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Exponential distance relation (aka Titius–Bode rule) in extra solar planetary systems

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ABSTRACT

In this paper we present phenomenological evidence for the validity of an exponential distance relation (also known as generalized Titius–Bode law) in the 32 planetary systems (31 extra solar, plus our Solar System) containing at least five planets each (known up to July 2023). We produce the semi-log fittings of the data, and we check them against the statistical indicators of R^2 and Median. Then we compare them with the data of 4000 artificial planetary systems created at random. In this way, a possible origin by chance of the Titius–Bode rule (TBR) is reasonably excluded. We also point out that in some systems the fittings can be definitely improved by the insertion of new planets into specific positions. We discuss the harmonic resonances method and fittings, and compare them with the Titius–Bode fittings. Moreover, for some specific systems, we compare the Titius–Bode fitting against a polynomial fitting ($r \sim n^2$). Further comparisons with previous relevant works are reported in the last section. It emerges that TBR describes 25 out of the 32 planetary systems (78 per cent) with a $R^2 \geq 0.95$. Further, it results to be the most economical (in terms of free parameters) and best-fitting law for the description of spacing among planetary orbits. This analysis allows us to conclude that an exponential distance relation can reasonably be considered as ‘valid’, or strongly corroborated, also in extra solar planetary systems.

Key words: planetary systems – exoplanets.

1 INTRODUCTION

Following the footsteps of the visionary work of Johannes Kepler (Kepler 1596), for more than 170 years scholars had been looking for a law able to encode in a formula the distances of the known planets from the Sun. Kepler proposed the platonic solids as a guide to the numerical progression of the major axis of the elliptic orbits he envisaged. Such an idea was quite rapidly abandoned, especially after the dynamics of the Solar System had been unveiled by Isaac Newton. Nevertheless the quest for a distance relation among the planets of the Solar System remained alive. And actually the quest soon extended also to the systems of moons of Jupiter and Saturn, discovered in the meanwhile. After several precursors had paved the way (among them we count Christian von Wolff, mathematician, physicist, philosopher, and his brilliant disciple Immanuel Kant), finally a rule for the distances of planets from the Sun was proposed by Johann Daniel Titius (Titius 1766), and published as a note in his German translation of the Charles Bonnet’s *Contemplation de la Nature* (1764). Soon the law was noticed and popularized by the much more famous astronomer Johann Elert Bode (Bode 1772).

The original formulation of the rule, proposed by Titius, can be expressed by the simple formula

$$r(n) = 0.4 + 0.3 \cdot 2^n, \quad (1)$$

where the distance $r(n)$ of the planet from the Sun is given in astronomical units, i.e. in terms of the radius of the Earth’s orbit (which defines 1 Astronomical Unit $\simeq 150 \times 10^6$ km). For $n = -\infty, 0, 1, 2, \dots$ this relation gives the distances $r(n)$ from the Sun, respectively, of Mercury, Venus, Earth, Mars, etc., including the asteroid belt (actually, Ceres, the heaviest asteroid, was discovered by Piazzi in 1801 by following the indication of this rule with $n = 3$), and Uranus ($n = 6$), which at the moment of the first formulation of the law (1766–1772) had not yet been discovered [see the book of Nieto (1972) for history, different explicit formulations, and extensive bibliography].

In its original form (1), the relation was not able to account for the distance of Neptune and Pluto. During the 20th century more refined versions of the rule were elaborated (Blagg 1913; Richardson 1945; Dermott 1968). The present-day versions are able to describe not only the planetary distances within the solar system, including planets like Neptune and Pluto, but also can be successfully applied to the systems of satellites orbiting Jupiter, Saturn, and Uranus (See Appendix D for the precise ways in which those moons obey TBR). The agreement between the predicted and the observed distances of the various satellites from the central body is really good, of the order of a few per cents (see e.g. again Nieto 1972). The modern version of the Titius–Bode rule can be expressed, if we neglect second-order corrections, by an exponential relation as

$$r(n) = a e^{2\lambda n}, \quad (2)$$

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where the factor 2 is introduced for convenience reasons and $n = 1, 2, 3, \dots$

For the Solar System we have

$$2\lambda = 0.53707, \quad e^{2\lambda} \simeq 1.7110, \\ a = 0.21363 \text{ A.U.}$$

The amazing thing found by Blagg was that the geometric progression ratio $e^{2\lambda}$ is roughly the same for the Solar System and for the satellite systems of Jupiter ($e^{2\lambda} \simeq 1.7277$), Saturn ($e^{2\lambda} \simeq 1.5967$), and Uranus ($e^{2\lambda} \simeq 1.4662$). The parameter λ is dimensionless, its value is inferred from the observed data, and it depends on the specific system considered (planetary or satellite system). Also the parameter a is in general obtained from observations, it has the dimension of a length and it is linked to the radius of the first orbit of the system considered, since $r(1) = ae^{2\lambda}$. With a slightly different formulation of the TB rule as $r(n) = ae^{2\lambda(n-1)}$, $n = 1, 2, 3, \dots$, then a coincides with the radius of the first orbit, $r(1) = a$.

Despite its quite evident successes in describing the spacing of the local planetary systems (Solar system, plus the satellites systems of Jupiter, Saturn, Uranus), the physical interpretation of the TB relation has been often questioned and remains a matter for heated debate.

Broadly speaking, there are at least three groups of opinions regarding the TBR and its physical significance: (i) the TBR constitutes just simply sheer numerology; (ii) the TBR is valid only for some specific systems due to particular conditions occurred during the planet formation process; (iii) the TBR is largely a direct consequence of (some) planetary stability requirements which should be satisfied during the course of the systems' existence.

Many criticisms have been risen against an effective physical meaning of the law. For example, according to Graner & Dubrulle (1994) the TB relation is probably just a consequence of the scale invariance of the disc which gave rise to the planets. However, according to Lynch (2003), it is not possible to conclude unequivocally that laws of Titius–Bode type are, or are not, physically significant. In other words, Lynch convincingly argued that the agreement with the observations cannot be safely considered as a mere statistical chance. So, the possibility of a physical explanation for the observed distributions remains open.

A somehow similar conclusion also appears to be consistent with the work by Hayes and Tremaine (1998). Following an approach which involves some statistical numerical experiments, they chose to fit randomly selected artificial planetary systems to Titius–Bode type laws by considering a distance rule inspired by the Hill stability of adjacent planets. They did not identify a particularly profound significance of the TBR, except that its general meaning is that stable planetary systems tend to be spaced in a regular manner.

On the side of physical models, many theories have been developed during the last 250 years to explain the Titius–Bode relation. There have been dynamical models connected with the theory of the origin of the solar system (Alfven 1954), gravitational theories (Schmidt 1946), nebular theories (Weizsäcker 1943, Willerding 1992), just to cite some of them. Many of them have been excellently reviewed in the book of Nieto (1972).

Also the approaches involving ideas from scale relativity, or stochastic trajectories, or also Schrödinger-like equations, in order to give account of the rule (2) have a robust tradition. During the years many authors have suggested various models in this direction (for an incomplete list of papers see for example: Caswell 1929; Albeverio et al. 1983; Nottale 1996; Agnese & Festa 1997; Nottale et al. 1997; Reinisch 1998; de Oliveira Neto et al. 2004; Scardigli 2007).

Of course, in the last 15–20 years, with the discovery of a growing number of extra solar planetary systems, it has become increasingly important to check the TB relation in the new observed systems. Following the original tradition, many authors have applied the TB rule to exoplanetary systems in order to predict new planets. In this direction goes, for example, the paper of Bovaird & Lineweaver (2013). The exoplanetary system 55 Cancri (HD 75732), containing five planets, was investigated in some detail using the TBR by Chang (2008), Poveda & Lara (2008), and Cuntz (2012). Cuntz (2012), in particular, argued that new planetary candidates were predicted in the 55 Cancri system through the Titius–Bode's relation, and perhaps one of the new planets could be habitable. Also the planetary system revolving around the star HD10180 has been studied in some specific detail (Lovis et al. 2011), and it appears to satisfy the TB rule. According to Chang (2010) one cannot rule out the possibility that the distribution of the ratio of orbiting periods in multiple planet systems is consistent with that derived from Titius–Bode's relation. Lara et al. (2020) used data from 27 exoplanetary systems with at least five planets and showed that the planetary orbital periods in exoplanetary systems are not consistent with a random distribution.

The main purpose of this paper is to examine the 32 planetary systems that, up to today (July 2023, see NASA Archive), appear to host at least five planets each, all orbiting around a *single* star, and to see if and to what extent the TB rule is satisfied in such systems. We decided to consider systems with at least five planets in order to fit at best the two parameters of TB rule, and to minimize the possible (statistical) errors. Of course, this investigation has been made possible by the surveys and discoveries operated by *Kepler* and *TESS* satellites, in particular in the last ten years or so.

On the other hand, a survey aimed at checking the TB rule in extra solar planetary systems seems to be quite timely now, since the data released by *Kepler* and *TESS* satellites cover a significant number of systems, each one endowed with a significant number of confirmed planets (at least five, as said).

As for the source of our data, we decided to stick with the NASA Exoplanet Archive (NASA Archive), in order to have an up-to-dated source, as well as to enjoy a reliable uniformity in the presentation of the data.

In this paper we focus mainly on an exponential spacing law (aka *Titius–Bode rule*). In Section 2 we present the planetary systems under scrutiny, and discuss the analytical tools employed to produce fittings and graphs. In Section 3 (following the tradition which led to the proposal and discovery of the asteroid belt), we present the possibility to improve some fittings by inserting new planets to fill suspected gaps among existing planetary orbits. We discuss several examples and propose some 'predictions'. In Section 4 we discuss a possible relation between the TB rule and the age of the planetary systems. In Sections 5 and 6 we briefly discuss also different descriptions of the planetary spacing data, namely the harmonic resonances fitting (Section 5), and the polynomial fitting (Section 6). Finally, Section 7 is devoted to some further specific comparisons with previous works, and then to conclusions.

We are pretty confident in saying that the main result of our paper is that the TB law seems to be confirmed as the 'best' rule to describe planetary spacing also among extra solar planetary systems, at least for what concerns the best fitting obtained with a minimum number of free parameters (two).

2 FITTINGS AND GRAPHS

The list of planetary systems considered in this work is given in Table 1.

