, rarely seen in reverse. When examining systems at the mesoscopic scale, irreversibility is accompanied by fluctuations, which are more precisely captured by fluctuation theorems. Fluctuation theorems [1–7], one of the central achievements in stochastic thermodynamics [8–10], relate the probability ratio between forward and backward random trajectories to the exponential of thermodynamic quantities associated with those trajectories. At the ensemble average level, the second law of thermodynamics is a corollary of fluctuation theorems.

Stochastic thermodynamics and fluctuation theorems have been studied in various systems, including those where relativistic effects are prominent. Early works on Brownian motion for relativistic particles and relativistic kinetic theory [11–15] spawned discussions on nonequilibrium phenomena in relativistic systems. Most recently, attempts have been made to develop the framework of stochastic thermodynamics for a single-particle system in Minkowski spacetime [16,17] and in curved spacetime [18–20]. For specific relativistic systems, work distributions [21–24] and fluctuation theorems [16,21–28] have been studied in the rest reference frame.

Behind almost all the statements of fluctuation theorems, an implicit assumption is that the heat bath and the thermodynamic system are initially at rest. Definitions of stochastic work and heat, as well as the formulation of fluctuation theorems, have not been extended to the system and the heat bath in motion, let alone to them in relative motion. As a pillar of modern physics, the principle of covariance is a fundamental requirement for all physical laws. However, the existing fluctuation theorems do not satisfy the principle of covariance. Hence, a significant unsolved problem is to promote the existing fluctuation theorems into covariant forms.

In this Letter, we promote the existing fluctuation theorems into covariant forms within the context of special relativity. Nonrelativistic systems are treated as a low speed limit. We find that the Lorentz covariance, which links space and time, necessitates a new consistent definition of the backward process across all inertial frames. For moving systems, in accordance with the energy-momentum four-vector, the concepts of work and heat must be generalized to four-vector form. Consequently, the refined statements of the second law of thermodynamics and fluctuation theorems must account for momentum-related quantities. When multiple systems or heat baths initially move relative to each other, existing fluctuation theorems fail, but our

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covariant theorems remain valid, extending the study of irreversibility and fluctuations to broader contexts.

Covariant fluctuation theorems—Fluctuation theorems are universal relations in thermodynamics, independent of the underlying dynamics of the system. The detailed fluctuation theorems relate the probability ratio between forward and backward trajectories, $\mathrm{Pr}(\omega)/\tilde{\mathrm{Pr}}(\tilde{\omega})$, to thermodynamic quantities, and they can be expressed in the following form:

$$\frac{\mathrm{Pr}(\omega)}{\tilde{\mathrm{Pr}}(\tilde{\omega})} = \exp(\text{thermodynamic quantity}). \tag{1}$$

Before proceeding to covariant fluctuation theorems, we need to specify the backward process and find the corresponding thermodynamic quantity in the context of special relativity.

We adopt the four-dimensional spacetime notation $x = (x^0 = t, x^1, x^2, x^3)$ and the Einstein summation rule with metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Roman scripts i, $j = 1, 2, 3$ are used for spatial components. Let us consider a general thermodynamic process over a finite-time period (precisely speaking, a process between two spacelike hypersurfaces). The system is driven by a spacetime-dependent external control field $h(x)$, so that the Lagrangian (or Hamiltonian) of the system explicitly depends on $h(x)$. Meanwhile, the system is in contact with a heat bath at the inverse temperature β. Usually, these problems are discussed in the rest frame of the heat bath, but we can switch to different inertial reference frames where the heat bath is moving.

The backward process is constructed by inverting both the space and the time of the driving field, $h(x) \to h(-x)$. This differs slightly from the usual definition that reverses only time, in order to ensure that the backward process is frame independent. Note that, as an even variable under the spacetime reversal, the velocity of the heat bath does not change. The initial distribution of the backward process will be specified later. Meanwhile, we define a backward trajectory $\tilde{\omega}$ in the backward process as the spacetime reversal of the forward trajectory ω in the forward process. The cartoon in Fig. 1 schematically illustrates a gas-expansion process and its backward process viewed by an observer (the system and the heat bath are moving at different velocities).

