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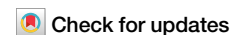
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All genuinely entangled stabilizer subspaces are multipartite fully nonlocal

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Understanding which entangled states give rise to Bell nonlocality and thus are resourceful in the device-independent framework is a long-standing unresolved problem. Here, we establish the equivalence between genuine entanglement and genuine nonlocality for a broad class of multipartite (pure and mixed) states originating from the stabilizer formalism. In fact, we prove that any (mixed) stabilizer state defined on a genuinely entangled subspace is multipartite fully nonlocal, meaning that it gives rise to correlations with no contribution from local hidden variable models of any type. Importantly, we also derive a lower bound on genuine nonlocality content of arbitrary multipartite states, opening the door to its experimental estimation.

Quantum entanglement and Bell nonlocality are some of the most characteristic features of quantum theory. Moreover, they are a cornerstone of quantum information science enabling several applications unachievable in classical physics such as quantum teleportation¹, quantum cryptography^{2,3} or certification of random numbers⁴. In some of these applications such as the latter two, Bell nonlocality proves to be a stronger resource as it allows to process information in the device-independent framework^{4,5} where no assumptions are made on the internal working on the devices except that they obey the rules of quantum theory.

It is well-known that both these resources are deeply related as Bell nonlocality can only be obtained from entangled states; yet they are not fully equivalent. Indeed, already in the bipartite case there exist mixed entangled states which do not violate any Bell inequality such as e.g. the well-known Werner states^{6,7}, despite the fact that in the case of pure states, entanglement implies Bell non-locality^{8,9}. In fact, understanding which entangled states can give rise to Bell violations and thus be a resource in the device-independent framework remains an unsolved and highly nontrivial problem.

While being already difficult to tackle in the bipartite case, the question which entangled states are nonlocal gets even more complicated when one enters the realm of multipartite quantum systems which are far more complex to handle. It was proved in refs. 10,11 as a generalization of the results of refs. 8,9 that all pure multipartite entangled states exhibit some form of nonlocality. However, an arguably more meaningful question in the multipartite scenario, i.e., whether all genuinely multipartite entangled (GME) states are genuinely multipartite nonlocal (GMNL)¹² remains unresolved even for the pure states. While nothing is known in the most general case, for some particular multipartite systems such as pure three-qubit¹³ or N -qubit symmetric¹⁴ states this equivalence was established. At the same time, there exist mixed GME states that are not GMNL¹². In fact, it is

even possible to construct GME mixed states that are fully local and thus do not exhibit any form of nonlocality¹⁵.

Here we establish the equivalence between genuine multipartite entanglement and nonlocality for a large class of pure and mixed states originating from the stabilizer formalism¹⁶. The latter provides a very convenient representation of multipartite states that encompass not only the well-known graph states, but also a broad class of genuinely entangled mixed states. This includes for instance mixed states corresponding to stabilizer quantum error correction codes¹⁶ such as the five-qubit¹⁷ or toric ones¹⁸. In this work, we exploit this representation to show that every genuinely entangled stabilizer subspace of the N -qubit Hilbert space is also genuinely nonlocal in the sense that every pure state belonging to it gives rise to genuinely nonlocal correlations. We thus introduce the first examples of multi-qubit Hilbert spaces composed of only genuine multipartite nonlocal pure states; see refs. 19,20 for constructions of genuinely entangled subspaces. We actually prove a much stronger result that any state (pure or mixed) belonging to a genuinely entangled stabilizer subspace is multipartite fully nonlocal (MFNL) which means that it gives rise to nonlocal correlations that are genuinely nonlocal in the strongest sense, i.e., they have no contribution coming from a local hidden variable model of any type (see refs. 21–24 for previous examples of fully nonlocal correlations). Importantly, this implication generalizes to mixed stabilizer states: any mixed state defined on a GME stabilizer subspace is also multipartite full nonlocal. We thus introduce here a broad class of mixed states that give rise to multipartite fully nonlocal correlations. Lastly, we derive a general lower bound on the genuine nonlocality content of any multipartite state or subspace (not necessarily stabilizer), based on the nonlocality contents of bipartite states obtained from local measurements performed by the remaining parties on the multipartite state and subspace respectively.

