Chapter 1

Introduction

The field of Noncommutative Geometry, initiated by A. Connes, has deeply influenced various branches of mathematics and theoretical physics by extending our geometric and topological intuition to the realm of operator algebras. A central philosophy in this theory involves translating geometric and topological ideas into the operator algebraic language. This provides new insights and powerful tools to tackle problems where the classical tools of measure theory and classical topology fail. One of the foundational examples of a construction connecting geometry and operator algebras is the classical Toeplitz extension. Recall that an extension of C^*-algebras is a short exact sequence of C^*-algebras. The classical Toeplitz extension reads

$$0 \to \mathcal{K} \to \mathcal{T} \to C(S^1) \to 0,$$

where $\mathcal{T}$ is the classical Toeplitz algebra, defined as the unital C^*-algebra generated by the unilateral shift on $\ell^2(\mathbb{N})$, $\mathcal{K}$ denotes the algebra of compact operators, and $C(S^1)$ is the commutative algebra of continuous functions on the unit circle. Identifying $\ell^2(\mathbb{N})$ with the Hardy space $H^2(S^1)$, the Toeplitz algebra can be represented as the unital C^*-algebra of Toeplitz operators T_f with continuous symbols f, where T_f is the compression of the multiplication operator M_f

to $H^2(S^1)$. One can view $\mathcal{T}$ as an extension of the algebra of continuous functions $C(S^1)$ by the ideal of compact operators $\mathcal{K}$, with the quotient map sending T_f to its symbol $f \in C(S^1)$. This setup provides a classical example of the Atiyah–Singer Index Theorem, known as the Noether–Gohberg–Krein Index Theorem: if f is a continuously differentiable function that vanishes nowhere on S^1, then T_f is a Fredholm operator and its index is equal to the minus of its winding number:

$$\mathrm{Ind}(T_f) = -\omega(f) = -\int_{S^1} \frac{f'(z)}{f(z)} dz.$$

A functional analytic quantity, the index of a Fredholm operator, is realised as a topological invariant, the winding number. In 1967, Coburn [19] proved that the Toeplitz algebra $\mathcal{T}$ is the universal C^*-algebra generated by an isometry. This deep connection between operator algebras and complex function theory establishes the Toeplitz extension as one of the cornerstones of noncommutative geometry.

Faced with the beauty and depth of the classical Toeplitz extension, it is natural to wonder if such an extension can be generalized beyond the classical Hardy space setting. Inspired by its construction, efforts were made to generalize it in two main directions: commutative and noncommutative.

Arveson's Toeplitz extension Theorem stands out as a natural generalization in the commutative setting. In his influential work [4], Arveson introduced the Drury–Arveson space H^2_d as a multivariable analogue of the Hardy space, and the unital C^*-algebra generated by its shift operators fits into the following extension:

$$0 \to \mathcal{K}(H^2_d) \to \mathcal{T}_{H^2_d} \to C(S^{2d-1}) \to 0.$$

In particular, he proved that the commutators of the shift operators on H^2_d are Schatten p-class operators for $p > d$. Motivated by a series of parallel work on curvatures of Hilbert modules (in the context of function theory) [5–8] and by Douglas' novel approach to index theory of Hilbert modules [21], the conjecture of essential normality of the submodules and quotient modules of H^2_d, was eventually formulated. This conjecture, known as the Arveson–Douglas conjecture, has become a central theme in modern function theory and operator theory.

Around the same time, Pimsner's work [51], published in 1997, is regarded as a natural and successful generalization in the noncommutative setting. Given a C^*-correspondence $\mathcal{E}$ over a C*-algebra A, Pimsner introduced the Toeplitz algebra $\mathcal{T}_{\mathcal{E}}$ and the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ associated with $\mathcal{E}$. In this setting, the Toeplitz extension becomes as follows:

$$0 \to \mathcal{K} \to \mathcal{T}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}} \to 0,$$

where $\mathcal{O}_{\mathcal{E}}$ is not commutative in general. Cuntz–Pimsner algebras $\mathcal{O}_{\mathcal{E}}$ generalize a broad class of dynamical C^*-algebras and provide a unified approach to studying a broad range of C^*-algebras arising from dynamical systems and graph theory.

Later, the concepts of the Toeplitz algebra and the Toeplitz extension have been extended and reinterpreted in the unifying framework of subproduct systems, formally introduced by O. Shalit and B. Solel in [58]. Their construction generalises Arveson's approach and Pimsner's approach simultaneously. Without going into the precise definition, given E a subproduct system of finite-dimensional Hilbert spaces, Viselter [61] proved the Toeplitz algebra $\mathcal{T}_E$ of E and the Cuntz–Pimsner algebra $\mathcal{O}_E$ of E fit into the following exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}_E \to \mathcal{O}_E \to 0.$$

The study of the K-theoretical aspects of $\mathcal{T}_E$ and $\mathcal{O}_E$ has gained increasing attention in the past five years, cf. [3], [31], and [30].

