Interactions of symplectic topology with singularity theory: Lagrangian pinwheels, Klein bottles and Milnor links of singularities
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Part I

Prelude

Chapter 1

Introduction

Ψάχνεις το μέλλον σου με μαθηματικά (ναι).

2ΚΓ-Μαθηματικά[1]

Δεν έχω διπλώματα, πτυχία, δεν έχω σπουδάσει,
είμαι αναλώσιμος, βρο, ανήκω στην εργατική τάξη.

ΔΠΘ-Κινούμενος στόχος[2]

The main chapters of this thesis consist of the four papers I completed as a PhD student at Leiden University. Despite being separate projects, there exists an underlying common theme in all of them, namely *the ramifications of singularity theory in contact and symplectic geometry.* The objects of study, e.g. the Lagrangian pinwheels or the links of singularities, and the methods of study, e.g. the symplectic rational blow-up/blow-down or the Milnor fibers, are directly coming from algebro-geometric settings. Lagrangian Klein bottles also make an appearance, and while this appearance is not directly related to algebraic geometry, we approach them with the same methodology which we used to approach the Lagrangian pinwheels.

There are many ways to group the contents of this thesis; I ordered them based on the dimensions of the objects. Parts II and III concern 4-dimensional symplectic manifolds and their Lagrangians. Papers [1] and [5] study Lagrangian pinwheels in symplectic rational manifolds while [2] studies squeezing constraints of Lagrangian Klein bottles in $S^2 \times S^2$. Part IV escapes the realm of low-dimensional topology and studies 5-dimensional contact manifolds and their six dimensional symplectic fillings, coming from the Milnor links and fibers of cA_n singularities.

1.1 Lagrangian $\mathbb{R}P^2$s and $L_{3,1}s$ in some rational symplectic manifolds

The study of Lagrangian submanifolds has a central place in symplectic geometry. Focusing on symplectic rational 4-manifolds, meaning the smooth manifolds $S^2 \times S^2$ or $X_k = \mathbb{C}P^2 \#_k \overline{\mathbb{C}P^2}$ for $k \geq 0$, equipped with a symplectic form, a lot of research has been carried out to understand their Lagrangian submanifolds [88], [31], [67], specifically on Lagrangian spheres and tori. By a theorem

[1] You are trying to make your future with mathematics-From: "Mathematics" by 2KG.

[2] I don't have certificates and degrees, I haven't studied/I am expendable, brother, I am working class- From: "Moving Target" by Deltapithita

of Viterbo-Eliashberg (Theorem 1.7.5 in [27]) hinging on Symplectic Field Theory, these are the only possible *orientable* Lagrangians of symplectic rational manifolds.

Recently, non-orientable Lagrangians have also started to gather attention [15],[89],[29], [24]. A particularly interesting feature of a non-orientable Lagrangian L in a symplectic 4-manifold (X,ω) is that it persists under small deformations in an **open** subset of $H^2_{dR}(M)$. This is in contract with the case when L is orientable where the condition $\int_L \omega = 0$ defines a **closed** subset of $H^2_{dR}(X)$ that may carry a Lagrangian L. Thus non-orientable Lagrangians give rise to non-squeezing phenomena, very similar to the classical non-squeezing results for symplectic balls in [43], [75].

The main theorem of Chapter 2 studies when two disjoint Lagrangian $\mathbb{R}P^2$s can be embedded, or, more colloquially, *squeezed*, in (X_3,ω). Let H, E_1, E_2, E_3 be the standard basis of $H_2(X_3,\mathbb{Z})$ and h, μ_1, μ_2, μ_3 be the periods of ω with respect to this basis. We show:

Theorem 1.1.1. *Consider (X_3,ω) and suppose that the $\mathbb{Z}_2$-homology classes H and $E_1+E_2+E_3$ carry Lagrangian $\mathbb{R}P^2$s. Then, these Lagrangians can be made disjoint via a Hamiltonian isotopy if and only if*

$$\sum \mu_i < h.$$

As an example, for a symplectic form ω with $\frac{h}{3} \leq \mu = \mu_1 = \mu_2 = \mu_3 < \frac{h}{2}$ the Lagrangian $\mathbb{R}P^2$s cannot be made disjoint by a Hamiltonian isotopy and therefore any two Lagrangian $\mathbb{R}P^2$s must intersect at least once. On the other hand, when $\mu < \frac{h}{3}$, there is enough room to Hamiltonianly isotope the Lagrangian $\mathbb{R}P^2$s away from each other. When $\mu > \frac{h}{2}$, we note (see Lemma 2.2.3) that there is not even a Lagrangian $\mathbb{R}P^2$ in the class H.

