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Type invariants for non-abelian odometers

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Abstract. In this work, we introduce the type and typeset invariants for equicontinuous group actions on Cantor sets; that is, for generalized odometers. These invariants are collections of equivalence classes of asymptotic Steinitz numbers associated to the action. We show the type is an invariant of the return equivalence class of the action. We introduce the notion of commensurable typesets and show that two actions which are return equivalent have commensurable typesets. Examples are given to illustrate the properties of the type and typeset invariants. The type and typeset invariants are used to define homeomorphism invariants for solenoidal manifolds.

Key words: solenoids, odometers, Cantor actions, profinite groups, Steinitz numbers
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1. Introduction

The odometer actions of $\mathbb{Z}$ are classified by an infinite sequence of positive integers, up to a suitable notion of equivalence. The odometer actions of the abelian group $\mathbb{Z}^n$ for $n > 1$ cannot be classified by any set of invariants [44], but the works of Arnold [2, 3], Butler [8], Fuchs [21], and Thomas [45], for example, have introduced invariants that can be used for distinguishing $\mathbb{Z}^n$ -odometers and classifying special subclasses of such actions. The odometer actions of non-abelian groups are less well understood, as many interesting new phenomena arise for such actions [24–27]. In this paper, we introduce the *type* and *typeset* invariants for odometers, which are invariants by conjugation and return equivalence. We give examples that show all possible types can be realized. In the last section of this work, we apply our results to the classification, up to homeomorphism, of solenoidal manifolds. We first introduce some basic notions.

1.1. *Types.* Let $\vec{m} = \{m_i \mid 1 \leq i < \infty\}$ be an infinite collection of positive integers. The *Steinitz number* (or sometimes called the *supernatural number*), defined by $\vec{m}$, is the infinite product

$$\xi(\vec{m}) = \text{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\}, \quad (1.1)$$

where lcm denotes the least common multiple of the collection of integers. A Steinitz number ξ can be uniquely written as the formal product over the set of primes Π ,

$$\xi = \prod_{p \in \Pi} p^{\chi_\xi(p)}, \quad (1.2)$$

where the *characteristic function* $\chi_\xi: \Pi \rightarrow \{0, 1, \dots, \infty\}$ counts the multiplicity with which a prime p appears in the infinite product ξ . Associated to a Steinitz number ξ is its *prime spectrum*.

Definition 1.1. Let $\Pi = \{2, 3, 5, \dots\}$ denote the set of primes. Given $\xi = \prod_{p \in \Pi} p^{\chi(p)}$, define:

$$\begin{aligned} \pi(\xi) &= \{p \in \Pi \mid \chi(p) > 0\} \quad \text{the prime spectrum of } \xi; \\ \pi_f(\xi) &= \{p \in \Pi \mid 0 < \chi(p) < \infty\} \quad \text{the finite prime spectrum of } \xi; \\ \pi_\infty(\xi) &= \{p \in \Pi \mid \chi(p) = \infty\} \quad \text{the infinite prime spectrum of } \xi. \end{aligned}$$

Definition 1.2. Two Steinitz numbers ξ and ξ' are said to be *asymptotically equivalent* if there exists finite integers $m, m' \geq 1$ such that $m \cdot \xi = m' \cdot \xi'$, and we then write $\xi \stackrel{a}{\sim} \xi'$. The asymptotic equivalence class of a Steinitz number ξ is called its *type* and is denoted by $\tau[\xi]$.

Note that if $\xi \stackrel{a}{\sim} \xi'$, then $\pi_\infty(\xi) = \pi_\infty(\xi')$. The property that $\pi_f(\xi)$ is an *infinite* set is also preserved by asymptotic equivalence of Steinitz numbers, so is an invariant of type.

The *type* of a Steinitz number was introduced by Baer in [5, §2], where the terminology *genus* was used for the type. Baer used the type to analyze the classification problem for rank n subgroups of $\mathbb{Q}^n$, where the rank of a subgroup $\mathcal{A} \subset \mathbb{Q}^n$ is the maximum cardinality of a linearly independent subset of $\mathcal{A}$. More detailed discussions can be found in the works by Arnold [2, §1], by Wilson [48, Ch. 2], by Ribes and Zalesskii [39, Ch. 2.3], and in §3.

1.2. *Cantor actions.* A Cantor action is a group action on a Cantor set by homeomorphisms. Let $(\mathfrak{X}, \Gamma, \Phi)$ denote a Cantor action $\Phi: \Gamma \times \mathfrak{X} \rightarrow \mathfrak{X}$. We write $g \cdot x$ for $\Phi(g)(x)$ when appropriate. The action is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x) = \{g \cdot x \mid g \in \Gamma\}$ is dense in $\mathfrak{X}$. The action $(\mathfrak{X}, \Gamma, \Phi)$ is *equicontinuous* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ and $g \in \Gamma$, we have that $d_{\mathfrak{X}}(x, y) < \delta$ implies $d_{\mathfrak{X}}(g \cdot x, g \cdot y) < \varepsilon$. The property of being equicontinuous is independent of the choice of the metric on $\mathfrak{X}$, which is compatible with the topology of $\mathfrak{X}$.

Definition 1.3. [13, 14] A minimal equicontinuous action $(\mathfrak{X}, \Gamma, \Phi)$ on a Cantor space $\mathfrak{X}$ is said to be a Γ -*odometer*, or just an *odometer* when the group Γ is clear from the context.

A Γ -odometer has an alternate description in terms of inverse limits of finite actions of Γ , which is the *algebraic model* for the action. Section 2 discusses in further detail the properties of Γ -odometers.

Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a descending chain of finite index subgroups. Note that the subgroups Γ_ℓ are *not assumed to be normal* in Γ . Let $X_\ell = \Gamma / \Gamma_\ell$ and note that Γ acts transitively on the left on the finite set X_ℓ . The inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a natural Γ -invariant quotient map $p_{\ell+1}: X_{\ell+1} \rightarrow X_\ell$. Introduce the inverse limit

$$\begin{aligned} X_\infty &\equiv \varprojlim \{p_{\ell+1}: X_{\ell+1} \rightarrow X_\ell \mid \ell \geq 0\} \\ &= \{(x_0, x_1, \dots) \in X_\infty \mid p_{\ell+1}(x_{\ell+1}) = x_\ell \text{ for all } \ell \geq 0\} \subset \prod_{\ell \geq 0} X_\ell. \end{aligned} \quad (1.3)$$

Then, X_∞ is a Cantor space with the Tychonoff topology, where the left actions of Γ on the factors X_ℓ induce a minimal isometric action denoted by $\Phi_\infty: \Gamma \times X_\infty \rightarrow X_\infty$, where X_∞ is given the metric induced by the discrete Γ -invariant metrics on factors X_ℓ . The following is a folklore result.

THEOREM 1.4. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer, then there exists a group chain $\mathcal{G}$ and a homeomorphism $\Psi: \mathfrak{X} \rightarrow X_\infty$ conjugating the action Φ with the action Φ_∞ .*

The choice of the odometer $(X_\infty, \Gamma, \Phi_\infty)$ derived from a group chain $\mathcal{G}$ in Theorem 1.4 is called an *algebraic model* for $(\mathfrak{X}, \Gamma, \Phi)$. The construction $\mathcal{G}$ uses an adapted neighborhood basis for the odometer, as defined by Definition 2.2. The choice of the group chain $\mathcal{G}$ is not unique, but two choices are related by a ‘tower equivalence’ of group chains as described in §4.

1.3. Results. We associate to a group chain $\mathcal{G}$ a Steinitz number

$$\xi(\mathcal{G}) = \text{lcm}\{\text{Index}[\Gamma : \Gamma_\ell] = \#(\Gamma / \Gamma_\ell) \mid \ell > 0\}. \quad (1.4)$$

Our first result is that $\xi(\mathcal{G})$ is independent of the choice of the group chain $\mathcal{G}$. In fact, more is true, as the asymptotic equivalence class of $\xi(\mathcal{G})$ —its typeset—is invariant under return equivalence. As explained in §2, the notion of return equivalence is the condition that the restrictions of the actions to adapted clopen subsets are isomorphic. It is the group action counterpart of Morita equivalence for groupoids and is a natural equivalence relation that arises in many contexts.

THEOREM 1.5. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer. There is a well-defined Steinitz number $\xi(\mathfrak{X}, \Gamma, \Phi)$ associated to the action, called its Steinitz order. Moreover, if the Γ' -odometer $(\mathfrak{X}', \Gamma', \Phi')$ is return equivalent to $(\mathfrak{X}, \Gamma, \Phi)$, then $\tau[\xi(\mathfrak{X}, \Gamma, \Phi)] = \tau[\xi(\mathfrak{X}', \Gamma', \Phi')]$; that is, their types are equal.*

The type of the action is defined to be $\tau[\mathfrak{X}, \Gamma, \Phi] = \tau[\xi(\mathcal{G})]$ for a choice of group chain $\mathcal{G}$.

The typeset invariants of an odometer are a more refined set of invariants that are used, in particular, to distinguish actions of abelian groups of rank greater than 1. When Γ is

non-abelian, their definition requires we introduce the normal chain associated to a group chain $\mathcal{G}$.

Recall that given a subgroup $H \subset G$, the normal core $C(H) \subset H$ is its largest normal subgroup. If H has finite index, then $C(H)$ also has finite index in G . We associate to $\mathcal{G}$ the group chain $\mathcal{G}^c = \{\Gamma = \Gamma_0 \supset C_1 \supset C_2 \supset \cdots\}$, where $C_\ell = C(\Gamma_\ell) \subset \Gamma_\ell$ is its normal core in Γ .

Let $\gamma \in \Gamma$ and let $\langle \gamma \rangle \subset \Gamma$ denote the subgroup it generates. For $\ell > 0$, the intersection $\langle \gamma \rangle_\ell = \langle \gamma \rangle \cap C_\ell$ is a subgroup of finite index in $\langle \gamma \rangle \cong \mathbb{Z}$. We thus obtain a group chain in $\langle \gamma \rangle$, denoted

$$\mathcal{C}_\gamma = \{\langle \gamma \rangle \supset \langle \gamma \rangle_1 \supset \langle \gamma \rangle_2 \supset \cdots\}. \quad (1.5)$$

Definition 1.6. Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a group chain. For $\gamma \in \Gamma$, the type $\tau[\gamma]$ of γ is the asymptotic equivalence class of the Steinitz order

$$\xi(\gamma) = \text{lcm}\{\#(\langle \gamma \rangle / \langle \gamma \rangle_\ell) \mid \ell > 0\}. \quad (1.6)$$

The *typeset* of $\mathcal{G}$ is the collection

$$\Xi[\mathcal{G}] = \{\tau[\gamma] \mid \gamma \in \Gamma\}. \quad (1.7)$$

Note that we allow $\gamma \in \Gamma$ to be the identity above, where $\tau[e] = \{0\}$.

Here are our next two results.

THEOREM 1.7. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer and let $\mathcal{G}$ be a group chain model. Then, the typeset $\Xi[\mathcal{G}]$ is independent of the choice of $\mathcal{G}$. Thus, $\Xi[\mathcal{G}]$ is an invariant under isomorphism of the action.*

The typeset of the action is defined to be $\Xi[\mathfrak{X}, \Gamma, \Phi] = \Xi[\mathcal{G}]$ for a choice of $\mathcal{G}$.

THEOREM 1.8. *If the Γ -odometer $(\mathfrak{X}, \Gamma, \Phi)$ and Γ' -odometer $(\mathfrak{X}', \Gamma', \Phi')$ are return equivalent, then their typesets $\Xi[\mathfrak{X}, \Gamma, \Phi]$ and $\Xi[\mathfrak{X}', \Gamma', \Phi']$ are commensurable.*

The notion of commensurable typesets is modeled on the notion of commensurable groups and is given in Definition 3.9. The type of an individual $\gamma \in \Gamma$ need not be preserved by return equivalence; rather, it is the collection of all these types—the typeset—that is preserved.

Various classes of Cantor actions admit stronger results. For example, we have the following corollary.

COROLLARY 1.9. *Suppose that Γ and Γ' are abelian. If the odometers $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are return equivalent, then their typesets $\Xi[\mathfrak{X}, \Gamma, \Phi]$ and $\Xi[\mathfrak{X}', \Gamma', \Phi']$ are equal.*

Example 5.3 shows that the typesets need not be equal for return equivalent odometers if one of the groups is abelian and the other is only virtually abelian. Section 5 also gives a selection of $\mathbb{Z}^n$ -odometers to illustrate the definitions of types and typesets. In particular, it is shown that all types can be realized for all $n \geq 1$.

A finitely generated group Γ is said to be *renormalizable* if there exists a proper self-embedding $\varphi: \Gamma \rightarrow \Gamma$ whose image has finite index [29]. Another name for this property is that Γ is *finitely non-co-Hopfian*. The embedding φ defines a group chain in Γ that gives rise to a Γ -odometer $(X_\varphi, \Gamma, \Phi_\varphi)$. The properties of the odometers obtained this way are studied in the work [29]. The classification problem for proper self-embeddings $\varphi: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ was related to the class field theory of number fields in the works of Sabitova [41, 42].

THEOREM 1.10. *Let $\varphi: \Gamma \rightarrow \Gamma$ be a renormalization with associated Γ -odometer $(X_\varphi, \Gamma, \Phi_\varphi)$. Then, the action has a well-defined type $\tau[X_\varphi, \Gamma, \Phi_\varphi]$ and typeset $\Xi[X_\varphi, \Gamma, \Phi_\varphi]$. Let $\varphi': \Gamma' \rightarrow \Gamma'$ be a renormalization of a second group Γ' , and assume the odometers $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X_{\varphi'}, \Gamma', \Phi_{\varphi'})$ are return equivalent, then their types and typesets are equal.*

Many finitely generated nilpotent groups admit renormalizations, as well as some other classes of groups, as discussed in [29]. Examples 5.4 and 5.5 give constructions using renormalizable groups which illustrate the conclusions of Theorem 1.10. For a finitely generated, torsion-free nilpotent group Γ , there are relations between type invariants and the dynamics of Γ -odometers [28].

Another source of examples arises from group actions on rooted trees, which induce odometer actions on the Cantor sets of ends of the trees. Section 6.1 discusses *d-regular odometers*, which are odometers that have faithful representations as actions on *d*-regular trees.

THEOREM 1.11. *A Γ -odometer $(\mathfrak{X}, \Gamma, \Phi)$ which is *d*-regular has finite typeset $\Xi[X_\infty, \Gamma, \Phi]$. More precisely, suppose $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to an action on a *d*-ary rooted tree, for some $d \geq 2$. Let P_d be the set of distinct prime divisors of the integers $\{2, \dots, d\}$, and let $N_d = |P_d|$. Then, the cardinality of the typeset satisfies*

$$|\Xi[X_\infty, \Gamma, \Phi]| \leq \sum_{k=0}^{N_d} \binom{N_d}{k} = \sum_{k=0}^{N_d} \frac{N_d!}{(N_d - k)! k!}. \quad (1.8)$$

Moreover, each type $\tau \in \Xi[X_\infty, \Gamma, \Phi]$ is represented by a Steinitz number ξ with empty finite prime spectrum $\pi_f(\xi)$, and so $\pi(\xi) = \pi_\infty(\xi)$.

Example 6.8 gives examples of *d*-regular odometers that realize the typesets in Theorem 1.11.

Finally, we mention two open problems about the type invariants for odometers.

Problem 1.12. (Realization) Given a finitely generated, torsion-free group Γ , are there restrictions on the types and typesets that can be realized by a Γ -odometer?

The work of Arnold [2] implies that every typeset can be realized by a $\mathbb{Z}^n$ -odometer. It seems likely that this is also true when Γ is a nilpotent group. The solution to Problem 1.12 for a general finitely generated group Γ is unknown.

