Subproduct systems and C*-algebras
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Part I

Preliminaries

Chapter 2

K-theory and K-homology

In this chapter, we recall operator K-theory and then discuss extensions of C^*-algebras and K-homology.

2.1 K-theory of C^*-algebras

Our main references for operator K-theory are [46], [12], and [62]. For clarity, we fix the following convention: we denote by $\mathrm{C}^*\mathrm{Alg}$ the category whose objects are C^*-algebras and whose morphisms are the $*$-homomorphisms between C^*-algebras.

The origins of K-theory trace back to A. Grothendieck's work in the late 1950s. His work was motivated by the study of coherent sheaves on varieties [14]. Building on Grothendieck's ideas, M. F. Atiyah and F. Hirzebruch extended K-theory to topological spaces [9]. Their generalization played an important role in formulating topological analogues of Riemann–Roch theorem for differentiable manifolds and in classifying vector bundles over topological spaces.

Roughly speaking, for a compact Hausdorff topological space X, the topological K-theory group $K^0(X)$ is defined to be the Grothendieck group of the monoid of isomorphism classes of complex vector bundles over X. The higher K-theory groups $K^{-n}(X)$ are defined by suspension, i.e., $K^{-n}(X) = K^0(\Sigma^n X)$. For topological complex K-theory, Bott periodicity reduces K-theory groups to only $K^0(X)$ and $K^{-1}(X)$, which leads to the six-term exact sequence and makes K-theory a computable homological theory.

The development of a K-theory for C^*-algebras was inspired by ideas from topological K-theory, together with the Gelfand–Naimark duality [46, Theorem 2.1.10], and the Serre–Swan theorem [49]. This is now an independent field of research known as *operator K-theory*.

In the categorical language, operator K-theory is a functor from the category of C^*-algebras $\mathrm{C^*Alg}$ to the category of abelian groups. That is, for each C^*-algebra A, operator K-theory associates two abelian groups, denoted by $K_0(A)$ and $K_1(A)$ such that for a $*$-homomorphism $f : A \to B$ between C^*-algebras A and B, there is an induced homomorphism f_* between abelian groups

$$f_* : K_i(A) \to K_i(B), \quad i = 0, 1.$$

There are three key properties of the operator K-theory: homotopy invariance, half-exactness, and stability, which are now introduced in turn as follows.

Let f and g be $*$-homomorphisms between C^*-algebras A and B. We say f is homotopic to g, if there exists a norm-continuous path of $*$-homomorphisms φ_t indexed by $t \in [0, 1]$, such that $\varphi_0 = f$ and $\varphi_1 = g$. The K-theory functor is homotopy invariant in the sense that if f and g are homotopic, then they induce the same homomorphism between K-theory groups, i.e., $f_* = g_*$.

Half exactness means the following: for any short exact sequence of C^*-algebras

$$0 \to J \to B \to A \to 0,$$

there is an induced sequence

$$K_i(J) \to K_i(B) \to K_i(A), \quad i = 0, 1,$$

that is exact at $K_i(B)$.

Lastly, the stability is the property that for any C^*-algebra A, one has $K_i(A) \cong K_i(A \otimes \mathcal{K})$, for $i = 0, 1$.

Concretely, for a C^*-algebra A, $K_0(A)$ is the Grothendieck group of equivalence classes of Murray–von Neumann projections in the matrix algebras over A, and the higher K-theory groups $K_n(A)$ are defined to be the K_0-group of the n-fold suspension of A, i.e.,

$$K_n(A) = K_0(S^n A),$$

where $SA := C_0(0,1) \otimes A$.

Let J be a closed two-sided ideal in A, there is a short exact sequence of C^*-algebras:

$$0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0.$$

By the half-exactness, we obtain sequences of K-theory groups

$$K_i(J) \to K_i(A) \to K_i(A/J), \tag{2.1}$$

that are exact at $K_0(A)$ and $K_1(A)$ respectively.

Operator K-theory also has Bott periodicity, i.e., $K_n(A) \cong K_{n+2}(A)$. Therefore, $K_0(A)$ and $K_1(A)$ are essentially the only two K-theory groups. A crucial consequence of Bott periodicity is the six-term exact sequence of K-theory groups, which is a powerful tool for computing K-theory groups of C^*-algebras.

