Subproduct systems and C*-algebras
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Chapter 3

Kasparov's bivariant K-theory

In his seminal work [36], Kasparov defines a bivariant theory assigning an abelian group $KK(A,B)$ to any pair of C^*-algebras A and B. This theory simultaneously generalizes both K-theory and K-homology. Together with the Kasparov product, a bilinear pairing between $KK(A,B)$ and $KK(B,C)$ that generalizes the index pairing of K-theory and K-homology, Kasparov's bivariant K-theory has become a fundamental tool in of noncommutative geometry.

In the following sections, we introduce Kasparov's bivariant K-theory. We begin by recalling the basic theory of Hilbert C^*-modules, which serve as models for Kasparov A-B-modules. In the next section, we introduce Kasparov's original approach to bivariant K-theory, followed by Cuntz's approach, which will be used in this thesis. As an application of KK-theory and the Kasparov product, we discuss extensions of C^*-algebras and K-homology.

3.1 Hilbert C^*-modules

Hilbert C^*-modules have become a fundamental language of noncommutative geometry today. Roughly speaking, a Hilbert C^*-module is a generalization of Hilbert space by allowing the inner product to take values in a C^*-algebra. The idea of modules over C^*-algebras as a generalization of Hilbert spaces began to appear in the 1950s. In 1953, I. Kaplansky published the first paper [35] to study modules over AW^*-algebras in order to solve problems in AW^*-algebras. A few years later, in 1973, W. Paschke published [50], the first paper formally studying the inner product modules over a C^*-algebra B, which are now called pre-Hilbert C^*-modules. One year later, in order to study the induced representations of C^*-algebras, M. Rieffel published [54], where he studied the completion of a pre-Hilbert C^*-module, and this was the first time Hilbert C^*-modules were defined.

In this section, we will briefly introduce the theory of Hilbert C^*-modules, serving as the language for the Kasparov bivariant K-theory. We will denote an arbitrary C^*-algebra with the letter B, and the main references for this section are [39] and [33].

Definition 3.1 (Pre-Hilbert C^*-module)**.** *A pre-Hilbert B-module is a complex vector space E that is also a right B-module equipped with a B-valued inner product $\langle \cdot, \cdot \rangle : E \otimes E \to B$, which is B-linear and $\mathbb{C}$-linear in the second variable such that for all $x, y \in E, b \in B$, and $\lambda \in \mathbb{C}$, satisfying*

(1) $(\lambda e)b = \lambda(eb) = e(\lambda b)$,

(2) $\langle x, yb \rangle = \langle x, y \rangle b$,

(3) $\langle x, y \rangle^* = \langle y, x \rangle$,

(4) $\langle x, x \rangle \geq 0$,

(5) $x \neq 0 \implies \langle x, x \rangle \neq 0$.

For a pre-Hilbert C^*-module E, one can define a norm using the norm of B, as follows,

$$\|e\|_E^2 = \|\langle e, e \rangle\|_B, \quad \forall e \in E. \tag{3.1}$$

Definition 3.2 (Hilbert C^*-module)**.** *A Hilbert B-module is a pre-Hilbert B-module E that is complete in the norm* (3.1)*. We say a closed subspace $F \subset E$ is a Hilbert submodule if F itself is a Hilbert B-module with respect to the B-valued inner product inherited from E.*

Hilbert C^*-modules naturally appear in operator algebra and geometry. We now present two examples below to revisit classical mathematical objects in the language of Hilbert C^*-modules.

Example 3.3. *The C^*-algebra B itself is a Hilbert B-module when equipped with the B-valued inner product $\langle b_1, b_2 \rangle = b_1^* b_2$, for $b_1, b_2 \in B$ and the right B-module action given by multiplication. Moreover, if $J \lhd B$ is a closed ideal, then J is also a Hilbert B-module with respect to the same B-valued inner product and right action.* □

We now turn to constructions with Hilbert B-modules as preparation for Kasparov A-B-modules.

Definition 3.4 (Direct sum of Hilbert B-modules)**.** *Let B be a C^*-algebra and $E_1, E_2, \dots, E_n$ be Hilbert B-modules. Then $E := \bigoplus_{k=1}^n E_k$ is a Hilbert B-module, whose B-valued inner product is defined as*

$$\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle_E = \sum_{k=1}^{n} \langle x_k, y_k \rangle_{E_k}.$$

It is readily seen that E is complete under this B-valued inner product.

