Gibbs states in statistical mechanics and dynamical systems
Makhmudov, M.

Chapter 5

The Eigenfunctions of the transfer operator for the Dyson model in a field

Abstract: In this chapter, we prove that, for high temperatures or strong magnetic fields, there exists a non-negative, integrable (with respect to the unique half-line Gibbs measure) eigenfunction of the transfer operator for the Dyson model if $\alpha \in \left(\frac{3}{2}, 2\right]$. However, unlike in the zero-magnetic-field case, this eigenfunction is not continuous.

This chapter is based on M. Makhmudov, *"The Eigenfunctions of the Transfer Operator for the Dyson model in a field"*, arXiv:2502.09588.

5.1 Introduction

The study of the so-called *equilibrium states*, specific types of invariant measures, has been a central topic in the ergodic theory of dynamical systems since the 1970s. The foundations of this theory were laid in the seminal works of Bowen and Ruelle at the end of the decade. Equilibrium states also play a crucial role in equilibrium statistical mechanics. A pivotal result in this context is the classical variational principle, proved by Lanford and Ruelle in 1969, which establishes that, under certain conditions, translation-invariant Gibbs measures coincide with equilibrium states. Recall that an equilibrium state μ for a continuous potential ϕ on the one-sided shift space (we also refer to it as the half-line shift space) $X_+ := E^{\mathbb{Z}_+}$, $|E| < \infty$ are probability measures on X_+ which are invariant under the shift (translation) map $S : X_+ \to X_+$, $(Sx)_j = x_{j+1}$, $j \in \mathbb{Z}_+$ and solves the variational equation:

$$\int_{X_+} \phi \, d\mu + \mathfrak{h}(\mu) = \sup\Big\{ \int_{X_+} \phi \, d\tau + \mathfrak{h}(\tau) : \tau \in \mathscr{M}_{1,S}(X_+) \Big\}, \tag{5.1}$$

where $\mathscr{M}_{1,S}(X_+)$ is the simplex of translation-invariant probability measures on X_+ and $\mathfrak{h}(\tau)$ denotes the measure-theoretic entropy of the translation-invariant measure τ.

A natural and effective approach to studying equilibrium states for a potential ϕ is through the transfer operator $\mathscr{L}_\phi$, which acts on the space of functions on X_+, particularly on continuous functions, via:

$$\mathscr{L}_\phi f(x) := \sum_{a \in E} e^{\phi(ax)} f(ax), \ \ f \in \mathbb{R}^{X_+}, \ \ x \in X_+. \tag{5.2}$$

In fact, if $\mathfrak{h}$ is an eigenfunction of $\mathscr{L}_\phi$, i.e., $\mathscr{L}_\phi \mathfrak{h} = \lambda \mathfrak{h}$, and ν is an eigenprobability for its adjoint $\mathscr{L}_\phi^* : C(X_+)^* \to C(X_+)^*$, i.e., $\mathscr{L}_\phi^* \nu = \lambda \nu$, where λ is the spectral radius of $\mathscr{L}_\phi$, then the measure $d\mu = \mathfrak{h} \cdot d\nu$ is an equilibrium state for ϕ and vice versa (see Proposition 3.1 in [5]). Classical fixed-point theorems ensure the existence of an eigenprobability for the adjoint $\mathscr{L}_\phi^*$, however, the existence of an eigenfunction for $\mathscr{L}_\phi$ is not always guaranteed. This issue has been extensively studied by Ruelle, Walters and others under various regularity conditions on ϕ [18, 20–23]. Notably, every potential ϕ satisfying these regularity conditions shares two key properties:

(P1) for all $\beta > 0$, the potential $\beta\phi$ admits a unique equilibrium state;

(P2) the transfer operator $\mathscr{L}_\phi$ is stable under "smooth" perturbations of ϕ in the sense that if $\mathscr{L}_\phi$ has a (principal) eigenfunction, then $\mathscr{L}_{\phi+\upsilon}$ also has a (principal) eigenfunction with the same regularity properties for every local function $\upsilon : X_+ \to \mathbb{R}$.