Primarily motivated by the $SU(2)$-subproduct systems studied in [3], this thesis introduces the notion of **quadratic subproduct systems** and develops a K-theoretic approach to study their associated C^*-algebras. Quadratic subproduct systems offer a framework that naturally generalizes the subproduct systems studied in [3], [31], and [30]. Motivated by the rich theory of quadratic algebras [52] we explore algebraic operations such as free products, Segre products, and Veronese subalgebras in the more general framework of subproduct systems. We investigate the properties of the associated C^*-algebras and their

K-theory groups. More precisely, our main contributions are summarized as follows.

Free product of subproduct systems

Our first contribution concerns the free product of subproduct systems. We address the first question posed in [3, Section 7.1], which asks about extending their construction to the case where the $SU(2)$-representation that induces the subproduct systems is no longer irreducible. We observe that, for multiplicity-free $SU(2)$-representations, the fibers of the associated subproduct system exhibit a free product-like structure akin to that of quadratic algebras. This is the first concrete and motivating example of a free product of subproduct systems, a novel construction, to the best of our knowledge, that has not been studied before.

We therefore introduce the notion of the free product of subproduct systems, motivated by the example of multiplicity-free $SU(2)$-representations. We discuss this construction in the context of quadratic subproduct systems, and study their Toeplitz algebras and Cuntz–Pimsner algebras. As an application, we construct explicit KK-equivalences for the Toeplitz algebras of free products of Temperley–Lieb subproduct systems defined in [31] and [30], which leads to the computation of K-theory groups of the Toeplitz algebras and Cuntz–Pimsner algebras. This generalizes the subproduct system associated with multiplicity-free $SU(2)$-representations, thus partially answering the question raised in [3, Section 7.1], that we mentioned at the start of this paragraph.

Our discussion is based on the recent preprint [2].

Segre subproduct systems

Motivated by the classical Segre embedding in algebraic geometry and the notion of the Segre product in quadratic algebras, our second contribution is to introduce a Segre product operation for subproduct systems. We show that the class of subproduct systems of finite-dimensional Hilbert spaces is closed under the Segre product, thus forming a monoid. For two generic quadratic subproduct systems, whose Hilbert series are explicitly determined by the number of generators and the number of quadratic relations, we derive an explicit recursive formula for the dimension sequence of their Segre product, which further allows us to compute their Hilbert series.

As an application, we investigate the Toeplitz algebra of the Segre product of two Temperley–Lieb subproduct systems. We study the commutation relations of the associated Toeplitz operators in detail and compute the K-theory groups, showing that the resulting Toeplitz algebras provide a novel example that is not KK-equivalent to $\mathbb{C}$, thus partially answering a problem proposed by M. Lesch in [40].

Veronese subproduct systems

Our third contribution involves the study of Veronese subproduct systems. Motivated by classical algebraic geometry in Veronese embedding, we consider Veronese subalgebras, an important construction relating graded algebras to projective geometry. Inspired by this classical operation, we introduce and investigate the concept of Veronese subproduct systems.

We begin by defining Veronese subproduct systems, mimicking the classical construction of Veronese subalgebras in algebraic geometry. We focus on their Hilbert series and algebraic properties, and we show that certain geometric properties are preserved under this operation. Our results include explicit characterizations of the C^*-algebras associated with some Veronese subproduct systems and computations of their K-theory groups, which leads to a partial answer to an aforementioned problem proposed by M. Lesch in [40].

Lastly, we prove that taking Veronese submodules preserves the Arveson–Douglas conjecture in the following sense: if a given quotient module satisfies the conjecture, then its Veronese submodules give rise to p-essentially normal Hilbert modules. This provides a novel tool for constructing new examples of p-essentially normal Hilbert modules that are not covered in the existing results.

Outline of the thesis

Part I covers some necessary prerequisites from subproduct systems, noncommutative geometry, and the Arveson–Douglas conjecture: Chapter 2 provides an overview of K-theory and K-homology of C^*-algebras. Afterwards, we present an overview of the theory of Hilbert C^*-modules, and Kasparov's bivariate K-theory in Chapter 3. These theories are fundamental tools for the study of C^*-algebras. Chapter 4 introduces subproduct systems and their C^*-algebras. As an application, we introduce Matsumoto's approach to subshifts of finite type in the framework of subproduct systems and comment on the relation between the dimension of the fibers and topological entropy. The enclosing chapter of this part is Chapter 5, where we introduce the theory of Hilbert modules, review the history of the Arveson–Douglas conjecture, and discuss its consequences.