Example 3.5. *Define E_B as the space of sequences $(b_1, b_2, \dots)$ that are eventually zero. Then E_B is a pre-Hilbert B-module, where the B-valued inner product is defined as*

$$\langle (a_i), (b_i) \rangle_{E_B} = \sum_{i=1}^{\infty} a_i^* b_i.$$

The completion of E_B with respect to this inner product is a Hilbert B-module, which we denote by H_B. □

Example 3.6. *Let X be a compact Hausdorff space and $E \to X$ be a complex vector bundle over X, equipped with a Hermitian metric $\langle \cdot, \cdot \rangle_E$. Denote by $\Gamma(E)$ the space of continuous sections of E. The space $\Gamma(E)$ is a Hilbert $C(X)$-module with the following structure:*

(1) The right $C(X)$-module structure of $\Gamma(E)$ is defined as follows. For $s \in \Gamma(E)$ and $f \in C(X)$, we define

$$(s \cdot f)(x) = s(x) f(x), \quad \forall x \in X.$$

(2) The $C(X)$-valued inner product is defined as follows:

$$\langle s_1, s_2\rangle(x) = \langle s_1(x), s_2(x)\rangle_E, \quad \forall x \in X.$$

The completion of $\Gamma(E)$ under the norm

$$\|s\| = \|\langle s, s\rangle\|^{\frac{1}{2}} = \sup_{x\in X} \sqrt{\langle s(x), s(x)\rangle_E},$$

is a Hilbert $C(X)$-module. $\square$

Unlike the case of Hilbert spaces, where every bounded linear map admits an adjoint, bounded linear maps between Hilbert C^*-modules do not necessarily have adjoints. This leads to the following definition.

Definition 3.7 (Adjointable operators)**.** *Let B be a C^*-algebra and E_1, E_2 be Hilbert B-modules. A map $T : E_1 \to E_2$ is adjointable if there exists a map $T^* : E_2 \to E_1$ such that $\langle Tx, y\rangle_{E_2} = \langle x, T^*y\rangle_{E_2}$ for all $x \in E_1, y \in E_2$. The set of adjointable maps is denoted by $\mathcal{L}_B(E_1, E_2)$.*

It is straightforward to prove that an adjointable map T between Hilbert B-modules is B-linear, and $\mathcal{L}_B(E_1, E_2)$ is a subspace of bounded linear maps from E_1 to E_2, equipped with the operator norm $\|T\| = \sup_{x\in E_1}\{\|Tx\| : \|x\| \leq 1\}$. As a consequence, we have:

Corollary 3.8. *Let B be a C^*-algebra and E be a Hilbert B-module. Then, $\mathcal{L}_B(E, E)$ is a C^*-algebra.*

Let E_1 and E_2 be Hilbert B-modules. We define the following class of operators in $\mathcal{L}_B(E_1, E_2)$, denoted by $\Theta_{x,y}$ for $x \in E_2, y \in E_1$:

$$\Theta_{x,y}(z) := x\langle y, z\rangle_{E_1}, \quad \forall z \in E_1.$$

These operators generalize rank-one operators in the setting of Hilbert spaces. Hence, we still call operators of the form $\Theta_{x,y}$ rank-one operators.

Let $\mathcal{K}_B(E_1, E_2)$ denote the closed linear span of operators of the form $\Theta_{x,y}$ for $x \in E_2, y \in E_1$, it follows directly that $\mathcal{K}_B(E, E)$ is a closed ideal in $\mathcal{L}_B(E, E)$. For simplicity, we denote $\mathcal{L}_B(E, E)$ by $\mathcal{L}_B(E)$ and $\mathcal{K}_B(E, E)$ by $\mathcal{K}_B(E)$.

Remark 3.9. *It is worth noting that the C^*-algebra $\mathcal{K}_B(E)$ generalizes the C^*-algebra of compact operators on a Hilbert space. Indeed, if we replace B with $\mathbb{C}$, then a Hilbert $\mathbb{C}$-module E reduces to a Hilbert space, and $\mathcal{K}_{\mathbb{C}}(E)$ coincides with the space of usual compact operators.*

Given a Hilbert B-module E, besides the norm topology on $\mathcal{L}_B(E)$, the strict topology is another important topology that appears in Cuntz's approach to the bivariant K-theory.

