Subproduct systems and C*-algebras

Ge, Y.

Chapter 10

Conclusion

In this thesis, we investigated quadratic subproduct systems and their C^*-algebras. In Chapter 4, we introduced subproduct systems and presented a categorical viewpoint of them. As an application, we discussed the Matsumoto subshift C^*-algebra and showcased a novel formula for the topological entropy of a subshift. Chapter 5 was devoted to an overview of the Arveson–Douglas conjecture, serving as a preliminary for Section 9.3.

In Part II, we focused on the quadratic subproduct systems and their C^*-algebras as well as their K-theory. In Chapter 6, we introduced the quadratic subproduct systems. In particular, we adopted the notion of genericity from quadratic algebras to quadratic subproduct systems and presented two motivating examples: the Temperley–Lieb subproduct systems and the $SU(2)$-subproduct system. Chapter 7 explored three operations on quadratic subproduct systems: free product, Segre product, and Veronese subproduct systems. These operations allowed us to construct novel examples of quadratic subproduct systems whose Toeplitz algebras behaved differently from $\mathbb{C}$ in the sense of K-theory.

In Chapter 8, we studied the Toeplitz algebras of quadratic subproduct systems arising from operations studied in Chapter 7. Our result showed that the free product structure descended naturally to the level of Toeplitz algebras, in terms of reduced free product. This generalized Speicher's construction in [16, Example 4.7.5]. Following this, we turned to the Segre product and computed the commutation relations for the Segre product of $SU(2)$-subproduct systems, which was initially computed in [3], as the first step toward understanding the structure of the associated Toeplitz algebra.

Chapter 9 brought together the core ideas and constructions throughout this thesis. We computed the K-theory groups of Toeplitz algebras arising from the free product of subproduct systems and derived a noncommutative Gysin sequence, thereby partially solving the first open problem in [3, Section 7.1]. Section 9.2 was devoted to a comprehensive case study of the Segre product. We focused on the Segre product of two $SU(2)$-subproduct systems and its Toeplitz algebra, and concluded this section by proving that this Toeplitz is **not** KK-equivalent to complex numbers. In Section 9.3, we derived a family of quotient modules, viewed as subproduct systems, whose Cuntz–Pimsner algebras are continuous functions on lens spaces, with explicitly known K-theory groups. This led to a concrete example that answered [40, Problem]. In the end, we proved that Veronese submodules of p-essentially normal modules give rise to p-essentially normal modules with higher dimensions and proposed a conjecture that if the original quotient module $\mathcal{Q}$ satisfies the Arveson–Douglas conjecture, then the quotient module $H^2_N/\mathcal{M}_J$ (which isomorphic to $\mathcal{Q}^{(k)}$, the k-th Veronese power of $\mathcal{Q}$) satisfies the Arveson–Douglas conjecture.

The construction in the final section gives rise to a family of quotient modules (see Example 9.19) that satisfy the Arveson–Douglas conjecture, which yields the following exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \to C(L(k,n)) \to 0. \tag{10.1}$$

In [21], the author conjectured that the odd K-homology class $[\tau]$ represented by (10.1) is the fundamental class of $L(k,n)$. This leads to a problem we find interesting: if $[\tau]$ was the fundamental class, what is the index pairing between $[\tau]$ and generators of the K-theory of $L(k,n)$? We leave this problem for future work.