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Chapter 5

The Arveson–Douglas conjecture

The Arveson–Douglas conjectures are a family of conjectures regarding the essential normality of submodules, and quotient modules of some analytic Arveson–Hilbert modules over polynomial rings. In the quest for a geometric invariant of commuting tuples of operators, Arveson formulated what is now known as the Arveson–Douglas conjecture in [7]. Later, Arveson proved it in [8] for special cases of Drury–Arveson spaces, Hardy spaces, and Bergman spaces. In 2005, R. Douglas showed that the conjecture could lead to a new kind of index theorem, which provides a framework to connect algebraic geometry and operator theory [21].

Three analytic function spaces are involved in the Arveson–Douglas conjecture: the Drury–Arveson space, the Hardy space, and the Bergman space. We will briefly discuss the Bergman space and mainly focus on the Drury–Arveson space, which can be viewed as a special example of a subproduct system.

In this chapter, we first discuss the significance of the Arveson–Douglas conjecture in functional analysis. We will connect this to subproduct systems and their associated Toeplitz algebras. Initially, we will revisit some funda-

mental concepts of analytic function spaces.

In this Chapter, the symbol $\mathbb{B}_n$ will always denote the n -dimensional open unit ball in $\mathbb{C}^n$.

5.1 Reproducing kernel Hilbert spaces

As anticipated, in order to discuss the Arveson–Douglas conjecture, we will focus on the Drury–Arveson spaces, Hardy spaces, and Bergman spaces. These are classical analytic function spaces and belong to a special family of Hilbert spaces of functions called reproducing kernel Hilbert spaces.

Definition 5.1 (Reproducing kernel Hilbert space). *Let X be a set, and $\mathcal{H}$ be a Hilbert space of functions on X . We say $\mathcal{H}$ is a reproducing kernel Hilbert space if the evaluation map*

$$ev_x : f \mapsto f(x), \quad \forall x \in X, f \in \mathcal{H}$$

is a continuous linear functional. For simplicity, we shall use the abbreviation RKHS when talking about reproducing kernel Hilbert spaces.

Let $\mathcal{H}$ be an RKHS. By the Riesz representation theorem, for each ev_x , there exists $K_x \in \mathcal{H}$ such that

$$ev_x(f) = f(x) = \langle f, K_x \rangle_{\mathcal{H}}.$$

We call K_x the reproducing kernel in the sense that f is determined (“reproduced”) by the function K_x and the inner product structure of $\mathcal{H}$.

Example 5.2. *Consider the standard separable Hilbert space $\ell^2(\mathbb{N})$, viewed as a Hilbert space of functions on $\mathbb{N}$, such that $f = (f_k)_k \in \ell^2(\mathbb{N})$ maps k to f_k . It is not hard to see that for all $k \in \mathbb{N}$, the reproducing kernel is represented by the standard orthonormal basis e_k , since $f(k) = f_k = \langle f, e_k \rangle_{\ell^2(\mathbb{N})}$ and e_k is viewed as a function on $\mathbb{N}$ represented by the Kronecker delta δ_k . $\square$*

Now that we have introduced the notion of a Reproducing kernel Hilbert space, we are ready to define the analytic function spaces that are central to the Arveson–Douglas conjecture.

Definition 5.3 (Drury–Arveson space). *The Drury–Arveson space H_n^2 is the reproducing kernel Hilbert space on $\mathbb{B}_n$ with reproducing kernel*

$$K_z(w) = \frac{1}{1 - \langle z, w \rangle}, \quad \forall z, w \in \mathbb{B}_n.$$

As a natural generalization of the Hardy space, other points of view of the Drury–Arveson space will be discussed in detail in the following section.

Definition 5.4 (Bergman space). *The Bergman space $L_a^2(\mathbb{B}_n)$ is a reproducing kernel Hilbert space on $\mathbb{B}_n$ with the reproducing kernel*

$$K_z(w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1}}, \quad \forall z, w \in \mathbb{B}_n.$$

It is worthwhile to note that one can define the Bergman space on any domain in $\mathbb{C}^n$. Indeed, for a domain $D \subset \mathbb{C}^n$, the Bergman space $L_a^2(D)$ can also be defined as the L^2 closure of analytic functions on D , although on a general domain D , we may lose the explicit description of reproducing kernels. However, for the sake of introducing the Arveson–Douglas conjecture, we won't go into further details. To provide the reader a general feeling of how the Bergman space is defined, we will define the weighted Bergman space on $\mathbb{B}_n$.