Note that the first property rules out the possibility of phase transitions, i.e., the existence of multiple equilibrium states for some β, a phenomenon of significant interest in statistical mechanics. Consequently, these classical conditions on ϕ exclude important one-dimensional models. Efforts to extend the theory to more general one-dimensional models have emerged in recent studies. In 2019, Cioletti et al. analyzed transfer operators for product-type potentials, which fall outside existing classes but still satisfy (P1) and (P2) [2]. A relevant study conducted in [1], though in a slightly different setup, provides a result that has implications for the transfer operator of the so-called Dyson potential. This potential corresponds to the 1D long-range Ising model [15] and is defined by:

$$\phi(x) = hx_0 + \beta \sum_{n=1}^{\infty} \frac{x_0 x_n}{n^\alpha}, \quad x \in X_+ := \pm 1^{\mathbb{Z}_+}, \tag{5.3}$$

where $\alpha > 1$ is the decay rate of the coupling, $\beta \geq 0$ is the inverse temperature, and $h \in \mathbb{R}$ represents the strength of the external field. The study in [1] shows that the transfer operator does not have a continuous eigenfunction when $h = 0$ and β is sufficiently large. A more notable development came in 2023 when Johansson, Öberg, and Pollicott analysed the transfer operator for the Dyson potential in the supercritical regime. Shortly thereafter, I, in collaboration with van Enter, Fernández, and Verbitskiy, introduced a new approach to addressing the eigenfunction problem for long-range models [5]. Following the developments in [5, 15], the following result on the Dyson model is known:

Theorem 5.1.1. [5, 15] *Let ϕ be the Dyson potential (5.3). Suppose $h = 0$, then:*

(i) for all $\alpha > 1$ and for $\beta \geq 0$ sufficiently small, the unique equilibrium state μ_+ for ϕ is equivalent to the unique half-line Gibbs state ν for ϕ, i.e., $\mu_+ \ll \nu$ and $\nu \ll \mu_+$. In particular, the Perron-Frobenius transfer operator $\mathscr{L}_\phi$ has an eigenfunction in $L^1(X_+, \nu)$;

(ii) furthermore, if $\alpha > \frac{3}{2}$, then for all $0 < \beta < \beta_c$ there exists a continuous version of the Radon-Nikodym density $\frac{d\mu_+}{d\nu}$, and thus the Perron-Frobenius transfer operator $\mathscr{L}_\phi$ has a positive continuous eigenfunction, where β_c is the critical temperature of the phase transitions for the standard Dyson model (see Theorem 5.2.1 in the next section).

However, both studies, [15] and [5], do not cover the case of the Dyson potential with a nonzero external field $h \neq 0$. The approach in [15] relies heavily on the random cluster representation of the Dyson model. The central obstacle to extending the method in [15] to non-zero external fields is the loss of symmetry, essential for the random cluster representation, which disrupts the cluster decay analysis. The method developed in [5] requires a certain sum of two-point functions to be uniformly bounded, a condition that fails for the Dyson model in a field. In this chapter, we adopt the technique from the previous chapter to treat the Dyson model in a nonzero field. The result of this chapter is as follows:

Theorem 5.A. *Suppose $\alpha \in (\frac{3}{2}, 2]$, $\beta \geq 0$ and $|h| > 0$ is sufficiently large* ($|h| > 2\beta\zeta(\alpha) + \log 4\beta\zeta(\alpha)$ *is enough, here ζ is the Riemann zeta function*). *Then*

(i) the Dyson potential ϕ has a unique equilibrium state μ_+ and there also exists a unique eigenprobability ν of $\mathscr{L}_\phi^*$;

(ii) μ_+ is absolutely continuous with respect to ν, i.e., $\mu_+ \ll \nu$. In particular, the Perron-Frobenius transfer operator $\mathscr{L}_\phi$ admits an integrable eigenfunction corresponding to the spectral radius $\lambda = e^{P(\phi)}$.

(iii) The Radon-Nikdoym derivative $\frac{d\mu_+}{d\nu}$ does not have a continuous version. In particular, the Perron-Frobenius transfer operator does not have a continuous principal eigenfunction.