Table 1. Each of the 32 systems considered in this work hosts at least five planets, revolving around a single star. The statistical indicators displayed are R^2 and Median (for definitions see the main text and appendices). We have a maximum agreement when $R^2_{\max} = 1$ and $\text{Median}_{\max} = 0$. For references on each single planetary system see Table E1.

System	Planets	R^2	Median	Notes
Sun	10	0.993448869	0.019581007	with Ceres
KOI-351	8	0.96322542	0.059404861	Kepler 90
Trappist 1	7	0.994255648	0.012727939	–
HD 10180	6	0.992725066	0.06644463	–
HD 191939	6	0.945551901	0.08825776	2022
HD 219134	6	0.913657699	0.126773358	–
HD 34445	6	0.974290137	0.108819429	–
K2-138	6	0.945458309	0.094321758	2021
Kepler 11	6	0.962373014	0.044210656	–
Kepler 80	6	0.955414548	0.04023616	–
TOI - 1136	6	0.987422135	0.031168586	2022
TOI - 178	6	0.983217838	0.047680406	2021
HD 108236	5	0.976381637	0.037194964	2021
HD 158259	5	0.999713447	0.003377513	2020
HD 23472	5	0.988496548	0.022330332	2022
HD 40307	5	0.984933377	0.046130252	–
K2-268	5	0.957182918	0.062647955	2019
K2-384	5	0.983020508	0.04991254	2022
Kepler 102	5	0.987900552	0.017713236	–
Kepler 122	5	0.98710756	0.026533536	–
Kepler-150	5	0.8605413	0.236636167	–
Kepler 154	5	0.985703723	0.045352455	–
Kepler 169	5	0.875354644	0.151151179	–
Kepler 186	5	0.92681234	0.093772029	–
Kepler 238	5	0.98771232	0.053557294	–
Kepler 292	5	0.995427692	0.008569513	–
Kepler 32	5	0.971262925	0.042153715	–
Kepler 33	5	0.948841017	0.054322078	–
Kepler 55	5	0.988367826	0.02874602	–
Kepler 62	5	0.950734271	0.15977283	–
Kepler 82	5	0.955569351	0.086643589	2019
Kepler 84	5	0.992394962	0.022666502	–

As we said, the data used to construct Table 1 are taken from NASA Exoplanet Archive.¹ Hereafter we discuss the reasons to consider that list, give comments, and explanations of parameters appearing in Table 1, and discuss anomalous cases.

First, in our analysis, we only consider extra solar planetary systems with at least *five* confirmed planets each (or more), revolving around a *single* star. The choice of the number ‘5’ for the population of the planetary systems examined, could appear quite arbitrary, and perhaps it is. However, simple practical considerations push towards that choice. Since our aim is to fit the distances/periods of the planets of a single system with a two-parameters exponential law, of course one would like to maximize the number of points contained in a single system, and therefore to choose systems with a large number of planets. To compute with accuracy the coefficients of a semi-log linear regression requires as many points as possible. That is even more true, if we consider that some of the TB regression lines will be used to ‘predict’ the presence of new planets in a given system. Obviously, the smaller the number of points you have at your disposal, the less performing is an exponential TB law in this prediction task. On the other hand, the known systems with

¹For this reason we excluded the system HIP 41378, which seems to have five planets, but the data of the fifth planet are not available on the NASA Archive.

Table 2. R^2 and Median analysis for planetary systems with two or three stars at their centre. Maximum agreement for $R^2_{\max} = 1$ and $\text{Median}_{\max} = 0$.

System	Planets	R^2	Median	Notes
Kepler 20	6	0.994934868	0.020480257	2 stars
55 Cancri	5	0.974846945	0.105266615	2 stars
GJ 667 C	5	0.940319108	0.085544299	3 stars
Kepler 296	5	0.999532566	0.006855603	2 stars
Kepler 444	5	0.996885042	0.007292345	3 stars

Table 3. Raw data for the planetary system KOI-351.

Planet name	Host name	N.Stars and N.Planets	Orbital period (days)
KOI-351 b	KOI-351	1 - 8	7.008151±0.000019
KOI-351 c	KOI-351	1 - 8	8.719375±0.000027
KOI-351 d	KOI-351	1 - 8	59.73667±0.00038
KOI-351 e	KOI-351	1 - 8	91.93913±0.00073
KOI-351 f	KOI-351	1 - 8	124.9144±0.0019
KOI-351 g	KOI-351	1 - 8	210.60697±0.00043
KOI-351 h	KOI-351	1 - 8	331.60059±0.00037
Kepler-90 i	KOI-351	1 - 8	14.44912±0.00020

many planets are not so many, and we also want to check the TB relation in as many different systems as possible, in order to strongly corroborate its statistical validity. The balance between these two opposite requirements has pushed us to choose systems populated with at least five planets.

For sake of completeness, in Table 2 we report also the statistical analysis for planetary systems containing at least five planets, but with two or three stars at their centre. Although some of these systems present very good statistical indicators (with the exception of GJ 667 C), it is also fair to say that the presence of two or three stars at the centre of these systems can complicate the understanding of their dynamics in unexpected ways. Therefore, as for the discussion of the phenomenology of the Titius–Bode rule in extra solar planetary systems, we think it is prudent to confine ourselves to single-star systems, at least in this paper, postponing the analysis of multiple-star systems to future works.²

2.1 Semi-log formulation of TB rule

In Table 3 we display a sample of the raw data of one of the 32 planetary systems under our scrutiny (precisely, KOI-351), data that we used to build the semi-log linear regressions, and hence to compute the R^2 and Median statistical indicators showed in Table 1. In particular, to our scope, pivotal data are: number of planets for each system; number of stars in each system (we considered planetary systems with only one single star at the centre); period and/or semimajor axis of each planetary orbit.

In order to produce a linear fitting of the data presented in Table 3, it is useful to take the log of relation (2) so that

$$\log r(n) = \log a + 2\lambda n, \quad (3)$$

²Further, we note that the system HD 20781-2, endowed with five planets, has not been reported neither in Table 1 nor in Table 2, since not only it contains two stars at its centre, but apparently *four* planets revolve around HD 20781, and *one* planet revolves around HD 20782... a really too complicated system!

which can be rewritten as

$$Y(n) = A + Bn, \quad (4)$$

where $Y(n) = \log r(n)$. The linear fittings will provide the coefficients A and B , which are linked to the TB parameters by the relations

$$\begin{aligned} a &= e^A \\ \lambda &= B/2. \end{aligned} \quad (5)$$

Since the planetary orbital periods are actually the variables directly measured by the observers, instead of semimajor axes, we can make use of Kepler third law, $T^2 = kr^3$, to reformulate the rule (2) in terms of orbital periods and integers³

$$T^2 = kr^3 = ka^3 e^{6\lambda n} \quad (6)$$

that is

$$T(n) = (ka^3)^{1/2} e^{3\lambda n}. \quad (7)$$

Taking the log of both members we have

$$Z(n) = C + Dn, \quad (8)$$

where

$$Z(n) = \log T(n), \quad C = \frac{1}{2} \log(ka^3), \quad D = 3\lambda. \quad (9)$$

Following equations (3)–(8), for each planetary system we proceed to order in an increasing way the periods $T(n)$, or better the $\log T(n)$, and we label each term of the resulting sequence with an increasing natural integer $i = 1, 2, 3, \dots$. In so doing we obtain, for each system, the set of data $\{i, \log T(i)\}$ to which the usual linear regression method is applied (the *least squared method*). Some of these plots are displayed, as examples, at the end of this Section in Fig. 1.

In our diagrams we do not report error bars. The reason is simple: from Table 3, and from the general tables in the NASA Archive, it appears clearly that the large majority of the errors on the periods (measured in days) are at most of the order of 10^{-2} or less. Since we are interested in displaying on the diagrams the logs of periods, any (already small) error bar would be further strongly suppressed. In formulae, if we have $T = T_0 \pm \varepsilon$, then

$$\log(T) = \log(T_0 \pm \varepsilon) \simeq \log T_0 \pm \frac{\varepsilon}{T_0}, \quad (10)$$

and displaying error bars of the order of 10^{-3} or less would be graphically tough, as well as useless.

2.2 Statistical tools

The tools we use to quantify the ‘goodness’ of the linear regressions, and consequently of the evaluations of the TB parameters, are the R^2 and Median statistical indicators (see Appendices A, B, C for definitions and properties). An inspection of Table 1 reveals that, of the 32 single-star planetary systems (including our Solar systems), at least six have an R^2 greater than 0.99, revealing a very good agreement between the astronomical data and the linear regression with the phenomenological TB rule.

One of the most common arguments used against the TB rule concerns a possible origin at random of the law itself. According

³A formulation of the exponential TB relation that uses orbital periods (in place of semimajor axis) is also known under the name of ‘Dermott’s law’ (Dermott 1968).