Definition 5.5 (Weighted Bergman space). *The weighted Bergman space $L_{a,s}^2(\mathbb{B}_n)$ with weight s is the closure of analytic functions on $\mathbb{B}_n$ with respect to the weighted L^2 -norm, that is,*

$$\|f\|_{L_{a,s}^2}^2 = \int_{\mathbb{B}_n} (1 - \|z\|^2)^s \cdot |f(z)|^2 dz,$$

where dz is the normalized Lebesgue measure on $\mathbb{B}_n$. In particular, when $s = 1$, $L_{a,s}^2(\mathbb{B}_n)$ is the classical Bergman space $L_a^2(\mathbb{B}_n)$.

5.1.1 The Drury–Arveson space as symmetric Fock space

In this subsection, we will discuss the Drury–Arveson space in detail and show that it is unitarily isomorphic to the symmetric Fock space that we introduced in Example 4.8.

Before diving into the details of Drury–Arveson spaces, we first recall the Hardy space, which is a classical analytic function space that motivates the definition of the Drury–Arveson space.

Definition 5.6 (Hardy space). *Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, and consider the algebra of analytic functions on $\mathbb{D}$, denoted by $\mathcal{O}(\mathbb{D})$. Let $f \in \mathcal{O}(\mathbb{D})$, and let $f = \sum_{n=0}^{\infty} a_n \cdot z^n$ be the Taylor expansion, the Hardy space H^2 is the closure of $\mathcal{O}(\mathbb{D})$ with respect to the norm*

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

Indeed, Hardy space on $\mathbb{D}$ also has a reproducing kernel Hilbert space description.

Definition 5.7 (Hardy space, RKHS point of view). *Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, the Hardy space H^2 is the reproducing kernel Hilbert space of functions on $\mathbb{D}$ with respect to the reproducing kernel*

$$K_z(w) = \frac{1}{1 - z \cdot \bar{w}}, \quad \forall z, w \in \mathbb{D}.$$

Remark 5.8. *Thanks to Fourier analysis, one can verify that the Hardy space is isomorphic to $\ell^2(\mathbb{N})$. More precisely, every function $f \in H^2$ admits a Taylor expansion at zero (see Definition 5.6):*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \forall f \in H^2,$$

where the sequence of coefficients $(a_n)_{n \in \mathbb{N}}$ belongs to $\ell^2(\mathbb{N})$. The coefficients $(a_n)_{n \in \mathbb{N}}$ coincide with the non-negative Fourier coefficients of $\tilde{f} \in L^2(S^1)$, the boundary function of f . Therefore, the identification, given by the Fourier transformation,

$$f \leftrightarrow (a_n)_{n \in \mathbb{N}}, \quad \forall f \in H^2$$

establishes a unitary isomorphism between H^2 and $\ell^2(\mathbb{N})$.

Recall the definition of the Drury–Arveson space in Definition 5.3 as an RKHS for the kernel

$$K_z(w) = \frac{1}{1 - \langle z, w \rangle}, \quad \forall z, w \in \mathbb{B}_d.$$

From the form of the reproducing kernel, it is clear that the Drury–Arveson space should serve as a natural generalization of the Hardy space in higher dimensions. In his original work [4], Arveson originally defined the Hilbert space H_d^2 as the completion of the space of polynomials with respect to the topology induced by a suitable inner product.

Definition 5.9 (Drury–Arveson space, power series point of view). *The Drury–Arveson space H_d^2 is defined to be the following*

$$H_d^2 = \left\{ f = \sum_{\alpha \in \mathbb{N}^d} a_\alpha \cdot z^\alpha : \|f\|^2 = \sum_{\alpha \in \mathbb{N}^d} |a_\alpha|^2 \cdot \frac{\alpha!}{|\alpha|!} < \infty \right\}.$$

From the norms defined for Drury–Arveson space and Hardy space, it is also clear that Drury–Arveson space is a natural generalization of Hardy space in higher dimensions from the power series point of view.