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Results

Preliminaries

Let us first provide some background information.

Stabilizer formalism. We begin with the stabilizer formalism and define the N -qubit Pauli group $\mathbb{P}_N$ to be one containing all N -fold tensor product of Pauli matrices $\mathbb{1}, X, Y, Z$ multiplied by ± 1 or $\pm i$. Let us then consider a subgroup $\mathbb{S}$ of $\mathbb{P}_N$. We call it a stabilizer if it is abelian and satisfies $-1 \notin \mathbb{S}$. The most interesting property of $\mathbb{S}$ is that, as the name suggests, its elements stabilize a non-empty subspace in the N -qubit Hilbert space $\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$. Precisely, for any stabilizer $\mathbb{S}$ there exists a subspace $V \subseteq \mathcal{H}_N$ such that $s|\psi\rangle = |\psi\rangle$ for any $|\psi\rangle \in V$ and any $s \in \mathbb{S}$. The largest subspace satisfying this condition is called a stabilizer subspace of $\mathbb{S}$ and is denoted $V_{\mathbb{S}}$. Clearly, this stabilizing property extends to all mixed states acting on a subspace $V_{\mathbb{S}}$, that is, for any $\rho : V_{\mathbb{S}} \rightarrow V_{\mathbb{S}}$ one has that $s\rho = \rho s$ for any $s \in \mathbb{S}$. Thus, the most basic function of a stabilizer $\mathbb{S}$, which we extensively use here, is to uniquely define a subspace in terms of a few algebraic relations that are convenient to handle.

Here we often need to consider matrices forming operators $s \in \mathbb{S}$ that correspond only to subsets of $[N] = \{1, \dots, N\}$. Hence, we denote by $s^{(Q)}$ a $|Q|$ -fold tensor product of the Pauli matrices from s that act on qubits belonging to the set $Q \subset [N]$. For instance, for $s = X \otimes Z \otimes \mathbb{1} \otimes XZ$, the sub-operator $s^{(Q)}$ corresponding to $Q = \{1, 3\}$ is $s^{(Q)} = X \otimes \mathbb{1}$.

Lastly, since for larger N the cardinality of $\mathbb{S}$ could substantially increase, it is convenient to represent $\mathbb{S}$ in terms of the minimal set of operators, called generators, that enable reproducing the other elements of $\mathbb{S}$. By writing $\mathbb{S} = \langle g_1, \dots, g_k \rangle$ we mean that $\{g_i\}_{i=1}^k$ is a generating set of $\mathbb{S}$. For instance, the stabilizer $\mathbb{S} = \{\mathbb{1} \otimes \mathbb{1}, X \otimes X, Z \otimes Z, -Y \otimes Y\}$ can be represented in this way as $\mathbb{S} = \langle X \otimes X, Z \otimes Z \rangle$.

Genuine multipartite entanglement. We can now move on to the definition of genuine entanglement in the multipartite setting. We consider an arbitrary N -partite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$ and denote by $\mathcal{B}(\mathcal{H})$ the operator space over $\mathcal{H}$. Let us then divide the set $[N] \equiv \{1, \dots, N\}$ into two disjoint and non-empty sets Q and $\bar{Q}$ and call it a bipartition $Q|\bar{Q}$. A state $|\psi\rangle \in \mathcal{H}$ is called genuinely multipartite entangled if for all bipartitions, $|\psi\rangle \neq |\psi_Q\rangle \otimes |\phi_{\bar{Q}}\rangle$ for any two pure states $|\psi_Q\rangle$ and $|\phi_{\bar{Q}}\rangle$ corresponding to the sets Q and $\bar{Q}$.