Notably, when $3\mu = h$, i.e. when (X_3,ω) is monotone, in which case it is also know as a *symplectic del Pezzo surface*, any two (non-homologous) Lagrangian $\mathbb{R}P^2$s must intersect, even though they can be made smoothly disjoint. In Section 2.2.2, we generalize this result for the monotone symplectic forms on X_k, for $k = 4, 5, 6$.

Theorem 1.1.2. *In (X_n,ω), for $n \leq 6$ and ω monotone, any two Lagrangian $\mathbb{R}P^2$s must intersect.*

The main technique we use to obstruct the existence of disjoint $\mathbb{R}P^2$s in (X_3,ω) is the *symplectic rational blow-up.* This surgery replaces the Lagrangian $\mathbb{R}P^2$s with symplectic (-4)-spheres, turning (X_3,ω) into $(X_5,\tilde{\omega})$. The constraints on the periods of $\tilde{\omega}$, for example that $\tilde{\mu}_i > 0$, give obstructions to the existence of disjoint Lagrangian projective spaces.

To construct the disjoint Lagrangians when $\mu_1 + \mu_2 + \mu_3 < h$ holds, we use a Nakai-Moishezon argument, inspired by [89]. Interestingly, we could not find such a general construction using the theory of almost toric fibrations. This is in contrast with the fact that almost toric fibrations can be used successfully to construct each Lagrangian $\mathbb{R}P^2$ separately.

In the last part of Chapter 2, we use again the symplectic rational blow-up to determine which symplectic forms on $S^2 \times S^2$ carry a Lagragian $L_{3,1}$ pinwheel, answering question *J.7* from J. Evans' book [32]. Lagrangian pinwheels are simple Lagrangian CW-complexes that were introduced by T. Khodorovskiy [49]; they arise as the Lagrangian skeletons to rational homology balls which are smoothings of cyclic quotient $\frac{1}{p^2}(1, pq-1)$ singularities. In particular, $L_{2,1}$ is the usual Lagrangian $\mathbb{R}P^2$. We prove:

Theorem 1.1.3. *Consider the symplectic manifold $(S^2 \times S^2, \omega_{a,b})$, where a, b are the symplectic areas on the product factors $A = [pt \times S^2]$ and $B = [S^2 \times pt]$, respectively. Then $(S^2 \times S^2, \omega_{a,b})$ carries a Lagrangian $L_{3,1}$ pinwheel in the $\mathbb{Z}/3$-homology class $A + B$ if and only if*

$$\frac{a}{2} < b < 2a.$$

Lagrangian pinwheels will be systematically studied in Chapter 3, generalizing the above non-squeezing to more general $L_{n,1}$ pinwheels.

1.2 Lagrangian $L_{n,1}$ Pinwheels in X_1 and $S^2 \times S^2$

In Chapter 3, we use the rational blow-up to understand which symplectic forms on $X_1 = \mathbb{C}P^2$ and $S^2 \times S^2$ carry Lagrangian $L_{n,1}$ pinwheels. For X_1, all possible symplectic forms are diffeomorphic to the forms $\omega_{h,\mu}$ which are obtained by blowing up a symplectic ball in $(\mathbb{C}P^2, \omega_{FS})$, where the line H has area $\int_H \omega_{FS}$ and the exceptional divisor of the blow-up has area μ. Similarly, any symplectic form on $S^2 \times S^2$ is diffeomorphic to a split symplectic form $\omega_{a,b} = a\sigma \oplus b\sigma$ where $\int_{S^2} \sigma = 1$

We restrict our attention to *liminal pinwheels*[3], i.e. pinwheels that are disjoint from a smoothly embedded sphere. This allows us to use results of [63] to determine how blowing up the pinwheel affects the rational ruled surface. Whether or not there exist pinwheels in ruled surfaces that are not liminal is subject of ongoing work.