Recall the symmetric subproduct system E^s defined in Example 4.8. The norm of the orthogonal basis of $\mathcal{F}_{E^s}$ was computed in Example 4.15, which coincides with the norm defined in the power series point of view. Therefore, there exists a unitary isomorphism between the Drury–Arveson space H_d^2 and the Fock space $\mathcal{F}_{E^s}$ of the symmetric subproduct system E^s , with $E_1^s \cong \mathbb{C}^d$, preserving the degree of tensor products. This motivates the subproduct system picture of Drury–Arveson space.

Definition 5.10 (Drury–Arveson space, subproduct systems point of view). *Let E^s be the symmetric subproduct system, with $E_1^s = \mathbb{C}^d$. The Drury–Arveson space H_d^2 is the Fock space $\mathcal{F}_{E^s}$ of the symmetric subproduct system E^s .*

In later chapters, we use the subproduct system picture to study the Arveson–Hilbert modules of Drury–Arveson spaces and quotient modules. Using this picture, we will construct novel Toeplitz algebras of such subproduct systems with different K -theoretical phenomena.

5.2 The Arveson–Douglas conjecture

The history of the Arveson–Douglas conjecture can be traced back to the paper [5], followed by a detailed version [6]. In these two papers, W. Arveson defined a geometric invariant, called the curvature of an Arveson–Hilbert module.

5.2.1 Arveson–Hilbert modules

Analytic function spaces play a central role in function theory. To study multivariate operator theory, in particular, commuting d -tuples

$$(T_1, T_2, \dots, T_d), \quad T_i \in \mathbb{B}(\mathcal{H}), \quad \forall i = 1, 2, \dots, d,$$

which act on a common Hilbert space $\mathcal{H}$, Arveson introduced a notion of function space equipped with a polynomial ring action. We shall call these objects Arveson–Hilbert modules to avoid confusion with Hilbert C^* -modules.

Definition 5.11 (Arveson–Hilbert module). *Let $\mathbb{C}[z_1, \dots, z_d]$ be the ring of polynomials with complex coefficients in d variables. An Arveson–Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$, also called a $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module, is a Hilbert space $\mathcal{H}$, equipped with a ring action given by a ring homomorphism*

$$\sigma : \mathbb{C}[z_1, \dots, z_d] \rightarrow \mathbb{B}(\mathcal{H}).$$

We will use the notation $(\mathcal{H}, \sigma)$, or simply $\mathcal{H}$ when the module action is clear.

Let $(\mathcal{H}, \rho)$ be a $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module, it is straightforward to verify that $(\rho(z_1), \dots, \rho(z_d))$ is a commuting d -tuple acting on $\mathcal{H}$. Conversely, given a commuting d -tuple $\bar{T} = (T_1, \dots, T_d)$ on $\mathcal{H}$, the assignment $z_i \mapsto T_i$ extends uniquely to a ring homomorphism $\rho : \mathbb{C}[z_1, \dots, z_d] \rightarrow \mathbb{B}(\mathcal{H})$.

As previously mentioned, this notion differs from that of a Hilbert C^* -module, since the coefficient ring $\mathbb{C}[z_1, \dots, z_d]$ is only viewed as a ring of polynomials instead of a C^* -algebra. Moreover, a $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module is a left $\mathbb{C}[z_1, \dots, z_d]$ -module, while for Hilbert C^* -modules it is customary to consider right actions.

Definition 5.12 (Submodule and quotient module). *Let $\mathcal{H}$ be an Arveson–Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$, and let $\mathcal{P}$ be a closed subspace of $\mathcal{H}$. We say that $\mathcal{P}$ is a submodule of $\mathcal{H}$ if*

$$\sigma(q)(\mathcal{P}) \subset \mathcal{P}, \quad \forall q \in \mathbb{C}[z_1, \dots, z_d]$$

where σ is the ring action inherited from the $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module $\mathcal{H}$.

Given a submodule $\mathcal{P}$, we say $\mathcal{Q}$ is a quotient module if, as a Hilbert space, $\mathcal{Q} = \mathcal{H}/\mathcal{P}$, and the left $\mathbb{C}[z_1, \dots, z_d]$ -action is given by

$$\sigma_{\mathcal{Q}}(q) = P_{\mathcal{Q}} \circ \sigma(q)|_{\mathcal{Q}}, \quad \forall q \in \mathbb{C}[z_1, \dots, z_d],$$

where $P_{\mathcal{Q}}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{Q}$, and $\sigma(q)|_{\mathcal{Q}}$ is the restriction of the $\mathbb{C}[z_1, \dots, z_d]$ -action on $\mathcal{H}$ to $\mathcal{Q}$.