Moving to the mixed-state case, we say that a mixed state $\rho \in \mathcal{B}(\mathcal{H})$ is genuinely multipartite entangled (GME)²⁵ if it does not admit a biseparable model, meaning that it cannot be written as a convex combination,

$$\rho = \sum_{Q \subset [N]} p_Q \sum_i q_{i;Q} \rho_Q^i \otimes \rho_{\bar{Q}}^i, \quad (1)$$

of states which are separable across various bipartitions $Q|\bar{Q}$. In an analogous manner, one can also define genuine entanglement for subspaces: we say that a subspace $V \subset \mathcal{H}$ is GME if all pure states belonging to it are GME¹⁹. Let us stress here that for a genuinely entangled subspace V , every mixed state defined on it is genuinely entangled too.

Our focus in this work is the stabilizer subspaces, and therefore we need a criterion that allows us to easily decide whether a given stabilizer subspace $V_{\mathbb{S}}$ is GME. Such a criterion was recently introduced in ref. 26 (see Theorem 1 therein), and for further purposes we recall it here as the following lemma.

Lemma 1. Given a stabilizer $\mathbb{S} = \langle g_1, \dots, g_k \rangle$, the corresponding stabilizer subspace $V_{\mathbb{S}}$ is GME iff for each bipartition $Q|\bar{Q}$ there exists a pair $i, j \in [k]$ for which:

$$[g_i^{(Q)}, g_j^{(\bar{Q})}] \neq 0. \quad (2)$$

Genuine multipartite nonlocality. Let us now consider a typical Bell scenario with N parties sharing a state $\rho \in \mathcal{B}(\mathcal{H})$ and performing

measurements on their shares of this state. Party i can freely choose to perform one of $x_i = 1, \dots, m$ measurements which yields an outcome $a_i = 0, \dots, d - 1$. By repeating these measurements many times, the parties create correlations that are described by a collection of joint probabilities $\mathcal{P} = \{P(\mathbf{a}|\mathbf{x})\}$, often referred to as behavior, where $P(\mathbf{a}|\mathbf{x})$ is a probability of obtaining outcomes $a_1, \dots, a_N = : \mathbf{a}$ after performing measurements $x_1, \dots, x_N = : \mathbf{x}$.

In order to formulate the notion of genuine multipartite nonlocality we need the concept of non-signaling behaviors. Let us for a moment abstract from quantum behaviors and introduce a broader class of behaviors for which a measurement choice made by parties belonging to a subset Q does not have an influence over the measurement result of the remaining parties. Formally, this is represented by a set of linear constraints,

$$P(\mathbf{a}_{\bar{Q}}|\mathbf{x}_{\bar{Q}}) = \sum_{i \in Q} \sum_{a_i} P(\mathbf{a}|\mathbf{x}) \quad (3)$$

for all $Q \subset [N]$, all $\mathbf{a}_Q$, and all $\mathbf{x}$, where $\mathbf{a}_Q = \{a_i\}_{i \in Q}$, $\mathbf{x}_Q = \{x_i\}_{i \in Q}$, and the sum $\sum_{a_i}$ is over all possible measurement results. We call behaviors satisfying the above constraints non-signaling. Note that all behaviors originating from quantum theory are non-signaling; yet there exist non-signaling correlations that are not quantum.

Let us now consider a behavior $\mathcal{P}$. We call it local with respect to a given bipartition $Q|\bar{Q}$ if

$$P(\mathbf{a}|\mathbf{x}) = \sum_{\lambda} q_{\lambda} P(\mathbf{a}_Q|\mathbf{x}_Q, \lambda) P(\mathbf{a}_{\bar{Q}}|\mathbf{x}_{\bar{Q}}, \lambda) \quad (4)$$

for all $P(\mathbf{a}|\mathbf{x}) \in \mathcal{P}$, where λ is the hidden variable with a distribution q_{λ} , and $P(\mathbf{a}_Q|\mathbf{x}_Q, \lambda)$ and $P(\mathbf{a}_{\bar{Q}}|\mathbf{x}_{\bar{Q}}, \lambda)$ are some non-signaling (in general non-quantum) behaviors corresponding to the disjoint sets Q and $\bar{Q}$ that satisfy the non-signaling conditions. Similarly to the case of entanglement, we call $\mathcal{P}$ genuinely multipartite non-local (GMNL)^{27,28} if it cannot be written as a convex combination of behaviors that are local with respect to various bipartitions.