Problem 1.13. (Classification) For an odometer $(\mathfrak{X}, \Gamma, \Phi)$ with type $\tau[\mathfrak{X}, \Gamma, \Phi]$ and typeset $\Xi[\mathfrak{X}, \Gamma, \Phi]$, classify the Γ -odometers with the same type and typeset.

1.4. Solenoidal manifolds. The classification of minimal equicontinuous Cantor actions is closely related to the classification problem of solenoidal manifolds, and the type invariants defined above yield homeomorphism invariants for solenoidal manifolds. We briefly recall their definition and state the applications, with details given in §7.

A *presentation* is a sequence of *proper finite covering* maps $\mathcal{P} = \{q_\ell: M_\ell \rightarrow M_{\ell-1} \mid \ell \geq 1\}$, where each M_ℓ is a compact connected manifold without boundary of dimension n . The inverse limit

$$\mathcal{M}_{\mathcal{P}} \equiv \varprojlim \{q_\ell: M_\ell \rightarrow M_{\ell-1}\} \subset \prod_{\ell \geq 0} M_\ell \quad (1.9)$$

is the *weak solenoid*, or *solenoidal manifold*, associated to $\mathcal{P}$. The set $\mathcal{M}_{\mathcal{P}}$ is given the relative topology, induced from the product topology, so that $\mathcal{M}_{\mathcal{P}}$ is compact and connected. The initial factor M_0 is called the *base manifold* of $\mathcal{M}_{\mathcal{P}}$.

McCord showed in [32] that a solenoidal manifold $\mathcal{M}_{\mathcal{P}}$ is a foliated space with foliation $\mathcal{F}_{\mathcal{P}}$, in the sense of [33], where the leaves of $\mathcal{F}_{\mathcal{P}}$ are coverings of the base manifold M_0 via the projection map onto the first factor, $\hat{q}_0: \mathcal{M}_{\mathcal{P}} \rightarrow M_0$, restricted to the path-connected components of $\mathcal{M}_{\mathcal{P}}$. Solenoidal manifolds are *matchbox manifolds* of dimension n in the terminology of [9, 10, 24]. Solenoidal manifolds have been studied for their geometric properties [43, 46, 47], for their analytic and index theory properties [11, 33], and from various number-theoretic viewpoints [41, 42].

Here are two applications of the type invariants. Details and proofs are given in §7.

THEOREM 1.14. *Associated to a presentation $\mathcal{P}$ is a well-defined Steinitz number $\xi(\mathcal{P})$, and the type $\tau[\xi(\mathcal{P})]$ depends only on the homeomorphism class of $\mathcal{M}_{\mathcal{P}}$. We denote this type by $\tau[\mathcal{M}_{\mathcal{P}}]$.*

For orientable one-dimensional solenoids, where each map $q_\ell: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is orientable, Bing observed in [7] that if $\tau[\mathcal{M}_{\mathcal{P}}] = \tau[\mathcal{M}_{\mathcal{P}'}]$, then $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic. McCord showed in [32, §2] the converse, that if $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic, then $\tau[\mathcal{M}_{\mathcal{P}}] = \tau[\mathcal{M}_{\mathcal{P}'}]$. (See also [1].) Together, these results yield the following well-known classification.

THEOREM 1.15. *For one-dimensional solenoidal manifolds, $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic if and only if $\tau[\mathcal{M}_{\mathcal{P}}] = \tau[\mathcal{M}_{\mathcal{P}'}]$.*

For solenoidal manifolds of dimension $n \geq 2$, no such classification by invariants can exist, even for the case when the base manifold $M_0 = \mathbb{T}^n$ (see [44].) The most one can hope for is to define invariants that distinguish various homeomorphism classes. Associated to a presentation $\mathcal{P}$ is a well-defined typeset $\Xi(\mathcal{P})$, which provides another invariant of the homeomorphism class of $\mathcal{M}_{\mathcal{P}}$.

THEOREM 1.16. *Assume that $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic solenoidal manifolds, then the typesets $\Xi[\mathcal{P}]$ and $\Xi[\mathcal{P}']$ are commensurable.*

1.5. *Structure.* The remainder of this paper proceeds as follows. Section 2 discusses the basic results about odometers required. Section 3 discusses the basic results about types and typesets required. Section 4 gives the proofs of the results above about classifying odometers. Section 5 gives some basic examples of odometers, and calculates their types and typesets. Section 6 discusses d -regular odometers, and calculates their types and typesets. Section 7 discusses solenoidal manifolds and their types and typesets.

2. Cantor actions

We recall some of the basic properties of Cantor actions, as required for the proofs of the results in §1. More complete discussions of the properties of equicontinuous Cantor actions are given in the text by Auslander [4], the papers by Cortez and Petite [14], Cortez and Medynets [13], and the authors' works, in particular [17, 18] and [26, §3].

2.1. *Basic concepts.* Assume that $\mathfrak{X}$ is a Cantor space. Let $\text{CO}(\mathfrak{X})$ denote the collection of all clopen (closed and open) subsets of $\mathfrak{X}$, which forms a basis for the topology of $\mathfrak{X}$. For $\phi \in \text{Homeo}(\mathfrak{X})$ and $U \in \text{CO}(\mathfrak{X})$, the image $\phi(U) \in \text{CO}(\mathfrak{X})$. Recall that the action $(\mathfrak{X}, \Gamma, \Phi)$ is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x) = \{g \cdot x \mid g \in \Gamma\}$ is dense in $\mathfrak{X}$.

The following result is folklore and a proof is given in [25, Proposition 3.1].

PROPOSITION 2.1. *For $\mathfrak{X}$ a Cantor space, a minimal action $\Phi: \Gamma \times \mathfrak{X} \rightarrow \mathfrak{X}$ is equicontinuous if and only if the Γ -orbit of every $U \in \text{CO}(\mathfrak{X})$ is finite for the induced action $\Phi_*: \Gamma \times \text{CO}(\mathfrak{X}) \rightarrow \text{CO}(\mathfrak{X})$.*

That is, given a clopen set $U \in \text{CO}(\mathfrak{X})$, Γ acts on the orbits of U by finite permutations. This justifies saying that a minimal equicontinuous Cantor action of Γ is a (generalized) odometer.

We say that $U \subset \mathfrak{X}$ is *adapted* to the action $(\mathfrak{X}, \Gamma, \Phi)$ if U is a *non-empty clopen* subset, and for any $g \in \Gamma$, if $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$. Given $x \in \mathfrak{X}$ and clopen set $x \in W$, there is an adapted clopen set U with $x \in U \subset W$. (For example, see the proof of [25, Proposition 3.1].)

For an adapted set U , the set of ‘return times’ to U ,

$$\Gamma_U = \{g \in \Gamma \mid g \cdot U \cap U \neq \emptyset\}, \quad (2.1)$$

is a subgroup of Γ , called the *stabilizer* of U . Then, for $g, g' \in \Gamma$ with $g \cdot U \cap g' \cdot U \neq \emptyset$, we have $g^{-1}g' \cdot U = U$, and hence $g^{-1}g' \in \Gamma_U$. Thus, the translates $\{g \cdot U \mid g \in \Gamma\}$ form a finite clopen partition of $\mathfrak{X}$ and are in 1–1 correspondence with the quotient space $X_U = \Gamma / \Gamma_U$. Then, Γ acts by permutations of the finite set X_U and so the stabilizer group $\Gamma_U \subset G$ has finite index. Note that this implies that if $V \subset U$ is a proper inclusion of adapted sets, then the inclusion $\Gamma_V \subset \Gamma_U$ is also proper.

Definition 2.2. Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer. A properly descending chain of clopen sets, $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell > 0\}$, is said to be an *adapted neighborhood basis* at $x \in \mathfrak{X}$ for the action Φ if $x \in U_{\ell+1} \subset U_\ell$ is a proper inclusion for all $\ell > 0$, with $\bigcap_{\ell > 0} U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ .

Given $x \in \mathfrak{X}$ and $\varepsilon > 0$, Proposition 2.1 implies there exists an adapted clopen set $U \in \text{CO}(\mathfrak{X})$ with $x \in U$ and $\text{diam}(U) < \varepsilon$. Thus, one can choose a descending chain $\mathcal{U}$ of adapted sets in $\text{CO}(\mathfrak{X})$ whose intersection is x , from which we obtain the following result.

PROPOSITION 2.3. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer. Given $x \in \mathfrak{X}$, there exists an adapted neighborhood basis $\mathcal{U}$ at x for the action Φ .*

COROLLARY 2.4. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer and $\mathcal{U}$ be an adapted neighborhood basis. Set $\Gamma_\ell = \Gamma_{U_\ell}$, with $\Gamma_0 = \Gamma$, then $\mathcal{G}_\mathcal{U} = \{\Gamma_0 \supset \Gamma_1 \supset \cdots\}$ is a descending chain of finite index subgroups.*

2.2. Equivalence of Cantor actions. We next recall the notions of equivalence of Cantor actions used in this work. The first and strongest is that of isomorphism, which is a generalization of the usual notion of conjugacy of topological actions. The definition below agrees with the usage in the papers [13, 25, 31].

Definition 2.5. Cantor actions $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ and $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ are said to be *isomorphic* if there is a homeomorphism $h: \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ and group isomorphism $\Theta: \Gamma_1 \rightarrow \Gamma_2$ so that

$$\Phi_1(g) = h^{-1} \circ \Phi_2(\Theta(g)) \circ h \in \text{Homeo}(\mathfrak{X}_1) \quad \text{for all } g \in \Gamma_1. \quad (2.2)$$

The notion of *return equivalence* for odometers is weaker than the notion of isomorphism, and is natural when considering the odometers defined by the holonomy actions for solenoidal manifolds, as considered in the works [24–26] and later in §4.

For an odometer $(\mathfrak{X}, \Gamma, \Phi)$ and an adapted set $U \subset \mathfrak{X}$, by an abuse of notation, we use Φ_U to denote both the restricted action $\Phi_U: \Gamma_U \times U \rightarrow U$ and the induced quotient action $\Phi_U: \mathcal{H}_U \times U \rightarrow U$, where $\mathcal{H}_U = \Phi(\Gamma_U) \subset \text{Homeo}(U)$. Then, $(U, \mathcal{H}_U, \Phi_U)$ is called the *restricted holonomy action* for Φ , in analogy with the case where U is a transversal to a solenoidal manifold, and $\mathcal{H}_U$ is the holonomy group for this transversal. A technical issue that often arises though is that while $\Gamma_U \subset \Gamma$ has finite index, the action map $\Phi_U: \Gamma_U \rightarrow \mathcal{H}_U$ need not be injective, and can in fact have a large kernel as is the case, for example, for the actions of weakly branch groups (see Example 6.10).

Definition 2.6. Odometers $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are *return equivalent* if there exists an adapted set $U \subset \mathfrak{X}$ for the action Φ and an adapted set $U' \subset \mathfrak{X}'$ for the action Φ' , such that the restricted actions $(U, \mathcal{H}_U, \Phi_U)$ and $(U', \mathcal{H}'_{U'}, \Phi'_{U'})$ are isomorphic.

Note that if we take $U = \mathfrak{X}$ and $U' = \mathfrak{X}'$ in Definition 2.6, then return equivalence may still be weaker than isomorphism in Definition 2.5, unless the actions Φ and Φ' are topologically free [25, 31, 37].

2.3. Algebraic Cantor actions. There is a natural basepoint $x_\infty \in X_\infty$ given by the cosets of the identity element $e \in \Gamma$, so $x_\infty = (e\Gamma_\ell)$. An adapted neighborhood basis of x_∞ is given by the clopen sets

$$U_\ell = \{x = (x_i) \in X_\infty \mid x_i = e\Gamma_i \in X_i, 0 \leq i \leq \ell\} \subset X_\infty. \quad (2.3)$$

Then, there is the tautological identity $\Gamma_\ell = \Gamma_{U_\ell}$.

Suppose that we are given an odometer $(\mathfrak{X}, \Gamma, \Phi)$ and an adapted neighborhood basis $\mathcal{U}$. Define subgroups $\Gamma_\ell = \Gamma_{U_\ell}$, with $\Gamma_0 = \Gamma$, which form the group chain $\mathcal{G}_\mathcal{U} = \{\Gamma_0 \supset \Gamma_1 \supset \cdots\}$. Then we have the following folklore result.

THEOREM 2.7. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer and $\mathcal{U}$ an adapted neighborhood basis. The action $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to the odometer $(X_\infty, \Gamma, \Phi_\infty)$ constructed from the group chain $\mathcal{G}_\mathcal{U}$.*

COROLLARY 2.8. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer, and assume that $\mathcal{G}_\mathcal{U}$ and $\mathcal{G}'_{\mathcal{U}'}$ are adapted neighborhood bases for the action. Then, the corresponding algebraic models of the action, $(X_\infty, \Gamma, \Phi_\infty)$ and $(X'_\infty, \Gamma, \Phi'_\infty)$, are isomorphic in the sense of Definition 2.5 with $\Theta: \Gamma \rightarrow \Gamma$ the identity map.*

3. Type and typeset

The notion of type was introduced in 1937 by Baer in [5, §2] as part of the study of the classification problem for rank n subgroups of $\mathbb{Q}^n$. The work of Butler [8] introduced a restricted class of subgroups in $\mathbb{Q}^n$, now called *Butler groups*. The classification theory for Butler groups was further developed by Richman [40] and Mutzbauer [34], and the works of Arnold (see [2, §1]), and Arnold and Vinsonhaler [3]. For a comprehensive treatment of these ideas, see the monograph by Fuchs [21]. Thomas applied the type invariants in his analysis of the classification complexity of these groups in the work [45, §3]. The applications of type invariants to profinite groups are discussed in the works by Ribes [38, Ch. 1, §4], Wilson [48, Ch. 2], and Ribes and Zalesskii [39, Ch. 2.3]. In this section, we recall basic notions and properties of Steinitz numbers and their types, and their definitions for Γ -odometers.

3.1. Types and typesets. Recall that a Steinitz number ξ can be written uniquely as the formal product over the set of primes,

$$\xi = \prod_{p \in \Pi} p^{\chi_\xi(p)}, \quad (3.1)$$

where the *characteristic function* $\chi_\xi: \Pi \rightarrow \{0, 1, \dots, \infty\}$ counts the multiplicity with which a prime p appears in the infinite product ξ . Note that multiplication of Steinitz numbers corresponds to the sum of their characteristic functions.

Recall from Definition 1.2 that two Steinitz numbers ξ and ξ' are said to be *asymptotically equivalent* if there exists finite integers $m, m' \geq 1$ such that $m \cdot \xi = m' \cdot \xi'$, and we then write $\xi \stackrel{a}{\sim} \xi'$. Recall that the type $\tau[\xi]$ associated to a Steinitz number ξ is the asymptotic equivalence class of ξ .

LEMMA 3.1. *ξ and ξ' satisfy $\xi \stackrel{a}{\sim} \xi'$ if and only if their characteristic functions χ_1, χ_2 satisfy:*

- $\chi_1(p) = \chi_2(p)$ for all but finitely many primes $p \in \Pi$; and
- $\chi_1(p) = \infty$ if and only if $\chi_2(p) = \infty$ for all primes $p \in \Pi$.

Given two types τ and τ' , we write $\tau \leq \tau'$ if there exists representatives $\xi \in \tau$ and $\xi' \in \tau'$ such that their characteristic functions satisfy $\chi_\xi(p) \leq \chi_{\xi'}(p)$ for all primes $p \in \Pi$. Then, two Steinitz numbers ξ and ξ' are asymptotically equivalent if and only if $\chi_\xi \leq \chi_{\xi'}$ and $\chi_{\xi'} \leq \chi_\xi$.

Definition 3.2. A *typeset* Ξ is a collection of types.

There are three operations on types τ and τ' : product, join, and intersection. Let χ (respectively χ') be the characteristic function for a representative $\xi \in \tau$ (respectively $\xi' \in \tau'$), then

Product: $\tau \cdot \tau'$ type defined by $\chi^+(p) = \chi(p) + \chi'(p)$;

Join: $\tau \vee \tau'$ type defined by $\chi^\vee(p) = \max\{\chi(p), \chi'(p)\}$;

Intersection: $\tau \wedge \tau'$ type defined by $\chi^\wedge(p) = \min\{\chi(p), \chi'(p)\}$.