Using the theory of almost toric fibrations we show that the obstructions we obtain are sharp, by constructing visible Lagrangian pinwheels for all cases that are not obstructed. These visible pinwheels are disjoint from a symplectic normal-crossings divisor and are, as such, liminal.

Combining the constructions of the visible pinwheels and the obstructions of liminal pinwheels we obtain:

Theorem 1.2.1.
- *The $\mathbb{Z}/(2k)$-homology class class $kH + (k+1)E$ of the symplectic manifold $(X_1, \omega_{h,\mu})$ carries a liminal $L_{2k,1}$ pinwheel if and only if*

$$\mu < \frac{k}{k+1}h.$$

- *The $\mathbb{Z}/(2k+1)$-homology class class $A + kB$ of the symplectic manifold $(S^2 \times S^2, \omega_{a,b})$ carries a liminal $L_{2k+1,1}$ pinwheel if and only if*

$$\frac{a}{k+1} < b < 2a.$$

The special case of $n = 2$, combined with results from [15] which show that every Lagrangian $\mathbb{R}P^2$ in X_1 is liminal, implies that there exists a Lagrangian $\mathbb{R}P^2$ in $(X_1, \omega_{h,\mu})$ if and only if $\mu < \frac{h}{2}$. In particular, this answers a question of Kronheimer in [53], by showing that the symplectic manifold $(X_1, \omega_{h,\mu})$, where $\frac{h}{2} < \mu < h$, does not carry a Lagrangian $\mathbb{R}P^2$.

Furthermore, we apply the above results for the Lagrangian pinwheel to obtain various obstructions for symplectic embeddings of the rational homology balls $B_{n,1}$ in open symplectic manifolds with boundary that can be compactified to rationally ruled surfaces.

The first class of such open symplectic manifolds are standard symplectic disc bundles (V_k, τ) over symplectic spheres with self-intersection k, as introduced by Biran in [14]. By projectivizing the bundle, or equivalently by adding a section at infinity, we obtain a rationally ruled surface. If there is a symplectic embedding of the rational homology ball $B_{n,1}$ into V_k, by compactifying V_k we obtain an embedding of a liminal Lagrangian pinwheel in a rational ruled surface, therefore we have:

[3]The name *liminal* comes from the fact that the pinwheels we study share some weak properties with visible pinwheels, allowing them to be treated as such after being blown-up. The word "liminal" is playfuly used as "almost visible".

Corollary 1.2.2. *Suppose that $n \geq 2$ is an integer. Then there are no symplectic embeddings of the rational homology ball $B_{n,1}$ into (V_{-n-1}, τ), even though smooth ones exist.*

In addition, there are no symplectic embeddings of $B_{2k+1,1}$ into (V_{-4}, τ), even though smooth ones exist.

In a similar manner, we examine embeddings of different shapes of $B_{n,1}$. Generalizing the usual definitions of symplectic ellipsoids $E(\alpha, \beta)$ and symplectic balls $B(\alpha) = E(\alpha, \alpha)$, we use almost toric fibrations to endow the rational homology balls $B_{n,1}$ with different symplectic forms, mimicking the properties of the usual ellipsoids and polydiscs. We call the resulting manifolds the *rational homology ellipsoids* $B_{n,1}(\alpha, \beta)$ and, in this context we call the distinguished ellipsoids $B_{n,1}(\alpha, \alpha)$ the *rational homology ball* $B_{n,1}(\alpha)$. Similarly, by $B_{n,1}(\alpha, \infty)$ we define the *rational homology cylinder*. We show that the classical non-squeezing theorem of Gromov between balls and cylinder extends to the "rational homology" case:

Corollary 1.2.3. *There exists a symplectic embedding $\iota : B_{n,1}(1) \hookrightarrow B_{n,1}(\alpha, \infty)$ if and only if $\alpha \geq 1$.*