While the above notion allows us to describe nonlocality in a quantitative way, it does not tell us much about how strong this nonlocality is. A possible approach to quantify GMNL in $\mathcal{P}$, put forward in ref. 29 as a multipartite generalization of the Elitzur-Popescu-Rohrlich (EPR-2) decomposition³⁰, is through the following convex decomposition

$$P(\mathbf{a}|\mathbf{x}) = \sum_{Q \in [N]} p_{Q|\bar{Q}} P_{Q|\bar{Q}}(\mathbf{a}|\mathbf{x}) + p_{NL} P_{NL}(\mathbf{a}|\mathbf{x}), \quad (5)$$

where $P_{Q|\bar{Q}}(\mathbf{a}|\mathbf{x})$ is a behavior that admits the decomposition (4) for a given bipartition $Q|\bar{Q}$, whereas $P_{NL}(\mathbf{a}|\mathbf{x})$ is one that is nonlocal across any bipartition, and, finally, $p_{Q|\bar{Q}}$ for all Q and p_{NL} form a probability distribution; in particular $\sum_{Q \in [N]} p_{Q|\bar{Q}} + p_{NL} = 1$.

Now, the minimal p_{NL} for which (5) holds true, denoted $\tilde{p}_{NL}$, is called *genuine entanglement content* of $\mathcal{P}$. A behavior $\mathcal{P}$ for which $\tilde{p}_{NL} > 0$ is genuinely entangled, and in the extreme case of $\tilde{p}_{NL} = 1$ we call it multipartite fully non-local²⁹. Thus, correlations that are MFNL are also GMNL, meaning that the former is a stronger form of multipartite nonlocality than the latter.

Notably, both these notions can also be applied to quantum states: we say that a state ρ is MFNL or GMNL if there exists a Bell scenario and a set of measurements such that the resulting behavior $\mathcal{P}$ is MFNL or GMNL, respectively.

Lastly, in order to detect MFNL (and thus also GMNL) of a given ρ , we will make use of Theorems 1 and 2 from ref. 29 which we below combine into a single lemma.

Lemma 2. A state ρ is said to be MFNL if for every bipartition $Q|\bar{Q}$ it is possible to create a maximally entangled state

$$|\phi_+\rangle_{ij} = \frac{1}{\sqrt{d}} \sum_{l=0}^{q-1} |ll\rangle_{ij} \quad (6)$$

for some positive integer $q \geq 2$, between $i \in Q$ and $j \in \bar{Q}$ for all possible outcomes of local measurements on the parties from $[N] \setminus \{i, j\}$.

Main result

Here, we present our main result that all genuinely entangled stabilizer subspaces are multipartite fully nonlocal. We thus establish, in particular, the equivalence between GME and GMNL for a large class of stabilizer states.

Our strategy to prove this statement is quite simple: we aim to show that for any genuinely entangled stabilizer subspace, it follows from Lemma 1 that the conditions of Lemma 2 are satisfied, that is, that one can create a maximally entangled state across any nontrivial bipartition $Q|\bar{Q}$ by local measurements on $N - 2$ parties. To achieve this we consider an interesting property of genuinely entangled stabilizer subspaces.

Lemma 3. Let $\mathbb{S}$ be a stabilizer. The corresponding subspace $V_{\mathbb{S}}$ is GME iff for every pair of qubits $\alpha_1, \alpha_2 \in [N]$ there exists a pair of stabilizing operators $s_i, s_j \in \mathbb{S}$ such that

$$[s_i^{(\alpha_1)}, s_j^{(\alpha_2)}] \neq 0, \quad [s_i^{(\alpha_2)}, s_j^{(\alpha_1)}] = 0 \quad (7)$$

for all $\alpha \in [N] \setminus \{\alpha_1, \alpha_2\}$ and all $i \in \{1, 2\}$.

