Subproduct systems and C*-algebras
Ge, Y.

Appendix A

Segre products beyond the quadratic case

In Section 7.2, we introduced the Segre product of subproduct systems. We now prove that the Segre product of subproduct systems is again a subproduct system, which implies that, together with the Segre product, the class of subproduct systems of finite-dimensional Hilbert spaces over $\mathbb{N}$ is a monoid.

Proposition A.1. *Let* $\mathrm{SPS}_f^{\mathbb{C}}$ *be the category of subproduct systems of finite-dimensional Hilbert spaces over* $\mathbb{N}$ *from Definition 4.22. The following holds:*

(1) If $E, F \in \mathrm{SPS}_f^{\mathbb{C}}$*, then the Segre product* $E \circ F \in \mathrm{SPS}_f^{\mathbb{C}}$*.*

(2) There exists a subproduct system (I, ι_I) *such that* $I \circ E = E$ *for all subproduct system* (E, ι_E)*.*

Proof. We prove the first claim as follows. By definition we have $(E \circ F)_0 := \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}$. Therefore, we only have to check the associativity of structure maps $\iota^{E \otimes_{\mathbb{C}} F}$, i.e.

$$(\iota_{s,t}^{E\circ F} \circ 1_r^{E\circ F}) \circ \iota_{s+t,r}^{E\circ F} = (1_s^{E\circ F} \circ \iota_{t,r}^{E\circ F}) \circ \iota_{s,t+r}^{E\circ F},$$

by a straightforward computation as follows,

$$
\begin{aligned}
&(\iota^{E\circ F}_{s,t} \circ 1^{E\circ F}_r) \circ \iota^{E\circ F}_{s+t,r}\\
&= (\iota^{E\circ F}_{s,t} \circ 1^{E\circ F}_r) \circ (1^E_{s+t} \otimes_{\mathbb{C}} \Gamma^{E,r}_{F,s+t} \otimes_{\mathbb{C}} 1^F_r \circ (\iota^E_{s+t,r} \otimes_{\mathbb{C}} \iota^F_{s+t,r}))\\
&= ((1^E_s \otimes_{\mathbb{C}} \Gamma^{E,t}_{F,s} \otimes_{\mathbb{C}} 1^F_t \circ (\iota^E_{s,t} \otimes_{\mathbb{C}} \iota^F_{s,t})) \otimes_{\mathbb{C}} 1^{E\otimes_{\mathbb{C}} F}_r)\\
&\circ (1^E_{s+t} \otimes_{\mathbb{C}} \Gamma^{E,r}_{F,s+t} \otimes_{\mathbb{C}} 1^F_r \circ (\iota^E_{s+t,r} \otimes_{\mathbb{C}} \iota^F_{s+t,r}))\\
&= (1^E_s \otimes_{\mathbb{C}} \Gamma^{E,t}_{F,s} \otimes_{\mathbb{C}} \Gamma^{E,r}_{E,t} \otimes_{\mathbb{C}} 1^F_r) \circ (1^E_s \otimes_{\mathbb{C}} 1^E_t \otimes_{\mathbb{C}} \Gamma^{E,r}_{E,s} \otimes_{\mathbb{C}} 1^F_t \otimes_{\mathbb{C}} 1^F_r)\circ\\
&(\iota^E_{s,t} \otimes_{\mathbb{C}} 1^E_r \otimes_{\mathbb{C}} \iota^F_{s,t} \otimes_{\mathbb{C}} 1^F_r) \circ (\iota^E_{s+t,r} \otimes_{\mathbb{C}} \iota^F_{s+t,r})\\
&= (1^E_s \otimes_{\mathbb{C}} \Gamma^{E,t}_{F,s} \otimes_{\mathbb{C}} \Gamma^{E,r}_{E,t} \otimes_{\mathbb{C}} 1^F_r) \circ (1^E_s \otimes_{\mathbb{C}} 1^E_t \otimes_{\mathbb{C}} \Gamma^{E,r}_{E,s} \otimes_{\mathbb{C}} 1^F_t \otimes_{\mathbb{C}} 1^F_r)\circ\\
&((\iota^E_{s,t} \otimes_{\mathbb{C}} 1^E_r \circ \iota^E_{s+t,r}) \otimes_{\mathbb{C}} (\iota^F_{s,t} \otimes_{\mathbb{C}} 1^F_r \circ \iota^F_{s+t,r}))\\
&= (1^E_s \otimes_{\mathbb{C}} \Gamma^{E,t}_{F,s} \otimes_{\mathbb{C}} \Gamma^{E,r}_{E,t} \otimes_{\mathbb{C}} 1^F_r) \circ (1^E_s \otimes_{\mathbb{C}} 1^E_t \otimes_{\mathbb{C}} \Gamma^{E,r}_{E,s} \otimes_{\mathbb{C}} 1^F_t \otimes_{\mathbb{C}} 1^F_r)\circ\\
&((1^E_s \otimes_{\mathbb{C}} \iota^E_{t,r}) \circ \iota^E_{s,t+r} \otimes_{\mathbb{C}} (1^F_s \otimes_{\mathbb{C}} \iota^F_{t,r}) \circ \iota^F_{s,t+r})\\
&= (1^{E\circ F}_s \otimes_{\mathbb{C}} \iota^{E\circ F}_{t,r}) \circ (1^E_s \otimes_{\mathbb{C}} \Gamma^{E,t+r}_{F,s} \otimes_{\mathbb{C}} 1^F_{t+r} \circ (\iota^E_{s,t+r} \otimes_{\mathbb{C}} \iota^F_{s,t+r}))\\
&= (1^{E\circ F}_s \otimes_{\mathbb{C}} \iota^{E\circ F}_{t,r}) \circ \iota^{E\circ F}_{s,t+r},
\end{aligned}
$$