We continue to identify stochastic thermodynamic quantities. Energy and momentum form a four-vector under the Lorentz transformation. The importance of 4-momentum is first reflected in the canonical equilibrium distribution for a moving object, $\exp(-\beta^{\mathrm{s}}_{\mu} P^{\mu})/Z$, where P^{μ} is the 4-momentum and Z is the partition function [14,29–33]. Here, the inverse temperature four-vector of the system $\beta^{\mathrm{s}}_{\mu} = \beta^{\mathrm{s}} u^{\mathrm{s}}_{\mu}$ is defined as the rest inverse temperature β^{s} (inverse temperature observed in the reference frame where the equilibrium system is at rest on average) times the four-velocity

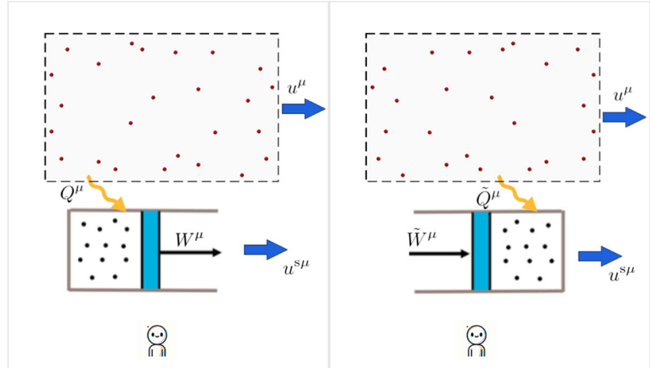

FIG. 1. A gas-expansion process and its backward process viewed by an observer: gas molecules (depicted by black dots) are in a cylinder. In the forward process (left panel), work is performed by expanding the piston, and the system exchanges heat with the heat bath (represented by red dots). The heat bath is moving at the four-velocity u^{μ}, and the initial four-velocity of the system is $u^{\mathrm{s}\mu}$. In the backward process (right panel), the piston is compressed, but the four-velocities do not change.

$u^{\mathrm{s}}_{\mu} = \eta_{\mu\nu} u^{\mathrm{s}\nu}$. We extend the stochastic work and heat [7–10] to stochastic work four-vector W^{μ} and heat four-vector Q^{μ}, corresponding to the 4-momentum change due to external driving and the 4-momentum exchange with the heat bath, respectively. These are stochastic generalizations of those discussed in the van Kampen formulation [29,34,35] and defined as functionals of individual stochastic trajectories. Furthermore, we define the stochastic entropy production of a trajectory ω as

$$\Sigma[\omega] = -\beta_{\nu} Q^{\nu}[\omega] + \Delta S[\omega]. \tag{2}$$

Here, $\beta_{\nu} = \beta \eta_{\mu\nu} u^{\mu}$ is the inverse temperature four-vector of the heat bath, with β the rest inverse temperature and u^{μ} the four-velocity of the heat bath; $\Delta S[\omega] = \ln \mathcal{P}_{\mathrm{i}}(\omega_{\mathrm{i}}) - \ln \mathcal{P}_{\mathrm{f}}(\omega_{\mathrm{f}})$ is the change of trajectory entropy (Boltzmann constant $k_{\mathrm{B}} = 1$) [6]; ω_{i} and ω_{f} are the initial and the final positions, respectively, of trajectory; $\mathcal{P}_{\mathrm{i}}$ and $\mathcal{P}_{\mathrm{f}}$ are the initial and the final distribution functions, respectively. The above entropy production is a Lorentz scalar.

We remark that there are several different formulations of relativistic thermodynamics [12,36,37], one of which, the van Kampen formulation, realizes a covariant theory; please see Supplemental Material [38] for an introduction of this formulation and its relation to other formulations. Our current theoretical framework can be viewed as a combination of the van Kampen formulation and stochastic thermodynamics.

Two ingredients are at the heart of all kinds of fluctuation theorems: initial Gibbs distribution and microreversibility [42,43]. In Supplemental Material [38], we show how these two ingredients, with some modifications, lead to covariant fluctuation theorems.

Different fluctuation theorems apply to different setups. First, the fluctuation theorem for stochastic entropy production is rather general, which holds true for arbitrary initial distribution of the forward process. The initial state of the backward process is chosen to be the spacetime reversal of the final state of the forward process. The fluctuation theorem for entropy production reads

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \mathrm{e}^{\Sigma[\omega]}. \tag{3}$$

Here, Pr and $\tilde{\Pr}$ are the trajectory probability in the forward and backward processes, respectively. ω and $\tilde{\omega}$ are a pair of forward and backward trajectories. Although this fluctuation theorem for entropy production appears similar to the usual version [6], the difference lies in the specific expression for entropy production in Eq. (2), where the momentum exchange with the heat bath must be taken into account.