Definition 3.10. *Let B be a C^*-algebra and E be a Hilbert B-module. The seminorms $\|\cdot\|_x, x \in E$ given by $\|T\|_x = \|Tx\| + \|T^*x\|, T \in \mathcal{L}_B(E)$ define a locally convex topology on $\mathcal{L}_B(E)$, which is called the strict topology.*

Definition 3.11 (Pushout of a Hilbert B-module)**.** *Let B and A be C^*-algebras, and let E be a Hilbert B-module. Suppose $f : B \to A$ is a surjective $*$-homomorphism. Define the following submodule of E:*

$$N_f := \{x \in E : f(\langle x, x\rangle_E) = 0\} \subset E.$$

Let $\pi : E \to E/N_f$ be the quotient map, and let $\pi(E)$ be the algebraic quotient. Then $\pi(E)$ is a right A-module with the action

$$\pi(x)f(b) := \pi(xb), \quad \forall x \in E, b \in B.$$

This action is well-defined since f and π are surjective and N_f is constructed such that the action on the quotient is consistent. The following A-valued sesquilinear form defined as

$$\langle \pi(x), \pi(y)\rangle_{E_f} := f(\langle x, y\rangle_E) \in A,$$

is an A-valued inner product on $\pi(E)$ making $\pi(E)$ a pre-Hilbert A-module. The completion E_f, sometimes denoted by $f_*(E)$ in the literature, is then a Hilbert A-module, called the pushout of f.

Definition 3.12 (Internal Tensor Product of Hilbert B-modules)**.** *Let B and A be C^*-algebras. Let E be a Hilbert B-module, and let F be a Hilbert A-module. Suppose* *there is a $*$-homomorphism $\phi : B \to \mathcal{L}_A(F)$. Then ϕ gives rise to a left B-module structure on F, namely,*

$$b.x := \phi(b)x, \quad \forall b \in B, x \in F.$$

Consider the algebraic tensor product $E \odot F$ and a subspace N_f defined as:

$$N_f = span\{xb \odot y - x \odot \phi(b)y : \forall x \in E, b \in B, y \in F\}.$$

Let π denote the quotient map $\pi : E \odot F \to E \odot F/N_f$. Then $\pi(E \odot F)$ is a right A-module with the action

$$\pi((x \odot y)a) := \pi(x \odot ya), \quad \forall x \in E, y \in F, a \in A,$$

which is well-defined because of the way N_f is defined. Then consider the following A-valued map:

$$\langle \pi(x_1 \odot y_1), \pi(x_2 \odot y_2) \rangle_{E \otimes_B F} := \langle y_1, \phi(\langle x_1, x_2 \rangle_E) y_2 \rangle_F \in A, \quad \forall x_i \in E, y_i \in F.$$

It is routine to check that this map vanishes on N_f. Therefore, it induces a well-defined A-valued inner product on the quotient space. The internal tensor product of E and F is the completion of the quotient algebraic tensor product $E \odot F/N_f$ with respect to the A-valued inner product defined above, and it is denoted by $E \otimes_B F$.

3.2 Kasparov's K-theory

In this section, we assume all C^*-algebras to be σ-unital.

Definition 3.13 (Graded C^*-algebras)**.** *Let A be a C^*-algebra. We say A is graded if there exists a $*$-automorphism $\beta_A : A \to A$ such that $\beta_A^2 = id_A$. We say β_A is the grading operator for A.*

For a graded C^*-algebra A, it is straightforward to verify that the idempotent $*$-automorphism β_A decomposes A into two eigenspaces A_0 and A_1 such that $\beta_A|_{A_0} = id_A$ and $\beta_A|_{A_1} = -id_A$. We call elements in $A_0 \cup A_1$ homogeneous elements. If $a \in A_0$, we say a is even, and if $a \in A_1$, we say a is odd.