1.3 Squeezing Lagrangian Klein bottles in $S^2 \times S^2$

In Part III we study Lagrangian Klein bottles in $S^2 \times S^2$. Considering the usual homological basis $A = [S^2 \times pt]$ and $B = [pt \times S^2]$, any symplectic form ω of $S^2 \times S^2$ is symplectomorphic, after scaling, to the product form ω_λ, where $\int_A \omega_\lambda = 1$ and $\int_B \omega_\lambda = \lambda$. An elementary but astute argument due to Evans [29], constructs a Lagrangian Klein bottle in the class B, when $\lambda < 2$, as a visible Lagrangian of the usual toric fibration on $(S^2 \times S^2, \omega_\lambda)$. Therefore it is natural to ask if there are any Lagrangian Klein bottles in B when $\lambda \geq 2$.

We show that, in fact, Lagrangian Klein bottles exist only when visible ones do:

Theorem 1.3.1. *There exists a Lagrangian Klein bottle in the $\mathbb{Z}/2$-homology class B in $(S^2 \times S^2, \omega_\lambda)$ if and only if*

$$\lambda < 2$$

Even though Klein bottles are rather different objects than pinwheels, the method of proof is very similar to those followed in the previous chapters: on the one hand, the constructive part comes from studying the moment polytopes of almost toric fibrations. On the other hand, the obstructive part follows by a topological argument, combined with the theory of pseudoholomorphic curves: assuming the existence of a Lagrangian Klein bottle L we perform Luttinger surgery around it, which was first introduced by Nemirovski in [79], to get a new symplectic manifold $(\widetilde{X}_L, \tilde{\omega})$. We can then study L by examining $(\widetilde{X}_L, \tilde{\omega})$.

We first determine the diffeomorphism type of $\widetilde{X}_L$, assuming that a Lagrangian Klein bottle L in B exists. According to results of Liu in [70], Seiberg-Witten theory dictates that X_L needs to be rational and since $b_2(X) = b_2(X_L)$, it can be only $S^2 \times S^2$ or X_1. Therefore we need to understand whether X_L is spin or not, or equivalently, whether all homology classes have even self intersection. To do that, we first show that, after the Luttinger surgery, there always exists some embedded surface representing a class $\Sigma \in H_2(X_L, \mathbb{Z})$ with $2\Sigma^2 = \mu(L) \mod 4$, where $\mu(L)$ is the Maslov index of L in $S^2 \times S^2$. We then appeal to a result of [25] that claims that all Lagrangian (in fact, totally real) Klein bottles in the class B must have Maslov index $2 \mod 4$, thus $\Sigma^2 = 1 \mod 2$ and so X_L cannot be spin. Therefore X_L is necessarily X_1.

We conclude by computing the cohomology class of $[\tilde{\omega}]$. This is done by using the fact that there is a $\mathbb{Q}$-homology basis of $H_2(S^2 \times S^2, \mathbb{Q})$ whose elements are represented by cycles disjoint from L. Thus, the same cycles form a $\mathbb{Q}$-basis of $H_2(X_L, \mathbb{Q})$. In particular, these cycles will have the same symplectic area as before, since Luttinger surgery is localized only around L. These considerations imply that the exceptional class E of X_1 has $\tilde{\omega}$-area $1 - \frac{\lambda}{2}$. Since Seiberg-Witten theory dictates that all such classes need to have positive area, we conclude that $\lambda < 2$.

1.4 Contact topology of Links of cA_n singularities

In Part IV, the final part of this thesis, we examine links of quasihomogeneous cA_n singularities, and the natural contact structure they are equipped with.

It is a well-known fact that given an isolated hypersurface singularity $X \subset \mathbb{C}^{N+1}$, its Milnor link

$$L_X = X \cap S^{2N+1}(\epsilon)$$

is a smooth $2N-1$ manifold, which inherits the natural contact structure ξ from (S^{2N+1}, ξ_{std}). The contact manifold (X_L, ξ) is an invariant of X in the sense that any two biholomorphic singularities have contactomorphic Milnor links [98].

