Interactions of symplectic topology with singularity theory: Lagrangian pinwheels, Klein bottles and Milnor links of singularities

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Summary

Lagrangian subspaces of symplectic 4-manifolds

The first three chapters of this thesis are devoted to the study of the behavior of Lagrangian submanifolds of symplectic 4-manifolds. We use mostly topological techniques, for example, different kinds of surgeries, and combine them with hard theorems of elliptic PDEs, usually in the guise of Taubes-Seiberg-Witten theory. In this direction, these parts of the thesis revolve around understanding different versions of the following question asked by J. Evans:

Question. *Let (X^4, ω) be a symplectic manifold. How does the behavior of specific Lagrangians change under deformations of ω in $H^2_{dR}(X)$?*

We examine this question for various different Lagrangians, in the case where X is a *symplectic rational manifold*, i.e. it is a symplectic 4-manifold diffeomorphic to $S^2 \times S^2$ or $X_k = \mathbb{C}P^2 \#_k \overline{\mathbb{C}P^2}$.

(A.-Evans) Let $X = S^2 \times S^2$ and let ω_λ be the product symplectic form which gives the factors $A = [S^2 \times \{p\}]$ and $B = [\{p\} \times S^2]$ areas 1 and λ respectively. The class B carries a Lagrangian Klein bottle if and only if

$$\lambda < 2.$$

(A.) Let L_1 and L_2 be Lagrangian $\mathbb{R}P^2$'s in (X_3, ω), i.e. $\mathbb{C}P^2$ after three symplectic blow-ups. They can be made disjoint with a Hamiltonian isotopy if and only if the periods h, μ_1, μ_2, μ_3 of ω satisfy the inequality

$$\mu_1 + \mu_2 + \mu_3 < h.$$

While we use different techniques to obtain these results, the underlying idea is universal: Given a Lagrangian L in (X, ω), we perform some surgery around it to construct a new symplectic manifold $(\widetilde{X}, \tilde{\omega})$ such that the properties of L in X are reflected in the symplectic topology of $(\widetilde{X}, \tilde{\omega})$. The restrictions that the theory of pseudo-holomorphic curves imposes on the symplectic manifold $(\widetilde{X}, \tilde{\omega})$ (for example, the necessary area positivity of certain *exceptional* homology classes) can then be probed to give surprising restrictions to the possible Lagrangian embeddings of L.

In the case of Theorem 5.7, the surgery we used is *Luttinger surgery* which was first introduced by Luttinger for Lagrangian tori and was later adapted by Nemirovski for Lagrangian Klein bottles. Luttinger surgery is essentially a higher dimensional analogue of the Dehn-Twist along knots in 3-dimensional topology and was used by Nemirovski to provide the first non-trivial obstructions for Lagrangian embeddings of Klein bottles to *positive* rational manifolds, i.e. rational manifolds with $c_1 \cdot \omega > 0$.

In the case that the Lagrangians are the projective spaces $\mathbb{R}P^2$s, we implemented the surgery known as the *symplectic rational blow-up*, introduced by T. Khodorovskiy, which replaces the Lagrangian $\mathbb{R}P^2$ with a symplectic (-4)-sphere. The systematic use of this technique appeared first in the work of Borman-Li-Wu and then, with much more attention to *non-squeezing* problems, in the beautiful work of Smirnov-Schevchishin.

We also studied embeddings of Lagrangian pinwheels $L_{p,q}$ in rational and ruled surfaces. The pinwheels $L_{p,q}$ come in many guises; practically they are the Lagrangian skeleton of certain rational homology balls $(B_{p,q}, \omega_{std})$ which appear as the Milnor Fibers of the $\frac{1}{p^2}(1, pq-1)$-quotient surface singularities. They are not submanifolds but, rather, well-embedded CW-complexes which enjoy many of the good properties that actual Lagrangian submanifolds have, such as having standard Weinstein neighborhoods. In particular, $L_{2,1}$ is just a Lagrangian $\mathbb{R}P^2$. Lagrangian pinwheels can also be *rationally blown-up*, making it possible to treat them similarly to real projective spaces.

Lagrangian $L_{p,q}$-pinwheels are the symplectic manifestation of the algebro-geometric phenomenon of degenerations of surfaces. As such, determining which Lagrangian pinwheels exist in a symplectic manifold helps us create bridges between the symplectic and algebraic geometry of this manifold, as was pioneered in the beautiful and deep papers of Evans, Smith, Urzua and others. While these papers are concerned only with the *existence* of lagrangian pinwheels, together with J. Hauber we also determined how much a natural class of Lagrangian pinwheels can be *squeezed*.

