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Letter

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Multipartite entanglement is an essential resource for quantum information theory and technologies, but its quantification has been a persistent challenge. Concentratable entanglement (CE), introduced recently, can be estimated from just two copies of a quantum state. Here, we propose generalized concentratable entanglement (GCE), a broader class of multipartite entanglement measures naturally tied to quantum Tsallis entropies, and present a parallelized protocol for estimating GCE across multiple state copies. Increasing the number of copies yields an improved error bound in the presence of imperfections. We prove that GCE is a well-defined entanglement monotone and conjecture some new entropic inequalities. Moreover, we demonstrate the concentration of entanglement into W states using three-state copies. Our results contribute to more robust and versatile characterizations of multipartite entanglement.

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Introduction. Quantum entanglement is one of the most intriguing and fundamental phenomena in quantum mechanics [1]. As a valuable resource in quantum information processing, entanglement is crucial for quantum networks [2–4], distributed quantum computing [5–7], and quantum sensing [8–10]. To certify the functionality of these applications, it is important to verify and quantify the degree of entanglement in quantum systems [11–15]. However, large-scale quantum networks and computers include numerous quantum subsystems, making this characterization challenging. Several multipartite entanglement measurement techniques have been proposed [16–22], but they are often impractical to estimate and limited by system size.

Recently, a family of multipartite entanglement measures called *concentratable entanglement* (CE) has been proposed [23–25]. Mathematically, it is the arithmetic mean of the linear entropy (the first-order approximation of von Neumann entropy), $1 - \mathrm{Tr}(\rho_\alpha^2)$ [26,27], of all possible subsystems α. In some special cases, CE can recover several well-known entanglement measures [18–22]. Moreover, it can be estimated efficiently via parallelized SWAP tests between two copies of a given quantum state $|\psi\rangle$ [28–30]. As collective measurements on multiple copies of a quantum system are advantageous for single-system property testing [31–33], it is compelling to consider the generalization of CE to more than two copies of $|\psi\rangle$, both mathematically and practically.

In this work, we introduce *generalized concentratable entanglement* (GCE) and reveal its close relation to quantum Tsallis entropies $T_K(\rho)$ [34,35], for any real $K > 1$. For any prime number K, we prove that GCE can be efficiently measured in a quantum computer using K copies of a state $|\psi\rangle$ and a parallelized permutation test [36,37]. We also illustrate the approach to compute GCE for nonprime number K by applying multiple different permutation tests. We demonstrate that, up to local unitaries, W states can be probabilistically extracted from the permutation test when $K = 3$, thereby concentrating the entanglement into W states. Furthermore, we analyze the errors in the estimated GCE for a constant number of noisy input states and show that they decrease with the number of state copies as $O(\frac{1}{K})$. We also prove that GCE is still a well-defined entanglement monotone as its value, on average, does not increase under local operations and classical communication (LOCC). In the end, we present several mathematical properties of GCE and, supported by strong numerical evidence, present two conjectures, which may provide deeper mathematical insights into the features of both GCE and Tsallis entropies. In addition, our definition of GCE, featuring efficient estimation and robust mathematical properties, is versatile and applicable to tasks like entanglement witnessing, multivariate trace estimation, and error mitigation, enhancing its utility beyond just entanglement measures.

Generalized concentratable entanglement. We consider the following definition of GCE.