There is a well-defined group homomorphism δ (see [12, Definition 8.3.1]),

$$\delta : K_1(A/J) \to K_0(J),$$

called the index map, that completes sequences (2.1) for $i = 0, 1$ into a long exact sequence for K-theory groups. This is formalized in the following theorem.

Theorem 2.1 ([62, Theorem 8.2.1]). *Let J be a two-sided ideal in A. Then the following sequence is exact everywhere:*

$$K_1(J) \xrightarrow{\iota_*} K_1(A) \xrightarrow{\pi_*} K_1(A/J) \xrightarrow{\delta} K_0(J) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J).$$

Since the higher K-theory groups are defined inductively by suspension, the above sequence extends into an infinite exact sequence. Bott periodicity reduces the infinite exact sequence to the following six-term exact sequence.

Theorem 2.2 (Six-term Exact Sequence)**.** *Let J be a closed two-sided ideal in A. Then the following six-term cyclic sequence is exact everywhere:*

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/J) \\ {\scriptstyle\delta}\big\uparrow & & & & \big\downarrow{\scriptstyle\delta} \\ K_1(A/J) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[\iota_*]{} & K_1(J) \end{array}.$$

2.1.1 Equivariant K-theory

To prepare for our applications in the study of symmetries and group actions on C*-algebras, we finish this section by introducing equivariant operator K-theory, which generalizes equivariant topological K-theory.

Let G be a compact group and X be a compact Hausdorff space with a continuous G-action α. The equivariant topological K-theory group $K_G^0(X)$ is defined to be the Grothendieck completion of the abelian semigroup of the isomorphism classes of equivariant G-vector bundles over X. Higher equivariant K-theory groups are defined via suspension as in the ordinary topological K-theory. The group action on a compact space X is dualized to a group action on the $C(X)$, which in turn inspired the construction of crossed products of C^*-algebras. In fact, the crossed product of $C(X)$ and G, denoted by $C(X) \rtimes_\alpha G$ encodes full information of the dynamical systems of the G-action on X. This idea motivates the definition of equivariant K-theory for arbitrary C^*-algebras with G-actions.

Definition 2.3 (Equivariant K-theory)**.** *Let G be a compact group and A be a unital C^*-algebra with a strongly continuous group action α. The equivariant K-theory group $K_0^G(A)$ is defined as the $K_0(A \rtimes_\alpha G)$, and the higher equivariant K-theory group $K_n^G(A)$ are defined via suspensions: set $K_n^G(A) := K_0^G(S^n A)$.*

We conclude this section with the following remarks.

(1) If $A = C(X)$ for some compact space X, then $K_0^G(A) \cong K_G^0(X)$, coinciding with equivariant topological K-theory. In particular, if $A = \mathbb{C}$, then $K_0^G(A) \cong R(G)$, the representation ring of G.

(2) Equivariant K-theory shares the same properties of ordinary K-theory, including stability, Bott periodicity, and the existence of a six-term exact sequence.

2.2 Extensions and K-homology

In this section, we introduce extensions of C^*-algebras and establish their connection to K-homology, discussing the foundational work [15], which examines extensions of C^*-algebras from continuous functions on compact metrizable spaces. Lastly, we present the index pairing between K-theory and K-homology.

2.2.1 General theory

R. Busby initiated a systematic study of extensions of C^*-algebras [17], where he defined an invariant to describe extensions, which later became known as the Busby invariant. However, his work did not attract much attention until 1977 (see [12, Chapter 15.2]).

In 1977, L. Brown, R. Douglas, and P. Fillmore published a famous paper [15] titled *Extensions of C^*-algebras and K-homology*, which is considered a cornerstone in the theory.

In this section, we denote the multiplier algebra of a C^*-algebra B by $\mathcal{M}(B)$, and the corona algebra $\mathcal{Q}(B)$ as the quotient $\mathcal{M}(B)/B$. Before introducing the Busby invariant, let us recall the classical notion of extensions of C^*-algebras.