A proof of this lemma can be found in Supplementary Note 1. It is easy to notice that Lemmas 1 and 3 are quite similar; in fact, the implication “ $\Leftarrow$ ” in Lemma 3 follows directly from Lemma 1. One important difference, however, is that Lemma 3 involves all operators from $\mathbb{S}$, whereas Lemma 1 is formulated only in terms of generators of $\mathbb{S}$. Still, the main advantage of Lemma 3 over Lemma 1 is that it tells us that for any pair of qubits one can always find two operators in $\mathbb{S}$ in which the local Pauli matrices anticommute exactly for those qubits, whereas they commute for the remaining ones.

Utilizing Lemma 3 we can now formulate our main result.

Theorem 1. A qubit stabilizer subspace $V_{\mathbb{S}}$ is multipartite fully nonlocal iff it is genuinely multipartite entangled.

Since the proof is quite technical, we have deferred it to Supplementary Note 2. Instead, here we provide a simple example illustrating its main idea. We consider the two-dimensional stabilizer subspace, denoted V_5 , corresponding to the five-qubit code^{17,31}, i.e., one stabilized by $\mathbb{S}_5 = \langle g_1, g_2, g_3, g_4 \rangle$, where

$$\begin{aligned} g_1 &= X_1 Z_2 Z_3 X_4, & g_2 &= X_2 Z_3 Z_4 X_5, \\ g_3 &= X_1 X_3 Z_4 Z_5, & g_4 &= Z_1 X_2 X_4 Z_5 \end{aligned} \quad (8)$$

with X_i, Z_i denoting the Pauli matrices that act on the party i .

Let us then consider a bipartition $\{1, 2\}|\{3, 4, 5\}$. The conditions of Lemma 3 for this bipartition are fulfilled with $\alpha_1 = 1, \alpha_2 = 4$, and $i = 3$ and $j = 4$, that is, the Pauli matrices at sites 1 and 4 of the stabilizing operators g_3 and g_4 anticommute, whereas those appearing at the remaining sites commute. This fact can be used to construct a maximally entangled state between parties 1 and 4 from any mixed state ρ defined on V_5 by performing suitable measurements on sites 2, 3, and 5. These measurements can be constructed from the joint eigenbases of the commuting Pauli operators appearing at those sites. For instance, for party 2 we have $g_3^{(2)} = \mathbb{1}$ and $g_4^{(2)} = X$, and so we pick the measurement for this party in the common eigenbasis of these operators,

i.e., $\{|+\rangle, |-\rangle\}$, where $|\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$. Analogously, for parties 3 and 5 we take the measurement bases $\{|+\rangle, |-\rangle\}$ and $\{|0\rangle, |1\rangle\}$, respectively.

Let us then consider a mixed state ρ defined on V_5 and assume that after performing the above measurements on it, the parties observe outcomes corresponding to $|+\rangle$ state for qubits 2 and 3, and $|0\rangle$ for qubit 5. Then, the post-measurement state corresponding to parties 1 and 4 reads

$$\sigma_{1,4} = \frac{1}{n} \text{Tr}_{2,3,5} [|+\rangle\langle+|_{2,3,5} (+ + 0) \rho], \quad (9)$$

where n is the normalization constant. Using the stabilizing relation $\rho = g_3 \rho$, we further obtain

$$\begin{aligned} \sigma_{1,4} &= \frac{1}{n} \text{Tr}_{2,3,5} [|+\rangle\langle+|_{2,3,5} (+ + 0) X_1 X_3 Z_4 Z_5 \rho] \\ &= X_1 Z_4 \frac{1}{n} \text{Tr}_{2,3,5} [|+\rangle\langle+|_{2,3,5} (+ + 0) \rho] = X_1 Z_4 \sigma_{1,4}. \end{aligned}$$

In a similar way, one can use the other stabilizing relation $\rho = g_4 \rho$ to show that $\sigma_{1,4} = Z_1 X_4 \sigma_{1,4}$. These two relations directly imply that, up to a local unitary, $\sigma_{1,4} = Z_1 X_4 \sigma_{1,4}$ is the maximally entangled state of two qubits $|\phi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$; recall that the latter is stabilized by $X_1 X_4, Z_1 Z_4$.