A graded $*$-homomorphism $f : A \to B$ between graded C^*-algebras A and B is a $*$-homomorphism that preserves the eigenspace decomposition, i.e., $f(A_i) \subset B_i$. If A is a graded C^*-algebra, we can define the graded commutator $[\cdot,\cdot]_g$ on A as follows:

$$[a,b]_g := ab - (-1)^{ij} ba, \quad \forall a \in A_i, b \in A_j.$$

Definition 3.14 (Graded Hilbert C^*-modules). *Let B be a graded C^*-algebra with grading operator β_B. We say a Hilbert B-module E is graded if there exists a linear surjective involution $S_E : E \to E$ such that*

(1) $S_E(xb) = S_E(x)\beta_B(b), \quad \forall x \in E, b \in B,$

(2) $\langle S_E(x), S_E(y)\rangle = \beta_B(\langle x,y\rangle), \quad \forall x,y \in E.$

Given graded C^*-algebras A and B, we are now ready to define the Kasparov A-B-module as follows.

Definition 3.15 (Kasparov A-B-module). *Let A and B be graded C^*-algebras. A Kasparov A-B-module is a triple $\mathcal{E} = (E,\phi,F)$, where E is a graded Hilbert B-module, $\phi : A \to \mathcal{L}_B(E)$ is a graded $*$-homomorphism, and $F \in \mathcal{L}_B(E)$ is an odd element, such that*

$$[F,\phi(a)]_g, \quad (F^2-1)\phi(a), \quad (F^*-F)\phi(a) \in \mathcal{K}_B(E), \quad \forall a \in A.$$

We call a Kasparov A-B-module degenerate if

$$[F,\phi(a)]_g = (F^2-1)\phi(a) = (F^*-F)\phi(a) = 0, \quad \forall a \in A.$$

The collection of Kasparov A-B-modules is denoted by $\mathbb{E}(A,B)$ and the collection of degenerate Kasparov A-B-modules is denoted by $\mathbb{D}(A,B)$.

Example 3.16. *Let A and B be graded C^*-algebras, and let $f : A \to B$ be a graded $*$-homomorphism. From Example 3.3, we know that B is a Hilbert B-module. It is straightforward to check that $(B,f,0)$ is a Kasparov A-B-module.* $\square$

The constructions of Hilbert C^*-modules introduced in Section 3.1 can be applied to Kasparov A-B-modules in a clear way.

Definition 3.17 (Isomorphism of Kasparov A-B-modules)**.** *Let A and B be graded C^*-algebras. Let $\mathcal{E}_1 = (E_1, \phi_1, F_1)$ and $\mathcal{E}_2 = (E_2, \phi_2, F_2)$ be two Kasparov A-B-modules. We say $\mathcal{E}_1$ is isomorphic to $\mathcal{E}_2$ if there exists an isomorphism $\psi : E_1 \to E_2$ of graded Hilbert B-modules such that,*

$$\psi \circ S_{E_1} = S_{E_2} \circ \psi, \quad \psi \circ \phi_1(a) = \phi_2(a) \circ \psi, \quad \psi \circ F_1 = F_2 \circ \psi.$$

We write $\mathcal{E}_1 \cong \mathcal{E}_2$ if $\mathcal{E}_1$ is isomorphic to $\mathcal{E}_2$.

We now introduce homotopy equivalence between Kasparov modules. Let B be a graded C^*-algebra, and let $IB := C([0,1], B) \cong C([0,1]) \otimes B$ be the C^*-algebra of B-valued continuous functions on $[0,1]$. There exists a continuous family of surjective $*$-homomorphisms $\{\pi_t : t \in [0,1]\}$ defined as follows:

$$\pi_t : IB \to B, \quad C([0,1]) \otimes B \ni f \mapsto f(t) \in B.$$

Define the grading operator on IB by $\beta_B \otimes id$. Then for each $t \in [0,1]$, π_t is a graded surjective $*$-homomorphism. If $\mathcal{E}$ is a Kasparov A-IB-module, then the pushout $\mathcal{E}_{\pi_t}$ is a Kasparov A-B-module.