Definition 5.13 (Graded Hilbert space and graded Arveson–Hilbert module). *A graded Hilbert space is a pair $(\mathcal{H}, \Gamma)$, consisting of a Hilbert space and a strongly continuous unitary representation Γ of the torus $\mathbb{T}$ on $\mathcal{H}$. For each $n \in \mathbb{Z}$, we can define the associated spectral space*

$$H_n = \{\xi \in \mathcal{H} : \Gamma(\lambda)\xi = \lambda^n \xi\},$$

and $\mathcal{H}$ is the direct sum of H_n .

A graded Arveson–Hilbert module is a graded Hilbert space $\mathcal{H}$, together with a contractive d -tuple $(T_1, \dots, T_d)$ that comprises the $\mathbb{C}[z_1, \dots, z_d]$ -module structure such that

$$\Gamma(\lambda)T_k\Gamma(\lambda)^{-1} = \lambda T_k, \quad \forall \lambda \in \mathbb{T}. \quad (5.1)$$

Remark 5.14. *Given a graded Arveson–Hilbert module $\mathcal{H}$, the associated d -tuple $(T_1, \dots, T_d)$ shifts the degree of $\mathcal{H}$, i.e., $T_k(H_n) \subset H_{n+1}$ for all $k = 1, 2, \dots, d$. Indeed, let $\xi \in H_n$. From equation (5.1), we have*

$$\Gamma(\lambda)T_k(\xi) = \lambda T_k\Gamma(\lambda)\xi = \lambda^{n+1}T_k(\xi).$$

Definition 5.15. Let $\mathcal{H}$ be a $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module such that the $\mathbb{C}[z_1, \dots, z_d]$ action is defined by a d -tuple of commuting operators $(T_1, \dots, T_d)$; that is,

$$\sigma(q)(\xi) = q(T_1, \dots, T_d)(\xi), \quad \forall q \in \mathbb{C}[z_1, \dots, z_d].$$

We call $\mathcal{H}$ a *contractive Arveson–Hilbert module* if

$$\left\| \sum_{i=1}^d T_i(\xi_i) \right\|^2 \leq \sum_{i=1}^d \|\xi_i\|^2, \quad \forall \xi_i \in \mathcal{H}.$$

Let $(T_1, \dots, T_d)$ be a commuting d -tuple acting on $\mathcal{H}$. Using $(T_1, \dots, T_d)$, we define a map $\phi : \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{H})$ by

$$\phi(A) = \sum_{i=1}^d T_i A T_i^*, \quad \forall A \in \mathbb{B}(\mathcal{H}).$$

We call $\mathcal{H}$ a **pure** Arveson–Hilbert module, if $\lim_{n \rightarrow \infty} \phi^n(1) = 0$, where 1 denotes the identity operator on $\mathcal{H}$.

Example 5.16. Let H_d^2 be the Drury–Arveson space. One can equip H_d^2 with a $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module structure via the ring homomorphism $\rho : \mathbb{C}[z_1, \dots, z_d] \rightarrow \mathbb{B}(H_d^2)$ defined as

$$\rho(p)f = p \cdot f, \quad \forall p \in \mathbb{C}[z_1, \dots, z_d], \forall f \in H_d^2.$$

The operator $\rho(p)$ is called the *shift operator* (or the *creation operator*) associated with the polynomial p and is denoted by M_p for simplicity.

We refer to the pair (H_d^2, ρ) (or simply H_d^2 when the module action is clear) as the Drury–Arveson module. The d -tuple $(M_1, \dots, M_d) := (\rho(z_1), \dots, \rho(z_d))$ is called the *standard d -tuple of multiplication operators* on H_d^2 . $\square$

The curvature of an Arveson–Hilbert module

The associated defect operator Δ is defined as $\Delta = (1 - \sum_{i=1}^d T_i T_i^*)^{\frac{1}{2}}$. The dimension of the range of Δ is called the *rank* of $\mathcal{H}$, denoted by $\text{rank}(\mathcal{H})$. The curvature of an Arveson–Hilbert module $\mathcal{H}$, denoted by $K(\mathcal{H})$, defined by Arveson is an extremal invariant subject to $\text{rank}(\mathcal{H})$, such that if $\mathcal{H}$ is a free module,

in the algebraic sense, $K(\mathcal{H}) = \text{rank}(\mathcal{H})$. Therefore, the curvature $K(\mathcal{H})$ can be viewed as an invariant that measures freeness, or flatness in the sense of geometry of an Arveson–Hilbert module.