Definition 1 (generalized concentratable entanglement). Consider an input n-qubit pure state $|\psi\rangle$ with labels $S = \{1, 2, \cdots, n\}$ for each qubit respectively, and one measures the entanglement of every non-empty subsystem s in the power set of S, i.e., $s \in \mathcal{P}(S)\backslash\{\varnothing\}$. The generalized concentratable rntanglement (GCE) is defined as

$$\mathcal{C}^{(K)}_{|\psi\rangle}(s) := \frac{1}{K-1}\left(1 - \frac{1}{2^{|s|}}\sum_{\alpha\in\mathcal{P}(s)} \mathrm{Tr}(\rho_\alpha^K)\right), \qquad (1)$$

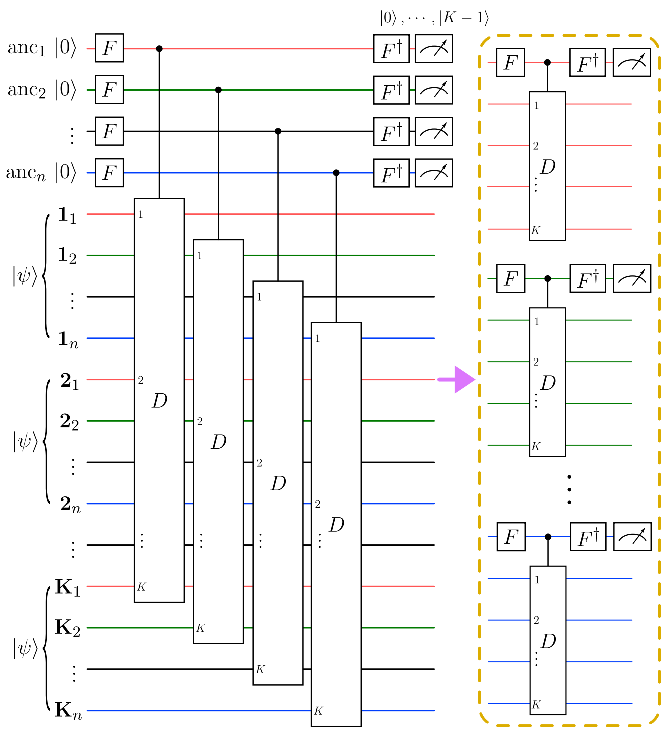

FIG. 1. Circuit structure for computing GCE with prime K. Multiple copies of the state $|\psi\rangle$ are prepared, and n permutation tests are performed on the state copies in parallel using n K-level qudit ancillas. The probability distribution of the measurement result on the ancillas can be efficiently sampled, simply by running the circuit, and hence one can estimate the GCE via Eq. (2). Notably, this only holds for any prime K. The analytical proof for this can be found in [39].

for any $K > 1$. Here, $|s|$ is the cardinality of s. ρ_α denotes the corresponding reduced density matrix of subsystem $\alpha \in \mathcal{P}(s)$, where $\rho = |\psi\rangle\langle\psi|$. We take $\mathrm{Tr}(\rho_\varnothing^K) = 1$.

Equivalently, the definition of $\mathcal{C}_{|\psi\rangle}^{(K)}(s)$ can be viewed as the arithmetic mean of the Tsallis entropies $T_K(\rho_\alpha) = \frac{1}{K-1}(1 - \mathrm{Tr}(\rho_\alpha^K))$ for all subsystems α of the measured system s. Notably, when $K = 2$, Eq. (1) recovers the original CE defined in [23]. Moreover, when $K \to 1$, Eq. (1) becomes the arithmetic mean of the von Neumann entropy [26].

Efficient estimation of GCE. Eq. (1) includes the term $\mathrm{Tr}(\rho_\alpha^K)$. When K is a positive integer, it can be estimated via a controlled-derangement operator, $|0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes D$, decomposed in controlled-SWAP operations [32,33,38], where a possible choice for D is the cyclic permutation operator such that $D|\psi_1\psi_2\cdots\psi_K\rangle = |\psi_2\psi_3\cdots\psi_1\rangle$. To estimate $\mathrm{Tr}(\rho_\alpha^K)$ in this way, one needs to prepare K copies of ρ_α for each α. For $K = 2$, the GCE can be estimated efficiently via a series of parallelized SWAP tests [23]. While an extension to multiple copies via a series of parallelized derangement operators might seem plausible, such an approach is not viable, since D is not Hermitian.