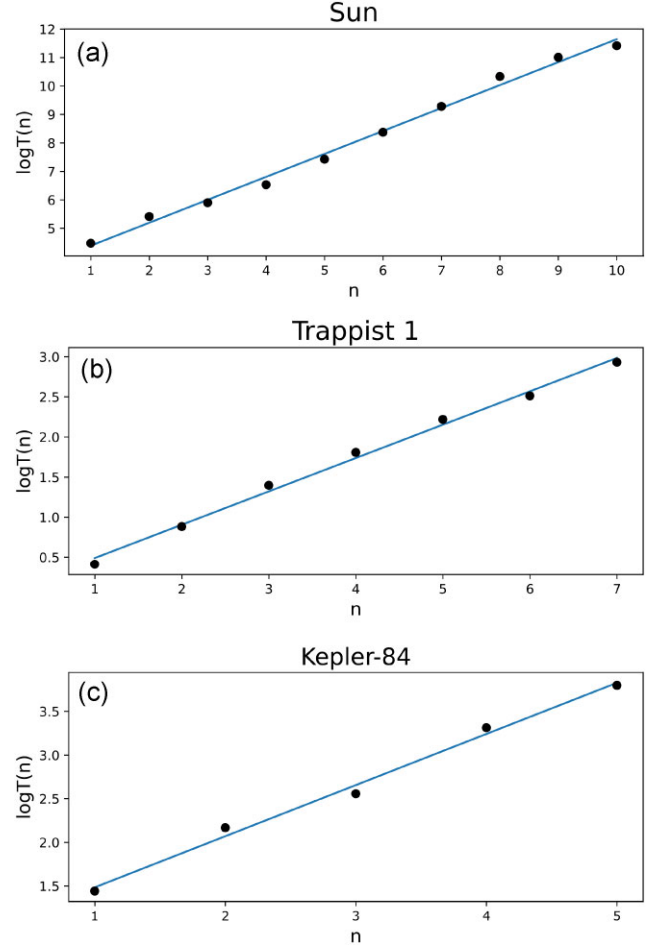


Figure 1. Examples of various TB fittings for different systems. (a) TB fitting of the Solar system, including Ceres and Pluto, namely 10 planets. The statistical indicators (see Table 1) are $R^2 = 0.993448869$ and Median = 0.019581007. (b) TB fitting of Trappist 1, a system with 7 confirmed planets. The statistical indicators (Table 1) are $R^2 = 0.994255648$ and Median = 0.012727939. (c) TB fitting of Kepler-84, a system with five confirmed planets. The statistical indicators (Table 1) are $R^2 = 0.992394962$ and Median = 0.022666502.

to this point of view, it would be statistically ‘easy’ to produce planetary systems at random, and they would naturally turn out to obey the TB relation, by pure chance. To explore this possibility, we created randomly 4000 artificial planetary systems: 1000 systems with 8 planets each, 1000 with 7, 1000 with 6, and 1000 systems with 5 planets each, respectively. In this first naive attempt we did not impose any particular further constraint. Essentially, referring for example to a system with 8 planets, we extracted at random 8 numbers T_i , with $0 \leq T_i \leq 1$, namely the periods of the 8 planets, and reordered them in an increasing way, so that $T_i < T_j$ iff $i < j$. Of course, through a trivial rescaling μT_i , these numbers can represent any period from zero to infinity. We then fit the points $\{i, \log T_i\}$, $i = 1, 2, \dots, 8$, with a linear regression, and compute the relevant R^2 and Median indicators. The averages over the 1000 random systems are displayed in Table 4.

Already a superficial glimpse to Table 1 shows that the R^2 ’s of ‘real’ systems are much closer to 1 than the R^2 ’s attained by the randomly created systems of Table 4. This first impression can be further substantiated by comparing ‘real’ and ‘random’ statistical indicators, as it is done in Table 5.

Table 4. R^2 and Median analysis for randomly created planetary systems. Maximum agreement $R^2_{\max} = 1$ and Median $_{\max} = 0$.

Systems with	R^2	Median	Notes
8 Planets	0.91183483	0.1416661	Averages over 1000 random systems
7 Planets	0.90784475	0.14490068	idem
6 Planets	0.90386207	0.15141145	idem
5 Planets	0.89719999	0.15552048	idem

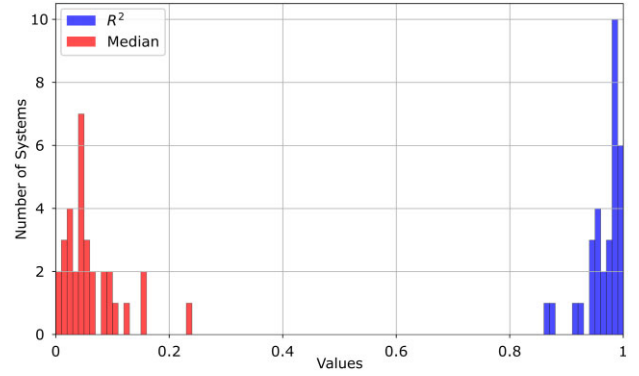
Table 5. Comparison between ‘Randomly created’ and ‘Real’ planetary systems, R^2 and Median analysis.

Systems	AVG of all R^2 's	AVG of all Medians
Randomly created	0.90518541	0.148374678
Real	0.966078109	0.06215032

There, we compare the averages of the statistical indicators R^2 and Median computed for artificial as well as for real planetary systems. From the figures, it appears extremely clear that an exponential distance relation (aka TB law) fits much better real planetary systems than artificial, randomly created, ones [see also the discussions in, e.g. Lecar (1973); Pletser (1988); Hayes & Tremaine (1998); Pletser (2017)]. Of course, the efficiency displayed by the TB rule in describing distances in real planetary systems calls for a (fully accepted and shared) theoretical explanation, which is perhaps still lacking, at present. However, from the phenomenological point of view, it is quite clear that the descriptive ability of the TB relation cannot be easily denied, even for extra solar planetary systems.

One can also refine the ‘quality’ of the artificial planetary systems, for example by inserting mechanisms which mimic, in principle, the gravitational evolution of the (proto)planetary system itself. A possibility followed by Hayes & Tremaine (1998) is to require that adjacent planets are separated by a minimum distance of k times the sum of their Hill radii, for opportune values of k ($0 \leq k \leq 8$). This originates more realistic planetary systems, and, according to the conclusion of Hayes & Tremaine (1998), these systems generally fit a Titius–Bode law better than the purely ‘random’ ones, and about as well as some of the real planetary systems (as for example our own Solar System). From our point of view, this reinforces our thesis that the TB rule truly describes an actual phenomenological property of real planetary systems (Lara et al. 2012).

To illustrate further the capacity of the TB relation to effectively describe the spacing in planetary systems, we provide hereafter a diagram where the TBR-valid systems are related to a ‘total number’ of observed exoplanetary systems. Of course, crucial is the definition of what ‘total number’ should mean in this context. Given our argued choice to restrict the investigation to systems with at least five planets, it would result into a nonsense to compare the number of TBR-valid systems with the totality of known planetary systems. In fact, we have sufficiently clarified that to include in our analysis also systems with two or three planets would produce meaningless results from the point of view of the semi-log linear regressions and the statistical indicators R^2 and Median. Obviously, there is only a single straight line passing through two points and in such two-planets systems we would get $R^2 = 1$, by definition. Just adding one or two points (namely, planets) would not produce yet meaningful results from the statistical point of view. Things become physically (and statistically) significant when the examined planetary system

**Figure 2.** Histogram of the R^2 and Median values. Distributions of the Median-values (red, on the left) and of the R^2 -values (blue, on the right) among the 32 exoplanetary systems listed in Table 1. The bin size is 0.01.

contains at least five planets. So, the ‘total number’ of exoplanet systems against which we compare the number of systems where TBR is satisfied (to certain degree) is the number of systems with at least five planets orbiting a single star, namely the 32 systems of Table 1.

Having stated the above, in Fig. 2 we show a histogram where we display the distribution of the R^2 -values among the 32 systems considered in Table 1 (namely how many systems have $R^2 > 0.99$, how many with $0.98 < R^2 < 0.99$, how many with $0.97 < R^2 < 0.98$, etc., until the worst $R^2 \simeq 0.86$ exhibited by Kepler 150). The R^2 histogram is depicted in blue, while an analogue histogram describing the distribution of the Median-values is depicted in red. From the histograms it is quite clear that the TB relation is very successful in mimicking the structure of real planetary systems. According to Table 1, 6 systems out of 32 (19 per cent) have an $R^2 > 0.99$, and 16 out of 32 (50 per cent) have an $R^2 > 0.98$. Only 7 systems out of 32 (22 per cent) present an $R^2 < 0.95$ (and they will be specifically studied in the next section, since they are very good candidates for the ‘prediction’, or insertion, of new planets). These numbers speak by themselves in favour of the remarkable efficiency of the Titius–Bode relation for the description of orbital spacing in planetary systems with at least five planets each.

3 PREDICTING NEW PLANETS WITH THE TITIUS–BODE RULE

From its conception, Titius–Bode rule was used as a predictive tool for discovering orbits of some celestial bodies in our Solar System. The original form of the law described well the distances of the known planets from the Sun, provided a gap between Mars and Jupiter was admitted. Since the law was working so well for all the known planets, and moreover it adapted almost perfectly to the ‘new comer’ Uranus (discovered in 1781), people started taking it seriously, and searched a planet in between Mars and Jupiter for about twenty years. The endeavor was crowned with success when finally Piazzi observed Ceres (the first and largest asteroid) in 1801.

In the previous sections we showed that the modern formulation of the law, equation (2), is in good agreement with the recent data of a large number of exoplanetary systems. Therefore it sounds natural to attempt harnessing TB rule, in order to successfully predict orbits of possible ‘missing’ celestial bodies, perhaps just in those systems where the agreement with the law is, apparently, less striking [on

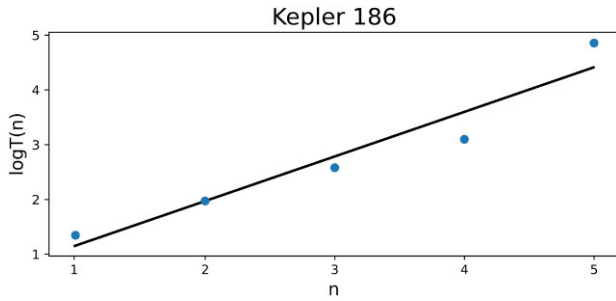


Figure 3. This is a ‘raw’ fitting of the Kepler 186 system, namely before the insertion of any gap. The statistical indicators (see Table 1) are $R^2 = 0.92681234$ and Median = 0.093772029. We can clearly identify a ‘big’ jump between the fourth and the fifth planet.

Table 6. R^2 and Median analysis before and after the insertion of one planet, for exoplanetary systems with not optimal agreement ($R^2 < 0.95$) according to Titius–Bode rule.

Systems	Old R^2	New R^2	Old Median	New Median
HD 219134	.913657699	.974791203	.126773358	.05855297
HD 191939	.945551901	.985952278	.08825776	.034601236
K2-138	.945458309	.991472463	.094321758	.035982039
Kepler-150	.8605413	.952818595	.236636167	.152566663
Kepler 169	.875354644	.960693503	.151151179	.091297799
Kepler 186	.92681234	.98794486	.093772029	.030801858
Kepler 33	.948841017	.991408774	.054322078	.023090953

this see also e.g. Bovaïrd & Lineweaver (2013); Scholkmann (2013); Huang & Bakos (2014); Bovaïrd et al. (2015); Lara et al. (2020)].

Our procedure for achieving such predictions is described in what follows, and it appears quite straightforward. To our understanding, the predictions are readily experimentally testable in present or future exploratory exoplanetary missions (see e.g. PLATO; ARIEL; TOLIMAN).