Remark 5.17. Let $\mathcal{H}$ be a rank n Arveson–Hilbert module over $\mathbb{C}[z_1, \dots, z_d]$. The purity of a d -tuple guarantees that the n -fold direct sum of H_d^2 is a rank n minimal dilation of $\mathcal{H}$, where H_d^2 is the Drury–Arveson space. The details of this result can be found in [6, Theorem 1.12].

To define the curvature, we need more preparations. Let $z = (z_1, \dots, z_d)$ be a point in $\mathbb{C}^d$, using the d -tuple $(T_1, \dots, T_d)$, we define

$$T(z) = \sum_{i=1}^d z_i \cdot T_i \in \mathbb{B}(\mathcal{H}).$$

Suppose $(T_1, \dots, T_d)$ is contractive, we have $\|T(z)\| \leq \|z\|$. In particular, for $z \in \mathbb{B}_d$, we have $\|T(z)\| < 1$. Therefore $(1 - T(z))$ is invertible. We now define a positive operator acting on the range of Δ for each $z \in \mathbb{B}_d$ as follows:

$$F(z) = \Delta(1 - T(z)^*)^{-1}(1 - T(z))^{-1}\Delta.$$

Definition 5.18 (Curvature, [5, Theorem A]). Let $\mathcal{H}$ be an Arveson–Hilbert module such that the $\mathbb{C}[z_1, \dots, z_d]$ action is given by a contractive d -tuple $(T_1, \dots, T_d)$. The curvature of $\mathcal{H}$ is the following non-negative quantity:

$$K(\mathcal{H}) = \int_{\partial\mathbb{B}_d} \lim_{r \rightarrow 1} (1 - r^2) \cdot \text{Tr}(F(r \cdot \xi)) d\xi \in [0, \text{rank}(\mathcal{H})].$$

From the definition, one can immediately observe that the curvature is invariant under unitary transformations since an operator’s trace is invariant under a unitary operator’s conjugation.

On the contrary, for a pure Arveson–Hilbert module, one can directly determine freeness from the curvature.

Theorem 5.19 ([5, First Extremal Theorem]). Let $\mathcal{H}$ be a pure Arveson–Hilbert module of finite rank r . Then $\mathcal{H}$ is unitarily equivalent to a free Arveson–Hilbert module if and only if $K(\mathcal{H}) = r$. In that case, $\mathcal{H} \cong (H_d^2)^{\oplus r}$, where H_d^2 denotes the Drury–Arveson space.

In general, as one can see from the definition, the curvature is hard to compute, and the integrality is also not ensured. However, for a graded pure finite rank Arveson–Hilbert module $\mathcal{H}$, Arveson proposed a beautiful approach to compute the curvature by introducing the Euler characteristic for Arveson–Hilbert modules.

Let $\mathcal{H}$ be a finite rank $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module. Consider the finitely generated $\mathbb{C}[z_1, \dots, z_d]$ -module

$$M_H = \mathbb{C}[z_1, \dots, z_d] \cdot \Delta(\mathcal{H}).$$

By Hilbert’s syzygy, there exists a finite free resolution of finitely generated free $\mathbb{C}[z_1, \dots, z_d]$ -modules

$$0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow M_H \rightarrow 0,$$

where $F_i \cong \mathbb{C}[z_1, \dots, z_d]^{\oplus \beta_i}$ for some non-negative integer β_i . It is well known that the alternating sum of multiplicities β_i is independent of the choice of resolution, which results in the following concept.

Definition 5.20 (Euler characteristic). *Let $\mathcal{H}$ be a finite rank $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module. The Euler characteristic of $\mathcal{H}$ is the following non-negative integer*

$$\chi(H) = \sum_{i=1}^d (-1)^{i+1} \beta_i.$$

The following theorem relates the analytic and algebraic invariants for certain $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert modules.