Note that $\tau \wedge \tau' \leq \tau \vee \tau' \leq \tau \cdot \tau'$. A typeset Ξ need not be closed under the operations of product, join, or intersection. However, a typeset Ξ always admits a partial ordering.

3.2. Type for profinite groups. A topological group $\mathfrak{G}$ is said to be profinite if it is isomorphic to the inverse limit of finite groups (see [48, Ch. 2] or [39, Ch. 2.3]).

The *Steinitz order* $\Pi[\mathfrak{G}]$ of a profinite group $\mathfrak{G}$ is defined by the supernatural number associated to a presentation of $\mathfrak{G}$, defined as follows. For a profinite group $\mathfrak{G}$, an open subgroup $\mathfrak{U} \subset \mathfrak{G}$ has finite index [39, Lemma 2.1.2]. Let $\mathfrak{D} \subset \mathfrak{G}$ be a closed subgroup and $\mathfrak{N} \subset \mathfrak{G}$ is an open normal subgroup, then $\mathfrak{N} \cdot \mathfrak{D}$ is an open subgroup of $\mathfrak{G}$ and $\mathfrak{N} \cap \mathfrak{D}$ is an open normal subgroup of $\mathfrak{D}$.

Definition 3.3. Let $\mathfrak{D} \subset \mathfrak{G}$ be a closed subgroup of the profinite group $\mathfrak{G}$. Define Steinitz orders as follows:

- (1) $\xi(\mathfrak{G}) = \text{lcm}\{\#\mathfrak{G}/\mathfrak{N} \mid \mathfrak{N} \subset \mathfrak{G} \text{ open normal subgroup}\}$;
- (2) $\xi(\mathfrak{D}) = \text{lcm}\{\#\mathfrak{D}/(\mathfrak{N} \cap \mathfrak{D}) \mid \mathfrak{N} \subset \mathfrak{G} \text{ open normal subgroup}\}$;
- (3) $\xi(\mathfrak{G} : \mathfrak{D}) = \text{lcm}\{\#\mathfrak{G}/(\mathfrak{N} \cdot \mathfrak{D}) \mid \mathfrak{N} \subset \mathfrak{G} \text{ open normal subgroup}\}$.

The Steinitz number $\xi(\mathfrak{G} : \mathfrak{D})$ is called the *relative Steinitz order* of the pair $(\mathfrak{G}, \mathfrak{D})$.

The Steinitz orders satisfy the Lagrange identity, where the multiplication is taken in the sense of supernatural numbers (see [39, 48]), and we have

$$\xi(\mathfrak{G}) = \xi(\mathfrak{G} : \mathfrak{D}) \cdot \xi(\mathfrak{D}). \quad (3.2)$$

In particular, we always have $\xi[\mathfrak{D}] \leq \xi[\mathfrak{G}]$.

Definition 3.4. The *relative type* $\Xi[\mathfrak{G} : \mathfrak{D}]$ is the type of $\xi(\mathfrak{G} : \mathfrak{D})$, where $\mathfrak{D} \subset \mathfrak{G}$ is a closed subgroup of the profinite group $\mathfrak{G}$.

3.3. Type for odometers. Let $(X_\infty, \Gamma, \Phi_\infty)$ be an odometer, defined by a group chain $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \dots\}$. Recall that the *normal core* of Γ_ℓ is the largest normal subgroup $C_\ell \subset \Gamma_\ell$. The profinite group $\widehat{\Gamma}(\mathcal{G})$ associated to the action $(X_\infty, \Gamma, \Phi_\infty)$ defined by $\mathcal{G}$ is given by the following definition.

Definition 3.5. Let $\mathcal{G}$ be a group chain with associated normal core chain $\{C_\ell \mid \ell \geq 0\}$, then

$$\widehat{\Gamma}(\mathcal{G}) = \varprojlim \{\Gamma/C_{\ell+1} \rightarrow \Gamma/C_\ell \mid \ell \geq 0\}. \quad (3.3)$$

The profinite group $\widehat{\Gamma}(\mathcal{G})$ is a quotient of the full profinite completion of Γ , but it is typically not equal to the profinite completion. The properties of the group $\widehat{\Gamma}(\mathcal{G})$ associated to a Cantor action are studied extensively in the authors' works [17, 18, 24, 26]. The Steinitz order of $\widehat{\Gamma}(\mathcal{G})$ is well defined and given by

$$\xi(\widehat{\Gamma}(\mathcal{G})) = \text{lcm}\{\#(\Gamma/C_\ell) \mid \ell > 0\}. \quad (3.4)$$

The type $\tau[\widehat{\Gamma}(\mathcal{G})] = \tau[\xi(\widehat{\Gamma}(\mathcal{G}))]$ need not be an invariant of return equivalence of the odometer $(X_\infty, \Gamma, \Phi_\infty)$, as explained below.

Definition 3.6. Let $(X_\infty, \Gamma, \Phi_\infty)$ be an odometer. The type $\tau[X_\infty, \Gamma, \Phi_\infty]$ of the action is the equivalence class of the Steinitz number

$$\xi(X_\infty, \Gamma, \Phi_\infty) = \text{lcm}\{\#X_\ell = \#(\Gamma/\Gamma_\ell) \mid \ell > 0\}. \quad (3.5)$$

Recall that there is a transitive action $\widehat{\Phi}_\infty: \widehat{\Gamma}(\mathcal{G}) \times X_\infty \rightarrow X_\infty$ induced by odometer $(X_\infty, \Gamma, \Phi_\infty)$. The action $(X_\infty, \widehat{\Gamma}(\mathcal{G}), \widehat{\Phi}_\infty)$ is free precisely when the isotropy subgroup $\mathcal{D}(\mathcal{G}) \subset \widehat{\Gamma}(\mathcal{G})$ of the action at the basepoint $x_\infty \in X_\infty$ is trivial. In this case, we have a homeomorphism $X_\infty \cong \widehat{\Gamma}(\mathcal{G})$ that commutes with the action, and so X_∞ inherits the structure of a Cantor group from the action.

However, when the action $(X_\infty, \widehat{\Gamma}(\mathcal{G}), \widehat{\Phi}_\infty)$ is not free, then $\mathcal{D}(\mathcal{G})$ is not trivial and we have the following proposition.

PROPOSITION 3.7. Let $(X_\infty, \Gamma, \Phi_\infty)$ be an odometer. Then, $\xi(X_\infty, \Gamma, \Phi_\infty) = \xi(\widehat{\Gamma}(\mathcal{G}) : \mathcal{D}(\mathcal{G}))$.

We omit the proof of this, as it is a direct consequence of the definitions, and the result is not needed for the proofs of our main theorems. However, Proposition 3.7 provides some insights on the properties of the type $\tau[X_\infty, \Gamma, \Phi_\infty]$ of an odometer with respect to return equivalence.

The Lagrange theorem for profinite groups, equation (3.2), implies that $\tau[\widehat{\Gamma}(\mathcal{G})] = \tau[X_\infty, \Gamma, \Phi_\infty] \cdot \tau[\mathcal{D}(\mathcal{G})]$. The type $\tau[\mathcal{D}(\mathcal{G})]$ need not be invariant under restriction to adapted subsets, and so $\tau[\mathcal{D}(\mathcal{G})]$ need not be invariant under the relation of return equivalence, and thus the same is true for $\tau[\widehat{\Gamma}(\mathcal{G})]$. However, the proof of Theorem 1.5 shows that the relative type $\tau[\widehat{\Gamma}(\mathcal{G}) : \mathcal{D}(\mathcal{G})] = \tau[\mathcal{G}]$ is invariant under restriction to adapted subsets.

3.4. Typesets for odometers. We next consider the properties of typeset under restriction, which leads to the notion of commensurable typesets. Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \dots\}$ be a group chain and $(X_\infty, \Gamma, \Phi_\infty)$ the associated odometer.

Let $H \subset \Gamma$ be a subgroup of finite index such that $\Gamma_\ell \subset H$ for some $\ell > 0$. By omitting some initial terms, we can assume that $\Gamma_1 \subset H$. Then, for each $\ell > 0$, define

$$C_\ell^H = \bigcap_{\delta \in H} \delta \Gamma_\ell \delta^{-1}. \quad (3.6)$$

Then, $C_\ell^H \subset \Gamma_\ell$ is a subgroup of finite index, as Γ_ℓ has finite index in Γ . In particular, $C_\ell = C_\ell^\Gamma$ is the normal core of Γ_ℓ and there is an inclusion $C_\ell \subset C_\ell^H$ for any choice of H . Also note that $C_{\ell+1}^H \subset C_\ell^H$ for all $\ell \geq 0$.

Given $\gamma \in \Gamma$, define $C_{\gamma,\ell}^H = \langle \gamma \rangle \cap C_\ell^H$, and let $C_\gamma^H = \{\langle \gamma \rangle \cap H = C_{\gamma,0}^H \supset C_{\gamma,1}^H \supset C_{\gamma,2}^H \supset \dots\}$ denote the resulting subgroup chain in $\langle \gamma \rangle$.

Definition 3.8. Given a group chain $\mathcal{G}$ and $H \subset \Gamma$ a subgroup with $\Gamma_\ell \subset H$ for some $\ell \geq 1$, the H -restricted order of $\gamma \in \Gamma$ is the Steinitz order with respect to the chain C_γ^H ,

$$\xi^H(\gamma) = \text{lcm}\{\#(\langle \gamma \rangle / C_{\gamma,\ell}^H) \mid \ell > 0\}. \quad (3.7)$$

The H -restricted typeset for the chain $\mathcal{G}$ is the collection

$$\Xi_H[\mathcal{G}] = \{\tau[\xi^H(\gamma)] \mid \gamma \in \Gamma\}. \quad (3.8)$$

Note that when $H = \Gamma$, we recover the typeset $\Xi[\mathcal{G}]$ in Definition 1.6.

We can now formulate the notion of commensurable typesets for group chains $\mathcal{G}$ and $\mathcal{G}'$.

Definition 3.9. Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \dots\}$ and $\mathcal{G}' = \{\Gamma' = \Gamma'_0 \supset \Gamma'_1 \supset \Gamma'_2 \supset \dots\}$ be group chains. We say that their typesets $\Xi[\mathcal{G}]$ and $\Xi[\mathcal{G}']$ are *commensurable* if there exists a finite index subgroup $H \subset \Gamma$ with $\Gamma_\ell \subset H$ for some $\ell \geq 1$, and $H' \subset \Gamma'$ with $\Gamma'_{\ell'} \subset H'$ for some $\ell' \geq 1$, such that $\Xi_H[\mathcal{G}] = \Xi_{H'}[\mathcal{G}']$.

Finally, we give an elementary result about the types of elements $\gamma \in \Gamma$ that follows from basic group theory. For all $\ell > 0$, we have that $\langle \gamma \rangle_\ell \subset C_\ell$, so the order of the subgroup $\langle \gamma \rangle / \langle \gamma \rangle_\ell$ divides the order of Γ / C_ℓ by Lagrange's theorem. Thus, we have the following proposition.

PROPOSITION 3.10. For a group chain $\mathcal{G}$, $\tau[\mathcal{G}] \leq \tau[\widehat{\Gamma}(\mathcal{G})]$, and for each $\gamma \in \Gamma$, $\tau[\gamma] \leq \tau[\widehat{\Gamma}(\mathcal{G})]$.

4. Invariance of types

In this section, we give the proofs of the results in §1 on odometers. The discussion of solenoidal manifolds and the proofs of Theorems 1.14 and 1.16 are given in §7.

The following notation is used. Let $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ denote odometers, and let $\mathcal{U}$ for $(\mathfrak{X}, \Gamma, \Phi)$ be a choice of an adapted neighborhood basis for $(\mathfrak{X}, \Gamma, \Phi)$ and $\mathcal{U}'$ a choice of an adapted neighborhood basis for $(\mathfrak{X}', \Gamma', \Phi')$. In some cases, we have $\mathfrak{X} = \mathfrak{X}'$.

Let $\mathcal{G}_\mathcal{U}$ be the group chain associated to $\mathcal{U}$, as in Corollary 2.4, and let $(X_\infty, \Gamma, \Phi_\infty)$ denote the algebraic model for $(\mathfrak{X}, \Gamma, \Phi)$ that it determines, as in §1.2. Similarly, let $\mathcal{G}'_{\mathcal{U}'}$ be the group chain associated to $\mathcal{U}'$ and $(X'_\infty, \Gamma', \Phi'_\infty)$ the algebraic model for $(\mathfrak{X}', \Gamma', \Phi')$.

4.1. *Proof of Theorem 1.5.* We must show that the Steinitz number $\xi(\mathfrak{X}, \Gamma, \Phi)$ of an odometer $(\mathfrak{X}, \Gamma, \Phi)$ is well defined and that its type $\tau[\xi(\mathfrak{X}, \Gamma, \Phi)]$ is an invariant under return equivalence. The first claim is a special case of the proof of invariance under return equivalence, as explained later.

Suppose that the odometers $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are return equivalent. Then, by Theorem 2.7, the odometers $(X_\infty, \Gamma, \Phi_\infty)$ and $(X'_\infty, \Gamma', \Phi'_\infty)$ are return equivalent, as in Definition 2.6.

Choose adapted sets $U \subset X_\infty$ and $U' \subset X'_\infty$ and a homeomorphism $h: U \rightarrow U'$ so that the induced homomorphism $h_*: \text{Homeo}(U) \rightarrow \text{Homeo}(U')$ restricts to an isomorphism between the image $\mathcal{H}_U = \Phi(\Gamma_U)$ with the image $\mathcal{H}'_{U'} = \Phi'(\Gamma'_{U'})$.

Recall a standard construction of a diagram of maps between adapted sets [20], [17, Theorem 3.3]:

$$\begin{array}{ccccccccccc}
 X_\infty & \supset & U & \supset & \gamma_1 \cdot U_{\ell_1} & \supset & h^{-1}(\gamma'_1 \cdot U'_{\ell'_1}) & \supset & \gamma_2 \cdot U_{\ell_2} & \supset & h^{-1}(\gamma'_2 \cdot U'_{\ell'_2}) & \supset & \cdots \\
 & & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & & \\
 X'_\infty & \supset & U' & \supset & h(\gamma_1 \cdot U_{\ell_1}) & \supset & \gamma'_1 \cdot U'_{\ell'_1} & \supset & h(\gamma_2 \cdot U_{\ell_2}) & \supset & \gamma'_2 \cdot U'_{\ell'_2} & \supset & \cdots
 \end{array} \quad (4.1)$$

The sets U_ℓ are defined in equation (2.3) and adapted to the action of Γ , and similarly for the sets $U'_{\ell'}$, which are adapted to the action of Γ' . The subscripts and intermediate adapted sets are defined iteratively in the following. We adopt the ‘ $\cdot$ ’ notation for the actions, as it is clear from the context which action is being applied. Recall that the action of Γ_U on U is minimal, as is the action of $\Gamma'_{U'}$ on U' . Denote $e_\infty = (e\Gamma_\ell) \in X_\infty$ and $e'_\infty = (e\Gamma'_{\ell'}) \in X'_\infty$, where Γ_ℓ is the isotropy subgroup of U_ℓ and $\Gamma'_{\ell'}$ is the isotropy subgroup of $U'_{\ell'}$, for each $\ell \geq 0$ and each $\ell' \geq 0$.

We define the maps and indices in diagram (4.1) recursively.

The set U is clopen, so there exists $\gamma_1 \in \Gamma$ such that $\gamma_1 \cdot e_\infty \in U$. Choose $\ell_1 > 0$ such that $\gamma_1 \cdot U_{\ell_1} \subset U$.