To start with, we give a quantitative definition of *optimal agreement* for a TB fitting by using the R^2 coefficient of determination: any system with an $R^2 \geq 0.95$ is said to be in *optimal agreement* with the TB rule. The idea now is that for the few systems where the agreement with TB rule is not *optimal*, we can quite easily identify at least a (big) ‘jump’ in the fitting graph (see e.g. the ‘jumps’ in Fig. 3). We then interpret the ‘jump’ as a missing celestial object, whose orbital period (or position) can be predicted using the TB fitting. Since the values of $\log T$ on the Y axis are fixed (namely given by the observations), then, in order to ‘smooth’ the jump in the graph (focus on Fig. 3 as an example), the only freedom we have is to insert a gap on the X axis, namely to shift the numerical places of the planets on the X axis. Below, we discuss an example and a detailed explanation of our predictive procedure for the few exoplanetary systems with a non-optimal agreement.

First, we collect in Table 6 the planetary systems (chosen from those listed in Table 1), whose agreement is non-optimal (namely $R^2 < 0.95$).

We consider then, as a first example, Kepler 186, a system with five planets. We compute the R^2 resulting from a direct linear regression of the five-planet system, without any gap inserted (Fig. 3). Then we check whether the insertion of one gap improves the R^2 and Median (significantly) or not. We note that if a gap (and therefore a planet) is placed at *any* position other than that of the big ‘jump’, the fit with TB becomes worse (R^2 worsens), as it can be seen from Fig. 4. Instead, if the gap is placed at the big ‘jump’ (between the fourth

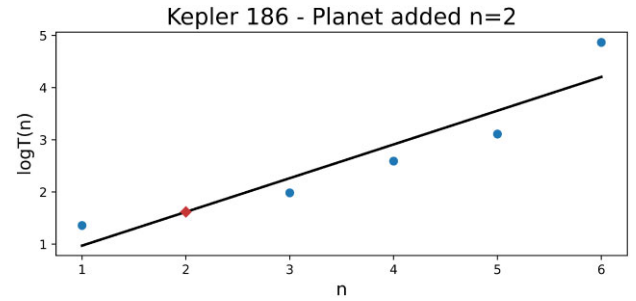


Figure 4. In the Kepler 186 system, inserting for example a planet between the first and second planet makes the fit to TB law worse, i.e. R^2 decreases and the Median increases significantly. The position of the hypothetical planet (represented by the red square) is $n = 2$. [$R^2 = 0.864350834$, Median = 0.141705383].

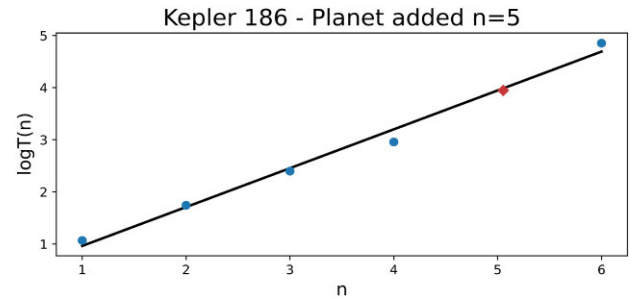


Figure 5. In the Kepler 186 system, inserting a planet between the fourth and fifth planet makes the fit to TB law much better, i.e. improves R^2 and Median significantly. The position of the hypothetical planet (represented by the red square) is $n = 5$. [$R^2 = 0.98794486$, Median = 0.030801858].

Table 7. Here we display *experimentally testable* predictions of the sequence positions and the periods of the planets ‘predicted’ with TB rule.

Systems	Position of predicted planet	Predicted period (days)
HD 219134	$n = 6$	488.3976385
HD 191939	$n = 6$	744.2371442
K2-138	$n = 6$	23.05233175
Kepler 150	$n = 5$	145.3213414
Kepler 169	$n = 5$	36.08528335
Kepler 186	$n = 5$	56.03297983
Kepler 33	$n = 2$	30.00633542

and the fifth planet), both R^2 and Median are significantly improved, hence the fit with TB becomes much better (Fig. 5). This point is very important: in most cases, the addition of a planet in an arbitrary place, even if the planet follows *exactly* the TB law, makes the overall fit worse. Only when the planet is placed at the right point of a (big) ‘jump’, then the fit becomes better; and significantly at that point.

Once the above procedure is applied to all the systems contained in Table 6, the experimentally testable outcome are the *orbital periods of the predicted planets*, displayed in Table 7. We point out that, using this method, it would be possible in principle to insert more than one gap (even not consecutive), and therefore more than one planet, in a TB fitting. We refrain however to do that, since it would probably increase the bias of the data too much. For example, adding two points to a set of five would affect the 40 per cent of the total data, or even adding two points to a set of seven would imply to bias almost the 30 per cent of the data. A further argument in favour of

being sparing in the addition of new gaps or planets in a planetary system is the following: the geometrical intuition suggests that, given two bunches of points on a diagram, the R^2 of a linear regression among those points will be in general improved by ‘moving away’ one bunch of points from the other. So, in general, it looks wise not to abuse with the insertion of ‘new’ planets. The possible reasons why these predicted missing planets have not yet been detected in previous surveys may be various, the first being, of course, the general difficulty of these measurements. A further realistic possibility is that the *missing planets* are not really planets, but instead for example *asteroid belts*, or clouds of dust (as it happens in our solar system). This possibility could be supported by the fact that, apart from Kepler 33, all the other systems allow for a gap at $n = 5$ or $n = 6$. Actually, this was the case of *our* solar system, where the gap was exactly at $n = 5$, between Mars and Jupiter. To showcase this possibility, we point out that the R^2 and Median of a TB fitting of our solar system without the inclusion of Ceres or the asteroid belt are, respectively, $R^2_{\text{old}} = 0.981440834$ and $\text{Median}_{\text{old}} = 0.034928964$; to be compared with the current values, which include the gap at $n = 5$ between Mars and Jupiter, i.e. $R^2_{\text{new}} = 0.993448869$ and $\text{Median}_{\text{new}} = 0.019581007$.

Finally, we would like to emphasize that the point of this section is *not* to improve the agreement of the TB rule with systems of sub-optimal agreement. The *sole* purpose of this section is to provide a testable prediction of the TB rule, which we believe may help in the search and discovery of new exoplanetary celestial objects.

4 TB RULE AND THE AGE OF THE PLANETARY SYSTEMS

It is clear that TB rule is intended to be obeyed on a statistical basis, by planetary systems which are already ‘running’ by a ‘fair’ amount of time. At the beginning of their history, protoplanetary systems are obviously full of dust, small debris, rocks etc. which can be found at any distance from the central star. With time, the planetary formation process goes on, matter aggregates onto specific bodies, and planets slowly emerge. In particular, preferred orbits, those described for example by TB rule, slowly emerge and become more precise and definite. On the other hand, it is well known that light objects, like comets, small debris, in general small objects, do not (and are not expected to) obey, singularly, the TB rule, at any time. Therefore, it seems reasonable to investigate a possible correlation between the age of the planetary system, i.e. the age of the central star, and the goodness to which the TB rule is obeyed in that planetary system. Hereafter the reader will find such analysis.

In Table 8 we report, like in Table 1, the name of the planetary system (i.e. the name of the central star), the number of planets in the system, the R^2 indicator, but in place of *Median* column we have the age of the central star in Gigayears. The age of the central star in general is not explicitly given on the NASA Exoplanet Archive (NASA Archive), however it can be obtained from literature dedicated to the specific planetary system under scrutiny.

According to the above argument, in a diagram ‘ R^2 versus Age of Star’, one would expect several points pretty far from 1 for young systems, and then for increasing ages, the points should accumulate nearby $R^2 = 1$. Instead, looking at Fig. 6, the situation is not such. We see young systems close to $R^2 = 1$, as well as older systems; and equally, young systems far from $R^2 = 1$, as well as old systems. Thus, according to the diagram in Fig. 6, no correlation emerges between the age of a planetary system, and the ‘goodness’ to which the system obeys TB rule.

Table 8. R^2 and age of the central star (in Gyr). Data source: NASA Exoplanet Archive, plus literature on the specific planetary system considered.

System	Planets	R^2	Age of Star (Gyr)	Notes
Sun	10	0.993448869	4.603	with Ceres
KOI-351	8	0.96322542	0.5	Kepler90
Trappist 1	7	0.994255648	7.2 ± 2.2	–
HD 10180	6	0.992725066	7.3	–
HD 191939	6	0.945551901	7 ± 3	2022
HD 219134	6	0.913657699	9.3	–
HD 34445	6	0.974290137	8.5 ± 2.0	–
K2-138	6	0.945458309	$2.3^{+0.44}_{-0.36}$	2021
Kepler 11	6	0.962373014	3.2 ± 0.9	–
Kepler 80	6	0.955414548	2	–
TOI - 1136	6	0.987422135	0.7	2022
TOI - 178	6	0.983217838	7.1 ± 5.3	2021
HD 108236	5	0.976381637	6.7 ± 4	2021
HD 158259	5	0.999713447	7.4	2020
HD 23472	5	0.988496548	–	2022
HD 40307	5	0.984933377	6.9 ± 4	–
K2-268	5	0.957182918	–	2019
K2-384	5	0.983020508	–	2022
Kepler 102	5	0.987900552	1.41	–
Kepler 122	5	0.98710756	3.89	–
Kepler-150	5	0.8605413	4.57	–
Kepler 154	5	0.985703723	4.47	–
Kepler 169	5	0.875354644	–	–
Kepler 186	5	0.92681234	4.0 ± 0.6	–
Kepler 238	5	0.98771232	6.76 ± 2	–
Kepler 292	5	0.995427692	5.13 ± 3	–
Kepler 32	5	0.971262925	2.7 ± 1	–
Kepler 33	5	0.948841017	4.27	–
Kepler 55	5	0.988367826	3.13	–
Kepler 62	5	0.950734271	7 ± 4	–
Kepler 82	5	0.955569351	5.1	2019
Kepler 84	5	0.992394962	4.9	–

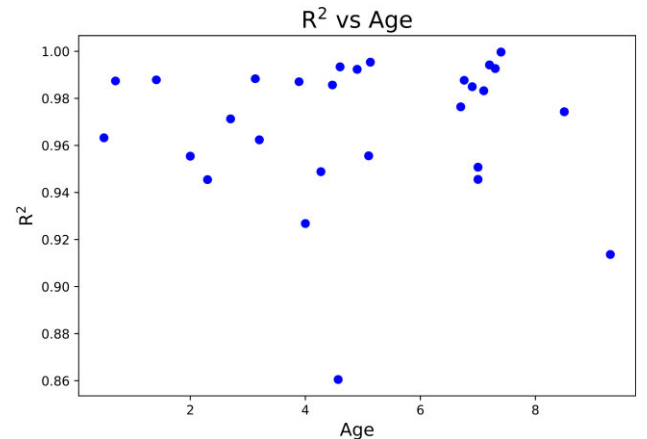


Figure 6. Diagram reporting R^2 versus age of the central star.