Theorem 5.21 ([5, Theorem B]). *For every pure finite rank graded $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module $\mathcal{H}$, we have*

$$K(\mathcal{H}) = \chi(\mathcal{H}).$$

In order to compute the curvature for more general classes of Arveson–Hilbert modules, Arveson initiated a new approach in [7] three years after [5].

Inspired by the role that the classical Dirac operator plays in the curvature, Arveson constructed a Dirac operator for any $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module in

[7]. We shall describe the construction of the Dirac operator of a d -tuple below, and the details can be found in [7].

We denote $\bar{T}$ as the d -tuple of commuting operators $(T_1, T_2, \dots, T_d)$ acting on a common Hilbert space $\mathcal{H}$. The construction imitates the Dirac operator in differential geometry. Suppose $\mathcal{H}$ is a Hilbert space of dimension d , let $\Lambda\mathcal{H}$ be the exterior algebra of $\mathcal{H}$, namely,

$$\Lambda\mathcal{H} = \sum_{i=0}^d \Lambda^i \mathcal{H},$$

where $\Lambda^i \mathcal{H}$ is the i -th exterior power of $\mathcal{H}$ and is spanned by the following form

$$z_1 \wedge z_2 \wedge \cdots \wedge z_d, \quad \forall z_i \in \mathcal{H}.$$

The inner product on $\Lambda^i \mathcal{H}$ is defined as follows:

$$\langle z_1 \wedge z_2 \wedge \cdots \wedge z_d, w_1 \wedge w_2 \wedge \cdots \wedge w_d \rangle = \det(\langle z_i, w_j \rangle_{\mathcal{H}}), \quad \forall z_i, w_j \in \mathcal{H}.$$

Let $e_1, \dots, e_d$ be an orthonormal basis of $\mathcal{H}$. For each e_i , we define the associated creation operator

$$C_i : z_1 \wedge z_2 \wedge \cdots \wedge z_d \mapsto e_i \wedge z_1 \wedge z_2 \wedge \cdots \wedge z_d,$$

which maps $\Lambda^i \mathcal{H}$ onto $\Lambda^{i+1} \mathcal{H}$ for $i < d$, and maps $\Lambda^d \mathcal{H}$ to zero.

Definition 5.22. Let $\bar{T}$ be a d -tuple acting on $\mathcal{H}$. The Dirac operator of $\bar{T}$ is a self-adjoint operator $D_{\bar{T}}$ acting on $\mathcal{H} \otimes \Lambda\mathbb{C}^d$, as follows:

$$D_{\bar{T}} = B + B^*, \quad B = \sum_{i=1}^d T_i \otimes C_i.$$

We say $\bar{T}$ is Fredholm if $D_{\bar{T}}$ is a Fredholm operator.

In [8], a criterion for determining whether a d -tuple $\bar{T}$ is Fredholm or not is given, where the essential normality property first appeared and showed importance.

Proposition 5.23 ([8, Proposition 1.1]). *Let $\bar{T}$ be a d -tuple on some Arveson–Hilbert module $\mathcal{H}$. If $\bar{T}$ satisfies*

- (1) $\bar{T}$ is essentially normal, that is, for all i, j , $[T_i, T_j^*] \in \mathcal{K}(\mathcal{H})$.
- (2) $\sum_{i=1}^d T_i T_i^*$ is Fredholm.

Then $\bar{T}$ is Fredholm.

Motivated by Proposition 5.23, we introduce the property of essential normality and p -essential normality as follows.

Definition 5.24 (Essential normality and p -essential normality). *Let $(\mathcal{H}, \rho)$ be a $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module. We say $(\mathcal{H}, \rho)$ is essentially normal if $[\rho(z_i), \rho(z_j)^*] \in \mathcal{K}(\mathcal{H})$ for all $i, j = 1, \dots, d$.*

Let $p \in \mathbb{N}$, we say $(\mathcal{H}, \rho)$ is p -essentially normal if $[\rho(z_i), \rho(z_j)^] \in \mathcal{S}^p(\mathcal{H})$ for all $i, j = 1, \dots, d$, where $\mathcal{S}^p(\mathcal{H})$ denotes the set of Schatten p -class operators.*