Definition 2.4 (Extension)**.** *Let A and B be C^*-algebras. An* extension *of A by B is a short exact sequence of C^*-algebras:*

$$0 \to B \xrightarrow{i} E \xrightarrow{p} A \to 0.$$

We denote an extension of A by B by (i, E, p), where $B \cong i(B) \subseteq E$ and $E/i(B) \cong A = p(E)$. The set of all extensions of A by B is denoted by $\mathcal{E}xt(A, B)$. We say two extensions (i_1, E_1, p_1) and (i_2, E_2, p_2) are isomorphic if there exists a $*$-isomorphism $\gamma : E_1 \to E_2$ such that $\gamma \circ i_1 = i_2$ and $p_2 \circ \gamma = p_1$. In other words, the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{i_1} & E_1 & \xrightarrow{p_1} & A & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & B & \xrightarrow{i_2} & E_2 & \xrightarrow{p_2} & A & \longrightarrow & 0.
\end{array}$$

Given an extension (i, E, p) of A by B, Busby proposed a way to transform it into a $*$-homomorphism $\tau : A \to \mathcal{M}(B)/B$, such that all essential information is preserved. The construction is as follows.

Let (i, E, p) be an extension of A by B, and let $(\iota, \mathcal{M}(B), \pi)$ be the extension of $\mathcal{Q}(B)$ by B, where ι is the natural embedding of B into $\mathcal{M}(B)$, and π is the quotient map. By the universal property of multiplier algebras, there exists a $*$-homomorphism $\varphi : E \to \mathcal{M}(B)$ such that $\iota(b) = \varphi \circ i(b)$. Since p is surjective, for each $a \in A$, there exists $e \in E$ such that $p(e) = a$. This allows us to define a map as follows,

$$\tau : a \mapsto \pi \circ \varphi(e).$$

The following proposition shows that τ is a well-defined $*$-homomorphism, independent of the choice of e.

Proposition 2.5. *Let A and B be C^*-algebras, and let (i, E, p) be an extension of A by B. Then there exist unique $*$-homomorphisms $\varphi : E \to \mathcal{M}(B), \tau : A \to \mathcal{Q}(B)$ such that the following diagram of C^*-algebras commutes:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{\quad i \quad} & E & \xrightarrow{\quad p \quad} & A & \xrightarrow{\quad i \quad} & 0 \\
 & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \tau} & & \\
0 & \longrightarrow & B & \xrightarrow{\iota} & \mathcal{M}(B) & \xrightarrow{\pi} & \mathcal{Q}(B) & \longrightarrow & 0,
\end{array} \tag{2.2}$$

where φ is defined by the universal property of the multiplier algebras and τ is constructed as above.

Definition 2.6 (The Busby invariant)**.** *Let (i, E, p) be the extension of A by B, and we define the Busby invariant of the extension to be the $*$-homomorphism $\tau : A \to \mathcal{Q}(B)$ given by*

$$\tau(a) = \pi(e), \quad p(e) = a.$$

The Busby invariant of an extension (i, E, p) contains all essential information, which means that two extensions are isomorphic if and only if their Busby invariants are identical. Indeed, given a $*$-homomorphism $\tau : A \to \mathcal{Q}(B)$, one can construct a canonical extension of A by B and show that this extension is isomorphic to any extension with the same Busby invariant.

Theorem 2.7. *Let A and B be C^*-algebras, there is a one-to-one correspondence between the set of $*$-homomorphisms $\tau : A \to \mathcal{Q}(B)$ and the set of isomorphism classes of extensions of A by B.*

Definition 2.8 (Split extension)**.** *Let (i, E, p) be an extension of A by B. It is said to be a split extension if there exists $*$-homomorphism $\gamma : A \to E$ such that $p \circ \gamma = id_A$.*

The following theorem provides another characterization of split extensions using Busby invariants.

Theorem 2.9. *Let (i, E, p) be an extension of A by B, and let $\tau : A \to \mathcal{Q}(B)$ be the corresponding Busby invariant. The following statements are equivalent:*

(1) There exists a $*$-homomorphism $\gamma : A \to E$ such that $p \circ \gamma = id_A$.