where the third last equality is due to the associativity of $\iota^E_{k,l}$ and $\iota^F_{k,l}$.

For the second claim, it is straightforward to verify that for each $E \in \mathrm{SPS}^{\mathbb{C}}_f$, we have $(I \circ E)_n = \mathbb{C} \otimes E_n = E_n$. This proves the proposition. $\square$

Appendix B

Compact quantum groups

This appendix provides a brief introduction to the compact quantum groups from the perspective of C^*-algebras. This theory, primarily developed by S.L. Woronowicz, generalizes the notion of classical compact groups. Our main reference is [47].

Definition B.1 ([47, Definition 1.1.1])**.** *A* compact quantum group *is a pair (A, Δ), where A is a unital C^*-algebra, $\Delta : A \to A \otimes A$ is a unital $*$-homomorphism, called the* comultiplication*, and $\iota : A \to A$ is the identity map such that:*

(1) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ are homomorphisms $A \to A \otimes A \otimes A$ (coassociativity);

(2) the spaces $(A \otimes 1)\Delta(A) = \mathrm{span}\{(a \otimes 1)\Delta(b) \mid a, b \in A\}$ and $(1 \otimes A)\Delta(A)$ are dense in $A \otimes A$ (cancellation property),

where the tensor product $\otimes$ means the minimal tensor product.

An important example of a compact quantum group is $SU_q(2)$, which can be viewed as a noncommutative deformation of the classical special unitary group $SU(2)$.

Example B.2 (Quantum $SU(2)$ group)**.** *Let $q \in [-1, 1]\backslash\{0\}$, the quantum group*

$SU_q(2)$ is the universal C^*-algebra generated by α and γ such that the matrix

$$u = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix} \text{ is a } \textbf{\textit{unitary}} \text{ matrix.}$$

Let 1 *denote the unit in* $SU_q(2)$*. From* $u^*u = uu^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$*, we obtain the following relations for the generators:*

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1, \quad \gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.$$

The comultiplication is defined as

$$\Delta(u_{ij}) = u_{i1} \otimes u_{1j} + u_{i2} \otimes u_{2j}.$$

Therefore, we have

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

It is routine to verify the coassociativity, and the cancellation property is proven in [47, Proposition 1.1.4] □

The compact quantum group $SU_q(2)$ generalizes the notion of $SU(2)$ when $q = 1$ in the following sense. Observe that if $q = 1$, the generators of $SU_1(2)$ satisfy the following relations:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + \gamma^*\gamma = 1, \quad \gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = \gamma\alpha, \quad \alpha\gamma^* = \gamma^*\alpha,$$

which shows that $SU_1(2)$ is commutative. Moreover, due to the universal property of $SU_1(2)$, it is straightforward to show that $SU_1(2) \cong C(SU(2))$, the C^*-algebra of continuous functions on the classical compact group $SU(2)$.

To conclude this appendix, we introduce the representation theory of compact quantum groups, which can be viewed as a dual theory to the representation theory of compact groups.