One can verify that for any other choice of the outcomes obtained by parties 2, 3, and 5, the post-measurement state $\sigma_{1,4}$ is stabilized by the same (up to the sign) operators, and thus, it is also the two-qubit maximally entangled state. Furthermore, the same algorithm can be repeated for all bipartitions, and therefore, the conditions of Lemma 2 are satisfied in this case. Consequently, the stabilizer subspace corresponding to the five-qubit code is multipartite fully nonlocal.

This theorem thus establishes an equivalence between genuine entanglement and nonlocality in the stabilizer formalism, generalizing results of ref. 29 derived for qubit graph states. Moreover, it also provides a convenient tool for constructing broad classes of mixed MFNL states for $N \geq 4$, as in fact any mixed state supported on a GME stabilizer subspace is MFNL. The above constraint follows from the fact that GME stabilizer subspaces of the smallest nontrivial dimension $\dim(V_{\mathbb{S}}) \geq 2$ exist in systems consisting of at least four qubits [cf. Theorem 3 in ref. 26]. Simultaneously, it is worth pointing out that in the qubit case, the maximal dimension of a GME stabilizer subspace for a given N is $2^{N-k(N)}$, where $k(N) = \lceil (1 + \sqrt{8N-7})/2 \rceil$ ²⁶.

For example, one of the GME stabilizer spaces of maximal dimension for $N = 7$ ²⁶ is stabilized by $\mathbb{S}_7 = \langle \tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4 \rangle$ with the generators given by

$$\begin{aligned} \tilde{g}_1 &= X_1 X_2 Z_3 Z_4 X_6 X_7, & \tilde{g}_2 &= Z_1 Z_2 Z_3 X_5 X_6, \\ \tilde{g}_3 &= X_3 X_4 X_5, & \tilde{g}_4 &= Z_4 Z_5 Z_6 Z_7. \end{aligned} \quad (10)$$

The corresponding stabilizer subspace $V_{\mathbb{S}_7}$ is eight-dimensional, meaning that, by Theorem 1, all of the states spanning $V_{\mathbb{S}_7}$, as well as every superposition of these states and their probabilistic mixtures, are MFNL. As an example of an MFNL mixed state, we can simply consider a normalized projection onto $V_{\mathbb{S}_7}$, which is the state given by

$$\rho_{\mathbb{S}_7} = \prod_{i=1}^4 \frac{\mathbb{1} + \tilde{g}_i}{2}. \quad (11)$$

However, there are also more well-known examples of stabilizer subspaces, which, by Theorem 1 can now be identified as MFNL. One of which we have already discussed above: the five-qubit code. Another example is the toric code¹⁸, which has been proven to be GME in ref. 32. In addition, the Steane³³ and the surface codes^{34,35} were long suspected to be GME based on their similarities to the five-qubit and toric codes. For completeness, we provide a proof that they are genuinely multipartite entangled in Supplementary Note 4. Our work thus demonstrates that the particular subspaces corresponding to the toric, Steane, and surface codes are all MFNL.

Thus, Theorem 1 establishes that GME and GMNL are equivalent notions for a large class of multipartite (mixed) states, and, moreover, it identifies a large class of states that are fully nonlocal. This is significant, as the only known example of a multipartite fully nonlocal mixed state is one composed of five copies of the Smolin state³⁶, provided in ref. 29.

Bounding the genuine nonlocality content. Lemma 2 relies on the proof of full nonlocality of the bipartite maximally entangled states which is achieved from violation of the chained Bell inequalities^{37,38} in the limit of the number of local measurements taken to infinity. What is more, it does not account for noises and experimental imperfections. All this makes our results hardly testable in experiments. Here, we address this issue by deriving a general bound on the genuine nonlocality content $\tilde{p}_{NL}$ of any behavior that tolerates noises and applies to scenarios with a finite number of local measurements.