5 COMPARISON BETWEEN THE TB RULE AND THE HARMONIC RESONANCES METHOD

Of course, the TB rule does not represent the only attempt to give a phenomenological description of planetary spacing. One of the most interesting and promising ways to arrive to such a description is the so-called *Harmonic Resonances* (HR) method. The idea comes from the observation that points of a periodic system, interacting

with each other (as for example the orbital systems), tend to manifest resonances, and resonances tend to stabilize the periodic system itself. In particular, computer simulations of planetary systems have shown that random injections of new planets tends to produce dynamically unstable orbits, until the planets surviving in the system settle their orbital periods into ‘harmonic ratios’, namely in (quasi) rational ratios, usually expressible in simple fractions [on this see Peale (1976); McFadden et al. (2007); Batygin, Morbidelli (2013); Aschwanden (2018)]. Historically, the HR idea seems to originate from a generalization of a resonance relation already known to Laplace (1829) among the orbital periods of three Galilean satellites of Jupiter, namely Io, Europa, Ganymede, which can be written as

$$\frac{1}{T_{\text{Io}}} - \frac{3}{T_{\text{Eur.}}} + \frac{2}{T_{\text{Gan.}}} \simeq 0. \quad (11)$$

Such relation is fulfilled with an accuracy of the order of $\approx 10^{-5}$, and, using more precise orbital periods, perhaps even to an astonishing accuracy of $\approx 10^{-9}$ (Peale 1976).

On the ground of the orbital periods known in the Solar System, Aschwanden (2018) proposed a generalization of equation (11) for a two-body resonance existing between *two neighboured planets* in stable long-term orbits, as

$$\frac{H_i}{T_i} - \frac{H_{i+1}}{T_{i+1}} \simeq \omega_{i,i+1}. \quad (12)$$

The H_i are (small) integers, T_i is the orbital period of the i -th planet, and $\omega_{i,i+1}$ is the residual that accounts for possible further resonances from third or more planets involved. As Aschwanden explains, once considered two neighboured planets, we know from observation T_i , T_{i+1} , and we choose the two smallest integers H_i , H_{i+1} (positive or negative) such that the modulus of the residual $|\omega_{i,i+1}|$ results to be minimal. The ratios H_{i+1}/H_i are called *harmonic ratios*. Of course, once a particular ratio between two periods is established, $T_{i+1}/T_i \simeq H_{i+1}/H_i$, this can be immediately transferred into a ratio between the distances of the planets from the central body, by virtue of Kepler’s third law, $R \propto T^{2/3}$. On the basis of previous empirical work (Peale 1976), the integers H_i are picked in the range 1,...,5, and therefore the possible distinct harmonic ratios considered are (5 : 4), (5 : 3), (5 : 2), (5 : 1), (4 : 3), (4 : 1), (3 : 2), (3 : 1), (2 : 1). This procedure is applied by Aschwanden to all the nine neighboured planet pairs that can be extracted from the Solar System, including Ceres (considered here as the main representative of the asteroids belt). So the pairs considered are: Mercury–Venus, Venus–Earth, Earth–Mars, Mars–Ceres, Ceres–Jupiter, Jupiter–Saturn, Saturn–Uranus, Uranus–Neptune, and Neptune–Pluto. Empirically, Aschwanden finds for the above nine planet pairs that the best-fitting resonances are confined to the five particular ratios (3 : 2), (5 : 3), (2 : 1), (5 : 2), (3 : 1), and the residuals are all quite small, being in the range $\omega_{i,i+1} \approx 0.005 - 0.06$. Moreover, it is found that the ratio (5 : 2) works for four different pairs Mercury–Venus, Mars–Ceres, Ceres–Jupiter, Jupiter–Saturn, the ratio (2 : 1) describes the pairs Earth–Mars and Uranus–Neptune, while the remaining three ratios fit one single pair each.

Also the role of the most massive planet, Jupiter, is investigated (Aschwanden 2018) by expanding the previous 2-body equation (12) into a 3-body resonance condition

$$\frac{H_i}{T_i} - \frac{H_{i+1}}{T_{i+1}} - \frac{H_{\text{Jup}}}{T_{\text{Jup}}} \simeq \omega_{i,i+1}. \quad (13)$$

However, identical results were found (yielding $H_{\text{Jup}} = 0$), except for the 3-body configuration Mars–Ceres–Jupiter. This supports the conclusion that the neighboured planet–planet interaction is more

important in shaping the resonance than the influence of the largest giant planet, with the exception perhaps of the planet–asteroid pairs.

Once, through empirical attempts, the harmonic ratios are established, they can be used to re-construct distances among planets in the Solar System, of course via the Kepler third law, namely

$$\frac{R_{i+1}}{R_i} = \left(\frac{T_{i+1}}{T_i} \right)^{2/3} \simeq \left(\frac{H_{i+1}}{H_i} \right)^{2/3}. \quad (14)$$

This procedure is applied by Aschwanden to the planets of the Solar System, as well as to 7 moons of Jupiter, to 13 moons of Saturn, to 8 moons of Uranus, and to 6 moons of Neptune. All the satellites are chosen according to the rule of having a diameter larger than 100 km, on the ground of the general rule that planetary spacing patterns (as TB rule, or HR rule) are in general obeyed by ‘enough big’ objects. It is in fact well-known that ‘small’ objects, like comets, small asteroids, debris, etc. do not follow, singularly, any particular pattern in the size of their orbits, and can be found at any distance (allowed by classical mechanics) from the central body.

For all the above planetary systems, HR model produces an agreement between observed planetary distances R_{obs} and model-predicted values R_{pred} significantly better than the Titius–Bode law (2). The reported quantities by Aschwanden (2018) are the *mean* and the *standard deviation* of the ratios $R_{\text{pred}}/R_{\text{obs}}$ computed for the satellites of a given planetary system. For example, for the Solar System the values for the TB law are $R_{\text{TB}}/R_{\text{obs}} = 0.95 \pm 0.13$, to be compared against a $R_{\text{HR}}/R_{\text{obs}} = 1.00 \pm 0.04$ given by the HR model. Similar good figures are described in Aschwanden (2018) for the satellite systems of Jupiter, Saturn, Uranus, Neptune, and even for two extra solar planetary systems, namely 55 Cnc, and HD 10180. Thus, Aschwanden (2018) concludes that the HR model is clearly better than the TB rule in describing planetary spacing, and can be also much more efficiently used to ‘predict’ possible missing planets or satellites in ‘empty’ places of the planetary sequences (Aschwanden & Scholkmann 2017; Scholkmann 2017).⁴

Although the above figures should not certainly be under-appreciated, some considerations on the comparison between the TB rule and the HR method are surely in order. HR method seems to have some grounds in the classical mechanics of periodic systems, perhaps firmer and more sounding than the grounds backing the TB rule. However, as a matter of fact, there is no definite theory, or bunch of theorems, in classical or celestial mechanics predicting that a periodic system should exhibit (harmonic) resonances, after that a certain amount of ‘running’ time has elapsed. And even less, no theory predicts what *specific* harmonic ratios should be used to describe a particular system. The integers entering the ratios in equations (11), (12), (13), (14) are chosen empirically, ad hoc for

⁴At a first sight, one further possibility to compare HR model and TB rule could be to consider a R^2 -test for the HR method applied to Solar System, so to compare it with the R^2 -test applied to the TB rule, also for the Solar System. However, R^2 -tests are well defined, in general, for predictive models encoded by a function, typically a linear regression, maybe coming from exponential rules through taking a log (see equations 2 and 3). On the contrary, from equation (14) it is clear that the HR method does not generate an exponential law, neither a definite function. This would happen if the ratios H_{i+1}/H_i were the same for all the planet pairs, say $(H_{i+1}/H_i)^{2/3} = \alpha$. Then from equation (14) one would infer $R_{i+1} = \alpha R_i$, and hence $R_n = \alpha^{n-1} R_1$, which is the TB rule. But in general the ratios H_{i+1}/H_i are not the same for different pairs, and actually they change from pair to pair in an (in principle) unpredictable way. Therefore the R^2 -test seems to be non-applicable to the HR rule. So, in order to compare TB and HR rules we have to stick on the Aschwanden method $R_{\text{pred}}/R_{\text{obs}}$, above already described.

any specific planetary pair, among ‘small’ integers. Of course, being a matter of rational numbers, if the ratio (5:2) describes well, say, the pair Jupiter–Saturn, nothing prevents us from thinking that perhaps the ratios (7:3), or (8:3), or maybe (51:22), or (48:19), can describe that pair even better. In other words, the arbitrariness in the choice of a given ratio is large. A further weak point of the HR method is that no theory predicts which ratio should be used for a particular planetary pair, or if/why ratios have to be repeatedly used (and how many times) to describe different pair of planets or satellites, as happens for example in the Solar System with the pairs Mercury–Venus, Mars–Ceres, Ceres–Jupiter, Jupiter–Saturn [and similarly in other (exo)planetary systems]. The lack of any explicit theoretical constraint for choosing the numbers H_i , makes the HR method equivalent to a model with five or even nine pairs of free parameters (namely, 10 to 18 free parameters in total). Then it comes with no surprise that the HR model fits the planetary distances (in our solar system, as well as in exoplanetary systems) with much higher accuracy than the TB law. Using a model that in principle has (at least) ten free parameters produces quite obviously better fittings than a model which has only two free parameters, as the TB rule in equation (2). As said above, Aschwanden claims to have singled five harmonic ratios out of nine which are particularly useful to describe planetary and satellite distances, both in the Solar and extra Solar systems. But, still, this is just an empirical, though extremely useful, observation. No reasons are given about why just those five ratios work well, or why the ratio (5:3) works better or worse than the ratio, say, (49:31).