Here we propose a quantum circuit building on the parallelized permutation test with K-level ancillas [36,37] to efficiently estimate the GCE for any prime K. The corresponding circuit is shown in Fig. 1. To begin with, K copies of $|\psi\rangle$ are prepared, and n ancillary K-level qudits are initialized in $|0\rangle$. Then, all ancillas are acted upon by a quantum Fourier transform $F: \ |z\rangle \to \frac{1}{\sqrt{K}}\sum_{k=0}^{K-1}\omega^{zk}|k\rangle$ where $\omega = e^{2\pi i/K}$. Subsequently, a multilevel controlled-D operator, $\sum_{z=0}^{K-1}|z\rangle\langle z| \otimes D^z$, is applied in parallel to qubits in each copy that share the same label. Finally, the inverse Fourier transforms $F^\dagger$ are applied to each ancilla and one measures them eventually. By running the circuit with sufficiently many repetitions, one can obtain the probability distribution of the resulting digit strings on the ancillas, *i.e.*, $p(\mathbf{z}) = p(z_1z_2\cdots z_n)$, where $\mathbf{z} \in \{0, 1, \cdots, K-1\}^n$. From $p(\mathbf{z})$, one can estimate the GCE in Eq. (1) for prime K. For non-prime K, however, a single permutation test circuit is insufficient since D^z for $z \in [1, K-1]$ related to single-cycic permutations only applies to prime K. This issue can be addressed by applying multiple different permutation tests with $\mathcal{K} < K$ state copies, where $\mathcal{K}$ is a divisor of K. Details are in [39], while the main text focuses on estimating GCE with prime K.

Proposition 1. Sampling from the probability distribution $p(\mathbf{z})$, we can estimate GCE for prime K as

$$\begin{aligned}\mathcal{C}_{|\psi\rangle}^{(K)}(s) &= \frac{K}{2^{|s|}(K-1)^2}\sum_{\alpha\in\mathcal{P}(s)}\left(1 - \sum_{\sum_{x\in\alpha} z_x \equiv 0 \pmod{K}} p(\mathbf{z})\right)\\ &= \frac{K}{2^{|s|}(K-1)}\sum_{\alpha\in\mathcal{P}(s)}\sum_{\sum_{x\in\alpha} z_x \not\equiv 0 \pmod{K}} p(\mathbf{z}). \qquad (2)\end{aligned}$$

From Proposition 1, one can find that:

Corollary 1. $p(\mathbf{z}') = 0$ for any $h(\mathbf{z}') \not\equiv 0 \pmod{K}$, where $h(\mathbf{z}')$ denotes the sum over the digits in the string $\mathbf{z}' = z_1'z_2'\cdots z_n'$, i.e., $h(\mathbf{z}') = \sum_{j=1}^n z_j'$.

This corollary states that by running the circuit in Fig. 1, certain measurement outcomes $\mathbf{z}'$ will never occur. As a result, there are only K^{n-1} nontrivial $p(\mathbf{z})$ instead of K^n.

Notably, when $K = 2$, the circuit in Fig. 1 recovers the parallelized SWAP test from Ref. [23], as the Fourier transforms become Hadamard gates, and controlled-D operators become Fredkin gates. In addition, the circuit in Fig. 1 can be encoded into a qubit system, possibly with redundant computing power, which may be more suitable for practical experiments.

Entanglement concentration for $K = 3$. When $K = 2$, there is a probabilistic way to concentrate the entanglement (Bell pairs) from the input states at the controlled system of each parallelized SWAP test. In this work, we provide a similar proposition for $K = 3$:

Proposition 2. For $K = 3$, there is a probabilistic entanglement concentration for each parallelized permutation test. Specifically, for each parallelized permutation test, when the top ancilla clicks at either $|1\rangle$ or $|2\rangle$. Then once we measure out the other target systems with the ancilla outcome being $|0\rangle$, there exists a set of local unitaries that convert the remaining controlled system into W states: $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$.