The image $h(\gamma_1 \cdot U_{\ell_1}) \subset U'$ is clopen, so choose $\gamma'_1 \in \Gamma'$ such that $\gamma'_1 \cdot e'_\infty \in h(\gamma_1 \cdot U_{\ell_1})$. Choose $\ell'_1 > 0$ such that $\gamma'_1 \cdot U'_{\ell'_1} \subset h(\gamma_1 \cdot U_{\ell_1})$.

The image $h^{-1}(\gamma'_1 \cdot U'_{\ell'_1}) \subset \gamma_1 \cdot U_{\ell_1}$ is clopen, so choose $\gamma_2 \in \Gamma$ such that $\gamma_2 \cdot e_\infty \in h^{-1}(\gamma'_1 \cdot U'_{\ell'_1}) \subset \gamma_1 \cdot U_{\ell_1}$. Choose $\ell_2 > \ell_1$ such that $\gamma_2 \cdot U_{\ell_2} \subset h^{-1}(\gamma'_1 \cdot U'_{\ell'_1})$.

The image $h(\gamma_2 \cdot U_{\ell_2}) \subset \gamma'_1 \cdot U'_{\ell'_1}$ is clopen, so choose $\gamma'_2 \in \Gamma'$ such that $\gamma'_2 \cdot e'_\infty \in h(\gamma_2 \cdot U_{\ell_2})$. Choose $\ell'_2 > \ell'_1$ such that $\gamma'_2 \cdot U'_{\ell'_2} \subset h(\gamma_2 \cdot U_{\ell_2})$.

Continue this procedure recursively to obtain diagram (4.1), where we have:

- increasing sequences $0 < \ell_1 < \ell_2 < \ell_3 < \cdots$ and $0 < \ell'_1 < \ell'_2 < \ell'_3 < \cdots$;
- a sequence $\{\gamma_1, \gamma_2, \gamma_3, \dots\} \subset \Gamma$;
- a sequence $\{\gamma'_1, \gamma'_2, \gamma'_3, \dots\} \subset \Gamma'$.

Observe that, by choice, we have $\gamma_{i+1} \cdot U_{\ell_{i+1}} \subset \gamma_i \cdot U_{\ell_i}$ and thus $\gamma_{\ell_i}^{-1} \gamma_{\ell_{i+1}} \in \Gamma_{\ell_i}$ for all $i \geq 1$. This recursion relation implies that the sequence $\{\gamma_i \mid i \geq 0\}$ converges in the

profinite topology on Γ induced by the subgroup chain $\{\Gamma_\ell \mid \ell \geq 0\}$, and similarly for $\{\gamma'_i \mid i \geq 0\}$ in the profinite topology on Γ' . If we denote the respective limits by $\{\overline{\gamma_i}\} \in \overline{\Gamma}$ and $\{\overline{\gamma'_i}\} \in \overline{\Gamma'}$, then we have

$$\overline{\{\gamma_i\}} \cdot e_\infty = h^{-1}(\overline{\{\gamma'_i\}} \cdot e'_\infty) \quad \text{and} \quad \overline{\{\gamma'_i\}} \cdot e'_\infty = h(\overline{\{\gamma_i\}} \cdot e_\infty).$$

All of the sets appearing in diagram (4.1) are adapted for their respective actions, so the set inclusions induce corresponding subgroup chains of their isotropy groups, and these chains are interlaced:

$$\begin{array}{ccccccccc} \Gamma & \supset & H = \Gamma_U & \supset & \gamma_1 \Gamma_{\ell_1} \gamma_1^{-1} & \supset & \Gamma_{h^{-1}(\gamma'_1 \cdot U'_{\ell'_1})} & \supset & \gamma_2 \Gamma_{\ell_2} \gamma_2^{-1} & \supset & \Gamma_{h^{-1}(\gamma'_2 \cdot U'_{\ell'_2})} & \supset & \cdots \\ & & \downarrow h_* & & \downarrow h_* & & \downarrow h_* & & \downarrow h_* & & \downarrow h_* & & \\ \Gamma' & \supset & H' = \Gamma'_{U'} & \supset & \Gamma'_{h(\gamma_1 \cdot U_{\ell_1})} & \supset & \gamma'_1 \Gamma'_{\ell'_1} (\gamma'_1)^{-1} & \supset & \Gamma'_{h(\gamma_2 \cdot U_{\ell_2})} & \supset & \gamma'_2 \Gamma'_{\ell'_2} (\gamma'_2)^{-1} & \supset & \cdots \end{array} \quad (4.2)$$

Conjugation does not change the index of a subgroup, so $[\Gamma : \Gamma_{\ell_j}] = [\Gamma : \gamma_j \Gamma_{\ell_j} \gamma_j^{-1}]$ for all $j \geq 1$, and likewise we have $[\Gamma' : \Gamma'_{\ell'_j}] = [\Gamma' : \gamma'_j \Gamma'_{\ell'_j} (\gamma'_j)^{-1}]$. It then follows that

$$[\Gamma : \Gamma_{\ell_j}] = [\Gamma : \Gamma_{\ell_1}] [\Gamma_{\ell_1} : \Gamma_{\ell_j}] = [\Gamma : \Gamma_{\ell_1}] [\gamma_1 \Gamma_{\ell_1} \gamma_1^{-1} : \gamma_j \Gamma_{\ell_j} \gamma_j^{-1}], \quad (4.3)$$

$$[\Gamma' : \Gamma'_{\ell'_j}] = [\Gamma' : \Gamma'_{\ell'_1}] [\Gamma'_{\ell'_1} : \Gamma'_{\ell'_j}] = [\Gamma' : \Gamma'_{\ell'_1}] [\gamma'_1 \Gamma'_{\ell'_1} (\gamma'_1)^{-1} : \gamma'_j \Gamma'_{\ell'_j} (\gamma'_j)^{-1}]. \quad (4.4)$$

We now show the key fact for the proof of Theorem 1.5.

LEMMA 4.1. *For all $j > i > 0$,*

$$[\Gamma_{\ell_i} : \Gamma_{\ell_j}] = [\Gamma'_{h(\gamma_i \cdot U_{\ell_i})} : \Gamma'_{h(\gamma_j \cdot U_{\ell_j})}]. \quad (4.5)$$

Proof. The isotropy group Γ_{ℓ_i} acts minimally on the clopen set U_{ℓ_i} , and its action translates the clopen subset $U_{\ell_j} \subset U_{\ell_i}$ to give a partition of this set. The index $[\Gamma_{\ell_i} : \Gamma_{\ell_j}]$ equals the number of clopen subsets in this partition. It follows that the action of $\gamma_i \Gamma_{\ell_i} \gamma_i^{-1}$ on $\gamma_i \cdot U_{\ell_i}$ partitions this set into $[\Gamma_{\ell_i} : \Gamma_{\ell_j}]$ translates of the clopen subset $\gamma_j \cdot U_{\ell_j}$.

The homeomorphism $h: U \rightarrow U'$ restricts to a homeomorphism $h: \gamma_i \cdot U_{\ell_i} \rightarrow h(\gamma_i \cdot U_{\ell_i})$, which induces a conjugacy of the action of $\Gamma_{\gamma_i \cdot U_{\ell_i}}$ with the action of $\Gamma'_{h(\gamma_i \cdot U_{\ell_i})}$ on $h(\gamma_i \cdot U_{\ell_i}) \subset U'$. Thus, the translates of the clopen subset $h(\gamma_j \cdot U_{\ell_j})$ partition $h(\gamma_i \cdot U_{\ell_i})$ into $[\Gamma_{\ell_i} : \Gamma_{\ell_j}]$ clopen subsets, and so the identity in equation (4.5) follows. $\square$

We now prove that $\tau[\xi(X_\infty, \Gamma, \Phi_\infty)] = \tau[\xi(X'_\infty, \Gamma', \Phi'_\infty)]$. Recall that

$$\xi(\mathcal{G}_U) = \text{lcm}\{[\Gamma : \Gamma_\ell] \mid \ell > 0\} = \text{lcm}\{[\Gamma : \Gamma_{\ell_j}] \mid j > 0\}, \quad (4.6)$$

$$\xi(\mathcal{G}'_{U'}) = \text{lcm}\{[\Gamma' : \Gamma'_\ell] \mid \ell > 0\} = \text{lcm}\{[\Gamma' : \Gamma'_{\ell'_j}] \mid j > 0\}. \quad (4.7)$$

Then, by equation (4.5), for $j > 1$, we have

$$[\Gamma : \Gamma_{\ell_j}] = [\Gamma : \Gamma_{\ell_1}] [\Gamma_{\ell_1} : \Gamma_{\ell_j}] = [\Gamma : \Gamma_{\ell_1}] [\Gamma'_{h(\gamma_1 \cdot U_{\ell_1})} : \Gamma'_{h(\gamma_j \cdot U_{\ell_j})}]. \quad (4.8)$$

Then calculate, recalling the inclusions $U' \supset h(\gamma_1 \cdot U_{\ell_1}) \supset \gamma'_1 \cdot U'_{\ell_1} \supset h(\gamma_j \cdot U_{\ell_j}) \supset \gamma'_j \cdot U'_{\ell'_j}$ from equation (4.1):

$$\begin{aligned} [\Gamma' : \Gamma'_{\ell'_j}] &= [\Gamma' : \gamma'_j \Gamma'_{\ell'_j} (\gamma'_j)^{-1}] \\ &= [\Gamma' : \Gamma'_{h(\gamma_1 \cdot U_{\ell_1})}] [\Gamma'_{h(\gamma_1 \cdot U_{\ell_1})} : \Gamma'_{\gamma'_1 \cdot U'_{\ell'_1}}] [\Gamma'_{\gamma'_1 \cdot U'_{\ell'_1}} : \Gamma'_{h(\gamma_j \cdot U_{\ell_j})}] [\Gamma'_{h(\gamma_j \cdot U_{\ell_j})} : \Gamma'_{\gamma'_j \cdot U'_{\ell'_j}}] \\ &= [\Gamma' : \Gamma'_{h(\gamma_1 \cdot U_{\ell_1})}] [\Gamma'_{h(\gamma_1 \cdot U_{\ell_1})} : \Gamma'_{h(\gamma_j \cdot U_{\ell_j})}] [\Gamma'_{h(\gamma_j \cdot U_{\ell_j})} : \Gamma'_{\gamma'_j \cdot U'_{\ell'_j}}]. \end{aligned} \quad (4.9)$$

Then, by equation (4.8), the last line of equation (4.9) yields

$$[\Gamma : \Gamma_{\ell_1}] [\Gamma' : \Gamma'_{\ell'_j}] = [\Gamma' : \Gamma'_{h(\gamma_1 \cdot U_{\ell_1})}] [\Gamma : \Gamma_{\ell_j}] [\Gamma'_{h(\gamma_j \cdot U_{\ell_j})} : \Gamma'_{\gamma'_j \cdot U'_{\ell'_j}}]. \quad (4.10)$$

Thus, the index $[\Gamma : \Gamma_{\ell_j}]$ divides $[\Gamma' : \Gamma'_{\ell'_j}]$ for all $j > 1$. In particular, $[\Gamma : \Gamma_{\ell_j}]$ divides $[\Gamma : \Gamma_{\ell_1}] \xi(\mathcal{P}')$ for all $j > 1$, and so $\xi(\mathcal{P})$ divides $[\Gamma : \Gamma_{\ell_1}] \xi(\mathcal{P}')$; hence $\tau[\mathcal{P}] \leq \tau[\mathcal{P}']$.

Next, repeat these calculations for $j > 1$, starting with

$$[\Gamma : \Gamma_{\ell_{j+1}}] = [\Gamma : \Gamma_{h^{-1}(\gamma'_1 \cdot U'_{\ell'_1})}] [\Gamma_{h^{-1}(\gamma'_1 \cdot U'_{\ell'_1})} : \Gamma_{h^{-1}(\gamma'_j \cdot U'_{\ell'_j})}] [\Gamma_{h^{-1}(\gamma'_j \cdot U'_{\ell'_j})} : \Gamma_{\gamma_{j+1} \cdot U_{\ell_{j+1}}}] \quad (4.11)$$

Lemma 4.1 can be applied to the inverse map $h^{-1} : U' \rightarrow U$ as well, to obtain that for all $j > i > 0$,

$$[\Gamma'_{\ell'_i} : \Gamma'_{\ell'_j}] = [\Gamma_{h^{-1}(\gamma'_i \cdot U'_{\ell'_i})} : \Gamma_{h^{-1}(\gamma'_j \cdot U'_{\ell'_j})}] \quad (4.12)$$

and so

$$[\Gamma : \Gamma_{\ell_{j+1}}] = [\Gamma : \Gamma_{h^{-1}(\gamma'_1 \cdot U'_{\ell'_1})}] [\Gamma'_{\ell'_1} : \Gamma'_{\ell'_j}] [\Gamma_{h^{-1}(\gamma'_j \cdot U'_{\ell'_j})} : \Gamma_{\gamma_{j+1} \cdot U_{\ell_{j+1}}}] \quad (4.13)$$

Thus, the index $[\Gamma'_{\ell'_1} : \Gamma'_{\ell'_j}]$ divides $[\Gamma : \Gamma_{\ell_{j+1}}]$ for all $j > 1$ and so $[\Gamma'_{\ell'_1} : \Gamma'_{\ell'_j}]$ divides $\xi(\mathcal{G}_{\mathcal{U}})$. It follows that $\xi(\mathcal{G}'_{\mathcal{U}'})$ divides $\xi(\mathcal{G}_{\mathcal{U}})$; hence, $\tau[\mathcal{G}'_{\mathcal{U}'}] \leq \tau[\mathcal{G}_{\mathcal{U}}]$. Reversing the roles of the group chains $\mathcal{G}_{\mathcal{U}}$ and $\mathcal{G}'_{\mathcal{U}'}$, we obtain that $\tau[\mathcal{G}_{\mathcal{U}}] = \tau[\mathcal{G}'_{\mathcal{U}'}]$, and thus $\tau[\mathfrak{X}, \Gamma, \Phi] = \tau[\mathfrak{X}', \Gamma', \Phi']$. This completes the proof that the types of return equivalent actions are equal.

Finally, to complete the proof of Theorem 1.5, we show that $\xi(\mathcal{G}_{\mathcal{U}}) = \xi(\mathcal{G}'_{\mathcal{U}'})$ for group chains $\mathcal{G}_{\mathcal{U}}$ and $\mathcal{G}'_{\mathcal{U}'}$ associated with $(\mathfrak{X}, \Gamma, \Phi)$. We proceed as in the above proof and note that while $\Gamma = \Gamma'$, the group chains define distinct inverse limit spaces X_{∞} and X'_{∞} . In the above calculations, take $U = X_{\infty}$ and $U' = X'_{\infty}$, $\gamma_1 = e \in \Gamma$ with $\ell_1 = 0$, and $\gamma'_1 = e' \in \Gamma'$ with $\ell'_1 = 0$. Then, the terms $[\Gamma : \Gamma_{\ell_1}] = 1$ and $[\Gamma' : \Gamma'_{\ell'_1}] = 1$. Then, by equation (4.10), we have $[\Gamma : \Gamma_{\ell_j}]$ divides $[\Gamma' : \Gamma'_{\ell'_j}]$ and by equation (4.13), we have $[\Gamma' : \Gamma'_{\ell'_j}]$ divides $[\Gamma : \Gamma_{\ell_{j+1}}]$. It follows that $\xi(\mathcal{G}_{\mathcal{U}}) = \xi(\mathcal{G}'_{\mathcal{U}'})$. That is, the Steinitz order is independent of the choice of a group associated to the action.

4.2. Proof of Theorem 1.8. We must show that the typesets of odometers are well defined and invariant under return equivalence modulo the equivalence relation on typesets

in Definition 3.9. As above, we show the result for return equivalence first, then deduce the result for conjugation from the proof of this. The proof begins as in §4.1.