(2) There exists a $*$-homomorphism $\sigma : A \to \mathcal{M}(B)$ such that $\tau = \pi \circ \sigma$.

In addition to the isomorphism of extensions, another equivalence relation, known as strong unitary equivalence, is used to define the extension group.

Definition 2.10 (Unitary equivalence)**.** *Two extensions τ_1 and τ_2 of A by B are said to be strongly unitarily equivalent if there is a unitary $u \in \mathcal{M}(B)$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u^*)$ for all $a \in A$, where $\pi : \mathcal{M}(B) \to \mathcal{Q}(B)$ denotes the quotient map. If τ_1 and τ_2 are strongly unitarily equivalent, we write $\tau_1 \approx_u \tau_2$.*

Let A and B be C^*-algebras, we write $E(A, B) = \mathcal{E}xt(A, B)/ \approx_u$. Assuming B is stable and σ-unital, and A is separable, one can endow $E(A, B)$ with an algebraic structure. Within this structure, two types of extensions, called degenerate extensions and semi-split extensions, play crucial roles.

Definition 2.11 (Degenerate extension)**.** *Let $\tau : A \to \mathcal{Q}(B)$ be a Busby invariant. We call τ degenerate if the corresponding extension is split.*

Definition 2.12 (Semi-split extension)**.** *Let (i, E, p) be an extension of A by B, we call it a semi-split extension if there exists a completely positive contraction $\gamma : A \to E$ such that $p \circ \gamma = id_A$.*

Since B is stable, there exists a map Θ_B implementing the isomorphism $M_2(B) \cong B$. In addition, Θ_B induces a $*$-isomorphism from $M_2(\mathcal{Q}(B))$ to $\mathcal{Q}(B)$, denoted by $\widetilde{\Theta}_B$ (see [33, Definition 1.3.8, Lemma 1.3.9]). For simplicity, we will omit the map Θ_B and identify $M_2(B)$ with B in the rest of the chapter. This allows us to define an addition on $E(A, B)$:

$$\tau_1 + \tau_2 = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}, \quad \forall \tau_1, \tau_2 \in E(A, B).$$

Let $\mathcal{D}(A, B)$ denote the set of degenerate Busby invariants $\tau : A \to \mathcal{Q}(B)$. We write $D(A, B) = \mathcal{D}(A, B)/ \approx_u$ as the collection of equivalence classes of degenerate Busby invariants. The extension semigroup is defined as follows.

Definition 2.13 (The extension semigroup)**.** *Let A and B be separable C^*-algebras with B stable. Then the extension semigroup $Ext(A, B)$ is defined as*

$$E(A, B)/D(A, B).$$

Let τ be a Busby invariant, we denote the representative of τ by $[\tau]$.

Note that the definition suggests that the split extensions are zero elements in $Ext(A, B)$, explaining why we call the Busby invariant of a split extension degenerate. The invertible elements in $Ext(A, B)$ form a group, called the extension group.

Definition 2.14 (The extension group)**.** *Let A and B be separable C^*-algebras with B being stable. Then the extension group $Ext^{-1}(A, B)$ is an abelian group consisting of invertible elements in $Ext(A, B)$.*

To ensure that the extension group $Ext^{-1}(A, B)$ can be nontrivial for some separable C^*-algebras A and stable separable C^*-algebra B, we have to find a more concrete description of invertible elements in $Ext(A, B)$.

Theorem 2.15 ([33, Theorem 3.2.9]). *Assume A and B are separable C^*-algebras with B stable. Let $\tau : A \to \mathcal{Q}(B)$ be a Busby invariant. The following conditions are equivalent:*

(1) $[\tau]$ is invertible in $Ext(A, B)$.

(2) τ corresponds to a semi-split extension.

Since the proof is technical and requires a nontrivial fact from the theory of completely positive linear maps between C^*-algebras, we omit the proof and recommend interested readers to read [33, Theorem 3.2.9].

We now introduce another description of the extension group. This approach allows us to relate the extension group $Ext^{-1}(A, B)$ to Kasparov's KK-group $KK^1(A, B)$, which will be introduced in Chapter 3.

