Gibbs states in statistical mechanics and dynamical systems

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Summary

This thesis explores the rigorous mathematical foundations of Gibbs measures, which are fundamental objects at the intersection of statistical mechanics and dynamical systems. Rooted in the pioneering works of Boltzmann and Gibbs, Gibbs measures provide a precise probabilistic description of the equilibrium states of systems with a vast number of interacting constituents. These ideas, formalised in the Dobrushin–Lanford–Ruelle (DLR) framework, lie at the heart of our understanding of phase transitions, long-term statistical behaviour, and the connections between microscopic interactions and macroscopic phenomena.

The thesis is structured into three interconnected parts. The first part develops the conceptual and technical foundations for several notions of Gibbsianity. It explores how Gibbs measures appear both as equilibrium distributions in statistical mechanics and as invariant measures in ergodic theory and dynamical systems. One of the key results is a constructive demonstration of a translation-invariant Gibbsian specification that can not be generated by any translation-invariant, uniformly absolutely summable interaction. This reveals an important structural mismatch and exposes conceptual limitations within the classical framework. To address this gap, the thesis establishes a new variational principle formulated entirely in terms of specifications, providing a robust alternative perspective that aligns more closely with the language of dynamical systems and does not rely on traditional interaction-based formulations. Moreover, it has been shown in the literature that certain long-range systems can admit equilibrium states that lack Gibbsian properties. To resolve this, the thesis identifies barely a minimal condition under which all equilibrium states of such long-range systems remain Gibbsian, thus clarifying the delicate boundary between general equilibrium measures and Gibbs states.

The second part turns to one-dimensional systems, which, despite their apparent simplicity, are a fertile ground for testing theoretical ideas about long-range interactions, phase transitions, and the spectral properties of transfer operators. Here, the Perron–Frobenius–Ruelle transfer operator plays a central role: it links the statistical structure of Gibbs measures with dynamical evolution and mixing properties. A significant contribution of this section is the development of new techniques for proving the existence and regularity of eigenfunctions of

transfer operators under minimal regularity assumptions – extending the classical theory to a class of long-range models for which previous results do not apply. These methods are applied in particular to the celebrated Dyson model, a prototypical example of a one-dimensional long-range Ising model that illustrates the rich interplay between statistical mechanics and dynamical systems.

The third part extends the study of Gibbs measures to multifractal analysis and large deviation principles in dynamical systems. Many phenomena of physical and mathematical interest depend on understanding rare events and the fine-scale statistical structure of systems over long time horizons. Using topological entropy as a dimension, the thesis develops a rigorous framework connecting the multifractal spectrum of general observables to large deviation properties. The results provide necessary and sufficient conditions for deducing the dimension and frequency of rare events directly from the large deviation behaviour of the system. This approach unifies ideas from ergodic theory, large deviations theory, and dimension theory, showing how the same core probabilistic structures underlie both typical and exceptional behaviour in complex systems.

Taken together, this thesis offers a cohesive and original development of modern Gibbs theory that bridges foundational concepts and contemporary challenges. By clarifying the subtle relation between specifications and interactions, establishing new results for transfer operators in long-range settings, and connecting multifractal properties to large deviation principles, it significantly advances our understanding of how local interactions, global symmetries, and rare events interplay in mathematical physics and dynamical systems.