Suppose that the odometers $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are return equivalent, then the odometers $(X_\infty, \Gamma, \Phi_\infty)$ and $(X'_\infty, \Gamma', \Phi'_\infty)$ are return equivalent. Choose adapted sets $U \subset X_\infty$ and $U' \subset X'_\infty$, and a homeomorphism $h: U \rightarrow U'$ so that the induced homomorphism $h_*: \text{Homeo}(U) \rightarrow \text{Homeo}(U')$ restricts to an isomorphism between the image $\mathcal{H}_U = \Phi(\Gamma_U)$ with the image $\mathcal{H}'_{U'} = \Phi'(\Gamma'_{U'})$ that induces an isomorphism between the actions $(U, \mathcal{H}_U, \Phi_U)$ and $(U', \mathcal{H}'_{U'}, \Phi'_{U'})$.

Next, choose basepoints $x \in U$ and $x' \in U'$. Choose an adapted neighborhood basis $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell > 0\}$ at x for the action Φ as in Definition 2.2, and an adapted neighborhood basis $\mathcal{U}' = \{U'_\ell \subset \mathfrak{X}' \mid \ell > 0\}$ at x' for the action Φ' . We can assume that $U_1 \subset U$ and $U'_1 \subset U'$.

Form the group chain $\mathcal{G} = \{\Gamma_0 \supset \Gamma_1 \supset \cdots\}$, where $\Gamma_\ell = \Gamma_{U_\ell}$ with $\Gamma_0 = \Gamma$, and the group chain $\mathcal{G}' = \{\Gamma'_0 \supset \Gamma'_1 \supset \cdots\}$, where $\Gamma'_\ell = \Gamma'_{U'_\ell}$ with $\Gamma'_0 = \Gamma'$.

Set $H = \Gamma_U$ and $H' = \Gamma'_{U'}$. We will show that the H -restricted typeset $\Xi_H[X_\infty, \Gamma, \Phi]$ and the H' -restricted typeset $\Xi_{H'}[X'_\infty, \Gamma', \Phi']$ are equal.

Given $\gamma \in \Gamma$, for any positive integer $m > 0$, the group $\langle \gamma^m \rangle$ is a subgroup of finite index in $\langle \gamma \rangle$. Thus, $\xi^H(\gamma) = m\xi^H(\gamma^m)$ and $\tau[\xi^H(\gamma)] = \tau[\xi^H(\gamma^m)]$. Since H has finite index in Γ , for any $\gamma \in \Gamma$, there exists $m > 0$ such that $\gamma^m \in H$. Thus, we have $\Xi_H[\mathcal{G}] = \Xi_H[H \cap \mathcal{G}]$, so it suffices to consider $\gamma \in H$, and likewise for $\gamma' \in H'$.

Let sequences $\{\gamma_i \mid i \geq 1\}$, $\{\gamma'_i \mid i \geq 1\}$, $\{\ell_i \mid i \geq 1\}$ and $\{\ell'_i \mid i \geq 1\}$ be chosen as in §4.1, resulting in diagram (4.1) of adapted sets, and diagram (4.2) of group inclusions. Note that as we assume $e_\infty \in U$, we can choose γ_1 to be the identity, and likewise as $e'_\infty \in U'$, we choose γ'_1 to be the identity. By choice, we have $\gamma_{i+1} \cdot U_{\ell_{i+1}} \subset \gamma_i \cdot U_{\ell_i}$ and thus $\gamma_{\ell_i}^{-1} \gamma_{\ell_{i+1}} \in \Gamma_{\ell_i}$ for all $i \geq 1$. In particular, this implies that $\gamma_i \in H$ for all $i \geq 1$. The analogous conclusion holds, that $\gamma'_i \in H'$ for all $i \geq 1$.

For notational convenience, set $H_i = \Gamma_{\gamma_i \cdot U_{\ell_i}} = \gamma_i \Gamma_{\ell_i} \gamma_i^{-1}$ and $H'_i = \Gamma'_{\gamma'_i \cdot U_{\ell'_i}}$ for $i > 0$. Then, $H_j \subset H_i \subset H = \Gamma_U$ and $H'_j \subset H'_i \subset H' = \Gamma'_{U'}$ for $j > i > 0$.

Let $\{X_\infty : U\}$ denote the set of translates of the adapted set U . Then, for $H = \Gamma_U$, we have the set equality $\{X_\infty : U\} = \Gamma/H$. Set $m = \#(\Gamma/H)!$, which is the order of the group of permutations on the set $\{X_\infty : U\}$. Then, for $\gamma \in \Gamma$, the action of γ^m on the set $\{X_\infty : U\}$ is the identity. This implies that $\gamma^m \in C_H^\Gamma$ the core of H .

Given a group chain $\mathcal{G}$ and $H \subset \Gamma$ a subgroup with $\Gamma_\ell \subset H$ for some $\ell \geq 1$, the H -restricted order of $\gamma \in \Gamma$ is the Steinitz order with respect to the chain $C_{\gamma, \ell}^H$,

$$\xi^H(\gamma) = \text{lcm}\{\#(\langle \gamma \rangle / C_{\gamma, \ell}^H) \mid \ell > 0\}. \quad (4.14)$$

The H -restricted typeset for the chain $\mathcal{G}$ is the collection

$$\Xi_H[\mathcal{G}] = \{\tau[\xi^H(\gamma)] \mid \gamma \in \Gamma\}. \quad (4.15)$$

LEMMA 4.2. $\tau[\xi^H(\gamma^m)] = \tau[\xi^H(\gamma)]$.

Proof. Let the cyclic group $\langle \gamma \rangle / C_{\gamma, \ell}^H$ have order m_ℓ . Then, γ^m generates a subgroup whose index is the greatest common divisor $n_\ell = \gcd(m, m_\ell)$. We have $m_\ell \mid m_{\ell+1}$; hence $n_\ell \leq n_{\ell+1}$, and also that $n_\ell \leq m$. Thus, the sequence $\{n_\ell \mid \ell \geq 1\}$ has an upper bound

$n_\infty \leq m$ which is realized for ℓ sufficiently large. It follows that $n_\infty \cdot \xi^H(\gamma^m) = \xi^H(\gamma)$; hence $\tau[\xi^H(\gamma^m)] = \tau[\xi^H(\gamma)]$. $\square$

So without loss of generality, it suffices to consider $\gamma \in C_H^\Gamma \subset H$.

The homeomorphism $h: U \rightarrow U'$ induces an isomorphism $h_*: \mathcal{H}_U \rightarrow \mathcal{H}_{U'}$. Thus, there exists $\gamma' \in H'$ whose action $\Phi'_{U'}(\gamma')$ on U' equals the image $\phi'_\gamma = h_*(\Phi_U(\gamma))$. We show that the H' -restricted type of γ' equals the H -restricted type of γ .

Let $V \subset W \subset X_\infty$ be adapted subsets. The restricted action of Γ_W on W is minimal, and hence the translates of V define a clopen partition of W . Let $\{W : V\}$ denote the set of elements in this partition and let $|W : V| = \#\{W : V\}$ denote the cardinality of the set of translates.

The action Φ induces a map $\Phi_V^W: \Gamma_W \rightarrow \text{Aut}(\{W : V\})$ into the permutations of the set $\{W : V\}$ of translates of V in W . The kernel of the map Φ_V^W is denoted by $C_V^W \subset \Gamma_V$ and equals the normal core of Γ_V as a subgroup of Γ_W . Thus, the index $[\Gamma_W : \Gamma_V] = |W : V|$ and $[\Gamma_W : C_V^W] = \#\text{Image}(\Phi_V^W(\Gamma_W))$.

Now, apply this observation to the action of $H = \Gamma_U$ on U . For each $j \geq 1$, the action Φ induces a map $\widehat{\Phi}_{U_{\ell_j}}^U: H \rightarrow \text{Aut}(\{U : \gamma_j \cdot U_{\ell_j}\})$, which permutes the elements of this partition. The kernel of the action map $\widehat{\Phi}_{U_{\ell_j}}^U$ is the subgroup

$$C_j^H = \bigcap_{\delta \in H} \delta \Gamma_{\gamma_j \cdot U_{\ell_j}} \delta^{-1} = \bigcap_{\delta \in H} (\delta \gamma_j) \Gamma_{\ell_j} (\delta \gamma_j)^{-1} = \bigcap_{\delta \in H} \delta \Gamma_{\ell_j} \delta^{-1}. \quad (4.16)$$

Set $C_{\gamma,j}^H = \langle \gamma \rangle \cap C_j^H$, then the subgroup $\langle \gamma \rangle / C_{\gamma,j}^H \subset H / C_j^H$ is mapped injectively into $\text{Aut}(\{U : U_{\ell_j}\})$. Thus, $\#(\langle \gamma \rangle / C_{\gamma,j}^H) = \#\text{Image}(\widehat{\Phi}_{U_{\ell_j}}^U(\langle \gamma \rangle))$, and so we have

$$\xi^H(\gamma) = \text{lcm}\{\#(\langle \gamma \rangle / C_{\gamma,j}^H) \mid j > 0\} = \text{lcm}\{\#\text{Image}(\widehat{\Phi}_{U_{\ell_j}}^U(\langle \gamma \rangle)) \mid j > 0\}. \quad (4.17)$$

We next use the conjugation by h to relate the order of a subgroup of $\text{Aut}(\{U : V\})$ with the order of a subgroup of $\text{Aut}(\{U' : V'\})$ for appropriate choices of adapted subsets V and V' . This is analogous to the idea behind the proof of Lemma 4.1.

From diagram (4.1), for $j \geq 1$, we have

$$U' \supset h(\gamma_1 \cdot U_{\ell_1}) \supset \gamma'_1 \cdot U'_{\ell'_1} \supset \cdots \supset h(\gamma_j \cdot U_{\ell_j}) \supset \gamma'_j \cdot U'_{\ell'_j} \supset \cdots. \quad (4.18)$$

Using that $\gamma'_j \cdot U'_{\ell'_j} \subset h(\gamma_j \cdot U_{\ell_j})$, we have

$$\gamma'_j \Gamma'_{\ell'_j} (\gamma'_j)^{-1} = \Gamma'_{\gamma'_j \cdot U'_{\ell'_j}} \subset \Gamma'_{h(U_{\ell_j})} = H'_j \subset H', \quad (4.19)$$

and so $C_j^{H'} = C_{\gamma'_j \Gamma'_{\ell'_j} (\gamma'_j)^{-1}}^{H'} \subset C_{H'_j}^{H'}$. Then,

$$\langle \gamma' \rangle \supset (\langle \gamma' \rangle \cap C_{H'_j}^{H'}) \supset (\langle \gamma' \rangle \cap C_j^{H'}) = C_{\gamma',j}^{H'}. \quad (4.20)$$

The action of $\Phi'_{U'}(\gamma')$ on U' translates the clopen set $h(\gamma_j \cdot U_{\ell_j})$ within the clopen set U' , and thus

$$\#(\langle \gamma \rangle / C_{\gamma,j}^H) = \# \text{Image}(\widehat{\Phi}_{U_{\ell_j}}^U(\langle \gamma \rangle)) = \# \text{Image}(\widehat{\Phi}_{h(U_{\ell_j})}^{U'}(\langle \gamma' \rangle)) = \#(\langle \gamma' \rangle / (\langle \gamma' \rangle \cap C_{H_j'}^{H'})). \quad (4.21)$$

Using that $\gamma'_j \cdot U'_{\ell'_j} \subset h(\gamma_j \cdot U_{\ell_j})$, we have

$$\gamma'_j \Gamma'_{\ell'_j} (\gamma'_j)^{-1} = \Gamma'_{\gamma'_j \cdot U'_{\ell'_j}} \subset \Gamma'_{h(U_{\ell_j})} = H'_j \subset H', \quad (4.22)$$

and so $C_j^{H'} = C_{\gamma'_j \Gamma'_{\ell'_j} (\gamma'_j)^{-1}}^{H'} \subset C_{H_j'}^{H'}$. We then have, using equation (4.20),

$$\begin{aligned} \#(\langle \gamma' \rangle / C_{\gamma',j}^{H'}) &= \#(\langle \gamma' \rangle / (\langle \gamma' \rangle \cap C_{H_j'}^{H'})) \cdot \#((\langle \gamma' \rangle \cap C_{H_j'}^{H'}) / C_{\gamma',j}^{H'}) \\ &= \#(\langle \gamma \rangle / C_{\gamma,j}^H) \cdot \#((\langle \gamma' \rangle \cap C_{H_j'}^{H'}) / (\langle \gamma' \rangle \cap C_j^{H'})). \end{aligned} \quad (4.23)$$

Thus, we have that $\#(\langle \gamma \rangle / C_{\gamma,j}^H)$ divides $\#(\langle \gamma' \rangle / C_{\gamma',j}^{H'})$ for all $j \geq 1$. It follows that $\tau_H[\gamma] = \tau[\xi^H(\gamma)] \leq \tau[\xi^{H'}(\gamma')] = \tau_{H'}[\gamma']$. Then, by reversing these calculations, starting with γ' chosen as above, we obtain $\tau_{H'}[\gamma'] \leq \tau_H[\gamma]$ and hence $\tau_H[\gamma] = \tau_{H'}[\gamma']$.

We have thus shown that for each $\gamma \in \Gamma$, there exists $\gamma' \in \Gamma'$ with the same restricted type. Reversing this process, we deduce that $\Xi_H[X_\infty, \Gamma, \Phi] = \Xi_{H'}[X'_\infty, \Gamma', \Phi]$. It follows that $\Xi[\mathcal{G}]$ and $\Xi[\mathcal{G}']$ are commensurable as in Definition 3.9, which proves Theorem 1.8.

4.3. Proof of Theorem 1.7. The claim of Theorem 1.7 is that for each $\gamma \in \Gamma$, there is a well-defined type $\tau[\gamma]$. In fact, we show the stronger result, that the Steinitz order $\xi(\gamma)$ is independent of the choice of an adapted basis $\mathcal{U}$ for the odometer $(\mathfrak{X}, \Gamma, \Phi)$. Given two adapted bases $\mathcal{U}$ and $\mathcal{U}'$, the algebraic models they define, $(X_\infty, \Gamma, \Phi_\infty)$ and $(X'_\infty, \Gamma, \Phi'_\infty)$, are isomorphic.

In the above proof of Theorem 1.8, we let $U = X_\infty$ and $U' = X'_\infty$, and so $H = \Gamma_U = \Gamma$ and $H' = \Gamma_{U'} = \Gamma'$. Then, observe that for $\ell > 0$, the normal subgroups $C_\ell^H = C_\ell$ and $C_\ell^{H'} = C'_\ell$. It follows that for $\gamma \in \Gamma$ and $\gamma' \in \Gamma'$ chosen as in the proof of Theorem 1.8, we have $\xi^H(\gamma) = \xi(\gamma)$ and $\xi^{H'}(\gamma') = \xi(\gamma')$, and the proof shows that $\xi(\gamma) = \xi(\gamma')$. It follows that the set of orders $\{\xi(\gamma) \mid \gamma \in \Gamma\}$ is independent of the choice of an algebraic model for the action, and so is an invariant of the isomorphism class of the action $(\mathfrak{X}, \Gamma, \Phi)$, as was to be shown.

4.4. Proof of Corollary 1.9. Suppose that both Γ and Γ' are abelian, and the actions are return equivalent. Then, every group Γ_ℓ in a subgroup chain $\mathcal{G}_\mathcal{U}$ in Γ is normal. Then, for any subgroup $H \subset \Gamma$, the normal cores satisfy $C_\ell^H = C_\ell = \Gamma_\ell$. Thus, in equation (4.17), the restricted type $\xi^H(\gamma) = \xi(\gamma)$. Similarly, we have $\xi^{H'}(\gamma') = \xi(\gamma')$ for $\gamma' \in \Gamma'$. Thus, the above proof shows in this case that the typesets $\Xi[\mathfrak{X}, \Gamma, \Phi]$ and $\Xi[\mathfrak{X}', \Gamma', \Phi']$ are equal, which is the claim of Corollary 1.9.