In Part II, we describe quadratic subproduct systems and the K-theoretical properties of their C^*-algebras. In Chapter 6, we outline the general framework of quadratic algebras, with an emphasis on their Hilbert series and how they inspire the definition of quadratic subproduct systems, which is introduced afterwards. In particular, we focus on the generic quadratic algebras and quadratic subproduct systems. Chapter 7 introduces three operations on subproduct systems: free products, Segre products, and Veronese subalgebras. We investigate the variation of the Hilbert series under these specific operations and describe the relationship between Veronese subalgebras and Veronese embedding within the framework of graded algebras, defined as coordinate rings. This chapter should serve as a preparation for the following chapters.

Chapter 8 connects the ideas of the previous chapters and combines the algebraic and analytic methods we introduced earlier. This chapter explores the Toeplitz and Cuntz–Pimsner algebras of quadratic subproduct systems. We prove that the Toeplitz algebra of the free product of subproduct systems is isomorphic to the reduced free product of the Toeplitz algebras of subproduct systems. As an application, we show how this theory works in the context of the free product of Temperley–Lieb subproduct systems and generalize the results in [3] and [31]. In addition, we explore $SU_q(2)$-subproduct systems, showing that multiplicity-free corepresentations give rise to Toeplitz algebras in the form of reduced free products, while isotypical corepresentations yield generic quadratic structures of the corresponding subproduct systems. Finally, we define the Segre product of two quadratic subproduct systems, showing that it can be described via explicit quadratic relations, and compute the commutation relations of Toeplitz operators for the Segre product of $SU(2)$-subproduct systems.

In the last chapter, Chapter 9, we investigate the K-theory of Toeplitz algebras and Cuntz–Pimsner algebras of subproduct systems. We first compute the K-theory groups of the Toeplitz algebras of free products of subproduct systems under certain conditions, applying six-term exact sequences. As an application, we obtain explicit formulas for the K-theory groups of Cuntz–Pimsner algebras of the free product of Temperley–Lieb subproduct systems. Next, we study a specific example of Segre product, the Segre product of $SU(2)$-subproduct systems, and determine the K-theory of the resulting Toeplitz algebra. Finally, we study the behaviour of Veronese subalgebras in the context of Hilbert modules, proving that the Veronese subalgebras of a quotient module satisfying the Arveson–Douglas conjecture give rise to p-essentially normal Hilbert modules. This theorem allows us to construct new classes of p-essentially normal Hilbert modules, which, to the best of our knowledge, are novel.

Finally, we conclude the thesis with three appendices. Appendix A is devoted to the discussion of the Segre product operation beyond the quadratic case, where we prove that the Segre product of subproduct systems of Hilbert spaces is again a subproduct system. To provide more background for Section 8.1.4, we briefly introduce the theory of compact quantum groups and their corepresentations in Appendix B.

Appendix C contains the computer programs developed and employed throughout this project. These codes aided us in checking our mathematical results and testing our intuitions. In Appendix C.1, we include three Magma programs: Appendix C.1.1 contains some pre-defined functions that are used in subsequent Appendices; Appendix C.1.2 consists of codes for constructing maximal subproduct systems; and Appendix C.1.3 demonstrates the way of computing Veronese powers of varieties. Appendix C.2 documents Macaulay 2 codes for computing the singular points of a variety, which proved more efficient than Magma in our experiments. Finally, Appendix C.3 offers a Python program that converts the results from Magma into symbolic form, thereby enhancing readability.

List of Notations

This section provides an overview of the notations and symbols used throughout this thesis for easy reference.

Symbol	Description
$\lfloor \cdot \rfloor$	The floor function.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{R}$	The set of all real numbers.
$\mathbb{N}$	The set of all natural numbers containing zero.
E^s	The symmetric subproduct system.
$\mathbb{B}^n$	The open unit ball in C^n.
$\mathrm{C}^*\mathrm{Alg}$	The category of C^*-algebras.
$\mathcal{K}$	The compact operators on a separable Hilbert space.
$\mathcal{S}^p(\mathcal{H})$	The p-th Schatten class operators of the Hilbert space $\mathcal{H}$.
$\mathcal{T}$	The classical Toeplitz algebra.
$\mathcal{T}_E$	The Toeplitz algebra of a subproduct system E.
$\mathcal{O}_E$	The Cuntz–Pimsner algebra of a subproduct system E.
$X \odot Y$	The algebraic tensor product of Hilbert C^*-modules X and Y.
$E \star F$	The free product of subproduct systems E and F.
$H \star_{\mathbb{C}} K$	The free product of Hilbert spaces H and K.
$A * C$	The full free product of C^*-algebras A and C.

Symbol	Description
$A *_B C$	The amalgamated free product of C^*-algebras A and C over B.
$K_i(A)$	The K_i-group of the C^*-algebra A.
$KK_i(A, B)$	The KK_i-group of C^*-algebras A and B.