Definition B.3 ([47, Definition 1.3.1])**.** *A corepresentation of a compact quantum group (A, Δ) on a finite-dimensional vector space $\mathcal{H}$ is an invertible element U of $\mathbb{B}(\mathcal{H}) \otimes A$ such that*

$$(\iota \otimes \Delta)(U) = U_{12}U_{13} \in \mathbb{B}(\mathcal{H}) \otimes A \otimes A, \tag{B.1}$$

where the notation U_{ij} is the leg notation. More precisely, suppose $U = u \otimes a \in \mathbb{B}(\mathcal{H}) \otimes A$, then $U_{12} = u \otimes a \otimes 1_A$ and $U_{13} = u \otimes 1_A \otimes a$. The corepresentation is called unitary *if $\mathcal{H}$ is a Hilbert space and U is unitary.*

Remark B.4. *Let $\mathcal{H}$ be a finite-dimensional vector space with basis $\{e_1, \ldots, e_n\}$. Suppose $\{m_{ij} : i, j = 1, \ldots, n\}$ is the corresponding set of matrix units in $\mathbb{B}(\mathcal{H})$, such that $m_{ij}(e_k) = \delta_{jk} e_i$, then the condition* (B.1) *reads as follows,*

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad \textit{for} \quad U = \sum_{i,j=1}^{n} m_{ij} \otimes u_{ij} \in \mathbb{B}(\mathcal{H}) \otimes A.$$

Definition B.5. *Assume that $U \in \mathbb{B}(\mathcal{H}_U) \otimes A$ and $V \in \mathbb{B}(\mathcal{H}_V) \otimes A$ are finite-dimensional corepresentations of a compact quantum group (A, Δ). We say a bounded linear operator $T : \mathcal{H}_U \to \mathcal{H}_V$ intertwines U and V if*

$$(T \otimes 1_A) U = V (T \otimes 1_A).$$

We denote by $\mathrm{Mor}(U, V)$ *the space of such intertwiners.*

We say U and V are equivalent if there is an invertible element in $\mathrm{Mor}(U, V)$. For simplicity, we write $\mathrm{Mor}(U, U)$ as $\mathrm{End}(U)$. A corepresentation U is called irreducible if $\mathrm{End}(U) \cong \mathbb{C}$.

Similar to the representation theory of compact groups, we conclude this appendix with Schur's lemma for corepresentation theory.

Proposition B.6 (Schur's lemma)**.** *Let U and V be irreducible corepresentations of a compact quantum group (A, Δ). Then, either* $\mathrm{Mor}(U, V) \cong \mathbb{C}$*, i.e., U is equivalent to V, or* $\mathrm{Mor}(U, V) = 0$.

Appendix C

Codes

This appendix presents the various codes used in this project.

C.1 Magma codes

It is important to note that all our codes were executed using Magma version *2.28-1* within the *Magma Notebook* environment in Visual Studio Code.

C.1.1 Some functions in Magma

In this section, we will define two Magma functions we use in this Ph.D. project.

```
Projection := func< Q, v | Solution(A*B,v*B)*A
where B is Transpose(A)
where A is BasisMatrix(Q) >;
```

Listing C.1: *Projection operator.*

C.1.2 Subproduct systems in Magma

```
V1:=VectorSpace(Rationals(),3);
T2:=TensorProduct(V1,V1);
B:=Basis(V1);

// Select the basis of V1
e1:=B[1];
e2:=B[2];
e3:=B[3];

// Define the Temperley--Lieb vector
e13 := TensorProduct(e1, e3);
e22 := TensorProduct(e2, e2);
e31 := TensorProduct(e3, e1);
v := e13 - e22 + e31;

// Define the determinant
Det := sub<T2 | [v]>;

// Define the second fibre of the subproduct system
V2 := OrthogonalComplement(T2, Det);

// Compute the dimension of V2
Dimension(V2);

// Define the third fibre of the subproduct system
V12:=TensorProduct(V1,V2);
V21:=TensorProduct(V2,V1);
V3:=V12 meet V21;
Dimension(V3);

// Define the fourth fibre of the subproduct system recursively
V13:=TensorProduct(V1,V3);
V22:=TensorProduct(V2,V2);
V31:=TensorProduct(V3,V1);
V4:=V13 meet V22 meet V31;
Dimension(V4);

// Compute the image of E1 under the map defined in [2]
Im_E1 := OrthogonalComplement(TensorProduct(V2, V1), V3);

// Compute the image of E1 under the map defined in [17]
```

```
B1 := Basis(V1);
v11 := TensorProduct(B1[1], v);
v12 := TensorProduct(B1[2], v);
v13 := TensorProduct(B1[3], v);
Imv11 := Projection(TensorProduct(V2, V1), v11);
Imv12 := Projection(TensorProduct(V2, V1), v12);
Imv13 := Projection(TensorProduct(V2, V1), v13);

// Define the subspace spanned by the image of E1 under Sergey's
   map
BM1 := Basis(Im_E1);
V_sub1 := sub<TensorProduct(V2, V1) | [Imv11, Imv12, Imv13]>;

// Compare the images of two maps
Im_E1 eq V_sub1;
```