It is surprising perhaps that the conclusion of Corollary 1.9 need not hold if one of the groups is virtually abelian, but not abelian, as illustrated in Example 5.3. The idea of Example 5.3 is that we add to an abelian group a single element that normalizes it, does

not commute with it, and destroys the equality $C_\ell^H = C_\ell$. Then, the types of elements are no longer equal to their H -restricted types.

4.5. Proof of Theorem 1.10. Let $\varphi: \Gamma \rightarrow \Gamma$ be a self-embedding with finite index image. Define a group chain $\mathcal{G}_\varphi$ by setting $\Gamma_0 = \Gamma$ and then inductively define $\Gamma_{\ell+1} = \varphi(\Gamma_\ell) \subset \Gamma_\ell$ for $\ell \geq 0$.

Then, with $H = \Gamma_k$ for $k > 0$ and $\ell > k$,

$$C_\ell^H = \bigcap_{\delta \in H} \delta^{-1} \Gamma_\ell \delta = \bigcap_{\delta \in \varphi^k(\Gamma)} \delta^{-1} \varphi^\ell(\Gamma) \delta = \varphi^k \left(\bigcap_{\delta \in \Gamma} \delta^{-1} \varphi^{\ell-k}(\Gamma) \delta \right) = \varphi^k(C_{\ell-k}), \quad (4.24)$$

and so $\Gamma_\ell / C_\ell^H = \varphi^k(\Gamma_{\ell-k} / C_{\ell-k})$. Then, for $\gamma \in H = \Gamma_k$, set $\bar{\gamma} = \varphi^{-k}(\gamma)$, then we have $\xi^H(\gamma) = \text{lcm}\{\#(\langle \gamma \rangle / C_{\gamma, \ell}^H) \mid \ell > k\} = \text{lcm}\{\#(\langle \bar{\gamma} \rangle / C_{\bar{\gamma}, \ell-k}) \mid \ell - k > 0\} = \xi(\bar{\gamma})$. (4.25)

Then, as $\Gamma = \varphi^{-k}(H)$, we have $\Xi_H[X_\varphi, \Gamma, \Phi_\varphi] = \Xi[X_\varphi, \Gamma, \Phi_\varphi]$.

Let $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X_{\varphi'}, \Gamma', \Phi_{\varphi'})$ be odometers associated to renormalizations $\varphi: \Gamma \rightarrow \Gamma$ and $\varphi': \Gamma' \rightarrow \Gamma'$. Assume the actions are return equivalent by a homeomorphism $h: U \rightarrow U'$. Then, choose $\delta \in \Gamma$ with $\delta \cdot e_\infty \in U$ and $\delta' \in \Gamma'$ with $\delta' \cdot e'_\infty \in U'$. Then, $\varphi^\delta = \Phi(\delta) \circ \varphi \circ \Phi(\delta^{-1})$ is a renormalization of Γ with group chain $\Gamma_\ell^\delta = \delta \Gamma_\ell \delta^{-1}$. The group Γ_ℓ^δ stabilizes the translate $\delta \cdot U_\ell$. As $\delta \cdot e_\infty \in U$, there exists $k > 0$ such that $\delta \cdot U_k \subset U$. Repeat this argument for the renormalization φ' to obtain a $\delta' \in \Gamma'$ and $k' > 0$ such that $\delta' \cdot U'_{k'} \subset U'$.

Then, proceed as in the proof in §4.2 with $H = \delta \Gamma_k \delta^{-1}$ and $H' = \delta' \Gamma'_{k'} (\delta')^{-1}$ to obtain that

$$\Xi[X_\varphi, \Gamma, \Phi_\varphi] = \Xi_H[X_\varphi, \Gamma, \Phi_\varphi] = \Xi_{H'}[X'_{\varphi'}, \Gamma', \Phi_{\varphi'}] = \Xi[X'_{\varphi'}, \Gamma', \Phi_{\varphi'}],$$

as claimed in Theorem 1.10. In particular, this implies that the typeset $\Xi[X_\varphi, \Gamma, \Phi_\varphi]$ is an invariant of the isomorphism class of the renormalizable Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$.

5. Basic examples

5.1. Virtually abelian actions. A group Γ is *virtually abelian* if it admits a finite-index subgroup $A \subset \Gamma$ which is abelian. It is straightforward to construct $\mathbb{Z}^n$ -odometer actions with prescribed spectra, and yet they illustrate several basic properties of the type and typeset invariants.

Example 5.1. Consider the case $n = 1$. Choose two disjoint sets of distinct primes,

$$\pi_f = \{q_1, q_2, \dots\}, \quad \pi_\infty = \{p_1, p_2, \dots\},$$

where π_f and π_∞ can be chosen to be finite or infinite sets, and either π_f is infinite or π_∞ is non-empty. Choose multiplicities $n(q_i) \geq 1$ for the primes in π_f . For each $\ell > 0$, define a subgroup of $\Gamma = \mathbb{Z}$ by

$$\Gamma_\ell = \{q_1^{n(q_1)} q_2^{n(q_2)} \cdots q_\ell^{n(q_\ell)} \cdot p_1^\ell p_2^\ell \cdots p_\ell^\ell \cdot n \mid n \in \mathbb{Z}\}. \quad (5.1)$$

If π_∞ is a finite set, then we use the convention that $p_\ell = 1$ in equation (5.1) when p_ℓ is not defined by the listing of π_∞ . The completion $\widehat{\Gamma}$ of $\mathbb{Z}$ with respect to this group chain admits a product decomposition into its Sylow p -subgroups

$$\widehat{\Gamma} \cong \prod_{i=1}^{\infty} \mathbb{Z}/q_i^{n(q_i)}\mathbb{Z} \cdot \prod_{p \in \pi_\infty} \widehat{\mathbb{Z}}_{(p)}, \quad (5.2)$$

where $\widehat{\mathbb{Z}}_{(p)}$ denotes the p -adic completion of $\mathbb{Z}$. Thus, $\pi(\xi(\widehat{\Gamma})) = \pi_f \cup \pi_\infty$. As $\mathbb{Z}$ is abelian, $X_\infty = \widehat{\Gamma}$. The type of this action classifies it up to return equivalence.

Example 5.2. The diagonal $\mathbb{Z}^n$ -odometer actions for $n \geq 2$ are direct extensions of Example 5.1. Make n choices of prime spectra as in Example 5.1, then take the product action on the individual factors. The type of the action no longer determines the isomorphism class of the $\mathbb{Z}^n$ -odometer actions obtained. The typeset is an invariant under return equivalence by Corollary 1.9, but only in special cases does the typeset determine the isomorphism class of the action. The interested reader can consult the works [2, 3, 8, 21, 34].

Example 5.3. We next give an example that illustrates that the commensurable relation on typesets is optimal. We construct the simplest example which shows this, and it is clear that many more similar constructions are possible.

Let $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}_2$ be the semi-direct product of $\mathbb{Z}^2$ with the order 2 group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, where the generator $\sigma \in \mathbb{Z}_2$ acts on $\mathbb{Z}^2$ by permuting the summands. Let $\Gamma' = \mathbb{Z}^2$, which is abelian.

Choose distinct primes $p, q > 1$. Define the subgroup chain in Γ and Γ' as follows:

$$\Gamma_\ell = \{(p^\ell k, q^\ell m, id) \mid (h, m) \in \mathbb{Z}^2\}, \quad \Gamma'_\ell = \{(p^\ell k, q^\ell m) \mid (k, m) \in \mathbb{Z}^2\}. \quad (5.3)$$

Note that the resulting actions (X_∞, Γ, Φ) and $(X'_\infty, \Gamma', \Phi')$ are return equivalent.

Observe that $C'_\ell = \Gamma'_\ell$. However, Γ_ℓ is not normal in Γ and we have

$$C_\ell = \{((pq)^\ell k, (pq)^\ell m, id) \mid (h, m) \in \mathbb{Z}^2\} \subset \Gamma_\ell. \quad (5.4)$$

The actions of Γ and Γ' have the same type, with characteristic functions $\chi(p) = \chi(q) = \infty$, and all other values are zero. However, we have the typesets

$$\Xi[X_\infty, \Gamma, \Phi] = \{[(pq)^\infty]\} \quad \text{and} \quad \Xi[X'_\infty, \Gamma', \Phi'] = \{[p^\infty], [q^\infty], [(pq)^\infty]\}, \quad (5.5)$$

so the typeset is not invariant under return equivalence.

This example easily generalizes, where we take $\Gamma' = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ to be the direct sum of n copies of $\mathbb{Z}$, and replace $\mathbb{Z}_2$ with any non-trivial subgroup $\Delta \subset \text{Perm}(n)$ of the permutation group on n elements. Let $\Gamma = \Gamma' \rtimes \Delta$ be the semi-direct product of Γ' with Δ . Choose the subgroup chain $\{\Gamma'_{\ell'}\}$ in Γ' as in Example 5.2 and use the chain $\{\Gamma_\ell = \Gamma'_{\ell'} \times id\}$ in Γ . Then, the resulting actions are return equivalent and one can obtain a wide variety of finite typesets $\Xi[X'_\infty, \Gamma', \Phi']$ for the abelian action. If the action of Δ is transitive, then we have that $\Xi[X_\infty, \Gamma, \Phi]$ consists of a single element. When Δ does not act transitively, there is even more variation on the typesets of the two actions.

5.2. Nilpotent odometers. The next examples use the actions associated to a renormalization of a finitely generated group Γ . Many finitely generated nilpotent groups admit a renormalization [12, 15, 16, 19, 30, 36], which yield many examples of odometer actions with well-defined typesets by Theorem 1.10.

The integer Heisenberg group is the simplest non-abelian nilpotent group, and is represented as the upper triangular matrices in $\mathrm{GL}(3, \mathbb{Z})$. That is,

$$\Gamma = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}. \quad (5.6)$$

We denote a 3×3 matrix in Γ by the coordinates as (a, b, c) .

Example 5.4. For a prime $p \geq 2$, define the self-embedding $\varphi_p: \Gamma \rightarrow \Gamma$ by $\varphi(a, b, c) = (pa, pb, p^2c)$. Then, define a group chain in Γ by setting

$$\Gamma_\ell = \varphi_p^\ell(\Gamma) = \{(p^\ell a, p^\ell b, p^{2\ell} c) \mid a, b, c \in \mathbb{Z}\}, \quad \bigcap_{\ell > 0} \Gamma_\ell = \{e\}.$$

For $\ell > 0$, the normal core for Γ_ℓ is given by $C_\ell = \mathrm{core}(\Gamma_\ell) = \{(p^{2\ell} a, p^{2\ell} b, p^{2\ell} c) \mid a, b, c \in \mathbb{Z}\}$, and so the quotient group $Q_\ell = \Gamma/C_\ell \cong \{(\bar{a}, \bar{b}, \bar{c}) \mid \bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}/p^{2\ell}\mathbb{Z}\}$. The profinite group $\widehat{\Gamma}_\infty$ is the inverse limit of the quotient groups Q_ℓ , so we have $\widehat{\Gamma}_\infty = \{(\widehat{a}, \widehat{b}, \widehat{c}) \mid \widehat{a}, \widehat{b}, \widehat{c} \in \widehat{\mathbb{Z}}_{p^2}\}$. Thus, every non-trivial $\gamma \in \Gamma$ has type $\tau[\gamma] = \tau[p^\infty]$.

Example 5.5. For distinct primes $p, q \geq 2$, define the self-embedding $\varphi_{p,q}: \Gamma \rightarrow \Gamma$ by $\varphi(a, b, c) = (pa, qb, pqc)$. Then, define a group chain in Γ by setting

$$\Gamma_\ell = \varphi_{p,q}^\ell(\Gamma) = \{(p^\ell a, q^\ell b, (pq)^\ell c) \mid a, b, c \in \mathbb{Z}\}, \quad \bigcap_{\ell > 0} \Gamma_\ell = \{e\}.$$

For $\ell > 0$, the normal core for Γ_ℓ is given by $C_\ell = \mathrm{core}(\Gamma_\ell) = \{((pq)^\ell a, (pq)^\ell b, (pq)^\ell c) \mid a, b, c \in \mathbb{Z}\}$, and so we obtain the quotient group $Q_\ell = \Gamma/C_\ell \cong \{(\bar{a}, \bar{b}, \bar{c}) \mid \bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}/(pq)^\ell \mathbb{Z}\}$. The profinite group $\widehat{\Gamma}_\infty$ is the inverse limit of the quotient groups Q_ℓ , so we have $\widehat{\Gamma}_\infty = \{(\widehat{a}, \widehat{b}, \widehat{c}) \mid \widehat{a}, \widehat{b}, \widehat{c} \in \widehat{\mathbb{Z}}_{pq}\}$. Thus, every non-trivial $\gamma \in \widehat{\Gamma}_\infty$ has type $\tau[\gamma] = \tau[(pq)^\infty]$.

Note that the typeset for the odometer defined by the $\varphi_{p,q}$ -renormalization equals the typeset for the abelian action in Example 5.3, but the two actions are clearly not return equivalent.

A second source of examples for odometer actions of nilpotent groups uses the decomposition of a profinite nilpotent group into its prime localizations, a technique that is especially adapted to realizing a given collection of primes as the spectrum of such an action. Moreover, these actions can be constructed to have special dynamical properties, as in [27]. The construction is necessarily more complex than for renormalizable actions, as there must be an infinite sequence of choices to make. These ideas were developed in [24, §9] for actions of $SL(n, \mathbb{Z})$, and the work [28] discusses this construction for nilpotent groups.

6. Examples: d -regular actions

It is well known that every odometer $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to an action of Γ on the boundary of a rooted tree. The study of actions on trees, especially the actions on d -ary (or d -regular) trees satisfying an additional condition of *self-similarity*, is an active topic in geometric group theory, see [23, 35] for surveys. The tree models for odometers are especially useful for constructing actions which are not topologically free, and thus for illustrating the commensurable relationship between types. In this section, we study the typesets for odometers on boundaries of d -regular rooted trees.

6.1. Actions on trees. A tree T is an infinite graph with the set of vertices $V = \bigsqcup_{\ell \geq 0} V_\ell$ and the set of edges E . Each V_ℓ , $\ell \geq 0$ is a finite set, called the *set of vertices at level ℓ* . Edges in E join pairs of vertices in consecutive level sets $V_{\ell+1}$ and V_ℓ , $\ell \geq 0$, so that a vertex in $V_{\ell+1}$ is connected to a single vertex in V_ℓ by a single edge. A tree is *rooted* if $|V_0| = 1$.

Definition 6.1. A tree T is *spherically homogeneous* if there is a sequence $n = (n_1, n_2, \dots)$, called the *spherical index* of T , such that for every $\ell \geq 1$, a vertex in $V_{\ell-1}$ is joined by edges to precisely n_ℓ vertices in V_ℓ . In addition, T is d -ary, or d -regular, if its spherical index $n = (n_1, n_2, \dots)$ is *constant*, that is, $n_\ell = d$ for some positive integer d .

We assume that $n_\ell \geq 2$ for $\ell \geq 1$. If $d = 2$, then a 2-ary tree T is also called a *binary* tree.

Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer and let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell > 0\}$ be a choice of an adapted neighborhood basis. By Corollary 2.4, there is a group chain $\mathcal{G}_\mathcal{U} = \{\Gamma_\ell = \Gamma_{U_\ell} \mid \ell \geq 0\}$ such that, associated to $\mathcal{G}_\mathcal{U}$ is an odometer $(X_\infty, \Gamma, \Phi_\infty)$ which is isomorphic to $(\mathfrak{X}, \Gamma, \Phi)$. Here, X_∞ is the inverse limit space of finite sets $X_\ell = \Gamma / \Gamma_\ell$, given by equation (1.3), see §2.3 for details.