Definition 2.16 (KK^1-cycle). *Let A and B be separable C^*-algebras with B being stable. A KK^1-cycle for A and B is a pair (v, λ) where*

$$v \in \mathcal{M}(B), \quad \lambda \in Hom(A, \mathcal{M}(B)),$$

such that

$$[v, \lambda(a)], \quad (v^* - v)\lambda(a), \quad (v^2 - v)\lambda(a) \in B, \quad \forall a \in A.$$

If $[v, \lambda(a)] = (v^* - v)\lambda(a) = (v^2 - v)\lambda(a) = 0$, we call (v, λ) a degenerate KK^1-cycle. The set of KK^1-cycles for A and B is denoted by $\mathbb{E}^1(A, B)$, and the set of degenerate KK^1-cycles for A and B is denoted by $\mathbb{D}^1(A, B)$.

Definition 2.17 (Homotopy). *Let A and B be separable C^*-algebras with B being stable. We say $(v_0, \lambda_0), (v_1, \lambda_1) \in \mathbb{E}^1(A, B)$ are homotopic if there exists a norm-bounded strictly continuous path $\omega_t \in \mathcal{M}(B)$ and a path $\phi_t \in Hom(A, \mathcal{M}(B))$ such that*

(1) $(\omega_t, \phi_t) \in \mathbb{E}^1(A, B)$ for each $t \in [0, 1]$, and $(\omega_i, \phi_i) = (v_i, \lambda_i)$ for $i = 0, 1$.

(2) $t \mapsto \phi_t(a)$ is strictly continuous for all $a \in A$, meaning that for every $\xi \in \mathcal{M}(B)$, the map $t \mapsto \phi_t(a)\xi$ is norm-continuous.

(3) *$t \mapsto [\omega_t, \phi_t(a)], t \mapsto (\omega_t^* - \omega_t)\phi_t(a)$, and $t \mapsto (\omega_t^2 - \omega_t)\phi_t(a)$ are norm continuous for all $a \in A$.*

If (v_0, λ_0) is homotopic to (v_1, λ_1), we write $(v_0, \lambda_0) \sim_h (v_1, \lambda_1)$.

This defines an equivalence relation on $\mathbb{E}^1(A, B)$. As the name "degenerate" suggests, degenerate KK^1-cycles are homotopic to $(0, 0) \in \mathbb{E}^1(A, B)$.

Lemma 2.18. *Let A and B be separable C^*-algebras with B being stable. If (v, λ) is degenerate, then (v, λ) is homotopic to $(0, 0)$.*

Definition 2.19. *Let A and B be separable C^*-algebras with B being stable. The $kK^1(A, B)$ is defined as $\mathbb{E}^1(A, B)/\sim_h$, and the representative of (v, λ) in $kK^1(A, B)$ is denoted by $[v, \lambda]$.*

The following lemma ensures that the $kK^1(A, B)$ does admit an abelian group structure.

Lemma 2.20. *Let A and B be separable C^*-algebras with B being stable. Then $kK^1(A, B)$ is an abelian group with zero elements represented by degenerate KK^1-cycles, where addition is defined as follows,*

$$(v_1, \lambda_1) + (v_2, \lambda_2) = (\Theta_{\mathcal{M}(B)}(\begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}), \Theta_{\mathcal{M}(B)}(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix})).$$

We conclude this section by identifying the extension group $Ext^{-1}(A, B)$ and $kK^1(A, B)$. To construct the isomorphism, we need the following Lemma.