Remark 5.1.2. *(i) Theorem 5.A is proven under a strong uniqueness condition for the Dyson interaction Φ (see (5.6) and Section 5.2), a concept introduced in* [5]. *This condition is implied by the Dobrushin uniqueness condition (DUC) on Φ (Section 5.3). The interaction Φ satisfies DUC if either 1) β is sufficiently small so that $\beta \leq \frac{1}{2\zeta(\alpha)}$ or 2) $|h|$ is sufficiently large so that $|h| > 2\beta\zeta(\alpha) + \log 4\beta\zeta(\alpha)$. Note that in the first version of the Dobrushin uniqueness condition, there is no constraint on the external field h. We prove Theorem 5.A for large $|h|$'s, however, the statement of Theorem 5.A remains valid and the same proof works for the first version of the DUC which covers the domain $0 \leq \beta < \frac{1}{2\zeta(\alpha)}$ and $h \neq 0$.*

We, in fact, believe that Theorem 5.A is true for all $h \in \mathbb{R}$ but $h = 0$, i.e., the transfer operator $\mathscr{L}_\phi$ has an integrable but not continuous eigenfunction for all $\beta \geq 0$, $h \neq 0$ (See Figure 5.1).

(ii) It is worth noting that Theorem 5.A suggests a certain similarity between the Dyson potential in a field and product-type potentials [2]. *Note that for a product-type potential, there exists a unique equilibrium state μ_+ and eigenprobability ν which both are Bernoulli measures such that $\nu = \prod_{n=0}^{\infty} \lambda_n$ with $\lambda_n(1) = p_n = \frac{\exp(\beta \sum_{i=1}^{n} i^{-\alpha})}{2\cosh(\beta \sum_{i=1}^{n} i^{-\alpha})}$ and $\mu_+ = \prod_{n=0}^{\infty} \lambda_\infty$* [2,5]. *Thus one can check*

that

$$\int_E \sqrt{\frac{d\lambda_\infty}{d\lambda_n}}\,d\lambda_n = \sqrt{p_\infty p_n} + \sqrt{(1-p_\infty)(1-p_n)} = 1 + O^*((p_n - p_\infty)^2),$$

therefore,

$$\log \int_E \sqrt{\frac{d\lambda_\infty}{d\lambda_n}}\,d\lambda_n = O^*((p_n - p_\infty)^2) = O^*(n^{-2\alpha+2}), \tag{5.4}$$

here $u(t) = O^*(v(t))$ *means the existence of* $C_1, C_2 > 0$ *such that* $C_1|u(t)| \le |v(t)| \le C_2|u(t)|$ *for all sufficiently small* t. *Then by (5.4), Kakutani's theorem implies that* $\mu_+ \ll \nu$ *if* $\alpha > \frac{3}{2}$ *and* $\mu_+ \perp \nu$ *if* $\alpha \le \frac{3}{2}$.

In the case of the Dyson potential, in the presence of a strong external field, spin-spin correlations become negligible, as they decay rapidly. Consequently, individual spins behave like independent random variables. Thus, we expect a phenomenon similar to that observed with product-type potentials, as described above, to occur in the Dyson model under an external field: for $\alpha \in \left(\frac{3}{2}, 2\right]$, *we have* $\mu_+ \ll \nu$, *while for* $\alpha \in \left(1, \frac{3}{2}\right]$, $\mu_+ \perp \nu$ *(See Figure 5.1).*

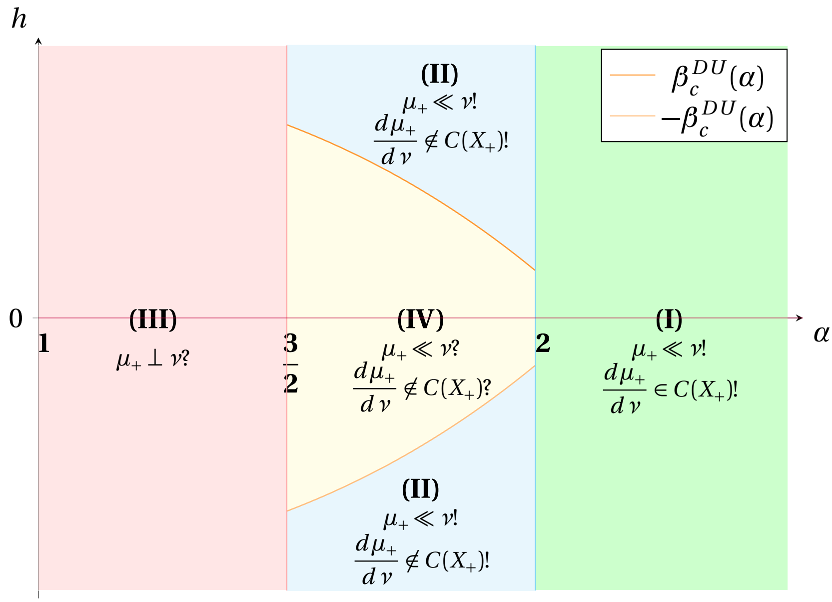

Figure 5.1: Eigenfunctions diagram for Dyson's model in a field for a fixed $\beta > 0$.