For $K > 3$, there does not exist a set of local unitaries that convert the controlled system into either W or Greenberger–Horne–Zeilinger (GHZ) states and the GCE of the output state from each parallelized permutation test is possibly larger than the values of GHZ or W states, which can be easily tested numerically. This is because for the GCE of the large system,

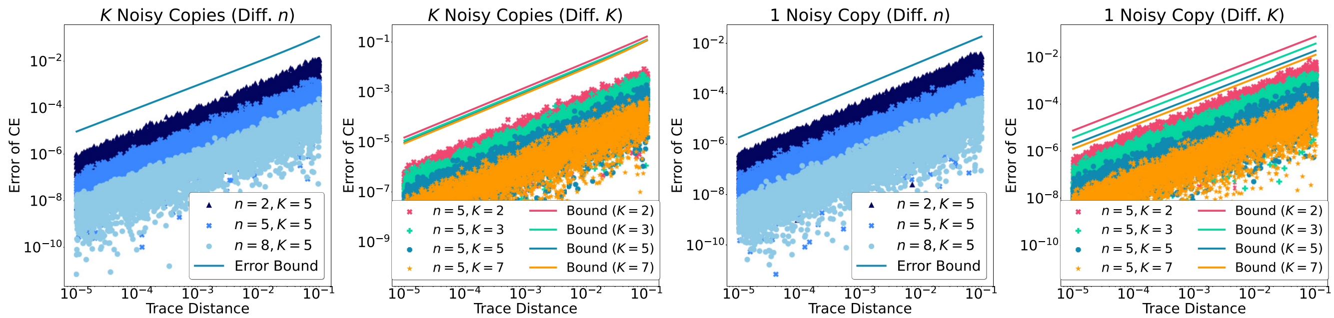

FIG. 2. Numerical simulations for GCE errors. We have considered 5000 samples of Haar-random input states $|\psi\rangle$ for each $\{K, n\}$ and $|s| = 2$ for all cases. The leftmost two plots show scenario 1, where there are K noisy copies of $|\psi\rangle$, each with trace distance ϵ to the perfect state. The rightmost two plots show scenario 2, where there is only 1 noisy copy. One can see that all the errors are upper-bounded by Eq. (3). Also, on average, the GCE errors decrease for larger K and n. Our study suggests there may exist sharper upper bounds.

its maximum value is far away from the case of GHZ or W states, as intensively discussed for $K = 2$ in Ref. [40]. Thus, other different entanglement structures may appear for the controlled system when $K > 3$.

Robustness analysis. In this section, we consider the scenario that only noisy states $|\psi'\rangle$ can be prepared such that there is a trace distance $\mathcal{D}(|\psi\rangle, |\psi'\rangle) = \frac{1}{2}\| |\psi\rangle\langle\psi| - |\psi'\rangle\langle\psi'| \|_1 = \epsilon$. In this case, the estimated GCE will also have errors accordingly, and here we propose an upper-bound for the errors of estimated GCE.

Proposition 3. Suppose there are prime K noisy input copies $|\psi_1'\rangle, |\psi_2'\rangle, \cdots, |\psi_K'\rangle$ with $\mathcal{D}(|\psi\rangle, |\psi_k'\rangle) = \epsilon_k$. Therefore, in general, via the approach shown in Fig. 1 and Eq. (2), there exists an upper bound for the estimated GCE error $\mathcal{E} = |\mathcal{C}^{(K)}_{|\psi_1'\rangle,|\psi_2'\rangle,\cdots,|\psi_K'\rangle}(s) - \mathcal{C}^{(K)}_{|\psi\rangle}(s)|$ [41]:

$$\mathcal{E} \leqslant \frac{2^{|s|}-1}{(K-1)2^{|s|}}\left(\sum_{k=1}^{K}\epsilon_k + \sum_{k<k'}\epsilon_k\epsilon_{k'} + \sum_{k<k'<k''}\epsilon_k\epsilon_{k'}\epsilon_{k''} + \cdots + 2\prod_{k=1}^{K}\epsilon_k\right). \tag{3}$$