A tree model for the action $(\mathfrak{X}, \Gamma, \Phi)$ is constructed using the group chain $\mathcal{G}_\mathcal{U}$. For $\ell \geq 0$, let $V_\ell = X_\ell$, and join $v_\ell \in V_\ell$ and $v_{\ell+1} \in V_{\ell+1}$ by an edge if and only if $v_{\ell+1} \subset v_\ell$ as cosets. The tree so constructed is spherically homogeneous, with spherical index entries $n_\ell = |\Gamma_{\ell-1} : \Gamma_\ell|$ for $\ell \geq 1$. The boundary ∂T of T is the collection of all infinite paths in T , that is,

$$\partial T = \left\{ (v_\ell)_{\ell \geq 0} \subset \prod_{\ell \geq 0} V_\ell \mid v_{\ell+1} \text{ and } v_\ell \text{ are joined by an edge} \right\} \cong X_\infty,$$

and the induced action of Γ on ∂T , which we also denote by (X_∞, Γ, Φ) .

Definition 6.2. An odometer $(\mathfrak{X}, \Gamma, \Phi)$ is d -regular, or just *regular*, if there exists $d \geq 2$ such that $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to an action of Γ on a rooted d -ary tree.

Remark 6.3. An odometer $(\mathfrak{X}, \Gamma, \Phi)$ is d -regular if the group chain $\mathcal{G}_\mathcal{U}$ above can be chosen so that each subgroup index $|\Gamma_\ell : \Gamma_{\ell-1}| = d$ for some $d \geq 2$ and all $\ell \geq 1$. Nilpotent actions given by a self-embedding $\varphi_p : \Gamma \rightarrow \Gamma$ in Example 5.4 are d -regular with $d = p^4$, and those in Example 5.5 are d -regular with $d = p^2 q^2$.

6.2. *Typesets of regular actions.* We consider next the typeset of a d -regular odometer.

THEOREM 6.4. *An d -regular odometer $(\mathfrak{X}, \Gamma, \Phi)$ has finite typeset $\Xi[\mathfrak{X}, \Gamma, \Phi]$. More precisely, suppose $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to an action on a d -ary rooted tree for some $d \geq 2$. Let P_d be the set of distinct prime divisors of the elements in the collection $\{2, 3, \dots, d\}$ and let $N_d = |P_d|$. Then,*

$$|\Xi[\mathfrak{X}, \Gamma, \Phi]| \leq \sum_{k=0}^{N_d} \binom{N_d}{k} = \sum_{k=0}^{N_d} \frac{N_d!}{(N_d - k)! k!}. \quad (6.1)$$

Moreover, each equivalence class $\tau \in \Xi[\mathfrak{X}, \Gamma, \Phi]$ is represented by a Steinitz number ξ with empty finite prime spectrum, $\pi_f(\xi) = \emptyset$, and so $\pi(\xi) = \pi_\infty(\xi)$.

Proof. Let N_d and P_d be as in the statement of the theorem and let $L_d = \text{lcm}\{P_d\}$.

Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell > 0\}$ be a choice of an adapted neighborhood basis with associated group chain $\mathcal{G}_\mathcal{U} = \{\Gamma_\ell = \Gamma_{U_\ell} \mid \ell \geq 0\}$ and Γ -odometer $(X_\infty, \Gamma, \Phi_\infty)$ isomorphic to an action of Γ on the boundary ∂T of a rooted tree T as constructed above.

LEMMA 6.5. *Let $\gamma \in \Gamma$, let $\xi(\gamma)$ be the Steinitz order of γ as defined in Definition 1.6, and let p be such that $\chi_\xi(p) \neq 0$. Then, p divides L_d .*

Proof. If $\chi_\xi(p) \neq 0$, then there exists the smallest $\ell \geq 1$ such that p divides the order of the group $\langle \gamma \rangle / \langle \gamma \rangle_\ell$. The group $\langle \gamma \rangle / \langle \gamma \rangle_\ell = \langle \gamma \rangle / \langle \gamma \rangle \cap C_\ell$ is isomorphic to a subgroup of Γ / C_ℓ , where C_ℓ is the normal core of Γ_ℓ , and acts on the coset space X_ℓ by permutations. Let $\lambda_{\gamma, \ell}$ be the permutation of X_ℓ induced by γ . Then, the order of $\langle \gamma \rangle / \langle \gamma \rangle_\ell$ is equal to the order of $\lambda_{\gamma, \ell}$ and so equal to the least common multiple of the length of the cycles in $\lambda_{\gamma, \ell}$.

Similarly, the order of $\langle \gamma \rangle / \langle \gamma \rangle_{\ell-1}$ is equal to the least common multiple of the length of the cycles in the permutation $\lambda_{\gamma, \ell-1}$ of $X_{\ell-1}$ induced by the action of γ . By the choice of ℓ , the order of $\langle \gamma \rangle / \langle \gamma \rangle_{\ell-1}$ is not divisible by p . Therefore, for any cycle $c_{\ell-1}$ in $\lambda_{\gamma, \ell-1}$, the length $|c_{\ell-1}|$ is not divisible by p .

Consider the preimage $S_{c_{\ell-1}}$ of the set of elements in $c_{\ell-1}$ under the inclusion of cosets $X_\ell \rightarrow X_{\ell-1}$. Then, $|S_{c_{\ell-1}}| = d|c_{\ell-1}|$ and γ permutes the elements in $S_{c_{\ell-1}}$. Let c_ℓ be a cycle in the permutation $\mu_{c_{\ell-1}}$ of $S_{c_{\ell-1}}$ induced by γ . Since the action of γ commutes with coset inclusions, $|c_\ell| = \alpha|c_{\ell-1}|$ for some $1 \leq \alpha \leq d$. Then, p must divide such an α for one of the cycles in $\mu_{c_{\ell-1}}$, for some cycle $c_{\ell-1}$ in $\lambda_{\gamma, \ell-1}$. It follows that p divides L_d . $\square$

Let $\xi(\gamma)$ be the Steinitz order of γ . Since every p for which $0 < \chi_\xi(p) < \infty$ divides L_d by Lemma 6.5, then the finite prime spectrum $\pi(\xi(\gamma))$ is finite, and the type $\tau(\gamma)$ has a representative $\widehat{\xi}$, such that if $\chi_{\widehat{\xi}}(p) \neq 0$, then $\chi_{\widehat{\xi}}(p) = \infty$. It follows that two types $\tau(\gamma)$ and $\tau(\gamma')$ with respective Steinitz orders ξ and ξ' are distinct if and only if there exists a prime p such that $\chi_\xi(p) = 0$ and $\chi_{\xi'}(p) \neq 0$. The bound in equation (6.1), which is the number of distinct collections of prime divisors of L_d , follows. $\square$

Example 6.6. Let $d = 2$, then $L_2 = 2$ and $N_2 = 1$, and the upper bound on the cardinality of the typeset for 2-regular odometers is 2, with possible types $\{[1], [2^\infty]\}$, where 1 denotes the type of the identity element.

Let $d = 3$, then $L_3 = 6$ and $N_3 = 2$. Then, the upper bound on the cardinality of the typeset for 3-regular odometers is 4, with possible types $\{[1], [2^\infty], [3^\infty], [(2 * 3)^\infty]\}$.

Let $d = 4$, then $P_4 = \{2, 3\}$, $L_4 = 6$, and $N_4 = 2$. Then, the upper bound on the cardinality of the typeset for 4-regular odometers is 4, with possible types $\{[1], [2^\infty], [3^\infty], [(2 * 3)^\infty]\}$.

Example 6.7. Let p be an odd prime. The Gupta–Sidki p -group $GS(p)$ [35, §1.8.1] is a group acting on the rooted p -ary tree. The group is generated by a cyclic permutation of the set of p elements $\sigma = (0, 1, \dots, p - 1)$ and the recursively defined map

$$\tau = (\sigma, \sigma^{-1}, 1, \dots, \tau).$$

Every element in the Gupta–Sidki p -group has finite order and so $\Xi[X_\infty, GS(p), \Phi_\infty] = \{[1]\}$, where $[1]$ is the trivial type, that is, the type of the identity element.

Example 6.8. Let $d = 2$ and let $(\mathfrak{X}, \Gamma, \Phi)$ be a 2-regular odometer action. Then, $|\Xi[\mathfrak{X}, \Gamma, \Phi]| \leq 2$, and either $\Xi[\mathfrak{X}, \Gamma, \Phi] = \{[1]\}$ or $\Xi[\mathfrak{X}, \Gamma, \Phi] = \{[1], [2^\infty]\}$.

An example of an action where the typeset $\Xi[\mathfrak{X}, \Gamma, \Phi] = \{[1]\}$ is the action of the Grigorchuk group, see for instance [23]. This group is an example of a Burnside group, which is an infinite group where every element has finite order, and so it has trivial typeset.

For the case $\Xi[\mathfrak{X}, \Gamma, \Phi] = \{[1], [2^\infty]\}$, we have two classes of examples. First, the examples of odometers where $\tau(\gamma) = [2^\infty]$ for any non-trivial $\gamma \in \Gamma$ are the actions of iterated monodromy groups of quadratic polynomials for which the orbit of the critical point is periodic. Such groups are torsion-free, see [6], and so they do not have any finite-order element except the identity in Γ . Such actions include, but are not limited to, the action of the odometer on the binary tree and the Basilica group.

Second, the action may have non-trivial elements that have trivial type, and also non-trivial elements with type $\{[2^\infty]\}$. These are given by the actions of the iterated monodromy groups of quadratic polynomials with strictly pre-periodic orbits of the critical point, see [6] for details.

6.3. Typeset under the commensuration relation. We now show that H -restricted types of a group element and of its conjugate need not coincide.

Given a tree T and a vertex $v \in V$, denote by T_v the subtree of T with root v .

Definition 6.9. [23] Let T be a spherically homogeneous tree and let $\Gamma \subset \text{Aut}(T)$. The action of Γ on ∂T is *weakly branch* if the following conditions hold:

- (1) the restriction of the action of Γ to each vertex level set V_n , $n \geq 1$, is transitive;
- (2) for every $\ell \geq 1$ and every $v \in V_\ell$, there exists $g \in \Gamma$ such that $g \cdot v = v$, the restriction of the action of g on the subtree T_v is non-trivial, and the restriction of g to the complement $\partial T - \partial T_v$ is the identity map.

Example 6.10. Let T be a spherically homogeneous tree and let $\Gamma \subset \text{Aut}(T)$ be so that the action of Γ on ∂T is weakly branch. Choose a sequence (v_ℓ) of vertices in T , and let Γ_ℓ be the isotropy group (equivalently, the stabilizer) of $v_\ell \in V_\ell$. Then, Γ_ℓ fixes v_ℓ while permuting other vertices in V_ℓ .

Choose $n \geq 1$. The normal core C_n of Γ_n consists of elements of Γ_n that fix every vertex in V_n . Let $g \in \Gamma_n$ be an element given by Definition 6.9 of a weakly branch group, namely, the restriction of γ to T_{v_n} is non-trivial, and γ is trivial on the complement of ∂T_{v_n} in ∂T . In particular, this means that $\gamma \in C_n$. However, since γ acts non-trivially on ∂T_{v_n} , then there exists $\ell > n$ such that $\gamma \notin C_\ell$, and then the Steinitz order $\xi(\gamma)$ is non-trivial.

Now, let $\delta \in \Gamma$ be such that $\delta \cdot v_n \neq v_n$. Then, the restriction of $\delta\gamma\delta^{-1}$ to ∂T_{v_n} is the identity map. Since the Steinitz order is invariant under conjugation, $\xi(\delta\gamma\delta^{-1})$ is non-trivial.

Now, consider the Γ_n -restricted Steinitz orders of γ and of $\delta\gamma\delta^{-1}$. The subgroup $C_\ell^{\Gamma_n}$ is the normal core of Γ_ℓ in Γ_n , so it fixes every vertex in the set $V_\ell \cap T_{v_n}$ and it may permute the vertices of V_ℓ that are not in T_{v_n} . In particular, for all $\ell \geq n$, we have $\delta\gamma\delta^{-1} \in C_\ell^{\Gamma_n}$, and the Γ_n -restricted Steinitz order $\xi^{\Gamma_n}(\delta\gamma\delta^{-1})$ is trivial. At the same time, $\xi^{\Gamma_n}(\gamma)$ is non-trivial, since the action of γ permutes the vertices in $V_\ell \cap T_{v_n}$ for some $\ell \geq n$. Depending on the group Γ , $\xi^{\Gamma_n}(\gamma)$ may have the trivial or non-trivial type; it is straightforward to construct examples of both situations.

7. Solenoidal manifolds

Solenoidal manifolds were introduced in §1.4. In this section, we discuss the relation between the classification of solenoidal manifolds up to homeomorphism and the classification of odometers up to return equivalence. We then give the proof of Theorems 1.14 and 1.16.

Recall that a *presentation* is a sequence of *proper finite covering maps* $\mathcal{P} = \{q_\ell: M_\ell \rightarrow M_{\ell-1} \mid \ell \geq 1\}$, where each M_ℓ is a compact connected manifold without boundary of dimension n . The inverse limit

$$\mathcal{M}_{\mathcal{P}} \equiv \varprojlim \{q_\ell: M_\ell \rightarrow M_{\ell-1}\} \subset \prod_{\ell \geq 0} M_\ell \quad (7.1)$$

is the *weak solenoid*, or *solenoidal manifold*, associated to $\mathcal{P}$.

By the definition of the inverse limit, for a sequence $\{x_\ell \in M_\ell \mid \ell \geq 0\}$, we have

$$x = (x_0, x_1, \dots) \in \mathcal{M}_{\mathcal{P}} \iff q_\ell(x_\ell) = x_{\ell-1} \quad \text{for all } \ell \geq 1. \quad (7.2)$$

For each $\ell \geq 0$, there is a fibration $\widehat{q}_\ell: \mathcal{M}_{\mathcal{P}} \rightarrow M_\ell$, given by projection onto the ℓ th factor in equation (7.1). Also, there is a covering map denoted by $\overline{q}_\ell = q_\ell \circ q_{\ell-1} \circ \dots \circ q_1: M_\ell \rightarrow M_0$, such that $\widehat{q}_0 = \overline{q}_\ell \circ \widehat{q}_\ell$. Choose a basepoint $x_0 \in M_0$ and basepoint $x_\infty \in \mathcal{X} = \widehat{q}_0^{-1}(x_0)$, the fiber over x_0 . Then, for each $\ell > 0$, this defines the basepoint $x_\ell = \widehat{q}_\ell(x_\infty) \in M_\ell$. Let $\Gamma_\ell = (\overline{q}_\ell)_\#(\Gamma_0)$ denote the image Γ_ℓ of the fundamental group $\pi_1(M_\ell, x_\ell)$ in $\Gamma = \pi_1(M_0, x_0)$.

Let $\mathcal{G}_{\mathcal{P}} = \{\Gamma_\ell \mid \ell \geq 0\}$ be the group chain determined by $\mathcal{P}$ and the choice of basepoint $x \in \mathcal{M}_{\mathcal{P}}$. Let $(X_\infty, \Gamma, \Phi_\infty)$ denote the odometer defined in equation (1.3) which is determined by $\mathcal{G}_{\mathcal{P}}$, which is called the monodromy action of the fibration $\widehat{q}_0: \mathcal{G}_{\mathcal{P}} \rightarrow M_0$. Note that we suppress the dependence on the choice of the basepoint x for convenience of notation.

The fiber $\overline{q}_\ell^{-1}(x_0) \subset M_\ell$ is identified with the quotient set $X_\ell = \Gamma/\Gamma_\ell$, which is a left Γ -space. In this way, the subspace $\mathfrak{X} = \widehat{q}_0^{-1}(x_0)$ of the inverse limit in equation (7.1) is identified with the Γ -space X_∞ . We then have the following result.