Note that the potential $\phi_0(x) = \beta \sum_{n=1}^{\infty} \frac{x_0 x_n}{n^\alpha}$, $x \in X_+ = \{\pm 1\}^{\mathbb{Z}_+}$ violates both (P1) and (P2) properties mentioned above. In fact, for all $\alpha \in (1,2]$, there exists $\beta_c(\alpha) > 0$ such that for all $\beta > \beta_c(\alpha)$, $\boldsymbol{\phi_0}$ has multiple equilibrium states [4, 8]. As regards the second property, the second parts of Theorem 5.1.1 and Theorem 5.A yields that the transfer operator $\mathscr{L}_{\phi_0}$ is unstable under "smooth" perturbations of ϕ_0 since for $\alpha \in (\frac{3}{2}, 2]$ and $\beta < \beta_c$, the principal eigenfunction of $\mathscr{L}_{\phi_0}$ is continuous whereas for $h \neq 0$, $\mathscr{L}_{\phi_0 + h\sigma_0}$ admits an integrable but not continuous eigenfunction, where $\sigma_0(x) = x_0$ for $x \in X_+$.

The chapter is organised as follows. In the Section 5.2 we define the notions of specifications and interactions. Here, we also introduce the interaction of the Dyson model. Section 5.3 is dedicated to the Dobrushin uniqueness condition and its consequences, which are important for this chapter. In Section 5.4, we discuss the construction of the intermediate interactions which are instrumental in the proof of Theorem 5.A. Section 5.5 is dedicated to the proof of Theorem 5.A.

5.2 Specifications and Interactions

In this chapter, we again consider the ferromagnetic spin space $E = \{\pm 1\}$. For a countable set $\mathbb{L}$ of sites, we consider a configuration space $\Omega = E^{\mathbb{L}}$. For $\mathbb{L}$, we either consider the set of integers $\mathbb{Z}$, or the set of negative integers $-\mathbb{N}$, or the set of non-negative integers $\mathbb{Z}_+$. As a product of compact sets, Ω is a compact and also a metrizable topological space. In fact, for any $\theta \in (0,1)$, $d(x,y) = \theta^{\mathbf{n}(x,y)}$ metrizes X_+, here for $x, y \in X_+$, $\mathbf{n}(x,y) = \min\{j \in \mathbb{Z}_+ : x_j \neq y_j\}$. $\mathscr{F}$ denotes the Borel sigma-algebra of X. For any $\Lambda \subset \mathbb{L}$, a sub sigma-algebra $\mathscr{F}_\Lambda \subset \mathscr{F}$ is defined as the minimal sigma-algebra so that for all $j \in \Lambda$, the functions $\sigma_j : \Omega \to E$ are measurable where $\xi \in \Omega \mapsto \sigma_j(\xi) = \xi_j \in \{\pm 1\}$. For configurations $\xi, \eta \in \Omega$ and a volume $\Lambda \subset \mathbb{L}$, $\xi_\Lambda \eta_{\Lambda^c}$ denotes the concatenated configuration, i.e., $(\xi_\Lambda \eta_{\Lambda^c})_i = \xi_i$ if $i \in \Lambda$ and $(\xi_\Lambda \eta_{\Lambda^c})_i = \eta_i$ if $i \in \Lambda^c = \mathbb{L} \setminus \Lambda$.

A *specification* γ, a regular family of conditional probabilities, is defined as a *consistent* family of proper probability kernels $\gamma_\Lambda : \mathscr{F} \times \Omega \to [0,1]$ indexed by finite subsets Λ of $\mathbb{L}$, denoted by $\Lambda \Subset \mathbb{L}$ [9, Chapter 1]. A probability measure $\mu \in \mathscr{M}_1(\Omega)$ is *Gibbs* for a specification γ if it is consistent with γ, i.e., for all $\Lambda \Subset \mathbb{L}$, $\mu = \mu\gamma_\Lambda$, here $B \in \mathscr{F} \mapsto \mu\gamma_\Lambda(B) := \int_\Omega \gamma_\Lambda(B|\eta)\mu(d\eta)$. The set of Gibbs measures for the specification γ is denoted by $\mathscr{G}(\gamma)$.