Listing C.2: *Construction of Temperley–Lieb subproduct systems and their structure maps.*

C.1.3 Computing the Veronese powers of varieties in Magma

In this section, we define a function that computes the Veronese power of a given coordinate ring. This function helps us verify the generating relations in the new coordinate ring of the Verones power so that we can analyze the smoothness of the variety of this new coordinate ring. This shows that our results extend the boundary of the known results of the Arveson–Douglas conjecture.

Before showing the Magma codes, it is worth mentioning that the underlying field in Magma we work with is the rational field $\mathbb{Q}$. Since the computer is only able to analyze the discrete fields like rational field and finite field, this limitation forces us to work with the rational field. On the other hand, in the mathematical experiment, we consider the variety $V(I)$ defined by homogeneous polynomials with rational coefficients. Under this setting, if a point p on a variety $V(I)$ over the rational field is singular, p is also singular over a field of complex numbers since the rank of the Jacobian matrix on p is independent of the underlying field.

```
function VeroneseIdeal(R, I, k)
    // Compute the quotient ring Q = R/I
    Q := quo< R | I >;

    // Compute a Groebner basis for I
    G := GroebnerBasis(I);

    // Get all monomials of degree k in R
    mons := MonomialsOfDegree(R, k);

    // Now select only those monomials that are standard
    // That is, m is standard if its normal form (with respect to G) equals m.
    basis := [];
    for m in mons do
        if m eq NormalForm(m, G) then
            Append(~basis, Q!m);
        end if;
    end for;

    // Now basis is a genuine vector space basis for Q_k.
    d := #basis;
    // Create a new polynomial ring with d variables over the base field
    S<[x]> := PolynomialRing(BaseRing(R), d);

    // Define the homomorphism from S to Q mapping each x[i] to basis[i]
    f := hom< S -> Q | basis >;

    // Compute the kernel of the homomorphism; this is the ideal of relations.
    J := AffineAlgebraMapKernel(f);
    return S, J;
end function;
QQ := RationalField();
P<x,y,z> := PolynomialRing(QQ,3);
I:=ideal<P | x^3-z*y^2>;

// The output 1
VeroneseIdeal(P, I, 2);
```

```
Polynomial ring of rank 6 over Rational Field
Order: Lexicographical
Variables: x[1], x[2], x[3], x[4], x[5], x[6]
Ideal of Polynomial ring of rank 6 over Rational Field
Order: Lexicographical
Variables: x[1], x[2], x[3], x[4], x[5], x[6]
Homogeneous
Basis:
[
    -x[4]*x[6] + x[5]^2,
    -x[2]*x[6] + x[3]*x[5],
    -x[2]*x[5] + x[3]*x[4],
    -x[1]*x[6] + x[3]^2,
    -x[1]*x[5] + x[2]*x[3],
    x[1]*x[3] - x[4]*x[6],
    -x[1]*x[4] + x[2]^2,
    x[1]*x[2] - x[4]*x[5],
    x[1]^2 - x[2]*x[5]
]

// The output 2
VeroneseIdeal(P, I, 4);

Polynomial ring of rank 12 over Rational Field
Order: Lexicographical
Variables: x[1], x[2], x[3], x[4], x[5], x[6], x[7], x[8], x[9], x
    [10], x[11], x[12]
Ideal of Polynomial ring of rank 12 over Rational Field
Order: Lexicographical
Variables: x[1], x[2], x[3], x[4], x[5], x[6], x[7], x[8], x[9], x
    [10], x[11], x[12]
Homogeneous
Basis:
[
    -x[8]*x[12] + x[10]*x[11],
    -x[2]*x[12] + x[9]*x[11],
    x[6]*x[11] - x[10]*x[12],
    -x[4]*x[12] + x[5]*x[11],
    x[4]*x[11] - x[7]*x[12],
    -x[1]*x[12] + x[3]*x[11],
    x[2]*x[11] - x[12]^2,
```

```
-x[4]*x[12] + x[10]^2,
-x[7]*x[12] + x[8]*x[10],
-x[1]*x[12] + x[7]*x[10],
-x[5]*x[12] + x[6]*x[10],
-x[3]*x[12] + x[4]*x[10],
-x[2]*x[12] + x[3]*x[10],
x[2]*x[10] - x[6]*x[12],
x[1]*x[10] - x[12]^2,
-x[6]*x[12] + x[8]*x[9],
-x[5]*x[12] + x[7]*x[9],
x[1]*x[9] - x[5]*x[10],
-x[7]*x[11] + x[8]^2,
-x[1]*x[11] + x[7]*x[8],
-x[4]*x[12] + x[6]*x[8],
-x[3]*x[12] + x[5]*x[8],
-x[1]*x[12] + x[4]*x[8],
x[3]*x[8] - x[12]^2,
x[2]*x[8] - x[10]*x[12],
x[1]*x[8] - x[11]*x[12],
x[7]^2 - x[11]*x[12],
-x[3]*x[12] + x[6]*x[7],
-x[2]*x[12] + x[5]*x[7],
x[4]*x[7] - x[12]^2,
x[3]*x[7] - x[10]*x[12],
x[2]*x[7] - x[4]*x[12],
x[1]*x[7] - x[8]*x[12],
-x[4]*x[9] + x[6]^2,
-x[3]*x[9] + x[5]*x[6],
x[4]*x[6] - x[5]*x[10],
x[3]*x[6] - x[9]*x[12],
x[2]*x[6] - x[9]*x[10],
x[1]*x[6] - x[2]*x[12],
-x[2]*x[9] + x[5]^2,
x[4]*x[5] - x[9]*x[12],
x[3]*x[5] - x[9]*x[10],
x[2]*x[5] - x[4]*x[9],
x[1]*x[5] - x[6]*x[12],
-x[2]*x[12] + x[4]^2,
x[3]*x[4] - x[6]*x[12],
x[2]*x[4] - x[5]*x[12],
x[1]*x[4] - x[10]*x[12],
```

```
    x[3]^2 - x[5]*x[12],
    x[2]*x[3] - x[5]*x[10],
    x[1]*x[3] - x[4]*x[12],
    x[2]^2 - x[9]*x[12],
    x[1]*x[2] - x[3]*x[12],
    x[1]^2 - x[7]*x[12]
]
```