For simplicity, we assume each copy of $|\psi\rangle$ is either noisy with ϵ error or perfect. For illustration, we consider the following two scenarios. First, suppose there are K noisy copies, then the GCE error is upper bounded by $\mathcal{E} \leqslant (1 - \frac{1}{2^{|s|}})\frac{1}{K-1}((1+\epsilon)^K + \epsilon^K - 1)$. Second, if there is only one noisy copy, the GCE error is upper bounded by $\mathcal{E} \leqslant (1 - \frac{1}{2^{|s|}})\frac{1}{K-1}\epsilon$. In this case, the upper bound of $\mathcal{E}$ decreases strongly reciprocally with K.

Figure 2 depicts numerical results for these two error scenarios. We find that the errors $\mathcal{E}$ will decrease on average under a larger number of copies K or with a larger state size n. Moreover, the errors are always below the error bound in Eq. (3). Note that the error bound in Eq. (3) is the analytical result and may not be sharp enough. It is possible that there also exists a general sharper bound including the system size n as well.

Properties of GCE. Our definition of GCE $\mathcal{C}^{(K)}_{|\psi\rangle}(s)$ enjoys some convenient properties, which hold in the more general setting $K \in \mathbb{R}, K > 1$.

Theorem 1. The GCE has the following properties:

(1) *Pure state entanglement measures: $\mathcal{C}^{(K)}_{|\psi\rangle}(s)$ is nonincreasing on average under LOCC.*

(2) *$\mathcal{C}^{(K)}_{|\psi\rangle}(s) = 0$ for fully product states $|\psi\rangle = \otimes_{j=1}^{n}|\phi_j\rangle$ and $\forall s \in \mathcal{P}(S)\backslash\{\varnothing\}$.*

(3) *Continuity: For two pure states $|\psi\rangle$ and $|\psi'\rangle$ that satisfy $\mathcal{D}(|\psi\rangle, |\psi'\rangle) \leqslant \epsilon$, then $|\mathcal{C}^{(K)}_{|\psi\rangle}(s) - \mathcal{C}^{(K)}_{|\psi'\rangle}(s)| \leqslant \frac{2K}{K-1}\epsilon$.*

(4) *$\mathcal{C}^{(K)}_{|\psi\rangle}(S) = \mathcal{C}^{(K)}_{|\psi\rangle}(S\backslash\{n_0\})$ for any single subsystem label $n_0 \in \{1, 2, 3, \ldots, n\}$.*

Theorem 1 highlights key properties of GCE, establishing it as a well-defined entanglement measure. Specifically, Theorem 1.1 shows that $\mathcal{C}^{(K)}_{|\psi\rangle}(s)$ is a well-defined entanglement measure for any $K > 1$. Also, one should not confuse Theorem 1.3 with Proposition 2, as Theorem 1.3 can be regarded as the special case of Proposition 2 where the input noisy states are exactly the same. The proof of the theorem can be found [39].

Furthermore, our numerical studies (c.f. [39]) provide evidence for the following conjecture:

Conjecture 1. The GCE has the following properties:

(1) *$\mathcal{C}^{(K)}_{|\psi\rangle}(s') \leqslant \mathcal{C}^{(K)}_{|\psi\rangle}(s)$ if $s' \subseteq s$.*

(2) *Subadditivity: $\mathcal{C}^{(K)}_{|\psi\rangle}(s \cup s') \leqslant \mathcal{C}^{(K)}_{|\psi\rangle}(s) + \mathcal{C}^{(K)}_{|\psi\rangle}(s')$ for $s \cap s' = \varnothing$.*