THEOREM 7.1. [10, Theorem 1.1] *Let $\mathcal{M}_\mathcal{P}$ and $\mathcal{M}_{\mathcal{P}'}$ be homeomorphic solenoidal manifolds. Then, the corresponding odometers $(X_\infty, \Gamma, \Phi_\infty)$ and $(X'_\infty, \Gamma', \Phi'_\infty)$ are return equivalent.*

Thus, an invariant of odometers modulo return equivalence is an invariant for solenoidal manifolds modulo homeomorphism. There are various classes of solenoidal manifolds where the converse statement can be shown, as discussed in [10]. Here is one example.

THEOREM 7.2. [10, Theorem 1.3] *Let $\mathcal{M}_\mathcal{P}$ and $\mathcal{M}_{\mathcal{P}'}$ be toroidal solenoidal of the same dimension; that is, both presentations $\mathcal{P}$ and $\mathcal{P}'$ have base manifold the n -torus $\mathbb{T}^n$. If the monodromy actions $(X_\infty, \Gamma, \Phi_\infty)$ and $(X'_\infty, \Gamma', \Phi'_\infty)$ are return equivalent, then $\mathcal{M}_\mathcal{P}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic.*

7.1. Type invariants for solenoidal manifolds. The covering degree m_ℓ of $q_\ell: M_\ell \rightarrow M_{\ell-1}$ equals the index of the subgroup $[\Gamma_{\ell-1} : \Gamma_\ell]$, and so the covering degree of $\overline{q}_\ell: M_\ell \rightarrow M_0$ is given by

$$\deg(\overline{q}_\ell) = m_\ell \cdot m_{\ell-1} \cdots m_1 = [\Gamma : \Gamma_\ell]. \quad (7.3)$$

The Steinitz order of a presentation $\mathcal{P}$ is

$$\xi(\mathcal{P}) = \text{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\} = \text{lcm}\{[\Gamma : \Gamma_\ell] \mid \ell > 0\} = \xi(\mathcal{G}_\mathcal{P}). \quad (7.4)$$

The Steinitz number $\xi(\mathcal{P}')$ can be thought of as the covering degree of the fibration $\widehat{q}_0: \mathcal{M}_\mathcal{P} \rightarrow M_0$.

Given a second presentation $\mathcal{P}'$ with solenoidal manifold $\mathcal{M}'_{\mathcal{P}'}$ and choice of basepoint $x'_0 \in \mathcal{M}'_{\mathcal{P}'}$, we similarly obtain subgroups Γ'_ℓ defining the group chain $\mathcal{G}'_{\mathcal{U}'}$, and we have $\xi(\mathcal{P}') = \xi(\mathcal{G}'_{\mathcal{U}'})$.

Suppose that the solenoidal manifolds $\mathcal{M}_\mathcal{P}$ and $\mathcal{M}'_{\mathcal{P}'}$ are homeomorphic, then by Theorem 7.1, the corresponding odometers $(X_\infty, \Gamma, \Phi_\infty)$ and $(X'_\infty, \Gamma', \Phi'_\infty)$ are return equivalent, and hence they have the equal types $\tau[X_\infty, \Gamma, \Phi_\infty] = \tau[X'_\infty, \Gamma', \Phi'_\infty]$ by Theorem 1.5. We thus obtain the following theorem.

THEOREM 7.3. *Associated to a presentation $\mathcal{P}$ is a well-defined Steinitz number $\xi(\mathcal{P})$, and the type $\tau[\xi(\mathcal{P})]$ depends only on the homeomorphism class of $\mathcal{M}_\mathcal{P}$, which is denoted by $\tau[\mathcal{M}_\mathcal{P}]$.*

We thus obtain the following well-defined homeomorphism invariants for solenoidal manifolds.

COROLLARY 7.4. *The infinite prime spectrum $\pi_\infty(\xi(\mathcal{P}))$ of a solenoidal manifold $\mathcal{M}_\mathcal{P}$ depends only on the homeomorphism class of $\mathcal{M}_\mathcal{P}$, as does also the property that $\pi_f(\xi(\mathcal{P}))$ is infinite.*

For orientable one-dimensional solenoids, where each map $q_\ell: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is orientable, Bing observed in [7] that if $\tau[\mathcal{M}_{\mathcal{P}}] = \tau[\mathcal{M}_{\mathcal{P}'}]$, then $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic. McCord showed in [32, §2] the converse, that if $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic, then $\tau[\mathcal{M}_{\mathcal{P}}] = \tau[\mathcal{M}_{\mathcal{P}'}]$. (See also [1].) Together, these results yield the following well-known classification.

THEOREM 7.5. *For orientable one-dimensional solenoidal manifolds, $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic if and only if $\tau[\mathcal{M}_{\mathcal{P}}] = \tau[\mathcal{M}_{\mathcal{P}'}]$.*

For solenoidal manifolds of dimension $n \geq 2$, no such classification by invariants can exist, even when the base manifold $M_0 = \mathbb{T}^n$. The most one can hope for is to define invariants which distinguish various homeomorphism classes, such as the type and typeset for a solenoidal manifold, invariants derived from the Cantor action associated to the monodromy of the fibration map $\widehat{q}_0: \mathcal{M}_{\mathcal{P}} \rightarrow M_0$.

Note that by Theorem 7.2, if two toroidal solenoids have return equivalent monodromy odometers, then they are homeomorphic. Thus, in this special case, the classification problem for solenoidal manifolds is equivalent to the classification problem of $\mathbb{Z}^n$ -odometers modulo return equivalence. Giordano, Putnam, and Skau discuss in [22] this classification problem.

7.2. Typeset invariants for solenoidal manifolds. The typeset invariants for solenoidal manifolds provide more refined invariants of their homeomorphism class. Before stating the precise definition, we give an intuitive definition.

Let M_0 be the base manifold of a presentation $\mathcal{P}$. Let $x_\infty \in \mathcal{M}_{\mathcal{P}}$ be a choice of a basepoint, which determines the basepoint $x_0 \in M_0$ and fundamental group $\Gamma = \pi_1(M_0, x_0)$. Suppose that $\gamma \in \Gamma$ is represented by a simple closed curve $c_\gamma: \mathbb{S}^1 \rightarrow M_0$. The projection map $\widehat{q}_0: \mathcal{M}_{\mathcal{P}} \rightarrow M_0$ restricts to a covering map on each leaf of the foliation $\mathcal{F}_{\mathcal{P}}$ of $\mathcal{M}_{\mathcal{P}}$. When the preimage $\mathcal{M}_\gamma = \widehat{q}_0^{-1}(c_\gamma(\mathbb{S}^1))$ is connected, it is a one-dimensional solenoid and so has a well-defined type by Theorem 7.3. Denote this type by $\tau[\gamma]$. The collection of these types is the ‘intuitive typeset’ of $\mathcal{M}_{\mathcal{P}}$.

The formal definition of the typeset proceeds as in §7.1. Associate to a presentation $\mathcal{P}$ the monodromy odometer $(X_\infty, \Gamma, \Phi_\infty)$. Then, for each $\gamma \in \Gamma$, there is the type $\tau[\gamma]$ as in Definition 1.6.

Definition 7.6. The typeset of $\mathcal{P}$ is the countable collection of types $\Xi[\mathcal{P}] = \{\tau[\gamma] \mid \gamma \in \Gamma\}$.

THEOREM 7.7. *Suppose that the solenoidal manifolds $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic, then their typesets $\tau[\mathcal{P}]$ and $\tau[\mathcal{P}']$ are commensurable.*

Proof. Suppose that $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{M}_{\mathcal{P}'}$ are homeomorphic, then by Theorem 7.3, their monodromy odometers $(X_\infty, \Gamma, \Phi_\infty)$ and $(X'_\infty, \Gamma', \Phi'_\infty)$ are return equivalent. Then, by Theorem 1.8, the typesets $\Xi[\mathcal{P}]$ and $\Xi[\mathcal{P}']$ are commensurable. $\square$

The type and typeset invariants suggest numerous questions about their relation to the analytic and geometric properties of solenoidal manifolds [43, 46, 47].

REFERENCES

- [1] J. M. Aarts and R. J. Fokkink. The classification of solenoids. *Proc. Amer. Math. Soc.* **111**(4) (1991), 1161–1163.
- [2] D. M. Arnold. *Finite Rank Torsion Free Abelian Groups and Rings (Lecture Notes in Mathematics, 931)*. Springer-Verlag, Berlin–New York, 1982.
- [3] D. M. Arnold and C. I. Vinsonhaler. Isomorphism invariants for abelian groups. *Trans. Amer. Math. Soc.* **330**(2) (1992), 711–724.
- [4] J. Auslander. *Minimal Flows and their Extensions (North-Holland Mathematics Studies, 153)*. North-Holland Publishing Co., Amsterdam, 1988 (*Notas de Matemática [Mathematical Notes]*, 122).
- [5] R. Baer. Abelian groups without elements of finite order. *Duke Math. J.* **3**(1) (1937), 68–122.
- [6] L. Bartholdi and V. V. Nekrashevych. Iterated monodromy groups of quadratic polynomials. I. *Groups Geom. Dyn.* **2**(3) (2008), 309–336.
- [7] R. H. Bing. A simple closed curve is the only homogeneous bounded plane continuum that contains an arc. *Canad. J. Math.* **12** (1960), 209–230.
- [8] M. C. R. Butler. A class of torsion-free abelian groups of finite rank. *Proc. Lond. Math. Soc.* (3) **s3-15** (1965), 680–698.
- [9] A. Clark and S. Hurder. Homogeneous matchbox manifolds. *Trans. Amer. Math. Soc.* **365**(6) (2013), 3151–3191.
- [10] A. Clark, S. Hurder and O. Lukina. Classifying matchbox manifolds. *Geom. Topol.* **23**(1) (2019), 1–27.
- [11] A. Connes. *Noncommutative Geometry*. Academic Press, San Diego, CA, 1994.
- [12] Y. Cornuier. Gradings on lie algebras, systolic growth, and cohopfian properties of nilpotent groups. *Bull. Soc. Math. France* **144**(4) (2016), 693–744.
- [13] M. I. Cortez and K. Medynets. Orbit equivalence rigidity of equicontinuous systems. *J. Lond. Math. Soc.* (2) **94**(2) (2016), 545–556.
- [14] M. I. Cortez and S. Petite. G -odometers and their almost one-to-one extensions. *J. Lond. Math. Soc.* (2) **78**(1) (2008), 1–20.
- [15] K. Dekimpe and J. Deré. Expanding maps and non-trivial self-covers on infra-nilmanifolds. *Topol. Methods Nonlinear Anal.* **47**(1) (2016), 347–368.
- [16] K. Dekimpe and K. B. Lee. Expanding maps on infra-nilmanifolds of homogeneous type. *Trans. Amer. Math. Soc.* **355**(3) (2003), 1067–1077.
- [17] J. Dyer, S. Hurder and O. Lukina. The discriminant invariant of Cantor group actions. *Topology Appl.* **208** (2016), 64–92.
- [18] J. Dyer, S. Hurder and O. Lukina. Molino theory for matchbox manifolds. *Pacific J. Math.* **289**(1) (2017), 91–151.
- [19] G. Endimioni and D. J. S. Robinson. On co-Hopfian groups. *Publ. Math. Debrecen* **67**(3–4) (2005), 423–436.
- [20] R. Fokkink and L. Oversteegen. Homogeneous weak solenoids. *Trans. Amer. Math. Soc.* **354**(9) (2002), 3743–3755.
- [21] L. Fuchs. *Abelian Groups (Springer Monographs in Mathematics)*. Springer, Cham, 2015.
- [22] T. Giordano, I. F. Putnam and C. F. Skau. $\mathbb{Z}^d$ -odometers and cohomology. *Groups Geom. Dyn.* **13**(3) (2019), 909–938.
- [23] R. I. Grigorchuk. Some problems of the dynamics of group actions on rooted trees. *Tr. Mat. Inst. Steklova* **273**(Sovremennyye Problemy Matematiki) (2011), 72–191.
- [24] S. Hurder and O. Lukina. Wild solenoids. *Trans. Amer. Math. Soc.* **371**(7) (2019), 4493–4533.
- [25] S. Hurder and O. Lukina. Orbit equivalence and classification of weak solenoids. *Indiana Univ. Math. J.* **69**(7) (2020), 2339–2363.
- [26] S. Hurder and O. Lukina. Limit group invariants for non-free Cantor actions. *Ergod. Th. & Dynam. Sys.* **41**(6) (2021), 1751–1794.
- [27] S. Hurder and O. Lukina. Nilpotent Cantor actions. *Proc. Amer. Math. Soc.* **150**(1) (2022), 289–304.
- [28] S. Hurder and O. Lukina. Prime spectrum and dynamics for nilpotent Cantor actions. *Pacific J. Math.* **327**(1) (2023), 107–128.
- [29] S. Hurder, O. Lukina and W. van Limbeek. Cantor dynamics of renormalizable groups. *Groups Geom. Dyn.* **15**(4) (2021), 1449–1487.
- [30] H. Lee and K. B. Lee. Expanding maps on 2-step infra-nilmanifolds. *Topology Appl.* **117**(1) (2002), 45–58.
- [31] X. Li. Continuous orbit equivalence rigidity. *Ergod. Th. & Dynam. Sys.* **38**(4) (2018), 1543–1563.
- [32] M. C. McCord. Inverse limit sequences with covering maps. *Trans. Amer. Math. Soc.* **114** (1965), 197–209.
- [33] C. C. Moore and C. L. Schochet. *Global Analysis on Foliated Spaces (Mathematical Sciences Research Institute Publications, 9)*, 2nd edn. Cambridge University Press, New York, 2006.

- [34] O. Mutzbauer. Type graph. *Abelian Group Theory (Honolulu, Hawaii, 1983) (Lecture Notes in Mathematics, 1006)*. Ed. R. Göbel, L. Lady and A. Mader. Springer, Berlin, 1983, pp. 228–252.
- [35] V. Nekrashevych. *Self-Similar Groups (Mathematical Surveys and Monographs, 117)*. American Mathematical Society, Providence, RI, 2005.
- [36] V. Nekrashevych and G. Pete. Scale-invariant groups. *Groups Geom. Dyn.* **5**(1) (2011), 139–167.
- [37] J. Renault. Cartan subalgebras in C^* -algebras. *Irish Math. Soc. Bull.* **61** (2008), 29–63.
- [38] L. Ribes. *Introduction to Profinite Groups and Galois Cohomology (Queen's Papers in Pure and Applied Mathematics, 24)*. Queen's University, Kingston, ON, 1970.
- [39] L. Ribes and P. Zalesskii. *Profinite Groups (Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 40)*. Springer-Verlag, Berlin, 2000.
- [40] F. Richman. An extension of the theory of completely decomposable torsion-free abelian groups. *Trans. Amer. Math. Soc.* **279**(1) (1983), 175–185.
- [41] M. Sabitova. Generalized ideal classes in application to toroidal solenoids. *Pacific J. Math.* **318**(1) (2022), 189–228.
- [42] M. Sabitova. A number-theoretic classification of toroidal solenoids. *Proc. Lond. Math. Soc. (3)* **129**(3) (2024), e12628.
- [43] D. Sullivan. Solenoidal manifolds. *J. Singul.* **9** (2014), 203–205.
- [44] S. Thomas. The classification problem for torsion-free abelian groups of finite rank. *J. Amer. Math. Soc.* **16**(1) (2003), 233–258.
- [45] S. Thomas. The classification problem for finite rank Butler groups. *Models, Modules and Abelian Groups*. Ed. R. Göbel and B. Goldsmith. Walter de Gruyter, Berlin, 2008, pp. 329–338.
- [46] A. Verjovsky. Commentaries on the paper *solenoidal manifolds* by Dennis Sullivan. *J. Singul.* **9** (2014), 245–251.
- [47] A. Verjovsky. Low-dimensional solenoidal manifolds. *EMS Surv. Math. Sci.* **10**(1) (2023), 131–178.
- [48] J. S. Wilson. *Profinite Groups (London Mathematical Society Monographs, New Series, 19)*. The Clarendon Press; Oxford University Press, New York, 1998.