Remark 5.25. *Let (H_d^2, ρ) be a standard Drury–Arveson module as in Example 5.16. By [4, Proposition 5.3], the standard d -tuple of multiplication operators of H_d^2 is p -essentially normal for $p > d$. Note that it is not essentially normal for $p = d$.*

Suppose $\bar{T}$ is Fredholm, i.e., the Dirac operator $D_{\bar{T}}$ is Fredholm. There is an integer invariant associated with $\bar{T}$, called the index, which is stable under homotopy and compact perturbation. Note that $\mathcal{H} \otimes \mathbb{C}^d$ is naturally endowed with a $\mathbb{Z}_2$ grading $\mathcal{H} \otimes \mathbb{C}^d = H_+ \oplus H_-$ as follows:

$$H_+ = \mathcal{H} \otimes \left(\sum_{i \text{ is even}} \Lambda^i \mathbb{C}^d \right), \quad H_- = \mathcal{H} \otimes \left(\sum_{i \text{ is odd}} \Lambda^i \mathbb{C}^d \right).$$

Then $D_{\bar{T}}$ can be written as a 2×2 matrix with respect to the $\mathbb{Z}_2$ grading:

$$D_{\bar{T}} = \begin{bmatrix} 0 & D_+^* \\ D_+ & 0 \end{bmatrix},$$

where D_+ denotes the restriction of $D_{\bar{T}}$ to H_+ . Since $D_{\bar{T}}$ is Fredholm, we have that the kernel of D_+ and the cokernel of D_+ are finite-dimensional and the index of D_+ , namely,

$$\text{Ind}(D_+) = \dim(\ker(D_+)) - \dim(\ker(D_+^*))$$

is an integer that is invariant under homotopy and compact perturbations.

The relation between curvature and index of a $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert module whose action is given by a d -tuple $\bar{T}$, is given by the following intrinsic theorem.

Theorem 5.26 ([7, Theorem B, Corollary]). *Let $\bar{T}$ be a pure contractive d -tuple of finite rank acting on a graded Hilbert space $\mathcal{H} = \bigoplus_k \mathcal{H}_k$. Assume $\bar{T}$ is graded, i.e., $T_i(\mathcal{H}_k) \subset \mathcal{H}_{k+1}$, $\forall i = 1, 2, \dots, d$. Then, both the kernel and the cokernel of D_+ are finite-dimensional and*

$$(-1)^d \cdot K(\mathcal{H}) = \dim(\ker(D_+)) - \dim(\ker(D_+^*)).$$

In particular, if $\bar{T}$ is Fredholm, the curvature of $\mathcal{H}$ is stable in the following sense: if $\bar{T}' = (T'_1, T'_2, \dots, T'_d)$ is another d -tuple of $\mathcal{H}'$ such that $\mathcal{H}$ is unitarily equivalent to $\mathcal{H}'$ by the unitary U and $T_i U - U T'_i$ is compact for each $i = 1, \dots, d$, then

$$K(\mathcal{H}) = K(\mathcal{H}').$$

Theorem 5.26 gives a beautiful and promising approach to finding an analogue of Atiyah–Singer index theory for $\mathbb{C}[z_1, \dots, z_d]$ -Hilbert modules of analytic function spaces, which could possibly connect functional analysis and algebraic geometry. However, to make the curvature a well-posed invariant, Fredholmness is necessary.

Recall Proposition 5.23, the sufficient condition for a d -tuple to be Fredholm, in the case of finite rank Arveson–Hilbert modules, the defect operator $1 - \sum_{i=1}^d T_i T_i^*$ is of finite rank; hence $\sum_{i=1}^d T_i T_i^*$ is Fredholm. Therefore, it is a key problem to determine whether the commutators $[T_i, T_j^*]$ are compact or not. This motivated Arveson to study the essential normality problem and to propose the conjecture in [8], and he ultimately proved the monomial case.

Conjecture 5.27 (The Arveson’s Conjecture). *Suppose $\mathcal{P}$ is a graded submodule of the vector-valued Drury–Arveson module $H_d^2 \otimes \mathbb{C}^r$. Then $\mathcal{P}$ is p -essentially normal, i.e., $[T_i, T_j^*] \in \mathcal{S}^p(\mathcal{P})$ for all $p > d$.*

Recall Arveson’s Toeplitz extension in Example 4.15. In [4], Arveson proved that the Drury–Arveson module H_d^2 is p -essentially normal for $p > d$, i.e., Arve-

son’s conjecture holds for H_d^2 . This implies that the algebra of compact operators is a subalgebra of the Toeplitz algebra, and therefore the Toeplitz algebra is the extension of $\mathcal{K}$ by a commutative C^* -algebra $C(X)$. In this case, $X = \partial\mathbb{B}_d$. Indeed, this applies to more general submodules in the following way.