Specializing to the case that $N = 3$, a quasihomogeneous cA_n singularity is equivalent to one of the following three types:

$$w^{a,b}_{\text{chain}} \coloneqq x_1^2 + x_2^2 + x_3^a x_4 + x_4^b$$

$$w^{c,d}_{\text{loop}} \coloneqq x_1^2 + x_2^2 + x_3^c x_4 + x_3 x_4^d$$

$$w^{e,f}_{\text{Fermat}} \coloneqq x_1^2 + x_2^2 + x_3^e + x_4^f$$

By a result of Boyer-Galicki [16], one can use Smale's classification of simply connected 5-manifolds to show that any singularity of the above type has link diffeomorphic to S^5 or a k-fold connected sum $\#_k S^2 \times S^3$ for $k \geq 1$. For example, the Fermat type singularities for $gcd(e, f) = k$ have link diffeomorphic to $\#_{k-1} S^2 \times S^3$, where by convention $\#_0 S^2 \times S^3 = S^5$, with similar formulas for the other two families. From this, it is evident that many different singularities have diffeomorphic links and thus just knowing the diffeomorphism type of the link does not give us very interesting information about the geometry of the singularity.

On the other hand, recent developments such as [77] show that the contact structure of the link (X_L, ξ) *does* retains a lot of the rigidity of the singularity. An optimistic conjecture of Evans-Lekili in [35], claims that the contact structure of the link contains information about the *small resolutions* of the *cDV* singularity X, i.e. resolutions such that the exceptional locus consists only of curves.

Associated to a singularity X is also the Milnor fiber F_X, which is a generic smoothing of the singularity. F_X has the structure of a Stein manifold and, using standard arguments, can be viewed, as a Stein filling of the link (L_X, ξ).

A very strong invariant of F_X is the symplectic cohomology ring $SH^*(F_X)$, which, roughly, comes from counting certain pseudoholomorphic cylinders on F_X which connect different Reeb orbits on L_X. While SH^* is extremely hard to compute directly, [35] builds on existing results on *homological mirror symmetry* for quasihomogeneous cA_n singularities, to compute algorithmically $SH^*(F_X)$ in terms of a dual singularity $\check{X}$.

Since cA_n singularities are terminal, an application of McLean's theorem from [77] shows that the ranks of $SH^*(F_X)$ depend only the the contact structure of (L_X, ξ). In addition, $SH^*(F_X)$ has also a graded lie bracket, the *Gerstenhaber bracket*, and [35] shows that the action of this bracket also depends only on the link (L_X, ξ) and not on the chosen filling. Therefore, $SH^*(F_X)$ and its

Gerstenhaber structure are *contact invariants* of the link, even though to compute them we use the auxiliary data structure coming from F_X.

Having said that, the conjecture of Evans-Lekili claims:

Conjecture 1. *Let F be the Milnor fiber of an isolated cDV singularity. Then, the singularity admits a small resolution such that the exceptional set has ℓ irreducible components if and only if $SH^*(F)$ has rank ℓ in every negative degree.*

In the first part of Chapter IV we give a closed formula for the ranks of $SH^*(F_X)$ and confirm the conjecture for all invertible cA_n singularities.

In the second part of the chapter, we further compute the Gerstenhaber structure on $SH^*(F_X)$ and compare it between links of cA_n singularities which are diffeomorphic, to understand how their contact structures are related.

First we note that non-equivalent singularities might have contactomorphic links for soft reasons. We define a *smooth deformation* of singularities which implies that the smoothly deformed singularities have contactomorphic links, using Gray's stability theorem. Essentially, smooth deformations $X_{0\leq t\leq 1}$ of singularities are such that the families of links L_t and the fibers F_t give smooth isotopies between (L_0, F_0) and (L_1, F_1). There are many examples of smoothly deformation equivalent singularities, for example Fermat, chain and loop singularities with $\gcd(e, f) + 2 = \gcd(a, b-1) + 1 = \gcd(c-1, d-1)$.

Having established a criterion that provides a contactomorphism between links of different singularities, we want to further understand when two singularities do not have contactomorphic links. This is done by explicitly computing and comparing the Gerstenhaber structure on $SH^*(F_X)$ for all possible cA_n singularities, which leads to a complete classification of the contact type of the associated links:

Theorem 1.4.1. *The links of any two invertible cA_n singularities are contactomorphic if and only if the singularities are smoothly deformation equivalent.*