Theorem 2. Given a state ρ and two parties α and $\bar{\alpha}$, let us denote $\tilde{p}_{NL}^{\alpha\bar{\alpha}} = \max \min \tilde{p}_{NL}^{\alpha\bar{\alpha}}(\mathbf{a}_R, \mathbf{x}_R)$, where $\tilde{p}_{NL}^{\alpha\bar{\alpha}}(\mathbf{a}_R, \mathbf{x}_R)$ is the minimal nonlocality content p_{NL} of a bipartite state shared by $\alpha, \bar{\alpha}$ that was created by performing local measurements $\mathbf{x}_R$ by the remaining parties $R = [N] \setminus \{\alpha, \bar{\alpha}\}$ and corresponding to the outcomes $\mathbf{a}_R$. Then, $\tilde{p}_{NL}$ of $\mathcal{P}$ is lower-bounded as

$$\tilde{p}_{NL} \geq 1 - \frac{1}{N-1} \sum_{\alpha=1}^N \sum_{\bar{\alpha} > \alpha} \left(1 - \tilde{p}_{NL}^{\alpha\bar{\alpha}}\right). \quad (12)$$

We present the proof in Supplementary Note 3. Note, that the above can also be used to lower bound $\tilde{p}_{NL}$ of a state or a subspace $V \subseteq \mathcal{H}$; in the latter case one needs to minimize $\tilde{p}_{NL}^{\alpha\bar{\alpha}}$ over all $\rho \in \mathcal{B}(V)$.

There are multiple of ways in which Theorem 2 can be used. First, together with the results of ref. 38, it enables determining the minimal number of measurements necessary to detect GMNL of N -partite systems by violating the chained Bell inequality between every pair of parties. Let us consider for instance the two-dimensional subspace V_5 corresponding to the five-qubit code. If all $\tilde{p}_{NL}^{\alpha\bar{\alpha}}$ are equal for all pairs $\alpha > \bar{\alpha}$, it follows that to detect GMNL ($\tilde{p}_{NL} > 0$) of this subspace one needs $\tilde{p}_{NL}^{\alpha\bar{\alpha}} > 3/5$. This can be achieved from violation of the chained Bell inequality with at least four measurements per observer (see Supplementary Note 3 for more details).

On the other hand, Theorem 2 allows for an estimation of genuine nonlocality content $\tilde{p}_{NL}$ from the experimentally observed bipartite $\tilde{p}_{NL}^{\alpha\bar{\alpha}}$. For instance, in ref. 39 the value of $\tilde{p}_{NL}^{\alpha\bar{\alpha}} = 0.874 \pm 0.001$ has been achieved experimentally for the two-qubit maximally entangled state. Assuming that such $\tilde{p}_{NL}^{\alpha\bar{\alpha}}$ could be achieved for every pair of qubits, this would imply that the genuine nonlocality content of V_5 is $\tilde{p}_{NL} \gtrsim 0.685$.

However, this raises an important question: why estimate $\tilde{p}_{NL}$ from bipartite $\tilde{p}_{NL}^{\alpha\bar{\alpha}}$, rather than compute $\tilde{p}_{NL}$ directly from Eq. (5)? The utility of Theorem 2 in this case can be appreciated fully when considering the complexity of the computation of $\tilde{p}_{NL}$. First, the number of terms $P_{Q|\bar{Q}}(\mathbf{a}|\mathbf{x})$ scales exponentially with N (as $2^{N-1} - 1$ to be exact). Second, the dimension of the non-signaling set containing all $P_{Q|\bar{Q}}(\mathbf{a}|\mathbf{x})$ also scales exponentially: for N parties, m measurement settings, and Δ measurement outcomes its dimension equals $[m(d-1)+1]^N - 1$ ⁴⁰. From this perspective, estimating $\tilde{p}_{NL}$ by $\tilde{p}_{NL}^{\alpha\bar{\alpha}}$ is highly beneficial.