Second, the covariant fluctuation theorem for work applies to the following situation. The initial distributions of the forward and the backward processes are in canonical equilibrium $\exp[-\beta_\mu P^\mu(h_\mathrm{i})]/Z_\mathrm{i}$ and $\exp[-\beta_\mu P^\mu(\tilde{h}_\mathrm{i})]/\tilde{Z}_\mathrm{i}$, at the inverse temperature four-vector β_μ, the same as that of the heat bath (therefore, no relative motion between the system and the heat bath), but with respect to their own initial driving configurations, $h_\mathrm{i}(x)$ and $\tilde{h}_\mathrm{i}(x) = h_\mathrm{f}(-x)$, respectively. The covariant fluctuation theorem for work reads

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \mathrm{e}^{\beta_\mu W^\mu[\omega] - \beta\Delta F}. \tag{4}$$

Here, $\exp(-\beta\Delta F) = Z_\mathrm{f}/Z_\mathrm{i}$ is the ratio between equilibrium partition functions. ΔF is the free energy difference.

The heat exchange fluctuation theorem applies to pure relaxation processes without external driving. The initial state is in equilibrium $\exp(-\beta^\mathrm{s}_\mu P^\mu)/Z_i$ at an inverse temperature four-vector β^s_μ different from that β_μ of the heat bath. With no external driving, the system exchanges heat four-vector with the heat bath and evolves toward a new equilibrium state. The initial state of the backward process is identical to that of the forward process, so there is no need to distinguish Pr and $\tilde{\Pr}$. The covariant heat exchange fluctuation theorem reads

$$\frac{\Pr(\omega)}{\Pr(\tilde{\omega})} = \mathrm{e}^{-(\beta_\mu - \beta^\mathrm{s}_\mu) Q^\mu[\omega]}. \tag{5}$$

The initial inverse temperature four-vector of the system $\beta^\mathrm{s}_\mu = \beta^\mathrm{s} u^\mathrm{s}_\mu$ may be unparallel to $\beta_\mu = \beta u_\mu$ of the heat bath, meaning the system initially moves relative to the heat bath [44]. Equations (4) and (5) are covariant generalizations of detailed fluctuation theorems of work [4] and heat [45], respectively.

Discussion—The above fluctuation theorems are at the trajectory level. By rearranging and integrating over all trajectories, integral covariant fluctuation theorems follow as $\langle \exp(-\Sigma)\rangle = 1$, $\langle \exp(-\beta_\mu W^\mu + \beta\Delta F)\rangle = 1$, and $\langle \exp[(\beta_\mu - \beta^\mathrm{s}_\mu) Q^\mu]\rangle = 1$. These theorems apply to the forward processes under respective conditions, without involving the backward process. Furthermore, integral fluctuation theorems imply several modified statements of the second law for arbitrary inertial observers: $\langle \Sigma \rangle = \langle -\beta_\mu Q^\mu + \Delta S\rangle \geq 0$ for an arbitrary nonequilibrium process, $\langle \beta_\mu W^\mu \rangle \geq \beta \Delta F$ for an initial equilibrium system driven out of equilibrium, and $\langle (\beta_\mu - \beta^\mathrm{s}_\mu) Q^\mu \rangle \leq 0$ for heat exchange between the system and the heat bath. Accordingly, when considering moving systems or baths, all the original statements about the second law of thermodynamics must be modified to include the momentum effect.

For the case that the system is initially at rest relative to the heat bath, the quantities $\beta u_\mu W^\mu$ and $(\beta - \beta^\mathrm{s}) u_\mu Q^\mu$ will appear in the fluctuation theorems of work and heat exchange. Here, $u_\mu W^\mu = W^0_\mathrm{rest}$ and $u_\mu Q^\mu = Q^0_\mathrm{rest}$ are conventional (energy-related) thermodynamic quantities observed in the rest frame of the heat bath. This indicates that the rest frame of the heat bath is special, where only the energy-related term does not vanish, and the usual fluctuation theorems for work and heat exchange are recovered. Nevertheless, for moving observers to construct $u_\mu W^\mu$ and $u_\mu Q^\mu$, they usually need to measure all four components.

On the other hand, if initially the system is moving relative to the heat bath, the heat exchange fluctuation theorem (5) cannot be rewritten into a rest-frame version [45], as it is impossible to find a reference frame where both the system and the bath are at rest. Examples include near-field radiative heat transfer and noncontact friction between moving bodies [46,47] and Earth's velocity relative to the cosmic microwave background [33,48–51]. Our formula shows that $(\beta_\mu - \beta^\mathrm{s}_\mu) Q^\mu$ is the key quantity associated with irreversibility.