By comparison, TB rule has only two free parameters, that we indicate with a and λ . The two free parameters have to be adapted (usually with a regression method) to the specific planetary system under consideration, and in general they change from system to system. The five harmonic ratios above proposed by Aschwanden, on the contrary, seem to be valid (again on an empirical basis) for any planetary system. However also the HR ratios have to be chosen each time ad hoc for the specific system examined, and often with a priori unpredictable repetitions.

All in all, we conclude that the TB rule still appears to be the empirical, *most economic* and efficient rule on the ‘market’ to describe planetary distances or periods.

6 IN LASKAR’S FOOTSTEPS: SECOND DEGREE POLYNOMIAL AND EXPONENTIAL

Laskar (2000) presented an interesting model of planetary accretion, and consequent formation of a planetary system, based on the conservation of mass, momentum, and angular-momentum-deficit stability. Without entering in technical details, we can say that the model predicts the distribution of the final stable orbits of the planetary system as a function of the initial linear mass density $\rho(r)$ in the protoplanetary disc. For a mass density of the form $\rho(r) = \zeta r^p$, the distributions of the stable final orbits are given in Table 9, for some specific values of p .

The parameters A_i , B_i are in general functions of the angular momentum deficit, and of the (arbitrary) parameter ζ (see Laskar 2000 for more details). In the context of this work, the parameters A_i , B_i can be determined from the observational data via the usual linear regression method. We notice that all the planetary distribution laws predicted by the Laskar’s model contain *only* two (free) parameters A_i , B_i . From this point of view the TB rule and the other distributions proposed in Table 9 are similar. On the contrary, they differ blatantly from the HR rule, where the potentially free parameters are at least five or ten.

Table 9. Planetary distributions corresponding to different initial mass densities $\rho(r) = \zeta r^p$ of the protoplanetary discs. A constant distribution $\rho(r) = \zeta$, i.e. $p = 0$, gives a law in n^2 ; while $p = -3/2$ gives a Bode-like exponential law. Of course, $n = 0, 1, 2, 3, \dots$

p	$r(n)$
0	$r^{1/2} = A_1 + B_1 n$
-1/2	$r^{1/3} = A_2 + B_2 n$
-1	$r^{1/6} = A_3 + B_3 n$
-3/2	$\log(r) = A_4 + B_4 n$

Particularly significant is the law with $r \sim n^2$. In fact, power laws as n^2 were proposed in several works, even long ago, for the distribution of planets or satellites in the Solar System (see Caswell 1929; Schmidt 1946; Nottale 1996; Nottale et al. 1997; Agnese & Festa 1997; Reinisch 1998; de Oliveira Neto et al. 2004).⁵ Although at a first glance the n^2 rule seems to be able to fit some data, more attentive considerations reveal that authors in Caswell (1929), Schmidt (1946), Nottale (1996), Nottale et al. (1997), Agnese & Festa (1997), Reinisch (1998), and de Oliveira Neto et al. (2004) are in general forced to use non-consecutive integers in order to accommodate the planetary distances. Or, perhaps more often, they split the Solar System into two sets of inner and outer planets (i.e. terrestrial and gigantic planets); the semimajor axis in each set then follow a n^2 power law to a high degree of approximation, however with different coefficients A_i , B_i for each of the two planetary families. In other words, we have two different fitting parabolae, which in general do not join into a single smooth function.

However, the sounding theoretical basis of the n^2 rule (at least, in the Laskar formulation), and its partial efficiency in describing planetary distances, have pushed us to check such law also in extra solar planetary systems. Therefore we took the worst six systems in terms of R^2 and Median agreement with the TB rule, and we recomputed them with a polynomial $(A + Bn)^2$ fitting, to see if their new R^2 and Median indicators are better or worse than those obtained with the TB fitting.

Since, according to Table 9 (Laskar), we have

$$r \sim (A + Bn)^2, \quad (15)$$

and also, according to Kepler third law, $r^3 \sim T^2$, then it should hold

$$T^{1/3} \sim A + Bn. \quad (16)$$

This last relation is the one we checked against the observational data. The ‘goodness’ of the linear regressions in getting the coefficients A , B is displayed in Table 10, as usual in terms of the R^2 statistical indicator. The R^2 of the six worst TB systems (central column) are compared with the R^2 of the same systems, but re-computed via a n^2 fitting (right column). Just a simple glance is sufficient to conclude that the TB fitting, even in the worst six cases, works way better than the n^2 fitting. The TB rule seems to emerge, once again, as the most efficient, and most economical mathematical relation to

⁵ It is interesting to observe that, with the exception of Schmidt, all the authors in Caswell (1929), Schmidt (1946), Nottale (1996), Nottale et al. (1997), Agnese & Festa (1997), Reinisch (1998), and de Oliveira Neto et al. (2004) arrive to a quadratic law for the orbital radii distribution, $r \sim n^2$, because they consider a quantum-like description of the planetary systems, where they assume a Newtonian potential $1/r$ together with a Bohr-like quantization condition for the angular momentum (per unit mass). A different example of quantum-like description, that instead leads to an exponential rule $r \sim e^{2\lambda n}$, is presented in Scardigli (2007).

Table 10. R^2 of the six worst TB systems compared with the R^2 of the same systems, but re-computed via a n^2 fitting.

Worst TB Systems	R^2 TB	R^2 Laskar n^2
Kepler 150	0.8605413	0.696871005
Kepler 169	0.875354644	0.769928531
HD 219134	0.913657699	0.684612854
Kepler 186	0.92681234	0.819327821
K2-138	0.945458309	0.860764702
HD 191939	0.945551901	0.785594964

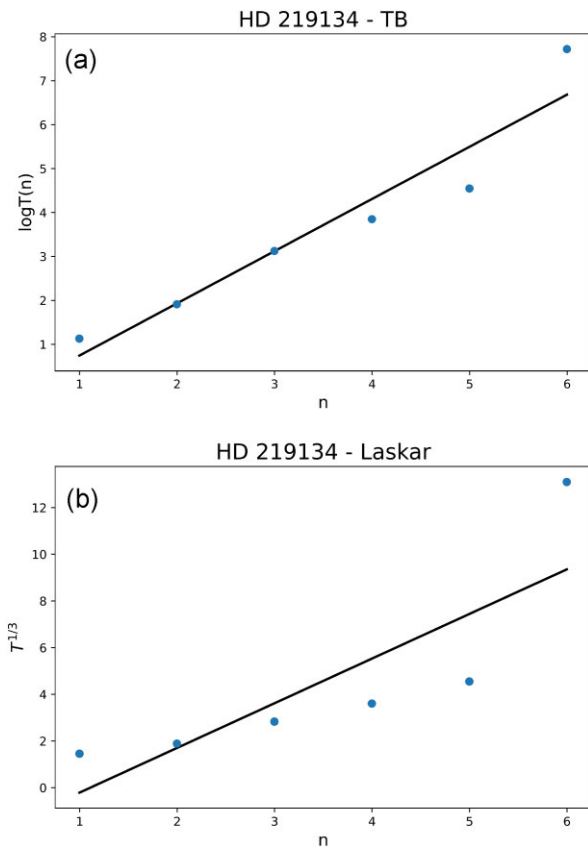


Figure 7. Graphic comparison of two different fittings for HD 219134, a system with six confirmed planets. (a) Titius–Bode fitting (namely semi-log) of HD 219134. The R^2 statistical indicator (Table 10) is $R^2 = 0.913657699$. (b) n^2 -fitting (Laskar model) of HD 219134. The R^2 statistical indicator is $R^2 = 0.684612854$.

describe planetary spacing, not only in the Solar System, but also in extra solar planetary systems.

For sake of completeness, we could perhaps check the R^2 of the worst six systems also against a model with $r \sim n^6$, listed also in the Laskar Table 9. However, it is quite clear that the larger is the exponent of the power law, $r \sim n^k$, the closer such law is to the exponential $r \sim e^n$, namely to the TB rule. Therefore, such further fitting and comparison with the TB rule probably would not result to be particularly insightful.

Finally, for a visual comparison, we consider one of the systems in Table 10, HD 219134, and we draw two diagrams of the same system: one displaying a TB-fitting and the other displaying a n^2 -fitting, see Fig. 7. Then, it appears extremely clear that a TB rule works way better than a n^2 -fitting (Laskar model).

7 DISCUSSION AND CONCLUSIONS

The aim of this paper is to check if the Titius–Bode rule can be considered a reliable description of the spacing among planets in a planetary system, with reference not only to our Solar system, but also in extra solar planetary systems. To this scope, after a general introduction to the Titius–Bode rule, we examined the 32 planetary systems (31 plus the Solar system) known to harbour at least five planets each (to date, July 2023). To keep the analysis as uniform as possible, we considered systems with a single central star only. We then computed the semi-log linear regressions, and used the statistical indicators R^2 and *Median* in order to measure how ‘good’ is an exponential distance relation, like the TB rule, in fitting the observational data. To clear the stage from the suspect of a purely random origin of the TB law, we created at random 4000 ‘artificial’ planetary systems and observed that, according to the statistical indicators, the TB rule fits real planetary systems much better than artificial ones. We found that the TB rule describes more than 78 per cent (25 out of 32) of the examined real systems with an accuracy better than $R^2 \geq 0.95$, and even more than 50 per cent with an $R^2 \geq 0.98$. By comparison, the ‘artificial’ random systems are fitted, on average, with an accuracy of $R^2 \simeq 0.90$.

Following a long tradition, we too use the TB rule to predict some new planets in systems that apparently present bad fittings. Our criterion to insert new planet(s) into a system is simple: the insertion should improve the fitting, namely the R^2 should getting closer to 1, and possibly the Median getting closer to 0. However, we also critically discuss the intrinsic limits that such a procedure seems to present naturally, and therefore the reasons of why it looks wise not to exceed in planetary insertions. A possible correlation between the age of a planetary system and the goodness to which TB rule is obeyed (which would seem a natural link), has been on the contrary ruled out by the observations.