Listing C.3: *Function of computing Veronese powers and examples.*

C.2 Macaulay2 codes

Macaulay2 is a powerful mathematical program that can process computations in Algebraic Geometry and Commutative Algebras. We shall use the Macaulay2 to compute the singular points on a variety, which verifies the computed example in the thesis.

```
VeroneseIdeal = (R, I, k) -> (
    -- Compute the quotient ring A = R/I
    A := R/I;

    -- Generate a basis for the degree k part of R (returns a matrix)
    monoms := basis(k, A);

    -- n is the number of monomials in degree k
    n := numcols monoms;

    -- Create a new polynomial ring S with n variables,
    -- each given degree k.

    S := QQ[w_1..w_n, Degrees => apply(n, i -> k)];
    -- Define the map phi: S -> A by sending the i-th variable of S
    -- to the i-th basis element (flatten the matrix to a list)
```

```
    phi := map(A, S, flatten entries monoms);

    -- Compute the kernel of phi (the defining ideal J)
    J := kernel phi;

    -- Return the new ring S and the ideal J
    (S, J)
);

    R = QQ[x,y,z];
    I1 = ideal(x^3 - z*y^2);
    I2 = ideal(x^2 + y^2 + z^2 - 2*x*z);
    A = R/I2;

    (S, J) = VeroneseIdeal(R, I2, 4)

    -- Output of S and J
    gens S
    gens J

      -- Compute the singular points
    singularLocus(J)

    -- The output is a long list of singular points with
    rational coordinates.
```

C.3 Python codes

As the dimensions of the subproduct systems increase, interpreting the results becomes challenging due to the large magnitude of the numbers involved. Fortunately, Python can be used to express these numbers as linear combinations of basis vectors.