The nonrelativistic limit of fluctuation theorems is nontrivial as well. When the three-velocity of the heat bath is relatively small, $v \ll c$, the relativistic dynamics recovers Newtonian dynamics, and the Lorentz transformation is reduced to the Galilean transformation. Specifically, the transformation rule for work (the same for heat) is $W^0 - \sum_{i=1}^3 v^i W^i = W^0_\mathrm{rest} + O(v^2/c^2)$. The momentum-related part appears in the leading order. For nonrelativistic systems, the covariant fluctuation theorems for work and heat exchange are reduced to the following formulas:

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \mathrm{e}^{\beta(W^0 - \sum_{i=1}^3 v^i W^i) - \beta\Delta F}, \tag{6}$$

$$\frac{\Pr(\omega)}{\Pr(\tilde{\omega})} = \mathrm{e}^{-(\beta-\beta^s)Q^0+\sum_{i=1}^{3}(\beta v^i-\beta^s v^{si})Q^i}. \tag{7}$$

Here, v^i and v^{si} are the three-velocities of the heat bath and the system relative to the observer, respectively. Even in the nonrelativistic limit, covariant fluctuation theorems must include the momentum-related components.

Two examples—In the following, we demonstrate the covariant fluctuation theorems in both field and particle systems, focusing on two examples: a driven relativistic stochastic field and relativistic Brownian motion of a charged particle.

First, we study field systems, which are important in relativity since interactions are transmitted via fields. Consider a classical massive scalar field ϕ driven by an external field h and in contact with a heat bath. We adopt the following relativistic covariant equation of motion:

$$\partial^\mu\partial_\mu\phi + \kappa^\mu\partial_\mu\phi + m^2\phi + \frac{\partial V(\phi)}{\partial\phi} = h + \sqrt{\frac{2\kappa}{\beta}}\xi(x), \tag{8}$$

where $V(\phi) = g\phi^4/4 + \cdots$ denotes the higher-order terms in the potential density; g and m are two parameters; β is the rest inverse temperature of the heat bath; $\kappa^\mu = \kappa u^\mu$, with κ being the friction coefficient and u^μ being the four-velocity of the heat bath. In the rest frame of the heat bath, $\kappa u^\mu\partial_\mu\phi = \kappa\dot{\phi}$ denotes the friction from the heat bath. $\xi(x)$ is the standard white noise with the correlation $\langle\xi(x)\xi(y)\rangle = \delta^4(x-y)$. The amplitude $\sqrt{2\kappa/\beta}$ of the noise ensures the detailed balance condition. The above equation is a modified model A dynamics [52,53] with inertia.

In relativity, the starting and ending time of the process are generalized to the spacelike hypersurfaces $\mathcal{A}_-$ and $\mathcal{A}_+$, respectively. The spacetime region between these two surfaces is denoted by $\mathcal{A}$. The backward process starts from $\tilde{\mathcal{A}}_- = -\mathcal{A}_+$ and ends at $\tilde{\mathcal{A}}_+ = -\mathcal{A}_-$. Here, the minus sign means multiplying every spacetime coordinate with -1. The driving field in the backward process is $\tilde{h}(x) = h(-x)$. Corresponding to a trajectory $\{\phi(x)\}_{x\in\mathcal{A}}$ in the forward process, the backward trajectory in the backward process is $\tilde{\phi}(x) = \phi(-x)$, $x \in -\mathcal{A}$.

For a field, the energy and momentum are characterized by the energy-momentum tensor $T^{\mu\nu} = [(\partial\mathcal{L})/(\partial\partial_\mu\phi)]\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$, where $\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi) + h\phi$ is the Lagrangian density. The energy-momentum tensor of the field is no longer a conserved current due to two distinct effects: the external driving h and the interaction with the heat bath, which are identified as the work four-vector and heat four-vector during the process:

$$W^\nu = \int_{\mathcal{A}} \mathrm{d}^4x \frac{\partial T^{\mu\nu}}{\partial h}\frac{\partial h}{\partial x^\mu}, \quad Q^\nu = \int_{\mathcal{A}} \mathrm{d}^4x \frac{\mathrm{d}T^{\mu\nu}}{\mathrm{d}x^\mu} - W^\nu. \tag{9}$$

The above definitions ensure that the work four-vector comes from the time-dependent external driving, while the heat four-vector comes from the 4-momentum exchange with the heat bath, in accord with the stochastic thermodynamics [54].