Our findings can be compared and contrasted with results of previous relevant works on the TB rule. Hayes and Tremaine (1998), for example, have considered, like us, randomly created planetary systems, but furthermore they selected specific systems by applying a conservative stability criterion which requires that adjacent planets are separated by a minimum distance of k times the sum of their Hill radii (for values of k ranging from 0 to 8). They performed least-squares fits of these systems to generalized Bode laws and compared them with the fit of our own Solar System. In so doing they found that this stability criterion generally produce geometrically spaced planets that fit a Titius–Bode law about as well as our own Solar System. From our point of view, this means that TB rule, far from being a mere product of chance, actually implies some kind of underlying physical mechanism at work. Thus, our findings which strongly support the validity of TB relation also in extra solar planetary systems, coherently fit with the results of Hayes and Tremaine, in pointing towards a definite universal physical law.

On a different line, Bovaïrd and Lineweaver (2013), building (also) on the early data collected by the *Kepler* mission, presented one of the first systematic studies on a possible validity of the TBR in extra solar planetary systems. They considered the 71 systems known in 2013 for harbouring at least 4 planets each. It should be said, however, that some of these systems contain double or triple stars at their centre, a circumstance that we were able to avoid in this paper, thanks of course to the much richer set of exosystems available to us, ten years later (2023). As statistical tool they employed a χ^2 analysis, which however, in case of uniform or constant error bars (a position adopted by Bovaïrd and Lineweaver), is essentially equivalent to the R^2 analysis used in our work (see Appendix A). These authors found

that a vast majority of exoplanetary systems of their set adhere to the TB relation to an extent similar or greater than our Solar system does. This finding is essentially confirmed by our work, ten years later, and perhaps even reinforced, given the more refined set of planetary systems we used (five or more planets for each system, and single-star systems only). Then Bovaïrd and Lineweaver, on the basis of the just verified TB law, dive into a whirlwind of predictions of new planets. They arrive at inserting up to 10 new planets for each system, compiling a list for the existence of 141 new exoplanets in 68 multiple-exoplanet systems. On this point, in Section 3 we have been much more prudent (for the geometrical reasons we explained there), and we ‘predicted’ at most one new planet per system. The criterium we adopted was however coincident with that of Bovaïrd and Lineweaver, namely the insertion of a new planet should improve the R^2 (or χ^2) distribution.

Finally, in Sections 5 and 6 we compared the TB rule with two of its major competitors, namely the Harmonic Resonances (HR) method (see e.g. Aschwanden 2018), and the polynomial fitting (see e.g. Laskar 2000). In both cases, although for different reasons, the TB rule, an exponential relation, emerges as the most ‘economical’ (in terms of free parameters) and best fitting law for the description of the spacing among planetary orbits.

In conclusion, we believe that our study contributes to definitively lift the TB rule out of the bag of conjectures and dubious numerical coincidences, and gives to the rule the status of a corroborated physical phenomenological law. Perhaps a bit like what the studies of Fraunhofer and Balmer did for the atomic spectra along the nineteenth century [for potential connections of TBR with atomic physics see Caswell (1929); Albeverio et al. (1983); Agnese & Festa (1997); Scardigli (2007); Batygin (2018)]. The theoretical interpretations and explanations of such an empirical law still remain a much debated matter of the present-day research.

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We dedicate this work to the cherished memory of Jan Zaanen. We thank Dr. Aravindh Swaminathan Shankar for his invaluable help with a Python code. FS also thanks G.’t Hooft and M. Arpino for useful discussions on the topic.

DATA AVAILABILITY

The data underlying this article are available in the NASA Exoplanet Archive, at <https://exoplanetarchive.ipac.caltech.edu/cgi-bin/TblView/nph-tblView?app=ExoTbls&config=PS> For each planetary system displayed in Table 1, the data are given in the specific references listed in Table E1.

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APPENDIX A: THE STATISTICAL TOOLS R^2 AND χ^2

The statistical indicator R^2 , called ‘coefficient of determination’, is defined in the following way. Given a set of points (x_i, y_i) , $i = 1, \dots, N$, the ordinary least-squares method tries to find the ‘best’ analytic function $y = f(x)$ that approximates the set (x_i, y_i) . Usually the function is a polynomial, and even more usually it is a straight line $y = A + Bx$. What one tries to minimize is the sum of the squares of the residuals

$$\varrho = \sum_i [y_i - f(x_i)]^2. \quad (\text{A1})$$

Of course, it would be useful to have ‘something’ with which to compare the sum ϱ , a sort of ‘maximum error’, in order to produce then a ‘percentage error’. The idea is then to consider the average

of the y_i data

$$\bar{y} = \frac{1}{N} \sum_i y_i \quad (\text{A2})$$

and to consider as maximum possible error the sum of the squares of the residuals in respect to this average $\bar{y}$

$$\varrho_{\max} = \sum_i (y_i - \bar{y})^2. \quad (\text{A3})$$

The percentage error is then clearly $\varrho/\varrho_{\max}$, and the R^2 indicator is defined as

$$R^2 = 1 - \frac{\varrho}{\varrho_{\max}}. \quad (\text{A4})$$

Therefore, the agreement between the fitting line and the data is as better as R^2 is closer to 1.

The statistical indicator χ^2 is defined, usually, as follow. Given a set of points (x_i, y_i) , $i = 1, \dots, N$, suppose that y_i indicate the observed data (each y_i is supposed to depend on the specific value x_i of the parameter x); each $\hat{y}_i$ indicates the datum predicted by the (regression) model for the specific value x_i of the parameter, namely $\hat{y}_i = f(x_i)$; each σ_i indicates the uncertainty, or error bar, associated with the specific observed datum y_i . Then χ^2 is defined as

$$\chi^2 = \frac{1}{N} \sum_i \frac{(y_i - \hat{y}_i)^2}{\sigma_i^2}. \quad (\text{A5})$$

In case of constant or uniform errors bars, namely $\sigma_i = \sigma$ for any i , then we can rescale $\sigma = 1$, and

$$\chi^2 = \frac{1}{N} \sum_i (y_i - \hat{y}_i)^2 = \frac{\varrho}{N}, \quad (\text{A6})$$

and therefore

$$R^2 = 1 - \frac{N\chi^2}{\varrho_{\max}}. \quad (\text{A7})$$

APPENDIX B: THE STATISTICAL TOOL Median

The definition of *Mean Absolute Deviation*, also known as *Average Absolute Deviation* in Excel (where it is produced by the statistical function AVEDEV) is the following: given a set of numbers $\{x_1, x_2, \dots, x_N\}$ and defined the *Mean* $\bar{x}$ in the usual way as

$$\bar{x} = \frac{1}{N} \sum_i x_i, \quad (\text{B1})$$

then the *Mean Absolute Deviation* referred to that set of numbers is

$$\text{MAD} = \frac{1}{N} \sum_i |x_i - \bar{x}|. \quad (\text{B2})$$

Clearly, MAD is a measure of how much ‘close’ the numbers x_i are to their arithmetical mean $\bar{x}$. The closer to zero is MAD, the less ‘dispersed’ are the numbers x_i around their mean $\bar{x}$. In the ideal case, when $\text{MAD} = 0$ then $x_i = \bar{x}$ for any i .

In our context of (linear) regressions, we call Median what is known as *Mean Average Error* (MAE). Given a set of points (x_i, y_i) , $i = 1, \dots, N$, where y_i are the observed data (supposed to depend on the values x_i of the parameter x), and $\hat{y}_i$ are the data predicted by the (regression) model, usually $\hat{y}_i = f(x_i)$, then the *Mean Average Error* is defined as

$$\text{MAE} = \frac{1}{N} \sum_i |y_i - \hat{y}_i|. \quad (\text{B3})$$

Obviously, the smaller is the MAE, the better is the fit by the model $y = f(x)$. The MAE can be compared with the *Mean Squared Error*, MSE, another statistical indicator of the ‘goodness’ of a fitting model, defined as

$$\text{MSE} = \frac{1}{N} \sum_i (y_i - \hat{y}_i)^2. \quad (\text{B4})$$

Of course, given the above definitions, the MSE and the sum of the squared residuals, equation (A1), are linked by the relation

$$\varrho = N \cdot \text{MSE} = \sum_i (y_i - \hat{y}_i)^2. \quad (\text{B5})$$

APPENDIX C: A THEOREM ON THE STATISTICAL INDICATORS R^2 , Median

Given a planetary system, the relevant TB fitting can be constructed by using, as starting data, either the orbital periods T_i , namely the set of points $\{i, \log T_i\}$, or the semimajor axis r_i , namely the set of points $\{i, \log r_i\}$, with $i = 1, 2, 3, \dots, N$. It is then quite easy to show that:

$$R^2(T) = R^2(r), \quad \text{Median}(T) = \frac{3}{2} \text{Median}(r) \quad (\text{C1})$$

namely, the R^2 of the data sets $\{i, \log T_i\}$ and $\{i, \log r_i\}$ are the same, while the Medians are proportional, through a factor $3/2$.

Proof. By the Kepler Third Law we have $T_i^2 = kr_i^3$. Defining $Y_i = \log r_i$, $Z_i = \log T_i$, we can write

$$Z_i = \frac{1}{2} \log k + \frac{3}{2} Y_i$$

and with reference to equations (3)–(9), a straightforward calculation yields

$$C = \frac{1}{2} \log k + \frac{3}{2} A; \quad D = \frac{3}{2} B.$$

Defining now $\bar{Z} = (\sum_j Z_j)/N$ and $\bar{Y} = (\sum_j Y_j)/N$, we can then write

$$\begin{aligned} Z_i - \bar{Z} &= \frac{3}{2} (Y_i - \bar{Y}) \\ Z_i - \bar{Z} &= \frac{3}{2} (Y_i - \bar{Y}) \end{aligned} \quad (\text{C2})$$

and these two can be used together with equations (A1), (A3) to arrive at

$$\varrho(T) = \frac{9}{4} \varrho(r), \quad \varrho_{\max}(T) = \frac{9}{4} \varrho_{\max}(r),$$

which yield the thesis $R^2(T) = R^2(r)$.

Using again equation (C2), and the definition (B3) of Median (i.e. MAE), we immediately also have $\text{Median}(T) = (3/2) \text{Median}(r)$. QED.

