Subproduct systems and C*-algebras
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Summary

In this thesis, we mainly study *subproduct systems* and their associated *C^*-algebras*. To make the topic accessible to a broader audience, we will provide some intuitive explanations of these core concepts.

Graded algebras and subproduct systems

In mathematics, complex structures can be understood by breaking them down into simpler and ordered layers. This is the core idea behind a *graded algebra*. First of all, an *algebra* is a mathematical structure in which the operations of addition and multiplication play a central role. The integers, rational, and real numbers form algebras, and so do sets of functions and matrices. An algebra is *commutative* if for any two of its elements a and b, the multiplication satisfies $ab = ba$. Algebras of numbers and functions are commutative, but algebras of matrices are not: It is well-known that for matrices A and B, the matrix product AB is in general not equal to the matrix product BA.

Roughly speaking, a graded algebra is an algebra partitioned into layers, in such a way that addition and multiplication are compatible with this partition. A simple example is the algebra of polynomials, where the grading is given by the degree of a given polynomial. Algebras of polynomials in which the variables represent numbers are commutative.

Now, what if we want to study polynomials in variables that are not complex or real numbers, but, for instance, matrices? This question naturally leads us to the study of *noncommutative polynomials*. A model for such algebras is given by *subproduct systems*, a concept that will play a central role in this Ph.D. thesis. Subproduct systems provide a concrete framework for building such noncommutative graded structures layer by layer. A subproduct system $E = \{E_n\}_{n\in\mathbb{N}}$ can be thought of as an ordered collection of Hilbert spaces representing the allowed noncommutative polynomials for each degree. The space E_1 is spanned by the generators, E_2 represents the allowed quadratic combinations, and so on.

A special class of subproduct systems called *quadratic subproduct systems* plays a central role in this thesis. Using the language of noncommutative polynomials, a quadratic subproduct system $E = \{E_n\}_{n\in\mathbb{N}}$ is defined by a simple yet powerful rule: all relations between the noncommutative polynomials are determined by the quadratic ones. Specifically, the entire structure is determined by the orthogonal complement $E_1 \otimes E_1 \ominus E_2$.

C^*-algebras as noncommutative spaces

In mathematics, a standard way to understand a topological space is via the collection of continuous functions defined on it. For instance, all properties of the circle S^1 can be fully recovered from the algebraic structure of its continuous functions. This collection of continuous functions, denoted by $C(S^1)$, is closed under point-wise addition, multiplication, and complex conjugation. In mathematical terms, one says that $C(S^1)$ is a *commutative* C^*-algebra, where the commutativity means that $fg = gf$ for all $f, g \in C(S^1)$.

A C^*-algebra generalizes such an idea to a non-commutative setting, i.e., the multiplication is not necessarily commutative. A guiding principle in noncommutative geometry is to think of a C^*-algebra as the noncommutative functions on some (non-existent) noncommutative space. The Gelfand–Naimark theorem makes this precise: every commutative C^*-algebra can be represented as the algebra of continuous functions on a compact topological space. More general non-commutative C^*-algebras can be described in terms of operators on some Hilbert space, thanks to the Gelfand–Naimark–Segal theorem.

Given a Hilbert space, we are usually interested in certain concrete operators on it. Therefore, it is natural to study the smallest C^*-algebra that contains those operators. Let E be a subproduct system of finite-dimensional Hilbert spaces and let $\mathcal{F}_E$ be the associated Fock space. The elements of E determine a family of creation operators on F_E. Together with the identity operator they generate the *Toeplitz algebra* $\mathcal{T}_E$, the C^*-algebra we mainly study in this thesis. Moreover, in our setting, the C^*-algebra of compact operators $\mathcal{K}(F_E)$ is an ideal contained in $\mathcal{T}_E$. The quotient C^*-algebra $\mathcal{T}_E/\mathcal{K}(F_E)$ (denoted by $\mathcal{O}_E$), called the *Cuntz–Pimsner algebra*, is another important C^*-algebra that we investigate in this thesis.

K-theory

Once we have constructed C^*-algebras from a subproduct system, a natural question arises: how can we describe them? Two such C^*-algebras might look very different but be "structurally equivalent". *Operator K-theory* provides a tool to classify the structure of C^*-algebras.

For a C^*-algebra A, K-theory associates to it two abelian groups, namely, $K_0(A)$ and $K_1(A)$. The K_0-group encodes information about its fundamental building blocks (projections), while the K_1-group captures its rotational properties (unitaries). Together, they provide a robust and computable invariant of the C^*-algebra at hand.

Take the algebra of complex numbers $\mathbb{C}$, the simplest C^*-algebra, as an example. Its K-theory groups are given by

$$K_0(\mathbb{C}) \cong \mathbb{Z}, \quad K_1(\mathbb{C}) = 0.$$

Here, $K_0(\mathbb{C}) \cong \mathbb{Z}$ reflects that every projection in a matrix algebra over $\mathbb{C}$ is classified by its rank, which is just an integer. The vanishing of $K_1(\mathbb{C})$ follows from the fact that all invertible matrices over $\mathbb{C}$ can be continuously deformed to the identity matrix.

By computing the K-theory of the Toeplitz and Cuntz–Pimsner algebras, we can associate two abelian groups to every subproduct system; each of these groups encodes certain properties of the subproduct system.

Contributions of this thesis

Now that we have introduced our main objects of study—namely subproduct systems, their C^*-algebras, and K-theory—we are ready to describe the main contributions of this thesis. These concern certain quadratic subproduct systems and their Toeplitz and Cuntz–Pimsner algebras.

Specifically, we investigate three operations on the class of quadratic subproduct systems: *free products*, *Segre products* and *Veronese powers*. These operations come from quadratic algebras and are compatible with the structure of quadratic subproduct systems. Our main contributions are as follows:

(1) We provide a detailed description of the structural properties of the new subproduct systems resulting from these operations.

(2) Subsequently, we compute the K-theory groups for both the Toeplitz algebras $\mathcal{T}_E$ and the Cuntz–Pimsner algebras $\mathcal{O}_E$ associated with these new subproduct systems constructed from free product and Segre product.

(3) Furthermore, we delve deeper into the analytical properties of the creation operators. For a specific class of subproduct systems (the Veronese powers of what are known as p-reductive systems), we examine their Schatten p-class properties. This is directly motivated by the celebrated Arveson–Douglas conjecture, an important conjecture that relates functional analysis and algebraic geometry. Our results contribute to a better understanding of this conjecture.

To sum up, this thesis contributes to our understanding of noncommutative geometry by constructing new examples of subproduct systems and computing homological properties of their associated C^*-algebras. Explaining such research results can be a challenge; nevertheless, I hope this summary offers a clearer feeling for the core concepts of this work and provides the reader with a better sense of the problems that currently interest mathematicians in this area.