Suppose $\mathcal{H}$ is a quotient module of H_d^2 whose d -tuple $(S_1, \dots, S_d)$ is given by the compression of the d -shift $(T_1, \dots, T_d)$ described in Example 4.15. If $\mathcal{H}$ is essentially normal, we have the following short exact sequence:

$$0 \rightarrow \mathcal{K} \rightarrow C^*(S_1, \dots, S_d) + \mathcal{K} \rightarrow C(X) \rightarrow 0,$$

for some nontrivial algebraic set X in the projective space. Moreover, this gives a K -homology class element in $K^1(X)$, denoted by $[\mathcal{H}]$.

Based on this observation and inspired by Arveson’s conjecture, one year later, R. Douglas proposed a family of conjectures on essential normality for more general Bergman spaces in [21], where he generalized Arveson’s conjecture to the Bergman spaces of pseudoconvex domains. Moreover, he showed how a positive answer would enable one to define the K -homology elements associated with the corresponding intersection of a pseudoconvex domain and the varieties that define the quotient Bergman module.

The following conjecture is the most well-known form of the Arveson–Douglas conjecture.

Conjecture 5.28 (The Arveson–Douglas Conjecture). *Let I be a homogeneous ideal in $\mathbb{C}[z_1, \dots, z_d]$ and let $[I]$ denote its closure in H_d^2 . Then the graded quotient module $\mathcal{Q}_I := [I]^\perp$ is p -essentially normal for all $p > \dim_{\mathbb{C}}(V(I))$, where*

$$V(I) = \{z \in \mathbb{C}^d : q(z) = 0, \forall q \in I\}.$$

Note that one can also consider the conjecture in the setting of the Bergman space $L_a^2(\mathbb{B}_n)$ or the Hardy space $H^2(\mathbb{B}_d)$.

Corollary 5.29. *Let I be a homogeneous ideal in $\mathbb{C}[z_1, \dots, z_d]$ and $[I]$ denote its closure in H_d^2 . If the graded quotient module $\mathcal{Q}_I := [I]^\perp$ is p -essentially normal for all $p > \dim_{\mathbb{C}}(V(I))$, one has the following exact sequence:*

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T}_I \rightarrow C(V(I) \cap \partial\mathbb{B}_d) \rightarrow 0, \quad (5.2)$$

which gives an odd K -homology element in $K^1(C(V(I) \cap \partial\mathbb{B}_d))$.

Based on Corollary 5.29, Douglas proposed a conjecture of K -homology in his paper *A New Kind of Index Theorem* [21].

Conjecture 5.30 (Douglas’ K -homology Conjecture). *Let I be a homogeneous ideal in $\mathbb{C}[z_1, \dots, z_d]$ and $[I]$ denote its closure in H_d^2 . If the graded quotient module $\mathcal{Q}_I := [I]^\perp$ is p -essentially normal for all $p > \dim_{\mathbb{C}}(V(I))$, then the K -homology element represented by the extension (5.2) coincides with the fundamental class of $V(I) \cap \partial\mathbb{B}_d$.*

We conclude this section by listing some positive results for the Arveson–Douglas conjecture.

Theorem 5.31. *Let I be a homogeneous ideal in $\mathbb{C}[z_1, \dots, z_d]$ and $[I]$ denote its closure in H_d^2 . The graded quotient module $\mathcal{Q}_I := [I]^\perp$ is p -essentially normal for all $p > \dim_{\mathbb{C}}(V(I))$ if one of the following conditions is satisfied:*

- (1) I is generated by monomials; see [8].
- (2) I is generated by a homogeneous polynomial or $d \leq 3$; see [29].
- (3) $V(I)$ decomposes into linear subspaces, and $V(I)$ decomposes into components with mutually disjoint linear spans; see [37].

In Section 9.3, we will discuss possible extensions of the p -essential normality results to a new class of varieties not covered by the conditions listed above.