APPENDIX D: TITUS-BODE RULE FOR THE SATELLITES OF JUPITER, SATURN, URANUS

We show here how and to what extent the TB rule is obeyed by the systems of moons of, respectively, Jupiter, Saturn, and Uranus. Although this subject is obviously outside of the main topic of this paper, from the historical point of view it presents a relevant interest, since it helps to depict the opinion that people formed about the TB law during the 20th century, before the discovery of exoplanetary systems. For the semimajor axis (or for the periods) of the satellites’ orbits we use recent data taken from Wikipedia (see there the links [Jupiter moons](#), [Saturn moons](#), [Uranus moons](#)) and also from the

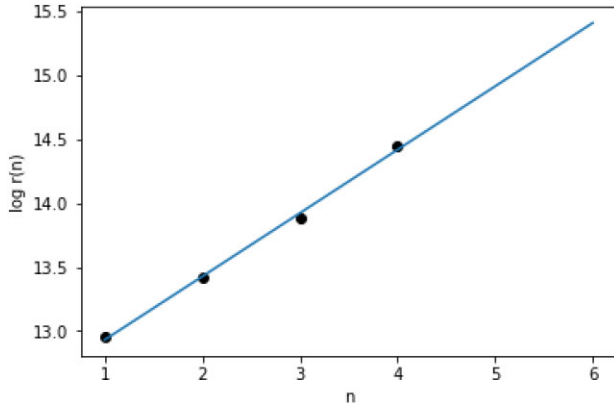


Figure D1. Fitting of the main satellites of Jupiter. The statistical indicators are $R^2 = 0.99759458$ and Median = 0.00181383.

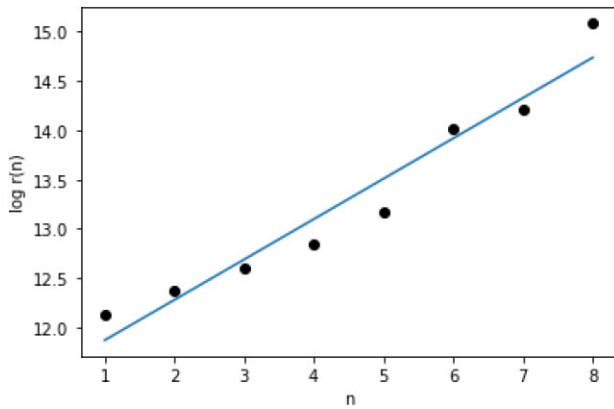


Figure D2. Fitting of the main satellites of Saturn. The statistical indicators are $R^2 = 0.94507382$ and Median = 0.014263996.

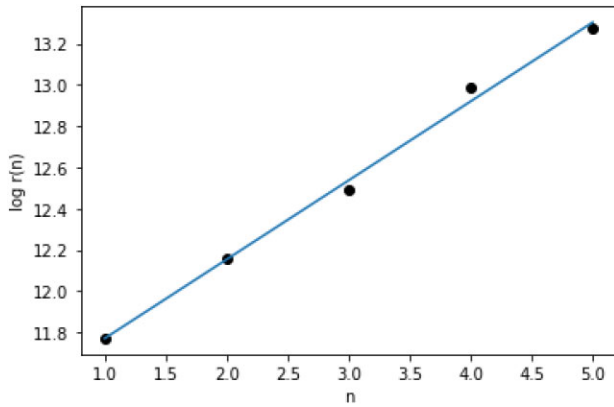


Figure D3. Fitting of the main satellites of Uranus. The statistical indicators are $R^2 = 0.99512428$ and Median = 0.00204344.

Encyclopedia of the Solar System (McFadden et al. 2007). In these tables and regression diagrams we do not include, on purpose, the satellites recently discovered only with spacecrafts (namely after *Pioneer* and *Voyager* missions), just because they are in general very light bodies, and it is well known that the TB rule works well only for quite large and quite massive objects (it does not work, for example, for comets or light asteroids).

In the case of Jupiter, for example, the satellites Metis, Adrastea, Amalthea, and Thebe, all orbiting at distances shorter than the Io's

Table D1. Main satellites of Jupiter.

Jupiter system	n	Orbital radius in km
Io	1	421 700
Europa	2	671 034
Ganymede	3	1070 412
Callisto	4	1882 709

Table D2. Main satellites of Saturn.

Saturn system	n	Orbital radius in km
Mimas	1	185 539
Enceladus	2	237 948
Tethys	3	294 619
Dione	4	377 396
Rhea	5	527 108
Titan	6	1221 870
Hyperion	7	1481 009
Iapetus	8	3560 820

Table D3. Main satellites of Uranus.

Uranus system	n	Orbital radius in km
Miranda	1	129 858
Ariel	2	190 930
Umbriel	3	265 982
Titania	4	436 282
Oberon	5	583 449

orbit, are all bodies with sizes of order 100 km, and masses of $10^{-7} - 10^{-4}$ times the mass of Io. Therefore, for the above reasons, they are not considered in the regression diagram. *A fortiori* the small moons orbiting at distances larger than Callisto orbit, all with sizes of 50 km or less, are ignored. The Jupiter diagram therefore displays only the Galilean satellites, Io, Europa, Ganymede, Callisto.

Analogue considerations hold for the satellites systems of Saturn and Uranus.

Hereafter the reader can find the regression diagrams for the main satellites of Jupiter 8, Saturn 9, and Uranus 10, each one equipped with the R^2 and Median values computed from the specific statistical analysis. As the reader can easily realize, the statistical indicators R^2 and Median assume really *good* values, in particular for Jupiter and Uranus. In fact, both the Jupiter's and Uranus' R^2 , as well as their Medians, are better than the Solar System R^2 and Median. Both Jupiter and Uranus systems fall in the set of $R^2 > 0.99$.

For the Saturn system the situation is clearly worse. This could perhaps be due to the fact that the system of Saturn contains more moons (146, May 2023) than any other planet in the Solar system. Moreover, the large rings might affect the distribution of the satellites' orbits in a still unknown way.

In any case, it would be interesting to define a (precise) quantitative criterion to understand when a moon should be expected to obey a TB relation, or not. At present, the empirical selection rule (that we used to compile the Tables D1, D2, D3) is just based on the size/mass of the considered moon. We followed the tradition, since this rule is also adopted in the Nieto book, as well as in most of the literature on the Titius-Bode relation. Possible quantitative formulations of this criterion will form the subject of future works.

APPENDIX E: REFERENCES FOR THE PLANETARY SYSTEMS IN TABLE 1

Table E1. References for planetary systems in Table 1.

System	Planets	References
Sun	9 + Ceres	https://doi.org/10.1016/C2010-0-67309-3
KOI-351	b,c,d,e,f,g,h i	https://doi.org/10.1088/0004-637X/781/1/18 https://doi.org/10.3847/1538-3881/aa9e09
Trappist 1	b,c,d,e,f,g,h	https://doi.org/10.3847/PSJ/abd022
HD 10180	b,c,d,e,f,g	https://doi.org/10.1088/0004-637X/792/2/111
HD 191939	b,c,d,e,f,g	https://doi.org/10.1051/0004-6361/202244120
HD 219134	b,c,d,e f,g	https://doi.org/10.1038/s41550-017-0056 https://doi.org/10.1088/0004-637X/814/1/12
HD 34445	b c,d,e,f,g	https://doi.org/10.3847/1538-3881/aa5df3 https://doi.org/10.3847/1538-3881/aa8b61
K2-138	b,c,d,e,f,g	https://doi.org/10.1051/0004-6361/201936267
Kepler 11	b,c,d,e,f,g	https://doi.org/10.1088/0004-637X/770/2/131
Kepler 80	b,c,d,e,f g	https://doi.org/10.3847/0004-6256/152/4/105 https://doi.org/10.3847/1538-3881/aa9e09
TOI - 1136	b,c,d,e,f,g	https://doi.org/10.3847/1538-3881/ad1330
TOI - 178	b,c,d,e,f,g	https://doi.org/10.1051/0004-6361/202039767
HD 108236	b,c,d,e,f	https://doi.org/10.1051/0004-6361/202039608
HD 158259	b,c,d,e,f	https://doi.org/10.1051/0004-6361/201937254
HD 23472	b,c,d,e,f	https://doi.org/10.1051/0004-6361/202244293
HD 40307	b,c,d,e,f	https://doi.org/10.1051/0004-6361/201220268
K2-268	b,c,d,e,f	https://doi.org/10.1093/mnras/stab2305
K2-384	b,c,d,e,f	https://doi.org/10.3847/1538-3881/ac5c4c
Kepler 102	b,c,d,e,f	https://doi.org/10.1051/0004-6361/202346211
Kepler 122	b,c,d,e f	https://doi.org/10.1088/0004-637X/784/1/45 https://doi.org/10.1088/0004-637X/787/1/80
Kepler-150	b,c,d,e f	https://doi.org/10.1088/0004-637X/784/1/45 https://doi.org/10.3847/1538-3881/aa62ad
Kepler 154	b,c d,e,f	https://doi.org/10.1088/0004-637X/784/1/45 https://doi.org/10.3847/0004-637X/822/2/86
Kepler 169	b,c,d,e,f	https://doi.org/10.1088/0004-637X/784/1/45
Kepler 186	b,c,d,e f	https://doi.org/10.1126/science.1249403 https://doi.org/10.1088/0004-637X/800/2/99
Kepler 238	b,c,d e,f	https://doi.org/10.1088/0004-637X/784/1/45 https://doi.org/10.1088/0067-0049/210/2/25
Kepler 292	b,c,d,e,f	https://doi.org/10.1088/0004-637X/784/1/45
Kepler 32	b c,d,e,f	https://doi.org/10.3847/1538-3881/abd93f https://doi.org/10.1088/0004-637X/750/2/114
Kepler 33	b,c,d,e,f	https://doi.org/10.1088/0004-637X/750/2/112
Kepler 55	b,c d,e,f	https://doi.org/10.1093/mnras/sts090 https://doi.org/10.1088/0004-637X/784/1/45
Kepler 62	b,c,d,e,f	https://doi.org/10.1126/science.1234702
Kepler 82	b,c,f d,e	https://doi.org/10.1051/0004-6361/201935879 https://doi.org/10.1088/0004-637X/784/1/45
Kepler 84	b,c d,e,f	https://doi.org/10.1088/0067-0049/208/2/22 https://doi.org/10.1088/0004-637X/784/1/45

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