It is important to note that all code was executed using Python version *3.7.2* within the Visual Studio Code environment.

```python
import numpy as np
import os
import re
import math

def printe(input_0, pow=4, dim=5):
    #pow = 4
    #dim = 5
    #input_0 = "0 1 0 0 0 -1/3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 -1/3 0 0 0 0 0 0 0 0 0 0 0 ... 0".split(' ')
    input_0 = input_0.split(' ')
    num_0 = [round(eval(i), 2) for i in input_0]
    array_0 = np.array(num_0).reshape([dim]*pow)
    non0_0 = np.nonzero(array_0)
    str_0 = []
    len_non0 = len(non0_0)
    for coord in zip(*[non0_0[i] for i in range(len_non0)]):
        idx = 0
        for i in range(len_non0):
            idx += coord[-i-1]*math.pow(dim, i)
            #idx = z*1 + y*dim + x*dim*dim + w*dim*dim*dim
        #print(idx)
        coef = input_0[int(idx)]
        s = str(coef)
        for i in range(len_non0):
            s += 'e'+str(coord[i]+1)
            #'e'+str(w+1)+'e'+str(x+1)+'e'+str(y+1)+'e'+str(z+1)
        str_0.append(s)
    #print(str_0)
    str_00 = ""
    for i in range(len(str_0)):
        s = str_0[i]
        if s[0] == '1' and s[1]=='e':
            s = s[1:]
        if i!=len(str_0)-1 and str_0[i+1][0] != '-':
            s += '+'
        #print(s)
```

```
        str_00 += s
    print(str_00)

#path = "The location where the document is saved"
path = "/Users/Sherlock/Desktop/sergey_Basis_V2.txt"

with open(path, 'r') as file:
    content = file.read()
    content = content.replace('\n', '')
    content = content.replace('[', '(')
    content = content.replace(']', ')')
    #if the out put is obtained in Magma Notebook, the data is in the form of [...], hence one should replace '[' and ']' by '(' and ')'.
    content = content.replace(',', '').
    #if the out put is of small size, the data is connected by ',', which is not required in the program.
    content = re.sub(r'\s+', ' ', content)
    content = content.split(')')
input_list = []
for c in content[:-1]:
    c = re.sub(r'\s*[\()]\s*', '', c)
    input_list.append(c)
for i in range(len(input_list)):
    #print("output {}".format(i))
    printe(input_list[i], pow=2, dim=5)
#pow is the index of the fibre e.g. if we want to transfer vectors in E_n, then pow = n.
#dim is the dimension of E_1.
```

Listing C.4: *Transfer Magma results into linear combination of basis vectors*

References

[1] D. J. Anick. Generic algebras and CW-complexes. Algebra, topology, and algebraic K-theory(Princeton, NJ, 1983), 247-321, Ann. of Math. Stud . 113, Princeton University Press, 1987.

[2] F. Arici and Y. Ge, *Quadratic subproduct systems, free products, and their C^*-algebras* (2025), preprint, ArXiv: 2504.12843.

[3] F. Arici and J. Kaad, Gysin sequences and $SU(2)$-symmetries of C^*-algebras, Trans. London Math. Soc. **8** (2021), no. 1, 440–492; MR4451552

[4] W. B. Arveson, Subalgebras of C^*-algebras. III. Multivariable operator theory, Acta Math. **181** (1998), no. 2, 159–228; MR1668582

[5] W. B. Arveson, The curvature of a Hilbert module over $\mathbf{C}[z_1, \cdots, z_d]$, Proc. Natl. Acad. Sci. USA **96** (1999), no. 20, 11096–11099; MR1712636

[6] W. B. Arveson, The curvature invariant of a Hilbert module over $\mathbf{C}[z_1, \cdots, z_d]$, J. Reine Angew. Math. **522** (2000), 173–236; MR1758582

[7] W. B. Arveson, The Dirac operator of a commuting d-tuple, J. Funct. Anal. **189** (2002), no. 1, 53–79; MR1887629

[8] W. B. Arveson, p-summable commutators in dimension d, J. Operator Theory **54** (2005), no. 1, 101–117; MR2168861

[9] M. F. Atiyah, F. Hirzebruch. Riemann-Roch theorems for differentiable manifolds. Bulletin of the American Mathematical Society, 65(4) 276-281 July 1959.

[10] D. Avitzour, Free products of C^*-algebras, Trans. Amer. Math. Soc. **271** (1982), no. 2, 423–435;

[11] B. V. R. Bhat and M. Mukherjee, Inclusion systems and amalgamated products of product systems, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **13** (2010), no. 1, 1–26; MR2646788

[12] B. E. Blackadar, *K-theory for operator algebras*, second edition, Mathematical Sciences Research Institute Publications, 5, Cambridge Univ. Press, Cambridge, 1998; MR1656031

[13] A. Borel. Sur la cohomologie des espaces fibres principaux et des espaces homogenes de groupes de Lie compacts. Ann. Math. (1953), 57 115–207. doi: 10.2307/1969728.