Definition 3.18 (Homotopy of Kasparov A-B-modules)**.** *Let A and B be graded C^*-algebras. Let $\mathcal{E}, \mathcal{E}'$ be two Kasparov A-B-modules. We say $\mathcal{E}$ is homotopic to $\mathcal{E}'$ if there exists a Kasparov A-IB-module $\mathcal{G}$ such that $\mathcal{G}_{\pi_0} \cong \mathcal{E}$ and $\mathcal{G}_{\pi_1} \cong \mathcal{E}'$. We write $\mathcal{E} \sim \mathcal{E}'$ if there is a finite set $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\} \subset \mathbb{E}(A,B)$ such that $\mathcal{E}_1 = \mathcal{E}$, $\mathcal{E}_n = \mathcal{E}'$, and $\mathcal{E}_t$ is homotopic to $\mathcal{E}_{t+1}$ for $t = 1, \ldots, n-1$.*

From Definition 3.18, it is straightforward to verify the following:

Proposition 3.19. *Let A, B, C be graded C^*-algebras, and let $\mathcal{E}, \mathcal{F}$ be Kasparov A-B-modules. We have the following properties:*

(1) The homotopy $\sim$ is an equivalence relation on $\mathbb{E}(A,B)$.

(2) Let $f : B \to C$ be a graded $*$-homomorphism. Then

$$f_*(\mathcal{E} \oplus \mathcal{F}) \sim f_*(\mathcal{E}) \oplus f_*(\mathcal{F}) \in \mathbb{E}(A, C).$$

(3) Suppose $\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1$, and $\mathcal{F}_2$ are Kasparov A-B-modules such that $\mathcal{E}_1 \sim \mathcal{F}_1$ and $\mathcal{E}_2 \sim \mathcal{F}_2$. Then

$$\mathcal{E}_1 \oplus \mathcal{E}_2 \sim \mathcal{F}_1 \oplus \mathcal{F}_2.$$

Definition 3.20 (Kasparov KK-group)**.** *Let A and B be graded C^*-algebras. The Kasparov KK-group, denoted by $KK(A,B)$, is defined to be $\mathbb{E}(A,B)/\sim$. It is a semigroup with the addition defined by the direct sum of Kasparov A-B-modules.*

It immediately follows from Proposition 3.19 that $KK(A,B)$ is an abelian semigroup, while showing $KK(A,B)$ is a group needs more work.

Lemma 3.21 ([33, Lemma 2.1.20])**.** *Let A and B be graded C^*-algebras. If $\mathcal{E} \in \mathbb{D}(A,B)$, then $\mathcal{E} \sim 0$ in $KK(A,B)$.*

Besides the homotopy of Kasparov A-B-modules, there is another equivalence relation that provides an alternative characterization of the Kasparov bivariant K-theory groups.

Definition 3.22 (Operator homotopy)**.** *Let A and B be graded C^*-algebras, and let $\mathcal{E}_1, \mathcal{E}_2$ be two Kasparov A-B-modules. We say $\mathcal{E}_1$ is operator homotopic to $\mathcal{E}_2$ if there exist a graded Hilbert B-module E, a graded $*$-homomorphism $\phi : A \to \mathcal{L}_B(E)$, and a norm continuous path $F_t \in \mathcal{L}_B(E)$, such that*

(1) $(E, \phi, F_t) \in \mathbb{E}(A,B)$ for each $t \in [0,1]$.

(2) $(E, \phi, F_0) \cong \mathcal{E}_1$, and $(E, \phi, F_1) \cong \mathcal{E}_2$.

We write $\mathcal{E}_1 \sim_{op} \mathcal{E}_2$, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are operator homotopic, and $\mathcal{E}_1 \approx \mathcal{E}_2$ if $\mathcal{E}_1 \sim_{op} \mathcal{E}_2$ up to degenerate Kasparov A-B-modules.

Lemma 3.23 ([33, Lemma 2.1.21])**.** *Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{E}(A,B)$. Then $\mathcal{E}_1 \approx \mathcal{E}_2$ implies $\mathcal{E}_1 \sim \mathcal{E}_2$.*

The Lemma 3.23 allows us to construct an inverse for $\mathcal{E} \in KK(A,B)$ via operator homotopy, which ensures the following theorem.

Theorem 3.24. *Let A and B be graded C^*-algebras. Then $KK(A,B)$ is an abelian group.*

Remark 3.25. *If A is separable, and B is σ-unital, then $\mathbb{E}(A,B)/\approx$ is isomorphic to $KK(A,B)$ (see [12, Theorem 18.5.3]).*

We now define the higher KK-groups, and show the Kasparov's bivariant K-theory provides a framework for both K-theory and K-homology.