Here, Conjecture 1.1 states that the GCE for any $K > 1$ of a given system is always larger than or equal to the GCE of its subsystems, which should hold for a well-defined entanglement measure. Conjecture 1.2 surmises that the GCE for any $K > 1$ obeys the subadditivity property, which means that the sum of two separate systems' GCEs should be larger than the GCE of the overall system. These two conjectures have been proven to be true for $K = 2$ [23]. Based on these arguments, we here conjecture that they also hold for $K > 1$. Moreover, we show that:

Proposition 4. Conjecture 1.1 is equivalent to a not-so-strong subadditivity (NSSSA) form of Tsallis entropy: for an n-partite (qubit) pure state $\rho = |\psi\rangle\langle\psi|$ and any of its tri-separation ABC s.t. B contains only one party (qubit), we have:

$$\sum_{\alpha_A \in \mathcal{P}(A)} (T_K(\rho_{\alpha_A BC}) + T_K(\rho_{\alpha_A}) - T_K(\rho_{\alpha_A B}) - T_K(\rho_{\alpha_A C})) \leqslant 0, \tag{4}$$

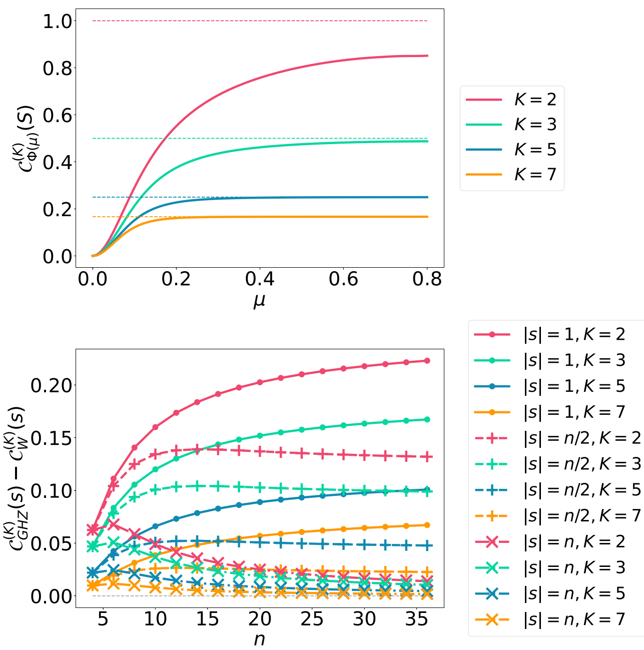

FIG. 3. GCE ($s = S$) of the 40-qubit spin-squeezed state $|\Phi(\mu)\rangle$ for different K and interaction strength μ (upper plot), and the GCE difference $\mathcal{C}^{(K)}_{\mathrm{GHZ}}(s) - \mathcal{C}^{(K)}_{W}(s)$ for different n, $|s|$ and K (lower plot). For $|\Phi(\mu)\rangle$, GCE measures more entanglement and goes asymptotically to $\sim(K-1)^{-1}$ with larger K and μ. For GHZ and W, GCE concludes more entanglement in GHZ state than in W state as $\mathcal{C}^{(K)}_{\mathrm{GHZ}}(s) > \mathcal{C}^{(K)}_{W}(s)$, and the GCE difference decreases for increasing K and $|s|$.

which is the sum over all possible strong subadditivity (SSA) of Tsallis entropy related to the subsets of A. Here, each α_A is in the power set of A, and $\alpha_A BC$, $\alpha_A B$, $\alpha_A C$ denote the union of the corresponding subsystems of ρ.

Proposition 4 may be interesting and important in its own right, because SSA of Tsallis entropy does not generally hold [42]. It is important to note that the equivalence between Conjecture 1.1 and NSSSA in Proposition 4 always holds true. However, the validity of the statement in Proposition 4 itself depends on the truth of Conjecture 1.1. We refer the reader to [39] for further discussions.