Qudit graph states. A natural extension of Theorem 1 would be to consider the stabilizer formalism with all local systems being d -dimensional. Unfortunately, it turns out that one cannot directly generalize Lemma 3 to higher $d > 2$. Nevertheless, using our approach we can still prove that simplest one-dimensional qudit stabilizer subspaces, which are in fact local-unitarily equivalent to multiqubit graph states, we can conclude that they are all MFNL; the same conclusion for graph states $d = 2$ was derived before in ref. 29.

The qudit graph states are defined as follows: let G be a multigraph. Then a graph state $|G\rangle$ associated to G is one that is stabilized by $\mathbb{S}_G = \langle g_1, \dots, g_N \rangle$ with

$$g_j = X_j \prod_{l=1}^N Z_l^{\Gamma_{jl}}, \quad (13)$$

where Γ_{jl} denotes the number of edges connecting vertices j and l in the graph G and

$$X = \sum_{j=0}^{d-1} |j+1\rangle\langle j|, \quad Z = \sum_{j=0}^{d-1} \exp(2\pi i j/d) |j\rangle\langle j| \quad (14)$$

are the generalized Pauli matrices. Notice that each qudit of $|G\rangle$ is associated with a vertex in G .

It is known that a graph state $|G\rangle$ is GME iff the graph G is connected⁴¹. This implies that for every bipartition $Q|\bar{Q}$, there exists a pair of vertices i, j such that $i \in Q, j \in \bar{Q}$ and $\Gamma_{ij} \neq 0$. To show that each GME graph state is MFNL, we begin by performing measurements in the computational basis $\{|j\rangle\}_{j=0}^{d-1}$ on every qudit apart from i and j . One then finds that the post-measurement state $|\psi_{ij}\rangle$ shared by parties i and j is stabilized by $X_i Z_j^{\Gamma_{ij}}$ and $Z_i^{\Gamma_{ij}} X_j$. One checks that these operators uniquely identify the state $|\psi\rangle$ as a maximally entangled state $|\phi_+\rangle$ (6) (again up to local unitaries) for $q = d/r$, where r is the greatest common divisor of d and Γ_{ij} . Since this procedure can be performed for any bipartition, Lemma 2 allows us to conclude that every qudit graph state is MFNL.

Discussion

There are still a few open questions to be explored. The most obvious one is whether qudit stabilizer subspaces that are genuinely multipartite entangled, are also multipartite fully nonlocal. As we discussed above, the approach used here which makes use of Lemma 3 is not suitable for that purpose, however, it is likely that this relationship between genuine multipartite entanglement and multipartite full nonlocality persists for any local dimension.

Alternatively, if one were interested only in testing genuine multipartite nonlocality, then a possibly better strategy would be to directly construct suitable Bell inequalities detecting it, such as those provided in refs. 22,42,43.

Another question to explore is whether it is possible to experimentally determine the genuine nonlocality content of multipartite stabilizer mixed states, building on our results. Multiple experiments aiming to determine the nonlocality content of bipartite states have already been performed (see, e.g., refs. 21,39,44,45 and references therein). The question is thus whether they can be combined with our Theorem 2 to provide results for the multipartite scenario.

One can finally ask whether a tighter bound on the genuine nonlocality content than that in Eq. (12) can be obtained, for instance throughout including genuine nonlocality contents of m -partite states ($m < N$) created by the remaining parties.

Lastly, let us note here that while finishing this manuscript, we became aware of ref. 46, in which a related problem has been studied but from the perspective of genuine multipartite entanglement.

Data availability

No datasets were generated or analysed during the current study.

Code availability

No codes were generated or used in the current study.

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Author contributions

R.A. conceived the project idea. O.M. formulated the proof of Lemma 3, Theorem 1 and 2. O.M. and R.A. discussed the results and formulated conclusions. O.M. and R.A. wrote the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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