Definition 3.26 (Higher KK-groups)**.** *Let A and B be graded C^*-algebras. The higher KK-group $KK^n(A,B)$ is defined to be $KK(A, B\otimes \mathbb{C}_n)$, where $\mathbb{C}_n$ denotes the complex Clifford algebra associated with an n-dimensional vector space.*

Like operator K-theory, the Kasparov's bivariant K-theory admits the Bott periodicity as well: $KK^{n+2}(A,B)\cong KK^n(A,B)$.

Proposition 3.27. *Let A and B be graded C^*-algebras. We have the following:*

$$K_i(A)\cong KK^i(\mathbb{C},A),\quad \forall i=0,1.$$

As introduced previously, Kasparov's bivariant K-theory also generalizes the K-homology. One approach to see this, is to define the K-homology for a C^*-algebra A by $K^i(A):=KK^i(A,\mathbb{C})$. Another approach to describe the odd Kasparov's bivariant K-theory is to use the extension groups of C^*-algebras introduced in the previous chapter.

Let A,B be separable C^*-algebras with B being stable. We now show that the extension group $Ext^{-1}(A,B)$ is isomorphic to the $KK^1(A,B)$-group by showing the isomorphisms:

$$Ext^{-1}(A,B)\cong kK^1(A,B)\cong KK^1(A,B).$$

The first isomorphism is due to Theorem 2.22. We now prove the second isomorphism. Let $[v,\lambda]\in kK^1(A,B)$, there is a map $\alpha: kK^1(A,B)\to KK^1(A,B)$ defined as follows: since $B\otimes\mathbb{C}_1\cong B\oplus B$, there exists an isomorphism $\phi: \mathcal{M}(B)\oplus\mathcal{M}(B)\to\mathcal{M}(B\otimes\mathbb{C}_1)$, which gives rise to a $*$-homomorphism

$$\phi\circ(\lambda,\lambda): A\to\mathcal{M}(B\otimes\mathbb{C}_1).$$

It is not hard to see that the triple

$$\mathcal{E}(v,\lambda)=(B\otimes\mathbb{C}_1,\phi\circ(\lambda,\lambda),\phi(2v-1,1-2v)),$$

represents a $KK^1(A,B)$-cycle. By the following theorem, this yields an isomorphism, which proves the isomorphism $Ext^{-1}(A,B) \cong KK^1(A,B)$.

Theorem 3.28 ([33, Proposition 1.3.6])**.** *Let A and B be separable C^*-algebras with B being stable. The map $\alpha : kK^1(A,B) \to KK^1(A,B)$ mapping $[v,\lambda]$ to $\mathcal{E}(v,\lambda)$ is a group isomorphism.*

Proof. The proof follows from routine checks of conditions and definitions. A full and detailed proof is available in [33, Proposition 1.3.6] for interested readers. □

We conclude this section by introducing the Kasparov product and its consequences. The heart of Kasparov's KK-theory is the Kasparov product, which provides a composition between KK-groups, generalizing the composition of $*$-homomorphisms and the index pairing between K-theory and K-homology.

Let A, B, C be separable graded C^*-algebras. The Kasparov product between $KK^i(A,B)$ and $KK^j(B,C)$ is an associative bilinear pairing $\otimes_B$:

$$\otimes_B : KK^i(A,B) \otimes_B KK^j(B,C) \to KK^{i+j}(A,C),$$

which is functorial in all variables. The existence of the Kasparov product is highly nontrivial. It was proven by G. Kasparov in [36] using a series of technical lemmas, which will not be discussed in this thesis.

To conclude this section, we briefly introduce the KK-equivalence and the universal coefficient theorem for C^*-algebras, as preparation for the later chapters.

Definition 3.29 (KK-equivalence)**.** *Let A and B be graded C^*-algebras. We say A is KK-equivalent to B if there exists $\boldsymbol{x} \in KK(A,B)$ and $\boldsymbol{y} \in KK(B,A)$ such that $\boldsymbol{x} \otimes_B \boldsymbol{y} = \boldsymbol{1}_A$ and $\boldsymbol{y} \otimes_B \boldsymbol{x} = \boldsymbol{1}_B$.*

3.2.1 The Universal Coefficient Theorem

The Universal Coefficient Theorem of Rosenberg and Schochet [55] is a useful theorem that allows us to compute KK-groups for a large class of C^*-algebras and to determine the KK-equivalence between C^*-algebras on the level of K-theory groups.