Examples. We now calculate GCE for selected quantum states that are of experimental interest. Let us first consider spin-squeezed states, $|\Phi(\mu)\rangle$, that can be prepared by evolving a coherent spin state under the one-axis twisting Hamiltonian operator $H_{OAT} = \chi \hat{S}_z^2$ for a time t and parametrized through the interaction strength $\mu = 2\chi t$ [43,44]. The GCE ($s = S$) of the 40-qubit state $|\Phi(\mu)\rangle$ is shown in Fig. 3 (upper plot). Notice that when the interaction strength μ becomes larger, the GCE predicts more entanglement in the state $|\Phi(\mu)\rangle$. Moreover, it goes asymptotically to $\sim(K-1)^{-1}$ for larger K, and larger K ends up with faster convergence w.r.t. the interaction strength μ.

Secondly, we consider W states ($|W\rangle = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} |0\cdots 1_j \cdots 0\rangle$) and GHZ states ($|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$). The GCE difference between them, $\mathcal{C}^{(K)}_{\mathrm{GHZ}}(s) - \mathcal{C}^{(K)}_{W}(s)$, is shown in Fig. 3 (lower plot). Note that $\mathcal{C}^{(K)}_{\mathrm{GHZ}}(s) > \mathcal{C}^{(K)}_{W}(s)$ for all cases, which illustrates that there is more entanglement in GHZ state than in W the state according to GCE measures. The GCE difference also decreases with increasing K and $|s|$. In addition, when $|s|$ is large enough (for example, $|s| \approx n$), increasing n makes the GCE difference go asymptotically to 0. More analytical details for these two examples can be found in [39].

Conclusions and outlook. In this work, we proposed a K-th order pure-state entanglement measure, generalized *concentratable entanglement*, for arbitrary real-valued $K > 1$. We provided an efficient way for estimating GCE on a quantum computer, through parallelized permutation tests given that the number of state copies, K, is prime. Using multiple parallelized permutation tests for both order K and its corresponding divisors enables the computation of GCE for non-prime K as well. We also showed that each parallelized permutation test can concentrate the entanglement into the W state with three state copies. In addition, we provided both analytical and numerical results for the errors of the estimated GCE when some of the state copies are imperfect. We demonstrated that these errors become smaller as the number of copies increases, for a constant number of imperfect copies. Then, we proved several mathematical properties of GCE, especially that GCE is a well-defined pure-state entanglement measure as it does not increase under LOCC on average. Backed up by strong numerical evidence, we also proposed two conjectures that may provide further mathematical insights, not only on GCE, but also of independent interest for the study of quantum Tsallis entropies. One of them hints at the existence of a weaker form of strong subadditivity of quantum Tsallis entropy (NSSSA), which lies between subadditivity (which holds [45]) and strong subadditivity (which does not hold in general [42]), and may serve as an interesting starting point for future investigations. Finally, we provided examples and explicitly calculated GCE for spin-squeezed, W, and GHZ states. We anticipate that our GCE measures could establish new directions in multipartite entanglement, enabling practical, noise-resilient implementation and making significant mathematical contributions to entanglement theories.

Notably, the original *concentratable entanglement* (CE) has natural connections to other various applications, including entanglement witnesses [46–51], multivariate trace estimation [38,52–57], probing the properties of many-body systems [58–60], error mitigation [32,33,52,61,62], quantum benchmarking [12,63–66], and inspiring other entanglement measures [67–69]. Given the wide-ranging applications of CE, exploring its generalization to GCE is highly meaningful, as GCE may offer deeper insights and further advance these applications even beyond the scope of GCE measure itself.

It would be interesting to explore the entanglement concentrated from the parallelized permutation test for $K > 3$, which remains open [40]. Moreover, deriving a sharper bound for the GCE errors would be useful to study the intrinsic error properties of larger n and K. Finally, a rigorous proof of Conjecture 1 should yield valuable insights in the quest to characterize all entropic inequalities in quantum information science [70–76].

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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