The classical approach to specifications is via interactions. An interaction Φ is a family $(\Phi_\Lambda)_{\Lambda \Subset \mathbb{L}}$ of functions $\Phi_\Lambda : \Omega \to \mathbb{R}$ such that $\Phi_\Lambda \in \mathscr{F}_\Lambda$, i.e., each Φ_Λ is independent of coordinates outside Λ. An interaction is called *Uniformly Absolutely*

Convergent (UAC) if for all $j \in \mathbb{L}$,

$$\sum_{j \in V \Subset \mathbb{L}} \sup_{\omega \in \Omega} |\Phi_V(\omega)| < \infty.$$

For a UAC interaction Φ, the Hamiltonian in a volume $\Lambda \Subset \mathbb{L}$ is $H_\Lambda := \sum_{V \cap \Lambda \neq \emptyset} \Phi_V$ and the associated specification (density) γ^Φ is defined by

$$\gamma_\Lambda^\Phi(\omega_\Lambda | \omega_{\Lambda^c}) := \frac{e^{-H_\Lambda(\omega)}}{Z_\Lambda(\omega_{\Lambda^c})}, \ \omega \in \Omega, \tag{5.5}$$

where Z_Λ is the normalization constant, also known as the partition function, which is $Z_\Lambda(\omega_{\Lambda^c}) := \sum_{\xi_\Lambda \in E^\Lambda} e^{-H_\Lambda(\xi_\Lambda \omega_{\Lambda^c})}$. Note that the specification $\gamma^\Phi = (\gamma_\Lambda^\Phi)_{\Lambda \Subset \mathbb{L}}$ is restored from (5.5) by

$$\gamma_\Lambda^\Phi(B|\omega) := \sum_{\xi_\Lambda \in E^\Lambda} \mathbb{1}_B(\xi_\Lambda \omega_{\Lambda^c}) \gamma_\Lambda^\Phi(\xi_\Lambda | \omega_{\Lambda^c}), \ B \in \mathscr{F}, \ \omega \in \Omega.$$

This chapter focuses on the Dyson model, a cornerstone of one-dimensional statistical mechanics renowned for its long-range interactions and critical phenomena. It is defined on the lattice $\mathbb{L} = \mathbb{Z}$ by

$$\Phi_\Lambda(\omega) := \begin{cases} -\dfrac{\beta \omega_i \omega_j}{|i-j|^\alpha}, & \text{if } \Lambda = \{i, j\} \subset \mathbb{Z}, \ i \neq j; \\ -h\omega_i, & \Lambda = \{i\}; \\ 0, & \text{otherwise}, \end{cases} \tag{5.6}$$

where $\alpha \in (1,2]$ is a parameter describing the decay of the interaction strength, $\beta \geq 0$ is the inverse temperature and $h \in \mathbb{R}$ represents the external field. In fact, the Dyson potential defined in (5.3) is related to the Dyson interaction Φ by

$$\phi = -\sum_{0 \in \Lambda \Subset \mathbb{Z}_+} \Phi_\Lambda.$$

The following theorem about the phase diagram of the Dyson model is well-known.

Theorem 5.2.1. [4, 7, 8] *Suppose Φ be the Dyson interaction given in (5.6).*

(i) If $h \neq 0$, then Φ has a unique Gibbs measure, i.e., $|\mathscr{G}(\gamma^\Phi)| = 1$;

(ii) If $h = 0$, there exists a finite critical temperature $\beta_c(\alpha) > 0$ such that for all $\beta \in [0, \beta_c(\alpha))$, $|\mathscr{G}(\gamma^\Phi)| = 1$ and for $\beta > \beta_c(\alpha)$, Φ exhibits phase transitions, i.e., $|\mathscr{G}(\gamma^\Phi)| > 1$.

5.3 The Dobrushin uniqueness condition for strong external fields

Dobrushin uniqueness condition is one of the most general criteria for the uniqueness of Gibbs states. We discuss it in the framework of a general countable set $\mathbb{L}$ of sites and a configuration space $\Omega := E^{\mathbb{L}}$. In fact, we use the Dobrushin uniqueness condition in the two cases of $\mathbb{L}$ which are $\mathbb{L} \in \{-\mathbb{N}, \mathbb{Z}\}$.