[14] A. Borel, J. P. Serre. The Riemann–Roch theorem. Bulletin of the Mathematical Society of France, Volume 86 (1958), pp. 97-136. doi: 10.24033

[15] L. G. Brown, R. G. Douglas and P. A. Fillmore, Extensions of C^*-algebras and K-homology, Ann. of Math. (2) **105** (1977), no. 2, 265–324; MR0458196

[16] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, 88, Amer. Math. Soc., Providence, RI, 2008; MR2391387

[17] R. C. Busby, Double centralizers and extensions of C^*-algebras, Trans. Amer. Math. Soc. **132** (1968), 79–99; MR0225175

[18] M. D. Choi and E. G. Effros, The completely positive lifting problem for C^*-algebras, Ann. of Math. (2) **104** (1976), no. 3, 585–609; MR0417795

[19] L. A. Coburn, The C^*-algebra generated by an isometry, Bull. Amer. Math. Soc. **73** (1967), 722–726; MR0213906

[20] J. Cuntz and W. Krieger, A class of C^*-algebras and topological Markov chains, Invent. Math. **56** (1980), no. 3, 251–268; MR0561974

[21] R. G. Douglas, A new kind of index theorem, in *Analysis, geometry and topology of elliptic operators*, 369–382, World Sci. Publ., Hackensack, NJ; MR2246775

[22] K. J. Dykema, Exactness of reduced amalgamated free product C^*-algebras, Forum Math. **16** (2004), no. 2, 161–180; MR2039095

[23] D. Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, 229, Springer, New York, 2005; MR2103875

[24] M. Engliš and J. Eschmeier, Geometric Arveson-Douglas conjecture, Adv. Math. **274** (2015), 606–630; MR3318162

[25] T. Gateva-Ivanova, Segre products and Segre morphisms in a class of Yang-Baxter algebras, Lett. Math. Phys. **113** (2023), no. 2, Paper No. 34, 34 pp.; MR4562210

[26] E. Germain, KK-theory of reduced free-product C^*-algebras, Duke Math. J. **82** (1996), no. 3, 707–723; MR1387690

[27] E. Germain, KK-theory of the full free product of unital C^*-algebras, J. Reine Angew. Math. **485** (1997), 1–10; MR1442186

[28] W. Gevaert, Algebraic subproduct systems of the Hopf algebra $O(SUq(2))$, Leiden University Student Repository (2022), https://hdl.handle.net/1887/4171449

[29] K. Y. Guo and K. Wang, Essentially normal Hilbert modules and K-homology, Math. Ann. **340** (2008), no. 4, 907–934; MR2372744

[30] E. Habbestad and S. Neshveyev, Subproduct systems with quantum group symmetry. II (2022), preprint, arXiv:2212.08512.

[31] E. Habbestad and S. V. Neshveyev, Subproduct systems with quantum group symmetry, J. Noncommut. Geom. **18** (2024), no. 1, 93–121; MR4705666

[32] F. Harary, *Graph theory*, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969; MR0256911

[33] K. K. Jensen and K. Thomsen, *Elements of KK-theory*, Mathematics: Theory & Applications, Birkhäuser Boston, Boston, MA, 1991; MR1124848

[34] E. T. A. Kakariadis and O. M. Shalit, Operator algebras of monomial ideals in noncommuting variables, J. Math. Anal. Appl. **472** (2019), no. 1, 738–813; MR3906397

[35] I. Kaplansky, Modules over operator algebras, Amer. J. Math. **75** (1953), 839–858; MR0058137

[36] G. G. Kasparov, The operator K-functor and extensions of C^*-algebras, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 3, 571–636, 719; MR0582160

[37] M. Kennedy and O. M. Shalit, Essential normality and the decomposability of algebraic varieties, New York J. Math. **18** (2012), 877–890; MR2991427

[38] A. U. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer, Berlin, 1997; MR1492989

[39] E. C. Lance, *Hilbert C^*-modules*, London Mathematical Society Lecture Note Series, 210, Cambridge Univ. Press, Cambridge, 1995; MR1325694