Lemma 2.21 ([33, Corollary 3.2.10])**.** *Assume A is a separable C^*-algebra, and let $[\tau] \in Ext^{-1}(A, B)$ be represented by a Busby invariant $\tau : A \to \mathcal{Q}(B)$. Then there exists a $*$-homomorphism $\lambda_\tau : A \to \mathcal{M}(B)$ and a fully complemented projection $v_\tau \in \mathcal{M}(B)$ such that $\tau = \pi(v_\tau \lambda_\tau)$.*

For each $\tau : A \to \mathcal{Q}(B)$, the above lemma ensures the existence of a projection v_τ and a $*$-homomorphism $\lambda_\tau : A \to \mathcal{M}(B)$. Since v_τ is a fully complemented projection, it is not hard to see that

$$(v_\tau^2 - v_\tau)\lambda_\tau(a) = 0 = (v_\tau^* - v_\tau)\lambda_\tau(a), \quad [v_\tau, \lambda_\tau(a)] \in B, \quad \forall a \in A.$$

Let A and B be separable C^*-algebras with B being stable. This construction provides a map between the extension group $Ext^{-1}(A, B)$ and the $kK^1(A, B)$ group, and this map is an isomorphism by the following theorem.

Theorem 2.22 ([33, Lemma 3.3.8, Theorem 3.3.10])**.** *Let A and B be separable C^*-algebras with B being stable. There is a group isomorphism $e : Ext^{-1}(A, B) \to kK^1(A, B)$ such that $e([\tau]) = [v_\tau, \lambda_\tau]$, where $v_\tau \in \mathcal{M}(B)$ is a projection, and λ_τ is a $*$-homomorphism from A to $\mathcal{M}(B)$.*

In the next section, we will apply this general theory of extensions to extensions of C^*-algebras of continuous functions on compact metric spaces.

2.2.2 Extensions of $C(X)$ by $\mathcal{K}$

Let $\mathcal{K}$ be the C^*-algebra of compact operators on $\ell^2(\mathbb{N})$, and let $C(X)$ be the C^*-algebra of continuous functions on some compact metrizable space X. In this section, we briefly introduce the theory of extension of $C(X)$ by $\mathcal{K}$, based on the paper [15].

The following theorem, proven by Brown, Douglas, and Fillmore, is a cornerstone of the theory: it shows that any two split extensions of $C(X)$ by $\mathcal{K}$ are unitarily equivalent and are determined by the topology of X. This result connects the structure of operator algebras to the underlying topology of X.

Theorem 2.23 ([15, Theorem 1.13])**.** *For any compact metrizable space X, there exists a split extension, and any two split extensions are unitarily equivalent.*

Sketch of proof. The existence follows by evaluating functions in $C(X)$ along a dense sequence $\{x_n\}$ in X and embedding the image into $\ell^\infty(\mathbb{N}) \subset \mathbb{B}(\ell^2(\mathbb{N}))$, yielding a split extension via the Busby invariant. Uniqueness up to unitary equivalence is shown by observing that any two such sequences are asymptotically equivalent and can be transformed to each other via a permutation unitary. Finally, using spectral theory and functional calculus, as in [15, Theorem 1.13], one shows that any split extension arises from such a construction. $\square$

Remark 2.24. *In the original paper [15], the authors use the terminology "trivial extension" instead of "split extension". This is because in the case of extensions of $C(X)$ by $\mathcal{K}$, the split extensions play the role of "zero elements" in the extension group, as discussed in the previous section.*

Theorem 2.23 shows the existence and uniqueness of split extensions of compact metrizable spaces. In fact, some compact metrizable spaces only admit split extensions up to unitary equivalence.

Lemma 2.25 ([15, Lemma 1.14])**.** *Let $X \subseteq \mathbb{R}$ be a compact space. Then, every extension of $C(X)$ splits.*

Theorem 2.26 ([15, Theorem 1.15])**.** *Let X be a compact, totally disconnected space. Then, every extension of $C(X)$ splits.*

Proof. Since X is totally disconnected, $C(X)$ is generated by projections. Therefore, $C(X)$ is generated by a non-negative function $p \in C(X)$ and X is homeomorphic to $p(X) \subseteq \mathbb{R}$. Applying Lemma 2.25 completes the proof. □

In [15], the authors proved that $Ext(\mathcal{K}, C(X)) = Ext(\mathcal{K}, C(X))^{-1}$ for compact metrizable spaces. In the original proof, the authors used an unpublished result of A. M. Davie, which claims the existence of a completely positive lifting of a unital positive linear map $\tau : C(X) \to E/J$ for some C^*-algebra E and a two-sided closed ideal $J \lhd E$. However, one year before [15], M. Choi and E. Effros published a paper containing a more general result.