Definition 3.30 (Bootstrap Category)**.** *Let $\mathcal{N}$ be the bootstrap category, which is the smallest class of separable nuclear C^*-algebras with the following properties:*

(1) $\mathbb{C} \in \mathcal{N}$.

(2) *$\mathcal{N}$ is closed under countable inductive limit.*

(3) *If $0 \to A \to E \to B \to 0$ is a short exact sequence of C^*-algebras, and two of the C^*-algebras are contained in $\mathcal{N}$, then the third one is also contained in $\mathcal{N}$.*

(4) *$\mathcal{N}$ is closed under KK-equivalence.*

For the C^*-algebras in the bootstrap category, we have the following Universal Coefficient Theorem (see [55, Theorem 1.17]).

Theorem 3.31 ([55, Theorem 1.17])**.** *Let A and B be separable C^*-algebras with $A \in \mathcal{N}$. Then, the following sequence is exact.*

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A,B) \xrightarrow{\gamma} Hom(K_*(A), K_*(B)) \to 0,$$

where $K_*(A) = K_0(A) \oplus K_1(A)$, $KK^*(A,B) = KK^0(A,B) \oplus KK^1(A,B)$, and δ shifts the degree by one.

Corollary 3.32 ([12, Corollary 23.10.2])**.** *Let A and B be C^*-algebras in the bootstrap category $\mathcal{N}$. If $K_*(A) \cong K_*(B)$, then A is KK-equivalent to B.*

3.2.2 Cuntz's picture

In this subsection, we introduce Cuntz's approach to Kasparov's bivariant K-theory, which views Kasparov A-B-modules as generalized homomorphisms. We will use this picture to prove certain KK-equivalences later. In Cuntz's picture, we only consider σ-unital C^*-algebras.

Definition 3.33 (KK_h-cycles)**.** *Let A and B be C^*-algebras. A $KK_h(A,B)$-cycle is a pair (ϕ_+, ϕ_-) of $*$-homomorphisms $\phi_\pm \in Hom(A, \mathcal{M}(\mathcal{K} \otimes B))$ such that*

$$\phi_+(a) - \phi_-(a) \in \mathcal{K} \otimes B = \mathcal{K}_{\mathcal{K}\otimes B}(\mathcal{K} \otimes B), \quad \forall a \in A.$$

The set of $KK_h(A,B)$-cycles is denoted by $\mathbb{F}(A,B)$.

As in the definition of the $KK(A,B)$ group, the $KK_h(A,B)$ is the group of equivalence classes.

Definition 3.34 (Homotopy)**.** *Let A and B be C^*-algebras. Two $KK_h(A,B)$-cycles (ϕ_+, ϕ_-) and (ψ_+, ψ_-) are homotopic if there exists a path*

$$(\lambda_+^t, \lambda_-^t) : [0,1] \to \mathbb{F}(A,B), t \in [0,1],$$

such that

(1) The maps $t \to \lambda_\pm^t(a)$ are strictly continuous for all $a \in A$.

(2) The maps $t \to \lambda_+^t(a) - \lambda_-^t(a)$ are norm-continuous for all $a \in A$.

(3) $(\lambda_+^0, \lambda_-^0) = (\phi_+, \phi_-)$ and $(\lambda_+^1, \lambda_-^1) = (\psi_+, \psi_-)$.

We write $(\phi_+, \phi_-) \sim (\psi_+, \psi_-)$ if (ϕ_+, ϕ_-) and (ψ_+, ψ_-) are homotopic.

It is straightforward to check that $\sim$ defines an equivalence relation. We now define the $KK_h(A,B)$ group.

Definition 3.35 ($KK_h(A,B)$ group)**.** *Let A and B be C^*-algebras. The $KK_h(A,B)$ group is defined as the set of homotopy classes $\mathbb{F}(A,B)/\sim$. The homotopy class represented by (ϕ_+, ϕ_-) is denoted by $[\phi_+, \phi_-]$.*

Let $\phi : A \to \mathcal{M}(\mathcal{K} \otimes B)$ be a $*$-homomorphism, it is clear that (ϕ, ϕ) defines a $KK_h(A,B)$-cycle, where difference between $*$-homomorphisms in the pair is zero. This is an analogue in the degenerate elements in $KK(A,B)$. Therefore, the following lemma is not a surprise.