Using path integral techniques for Eq. (8), the ratio of conditional probabilities between forward and backward trajectories satisfies the following relation [38]:

$$\frac{\Pr[\phi|\phi_{\mathcal{A}_-}, \pi_{\mathcal{A}_-}]}{\tilde{\Pr}[\tilde{\phi}|\tilde{\phi}_{\tilde{\mathcal{A}}_-}, \tilde{\pi}_{\tilde{\mathcal{A}}_-}]} = \mathrm{e}^{-\beta_\mu Q^\mu[\phi]}. \tag{10}$$

Here, $\pi = \partial_t\phi$ is the time derivative of ϕ, and subscript $\mathcal{A}_-$ indicates the values on the initial hypersurface. The above probabilities are conditioned on their initial field configurations of (ϕ, π) and $(\tilde{\phi}, \tilde{\pi})$, respectively. For different setups, multiplying with different initial distributions, Eq. (10) results in covariant fluctuation theorems (3)–(5) [8]. A detailed derivation can be found in Supplemental Material [38].

For a special case where higher-order terms $V = 0$ and the driving process starts from the infinite past and ends in the infinite future, the joint distribution of the work four-vector can be calculated explicitly (see Supplemental Material [38]). This distribution satisfies the covariant Jarzynski equality $\langle\exp(-\beta_\mu W^\mu + \beta\Delta F)\rangle = 1$.

Next, we study the particle system. Relativistic Brownian motion, a simple model for a single particle moving in the presence of a heat bath, attracts lots of attention [12]. Here, we focus on the relativistic Ornstein-Uhlenbeck process, one specific form of relativistic Brownian motion, in which the friction is proportional to the velocity.

Consider a particle with electric charge q and mass m, which undergoes a relativistic Ornstein-Uhlenbeck process and is driven by an external electromagnetic field with 4-potential $A^\mu(x)$. In the rest frame of the heat bath, the equation of motion is

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = f^i - \kappa\frac{p^i}{p^0} + \sqrt{\frac{2\kappa}{\beta}}\xi^i, \tag{11}$$

where f^i is the electromagnetic force, κ is the friction coefficient, and ξ^i is standard white noise with correlation $\langle\xi^i(t)\xi^j(t')\rangle = \delta_{ij}\delta(t-t')$. For a charged particle, the canonical 4-momentum is $P^\mu = p^\mu + qA^\mu$, where $p^\mu = m\mathrm{d}x^\mu/\mathrm{d}\tau$ is the kinetic momentum and τ is the proper time. If the external field is stationary in the rest frame of the heat bath, $A^\mu(t, \vec{x}) = A^\mu(0, \vec{x})$, the Gibbs distribution $\mathcal{P}^{\mathrm{eq}} \propto \exp(-\beta P^0)$ is the stationary solution of the above equation [14]. The corresponding covariant form is $\propto \exp(-\beta_\mu P^\mu)$ [14,29–33].

In a finite-time process, every worldline in the ensemble starts from spacelike hypersurfaces $\mathcal{A}_-$ and ends at $\mathcal{A}_+$. The backward process starts from $\tilde{\mathcal{A}}_- = -\mathcal{A}_+$ and ends at

$\tilde{\mathcal{A}}_+ = -\mathcal{A}_-$. The electromagnetic potential in the backward process is $\tilde{A}^\mu(x) = A^\mu(-x)$. Correspondingly, the field strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ gets a minus sign, $\tilde{F}^{\mu\nu}(x) = -F^{\mu\nu}(-x)$ [55]. The backward trajectory in the backward process is obtained by reversing the spacetime coordinate of the corresponding worldline, $\tilde{x}^\mu(s) = -x^\mu(-s)$, while keeping the 4-momentum unchanged.