Theorem 2.27 (Choi–Effros Lifting Theorem, [18, Corollary 3.11])**.** *Suppose A is a separable, nuclear C^*-algebra, and B is a unital C^*-algebra with a two-sided ideal $J \lhd B$. Then each completely positive contraction $\varphi : A \to B/J$ has a completely positive lifting $\tilde{\varphi} : A \to B$.*

By Theorem 2.27 above, one can prove the following corollary in a simpler way.

Corollary 2.28 ([15, Theorem 1.23])**.** *For any compact metrizable space X, the semigroup $Ext(\mathcal{K}, C(X))$ is a group.*

Proof. Let $0 \to \mathcal{K} \xrightarrow{i} E \xrightarrow{p} C(X)$ be an extension of $C(X)$ by $\mathcal{K}$. Since $C(X)$ is a nuclear C^*-algebra, by the Choi–Effros Lifting Theorem, the extension is semi-split. Indeed, we have $C(X) \cong E/\mathcal{K}$, and this isomorphism is completely positive and thus admits a completely positive lifting $\gamma : C(X) \to E$ acting as the right inverse for p. $\square$

At the end of this section, we will realize the index pairing between the odd K-homology group $K^1(C(X))$ and the odd K-theory group $K_1(C(X))$ by using the identification of the odd K-homology group and the extension group $Ext(C(X), \mathcal{K})$.

In Chapter 3, we will introduce Kasparov's KK-groups, which generalize K-theory groups and K-homology simultaneously. Furthermore, the Kasparov product provides a general framework for the index pairing between K-theory groups and K-homology. For a compact metrizable space X, there is an index pairing as follows,

$$\otimes_{C(X)} : K_1(C(X)) \times Ext(\mathcal{K}, C(X)) \to K_0(\mathbb{C}) \cong \mathbb{Z}.$$

This is a special case of the Kasparov product, which is, in general, very hard to compute. The group $Ext(\mathcal{K}, C(X))$ is isomorphic to the odd Kasparov KK-group $KK^1(\mathcal{K}, C(X))$, which will be defined in Chapter 3. We now present a concrete realization of the Kasparov product, using the extension group, via the Fredholm index of operators associated with extensions.

Let $[\tau] \in Ext(C(X), \mathcal{K})$ represented by the Busby invariant

$$\tau \in \mathrm{Hom}(C(X), \mathcal{Q}(\ell^2(\mathbb{N}))),$$

which extends to a unital $*$-homomorphism

$$\tau_n : C(X) \otimes M_n(\mathbb{C}) \to \mathcal{Q}(\ell^2(\mathbb{N})) \otimes M_n(\mathbb{C}) \cong \mathcal{Q}(\ell^2(\mathbb{N}) \otimes \mathbb{C}^n).$$

Recall that $K_1(C(X))$ is generated by homotopy classes of unitaries in $M_n(C(X))$ for $n \in \mathbb{N}$. Let $[f]$ be a nonzero element in $K_1(C(X))$ represented by a unitary

$$f \in U_n(C(X)) \subseteq M_n(C(X)) \cong C(X, M_n(\mathbb{C})) \cong C(X) \otimes M_n(\mathbb{C}),$$

we have that $\tau_n \circ f$ is an invertible element in $\mathcal{Q}(\ell^2(\mathbb{N})) \otimes M_n(\mathbb{C})$. Therefore, by Atkinson's theorem, there exists a lifting $\widetilde{\tau_n \circ f} \in \mathbb{B}(\ell^2(\mathbb{N}) \otimes \mathbb{C}^n)$ of $\tau_n \circ f$ such that $\widetilde{\tau_n \circ f}$ is Fredholm. Then, the pairing can be defined as follows:

$$[f] \otimes_{C(X)} [\tau] = \mathrm{ind}(\widetilde{\tau_n \circ f}) \in \mathbb{Z}.$$

Since the Fredholm is independent of the choice of the lifting and invariant under the homotopy, the index pairing is well-defined. This gives a concrete realization of the pairing between the K-theory groups and K-homology.