



Universiteit
Leiden

The Netherlands

Interactions of symplectic topology with singularity theory: Lagrangian pinwheels, Klein bottles and Milnor links of singularities

Adaloglou, N.

Citation

Adaloglou, N. (2025, October 10). *Interactions of symplectic topology with singularity theory: Lagrangian pinwheels, Klein bottles and Milnor links of singularities*. Retrieved from <https://hdl.handle.net/1887/4267032>

Version: Publisher's Version

License: [Licence agreement concerning inclusion of doctoral thesis in the Institutional Repository of the University of Leiden](#)

Downloaded from: <https://hdl.handle.net/1887/4267032>

Note: To cite this publication please use the final published version (if applicable).

Interactions of symplectic topology with singularity theory

Lagrangian pinwheels, Klein bottles and Milnor links of singularities

Nikolas Adaloglou

Mathematisch Instituut
Leiden University
The Netherlands
March 2025

**Interactions of symplectic geometry with
singularity theory**

Proefschrift

ter verkrijging van
de graad van doctor aan de Universiteit Leiden,
op gezag van rector magnificus prof.dr.ir. H. Bijl,
volgens besluit van het college voor promoties
te verdedigen op vrijdag 10 oktober 2025

klokke 11.30 uur

door

Nikolas Adaloglou

geboren te Athene, Griekenland

in 1996

Promotores:

Prof. dr. R.M. van Luijk

Prof. dr. S.J. Edixhoven †

Copromotor:

Dr. F. Pasquotto

Promotiecommissie:

Prof. dr. ir. G. L. A. Derks

Prod. dr. D. S. T. Holmes

Prof. dr. J. D. Evans (Lancaster University)

Prof. dr. A. Keating (Cambridge University)

Prof. dr. F. Schlenk (Université de Neuchâtel)

Contents

I	Prelude	7
1	Introduction	9
1.1	Lagrangian $\mathbb{R}P^2$ s and $L_{3,1}$ s in some rational symplectic manifolds	9
1.2	Lagrangian $L_{n,1}$ Pinwheels in X_1 and $S^2 \times S^2$	11
1.3	Squeezing Lagrangian Klein bottles in $S^2 \times S^2$	12
1.4	Contact topology of Links of cA_n singularities	13
II	Lagrangian pinwheels	15
2	Two $\mathbb{R}P^2$s and an $L_{3,1}$	17
2.1	Recollections	19
2.1.1	Lagrangian pinwheels and the symplectic rational blow-up/down	19
2.1.2	Basic Symplectic Geometry of rational surfaces	20
2.1.3	Homology classes of Lagrangian $\mathbb{R}P^2$ s.	22
2.2	Lagrangian projective planes in rational manifolds	24
2.2.1	In X_3	24
2.2.2	In \mathbb{D}_k	30
2.3	An $L_{3,1}$ pinwheel in $S^2 \times S^2$	31
3	$L_{n,1}$s in ruled surfaces	35
3.1	Introduction	35
3.1.1	Main results	36
3.1.2	Organization	38
3.1.3	Acknowledgments	38
3.2	Preliminaries	38
3.2.1	Notation	38
3.2.2	Recollections of almost toric fibrations	39
3.2.3	Lagrangian Pinwheels	40
3.3	Rational blow-ups on rational and ruled surfaces	43
3.3.1	Rational and ruled symplectic surfaces	43
3.3.2	Blowing-up Lagrangian $L_{n,1}$ -pinwheels	44
3.3.3	A motivating example: Lagrangian $\mathbb{R}P^2$ s and Kronheimer's question	46
3.4	Constructing Lagrangian pinwheels	49
3.4.1	Lagrangian $L_{2k+1,1}$ -Pinwheels in $S^2 \times S^2$	49
3.4.2	Lagrangian $L_{2k,1}$ -Pinwheels in X_1	51
3.4.3	Compactifications of rational homology balls	52
3.5	Obstructing liminal Pinwheels	56
3.5.1	Preparations	56
3.5.2	Necessity of the inequalities	61
3.6	Corollaries of A and B	62

3.6.1	Khodorovskiy's smooth embeddings of rational homology balls	62
3.6.2	A non-squeezing result	63
3.7	Appendix A: Solving for μ_{n-1} and μ_n	64
3.8	Appendix B: Blowing up while taking account of sizes	65
3.9	Appendix C: The orthogonal complement of a pinwheel	66
III Interlude: Klein bottles		69
4	Lagrangian Klein bottles	71
4.1	Introduction	71
4.1.1	Basics on Klein bottles and Maslov indexes	72
4.2	Klein bottle Luttinger surgery	74
4.2.1	Defining the surgery	74
4.2.2	Effect of Luttinger surgery on X	75
4.3	Chern and Maslov classes	77
4.3.1	The Maslov index for Lagrangian Klein bottles in $S^2 \times S^2$	77
4.4	Finishing the proof	78
IV Contact and symplectic topology of cA_n singularities		81
5	Contact geometry of links of singularities	83
5.1	Introduction	83
5.2	Background	88
5.2.1	Compound Du Val singularities and small resolutions	88
5.2.2	The link of a singularity and its contact structure	89
5.2.3	Symplectic cohomology and its algebraic structure	90
5.2.4	Mirror symmetry for invertible polynomials	91
5.3	The general framework for the computations	93
5.3.1	Setting and notation	93
5.3.2	Determining the good pairs	95
5.4	The main formulas	98
5.4.1	The chain-type polynomials	99
5.4.2	The loop-type polynomials	102
5.4.3	The Fermat-type polynomials	104
5.5	Applications of the bigrading to contact topology	108
5.5.1	Smooth deformations of hypersurface singularities	108
5.5.2	Bigradings on $HH^*(\mathcal{C}_w)$ and $SH^*(F)$	110
5.5.3	Some useful contact invariants	112
5.5.4	Comparing different links when $\rho \geq 2$	114
5.5.5	The remaining case: $\rho \leq 1$	116
5.6	Appendix: Index positivity	119
5.7	Appendix: A few explicit calculations	121
Summary		125
Samenvatting		129
Acknowledgments		133
Curriculum Vitae		135