In this thermodynamic process, the heat four-vector is identified as the 4-momentum change associated with the friction and noise. For every trajectory, we have $Q^0 = \int_{t_{\mathcal{A}_-}}^{t_{\mathcal{A}_+}} \sum_i (-\kappa v^i + \sqrt{2\kappa/\beta}\xi^i) v^i \mathrm{d}t$ and $Q^i = \int_{t_{\mathcal{A}_-}}^{t_{\mathcal{A}_+}} (-\kappa v^i + \sqrt{2\kappa/\beta}\xi^i) \mathrm{d}t$ in the rest frame of the heat bath. Using the equation of motion (11), the above definitions of heat can be converted into a covariant expression:

$$Q^\mu = \Delta p^\mu - q \int F^{\mu\nu} \mathrm{d}x_\nu. \tag{12}$$

The above line integral is along the worldline $x(s)$, from $x_{\mathcal{A}_-}$ to $x_{\mathcal{A}_+}$. By noticing the Euler-Lagrange equation $\mathrm{d}P^\mu/\mathrm{d}s = -\partial^\mu L$, with Lagrangian $L = -m\sqrt{[(\mathrm{d}x_\mu)/(\mathrm{d}s)][(\mathrm{d}x^\mu)/(\mathrm{d}s)]} - qA_\mu(x)[(\mathrm{d}x^\mu)/(\mathrm{d}s)]$, the 4-work is defined as $W^\mu = -\int \mathrm{d}s \partial^\mu L$ [56], which leads to

$$W^\mu = q \int \partial^\mu A^\nu \mathrm{d}x_\nu. \tag{13}$$

Q^μ and W^μ satisfy the first law of thermodynamics, $Q^\mu + W^\mu = \Delta P^\mu$.

According to path integral techniques for diffusion processes, the ratio of conditional trajectory probabilities between forward and backward trajectories satisfies [38]

$$\frac{\Pr[x|x_{\mathcal{A}_-}, p_{\mathcal{A}_-}]}{\tilde{\Pr}[\tilde{x}|\tilde{x}_{\tilde{\mathcal{A}}_-}, \tilde{p}_{\tilde{\mathcal{A}}_-}]} = \mathrm{e}^{-\beta_\mu Q^\mu[x]}. \tag{14}$$

The above trajectory probabilities are conditioned on their initial position and momentum, $(x_{\mathcal{A}_-}, p_{\mathcal{A}_-})$ and $(\tilde{x}_{\tilde{\mathcal{A}}_-}, \tilde{p}_{\tilde{\mathcal{A}}_-})$, respectively. Similar to the procedure for field systems, the covariant fluctuation theorems (3)–(5) are obtained by multiplying both sides of Eq. (14) with different initial distributions corresponding to different setups; see Supplemental Material [38].

In a simple case of a pure relaxation process for a $(1+1)$D ultrarelativistic free particle, the joint distribution of heat two-vector can be analytically solved [17,38], and it satisfies the covariant heat exchange fluctuation theorem, $\langle \exp[(\beta_\mu - \beta_\mu^{\mathrm{s}})Q^\mu] \rangle = 1$.

Conclusion—The existing fluctuation theorems are valid only when both the system and the heat bath are at rest relative to the observer and do not satisfy the principle of covariance, a fundamental guiding principle of physical laws. We introduce the concept of covariant stochastic work four-vector, heat four-vector, and entropy production and then successfully promote existing fluctuation theorems into covariant forms, which are applicable to moving systems and heat baths in both relativistic and nonrelativistic systems. We demonstrate the validity of the covariant fluctuation theorems in two examples.

By combining he van Kampen formulation and stochastic thermodynamics, our approach sheds new lights on the relativistic thermodynamics theory [38]. Also, our Letter bridges a gap between stochastic thermodynamics and the principle of covariance. The covariant fluctuation theorems are significant both theoretically and practically. They unify fluctuation theorems for different inertial observers and show that the second law should include the momentum-related quantities for moving systems and heat baths. Importantly, for systems in relative motion, existing fluctuation theorems fail. Only the covariant fluctuation theorems are valid.

The study of irreversibility and fluctuations is extended to broader contexts, for moving systems and heat baths. For example, Earth is moving relative to the cosmic microwave background (CMB) radiation [33,48–51,57], so the momentum-related component cannot be ignored when measuring CMB spectrum at a high precision. A precise CMB spectrum encodes important information about the physical properties of the early Universe. Besides, at nanoscales, near-field radiative heat transfer occurs between bodies in relative motion, accompanied by noncontact friction [46,47,58,59]. Theoretical predictions for the noncontact friction have varied widely [60]. By investigating the heat and momentum transfer via the noncontact friction, covariant fluctuation theorems can be used as a criterion to test the validity of those results. Furthermore, our Letter has potential applicability to heavy-ion physics and relativistic hydrodynamics theory when implementing thermodynamic fluctuations [61–64]. Applications of covariant fluctuation theorems to those systems will be given in our future study.

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