Consider a uniformly absolutely convergent (UAC) interaction $\Phi = \{\Phi_\Lambda(\cdot) : \Lambda \Subset \mathbb{L}\}$ on Ω and let γ^Φ be the corresponding Gibbsian specification. For any sites $i, j \in \mathbb{L}$, define

$$C(\gamma^\Phi)_{i,j} := \sup_{\eta_{\mathbb{L}\setminus\{j\}} = \overline{\eta}_{\mathbb{L}\setminus\{j\}}} ||\gamma^\Phi_{\{i\}}(\cdot|\eta) - \gamma^\Phi_{\{i\}}(\cdot|\overline{\eta})||_\infty,$$

where $||\cdot||_\infty$ is the supremum norm on $\mathscr{M}(\Omega)$ defined by $||\tau||_\infty := \sup_{B \in \mathscr{F}} |\tau(B)|$ for any finite signed Borel measure τ. The infinite matrix $C(\gamma^\Phi) := (C(\gamma^\Phi)_{i,j})_{i,j\in\mathbb{L}}$ is called the Dobrushin interdependence matrix.

Definition 5.3.1. [9] *The specification γ^Φ satisfies the **Dobrushin uniqueness condition** if*

$$c(\gamma^\Phi) := \sup_{i\in\mathbb{L}} \sum_{j\in\mathbb{L}} C(\gamma^\Phi)_{i,j} < 1. \tag{5.7}$$

The Dobrushin uniqueness condition admits slightly stronger — and easy to check— forms. One of them is a high-temperature condition, which is considered in [5]. Another version is known as the strong magnetic field condition, which we formulate here:

Proposition 5.3.2. [9, Example 8.13] *Let $\mathbb{L}$ be any countable set and $E = \{\pm 1\}$. Assume Φ be a UAC interaction on $\Omega = \{\pm 1\}^{\mathbb{L}}$ such that $\Phi_{\{i\}} = -h\sigma_i$ for all $i \in \mathbb{L}$ and for some $h \in \mathbb{R}$. Suppose*

$$|h| > \log \sup_{i\in\mathbb{L}} \Big\{ \exp\Big(\frac{1}{2} \sum_{A\ni i, |A|\geq 2} \delta(\Phi_A)\Big) \cdot \sum_{A\ni i} (|A|-1)\delta(\Phi_A) \Big\}, \tag{5.8}$$

where $\delta(f) := \sup\{|f(\xi) - f(\eta)| : \xi, \eta \in E^{\mathbb{L}}\}$ is the variation of $f : E^{\mathbb{L}} \to \mathbb{R}$. Then γ^Φ satisfies the Dobrushin uniqueness condition.

The proof of Proposition 5.8 boils down to showing that for all i, j, $i \neq j$,

$$C(\gamma^\Phi)_{ij} \leq \exp\Big(-|h| + \frac{1}{2} \sum_{A\ni i, |A|\geq 2} \delta(\Phi_A)\Big) \cdot \sum_{A\supset\{i,j\}} \delta(\Phi_A) =: \bar{C}(\Phi)_{ij},$$

and hence $c(\gamma^{\Phi}) = \sup_i \sum_j C(\gamma^{\Phi})_{ij} \le \sup_i \sum_j \bar{C}(\Phi)_{ij} = \bar{c}(\Phi)$. Note that, here without loss of generality, we set $\bar{C}(\Phi)_{ii} = 0$ for all $i \in \mathbb{L}$. Notice that the non-negative matrix $\bar{C}(\Phi) := (\bar{C}(\Phi)_{i,j})_{i,j\in\mathbb{L}}$ is symmetric.

The condition (5.8) is stable under a *perturbation* of the underlying model/interaction Φ. Indeed, let $\Psi = \{\Psi_V\}_{V \Subset \mathbb{L}}$ be an interaction such that for $V \Subset \mathbb{L}$, either $\Psi_V = \Phi_V$ or $\Psi_V = 0$, then it is straightforward to check that the matrix $\bar{C}(\Psi)$ is dominated by $\bar{C}(\Phi)$, i.e., for all $i, j \in \mathbb{L}$, $\bar{C}(\Psi)_{ij} \le \bar{C}(\Phi)_{ij}$, thus, in particular, $\bar{c}(\Psi) \le \bar{c}(\Phi)$. Hence Ψ inherits the Dobrushin uniqueness condition from Φ as long as Φ satisfies (5.8).