[40] M. Lesch, K-theory and Toeplitz C^*-algebras—a survey, in *Séminaire de Théorie Spectrale et Géométrie, No. 9, Année 1990–1991*, 119–132, Sémin. Théor. Spectr. Géom., 9, Univ. Grenoble I, Saint-Martin-d'Hères; MR1715935

[41] D. A. Lind and B. H. Marcus, *An introduction to symbolic dynamics and coding*, second edition, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 2021; MR4412543

[42] S. Mac Lane, *Categories for the working mathematician*, second edition, Graduate Texts in Mathematics, 5, Springer, New York, 1998; MR1712872

[43] Y. I. Manin, *Quantum groups and noncommutative geometry*, second edition, CRM Short Courses, Centre de Recherches Mathématiques, [Montreal], QC, 2018 Springer, Cham, 2018; MR3839605

[44] K. Matsumoto, On C^*-algebras associated with subshifts, Internat. J. Math. **8** (1997), no. 3, 357–374; MR1454478

[45] B. Mesland and A. Rennie, The Friedrichs angle and alternating projections in Hilbert C^*-modules, J. Math. Anal. Appl. **516** (2022), no. 1, Paper No. 126474, 19 pp.; MR4454287

[46] G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, Boston, MA, 1990; MR1074574

[47] S. V. Neshveyev and L. Tuset, *Compact quantum groups and their representation categories*, Cours Spécialisés, 20, Soc. Math. France, Paris, 2013; MR3204665

[48] N. Ozawa, Nuclearity of reduced amalgamated free product C^*-algebras, Sūrikaisekikenkyūsho Ko̱kyūroku No. 1250 (2002), 49–55; MR1915435

[49] E. Park, *Complex topological K-theory*, Cambridge Studies in Advanced Mathematics, 111, Cambridge Univ. Press, Cambridge, 2008; MR2397276

[50] W. L. Paschke, Inner product modules over B^*-algebras, Trans. Amer. Math. Soc. **182** (1973), 443–468; MR0355613

[51] M. V. Pimsner, *A class of C^*-algebras generalizing both Cuntz–Krieger algebras and crossed products by $\mathbb{Z}$*, Free probability theory, Fields Institute Communications 12 (ed. D.-V. Voiculescu; American Mathematical Society, Providence, RI, 1997) 189-212; MR1426840.

[52] A. Polishchuk and L. E. Positselski, *Quadratic algebras*, University Lecture Series, 37, Amer. Math. Soc., Providence, RI, 2005; MR2177131

[53] W. Pusz and S. L. Woronowicz, Twisted second quantization, Rep. Math. Phys. **27** (1989), no. 2, 231–257; MR1067498

[54] M. A. Rieffel, Induced representations of C^*-algebras, Advances in Math. **13** (1974), 176–257; MR0353003

[55] J. M. Rosenberg and C. L. Schochet, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor, Duke Math. J. **55** (1987), no. 2, 431–474; MR0894590

[56] G. Segal. The representation ring of a compact Lie group. Publications Mathématiques de l'Institut des Hautes Scientifiques 34, 113-128 (1968). doi: 10.1007/BF02684592

[57] O. Shalit. *Operator Theory and Function Theory in Drury–Arveson Space and Its Quotients*. In: Alpay, D. (eds) Operator Theory. Springer, Basel.

[58] O. M. Shalit and B. Solel, Subproduct systems, Doc. Math. **14** (2009), 801–868; MR2608451

[59] Snaith VP. Massey products in K-theory. II. Mathematical Proceedings of the Cambridge Philosophical Society. 1971;69(2):259-289. doi:10.1017/S0305004100046636.

[60] R. P. Stanley, *Enumerative combinatorics. Volume 1*, second edition, Cambridge Studies in Advanced Mathematics, 49, Cambridge Univ. Press, Cambridge, 2012; MR2868112

[61] A. Viselter, Cuntz-Pimsner algebras for subproduct systems, Internat. J. Math. **23** (2012), no. 8, 1250081, 32 pp.; MR2949219

[62] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Science Publications, Oxford Univ. Press, New York, 1993; MR1222415

[63] H. S. Wilf, *generatingfunctionology*, second edition, Academic Press, Boston, MA, 1994; MR1277813

[64] S. L. Woronowicz, Twisted SU(2) group. An example of a noncommutative differential calculus, Publ. Res. Inst. Math. Sci. **23** (1987), no. 1, 117–181; MR0890482

[65] J. J. Zhang, Quadratic algebras with few relations, Glasgow Math. J. **39** (1997), no. 3, 323–332; MR1484574