Lemma 3.36. *Let A and B be C^*-algebras. Let $(\phi, \phi) \in \mathbb{F}(A,B)$, then*

$$(\phi, \phi) \sim (0,0) \in \mathbb{F}(A,B).$$

Since the C^*-algebra $\mathcal{K} \otimes B$ is stable, there exists a $*$-isomorphism $\Theta_B : M_2(\mathcal{M}(\mathcal{K} \otimes B)) \to \mathcal{M}(\mathcal{K} \otimes B)$, cf. [33, Definition 1.3.8]. Using this isomorphism, we define addition $+$ in $KK_h(A, B)$ as follows, for $[\phi_+, \phi_-]$, and $[\psi_+, \psi_-] \in KK_h(A, B)$, define

$$[\phi_+, \phi_-] + [\psi_+, \psi_-] = \left[\Theta_B \circ \begin{bmatrix} \phi_+ & 0 \\ 0 & \psi_+ \end{bmatrix}, \Theta_B \circ \begin{bmatrix} \phi_- & 0 \\ 0 & \psi_- \end{bmatrix} \right] \in KK_h(A, B).$$

With this addition, $KK_h(A, B)$ is a semigroup. The following proposition shows that $KK_h(A, B)$ is an abelian group.

Proposition 3.37. *Let A and B be C^*-algebras. With the addition $+$ defined above, $KK_h(A, B)$ is an abelian group. The zero element is represented by $(0, 0)$, and*

$$-[\phi_+, \phi_-] = [\phi_-, \phi_+] \in KK_h(A, B).$$

We now briefly explain why $KK_h(A, B)$ is isomorphic to $KK(A, B)$ to conclude this chapter. To establish the connection to Kasparov's picture, we need the following preparations.

(1) Let $\{e_{ij} : i, j \in \mathbb{N}\}$ be the full system of matrix units, such that $e_{ij}^* = e_{ji}$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$, and $\mathrm{span}_{\mathbb{C}}\{e_{ij} : i, j \in \mathbb{N}\}$ is dense in $\mathcal{K}$. This is unique up to a unitary equivalence.

(2) Let B be a C^*-algebra and H_B be the Hilbert B-module defined in Example 3.5. Define the $*$-homomorphism $\Psi_B : \mathrm{span}_{\mathbb{C}}\{e_{ij} : i, j \in \mathbb{N}\} \otimes B \to \mathcal{L}_B(H_B)$, as follows:

$$\Psi_B(e_{ij} \otimes bc^*) = \Theta_{b_i, c_j} \in \mathcal{K}(H_B), \quad \forall b, c \in B, i, j \in \mathbb{N},$$

where b_i represents an element in H_B whose i-th entry is b and other entries are 0; similiarly for c_j. By [33, Lemma 1.1.14], Ψ_B is uniquely extended to a $*$-homomorphism $\widetilde{\Psi}_B : \mathcal{M}(\mathcal{K} \otimes B) \to \mathcal{L}_B(H_B)$.

Let $\widehat{H}_B$ denote the graded Hilbert B-module such that $\widehat{H}_B \cong H_B \oplus H_B$ and $S_{\widehat{H}_B} = \mathrm{id}_{H_B} \oplus -\mathrm{id}_{H_B}$. Then given a pair of $*$-homomorphisms $(\phi_+, \phi_-) \in$

$\mathbb{F}(A, B)$, we define the triple

$$\mathcal{E}(\phi_+, \phi_-) = \left(\widehat{H}_B, \begin{bmatrix} \widetilde{\Psi}_B \circ \phi_+ & 0 \\ 0 & \widetilde{\Psi}_B \circ \phi_- \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

It is straightforward to show that $\mathcal{E}(\phi_+, \phi_-)$ is a Kasparov A-B-module, and this correspondence gives rise to an isomorphism $\mu : KK_h(A, B) \to KK(A, B)$ (see [33, Theorem 4.1.8]).