(A.-Hauber) Let $(S^2 \times S^2, \omega_{a,b})$ be the symplectic manifold where $\omega_{a,b}(A) = a$ and $\omega_{a,b}(B) = B$. Analogously, let $(X_1, \omega_{h,\mu})$ be the symplectic manifold where $\omega_{h,\mu}(H) = h$ and $\omega_{h,\mu}(E) = \mu$. Then:

- There exists a $L_{n,1}$ Lagrangian pinwheel in $(S^2 \times S^2, \omega_{a,b})$, in the $\mathbb{Z}/n$-homology class $A+kB$ if and only if $n = 2k+1$ and
$$\frac{a}{k+1} < b < 2a.$$
- There exists a $L_{n,1}$ Lagrangian pinwheel in $(X_1, \omega_{h,\mu})$, in the $\mathbb{Z}/n$-homology class $kH + (k+1)E$, if and only if $n = 2k$ and
$$\mu < \frac{k}{k+1} h.$$

In particular, none of Khodorovskiy's smooth embeddings of rational homology balls from can be made symplectic.

An important corollary of our work is a *non-squeezing* type theorem for the rational homology balls $B_{n,1}$.

(A.-Hauber) The $B_{n,1}(\alpha)$, namely the rational homology ball of size 1, embeds symplectically into the rational homology cylinder $B_{n,1}(\alpha, \infty)$ if and only if $\alpha \geq 1$.

A common thread of the above results lies in their *constructive* part, meaning in showing that when the corresponding inequalities are satisfied the analogous Lagrangian indeed exists. In most cases, the theory of *almost toric fibrations*, or ATFs, is used. Given a symplectic 4-manifold (X^4, ω), an ATF on it is a fibration from X to some convex polytope of $\mathbb{R}^2$. Following Margaret Symington's "four from two" philosophy, much of the symplectic geometry of X can be read from the *affine geometry* of polytope of the ATF. In particular, one can also create a correspondence between certain lines in the polytope and Lagrangian subspaces in X. These are the so-called *visible Lagrangians.*

Since ATF's provide a very robust, yet flexible, way to represent lagrangians in symplectic 4-manifolds it is natural to ask:

Question. *Consider a symplectic four-manifold (X, ω) and all the possible compatible almost toric fibrations on (X, ω). Are there Lagrangians in (X, ω) that cannot be constructed as visible Lagrangians of some compatible almost toric fibration?*

In other words, the above question asks how much of the symplectic geometry of a four-manifold can be read from the base of an almost toric fibration. It is important to note that every known example shows that all the possible embedding obstructions of a Lagrangian can be read from the almost toric base diagram.

Contact geometry of Links of singularities

The final chapter of this thesis concerns the symplectic and contact topology of the links of isolated hypersurface singularities. This is a subject with very rich interdisciplinary interaction, mostly between algebraic geometry, symplectic/contact geometry and geometric topology; the central question is:

Question. *What algebraic properties of an isolated hypersurface singularity are reflected in the contact geometry of its link?*

Given a hypersurface $X \subset \mathbb{C}^{n+1}$ with an isolated singularity at the origin, the link of the singularity is $L_X = X \cap S^{2n+1}(r)$, for r small enough. In addition, the *field of complex tangencies* of L_X, i.e. the distribution $\xi = JTL_X \cap TL_X$ with respect to the standard complex structure J of $\mathbb{C}^{n+1}$, is a contact structure. Therefore, the pair (L_X, ξ) is a contact manifold and for small enough r the contactomorphism type of (L_X, ξ) is independent of r so (L_X, ξ) depends only on the singularity of X.

In complex dimensions 1 and 2 the topology of L_X alone carries a lot of information about the algebraic geometry of the corresponding singularity. However, in dimensions ≥ 3 the topology of the link becomes very simple and if one wants to retrieve interesting information one must consider more subtle geometric structures of the link, for instance, the natural contact structure ξ it carries. Surprisingly, this contact structure remembers quite a lot of the underlying algebraic geometry. My main results in this area are the following:

(A.-Pasquotto-Zanardini) Any two quasihomogeneous cA_n singularities have contactomorphic links if and only if, up to a holomorphic change of variables, their defining polynomials are deformation equivalent.

The tool used to distinguish the contact structure of the links is *symplectic cohomology.* Symplectic cohomology comes in various flavors; they are all Floer-type theories, having Gromov's theory of pseudoholomorphic curves as an underpinning. While symplectic cohomology can be a very strong invariant, it is notoriously difficult to compute. We relied on recent advancements in *homological mirror symmetry* to carry out these computations. In addition, our computations were thorough enough that we gave a partial answer to a surprising conjecture of Evans-Lekili, relating the existence of small resolutions of *cDV* singularity to the symplectic cohomology of its smoothing.

(A.-Pasquotto-Zanardini) A quasihomogeneous cA_n singularity admits a small resolution if and only if the symplectic cohomology of its smoothing has constant rank on negative degrees.

Theorems 5.7 and 5.7 reflect the flexibility versus rigidity dichotomy that lies in the heart of symplectic and contact geometry; the link of the singularity, viewed as a contact manifold, retains a lot of the rigidity of the algebraic geometry yet, because of its flexibility, can reflect only what algebro-geometric quantity remains invariant under deformation.