The crucial property of the Dobrushin uniqueness condition is that it provides the uniqueness of the compatible probability measures with the specification γ^{Φ}. In fact, we have the following theorem.

Theorem 5.3.3. [9, Chapter 8] *If γ^{Φ} satisfies the Dobrushin uniqueness condition (5.7), then $\left|\mathscr{G}(\gamma^{\Phi})\right| = 1$.*

The validity of Dobrushin's criterion yields several important properties of the unique Gibbs state such as concentration inequalities and explicit bounds on the decay of correlations.
The first property involves the coefficient $\bar{c}(\Phi)$, which governs the deviation behaviour of the unique Gibbs measure. Let

$$\delta_k F := \sup\left\{F(\xi) - F(\eta) : \xi_j = \eta_j,\ j \in \mathbb{L} \setminus \{k\}\right\} \tag{5.9}$$

denote the oscillation of a local function $F : \Omega \to \mathbb{R}$ at a site $k \in \mathbb{L}$ and $\underline{\delta}(F) = (\delta_k F)_{k\in\mathbb{L}}$ is the oscillation vector, where $\Omega = E^{\mathbb{L}}$.

Theorem 5.3.4. [17][19] *Suppose Φ is a UAC interaction satisfying (5.8) and let μ_{Φ} be its unique Gibbs measure. Set*

$$D := \frac{4}{(1-\bar{c}(\Phi))^2}\,. \tag{5.10}$$

Then μ_{Φ} satisfies a ***Gaussian Concentration Bound*** *with the constant D, i.e., for any continuous function F on $\Omega = E^{\mathbb{L}}$, one has*

$$\int_{\Omega} e^{F - \int_{\Omega} F\, d\mu_{\Phi}}\, d\mu_{\Phi} \le e^{D\|\underline{\delta}(F)\|_2^2}.$$

The second important consequence of the Dobrushin's uniqueness condition is the following:

Theorem 5.3.5. [9, Theorem 8.20] *Let γ and $\tilde{\gamma}$ be two specifications on $\Omega = E^{\mathbb{L}}$. Suppose γ satisfies Dobrushin's condition. For each $i \in \mathbb{L}$, we let b_i be a measurable function on Ω such that*

$$\|\gamma_{\{i\}}(\cdot|\omega_{\{i\}^c}) - \tilde{\gamma}_{\{i\}}(\cdot|\omega_{\{i\}^c})\|_{\infty} \le b_i(\omega) \tag{5.11}$$

for all $\omega \in \Omega$. If $\mu \in \mathscr{G}(\gamma)$ and $\tilde{\mu} \in \mathscr{G}(\tilde{\gamma})$ then for all $f \in C(\Omega)$,

$$|\mu(f) - \tilde{\mu}(f)| \leq \sum_{i,j \in \mathbb{L}} \delta_i(f) D(\gamma)_{i,j} \tilde{\mu}(b_j). \tag{5.12}$$

We present the third property in the particular setup $\mathbb{L} = \mathbb{Z}$, and it involves the $\mathbb{Z} \times \mathbb{Z}$ matrix

$$D(\gamma^{\Phi}) = \sum_{n=0}^{\infty} C(\gamma^{\Phi})^n \,. \tag{5.13}$$

The sum of the $\mathbb{Z} \times \mathbb{Z}$ matrices in the right-hand side converges due to the Dobrushin condition (5.7).

Proposition 5.3.6. [6, 9] *Consider a UAC interaction Φ on $X = E^{\mathbb{Z}}$. Assume the specification γ^{Φ} satisfies the Dobrushin condition (5.7) and let μ be its unique Gibbs measure. Then, for all $f, g \in C(X)$ and $i \in \mathbb{Z}$,*

$$\left| cov_{\mu}(f, g \circ S^i) \right| \leq \frac{1}{4} \sum_{k,j \in \mathbb{Z}} D(\gamma^{\Phi})_{jk} \cdot \delta_k f \cdot \delta_{j-i} g. \tag{5.14}$$

Suppose Φ satisfies the Dobrushin condition (5.8) and define the non-negative symmetric $\mathbb{Z} \times \mathbb{Z}$–matrix by

$$\bar{D}(\Phi) := \sum_{n \geq 0} \bar{C}(\Phi)^n \,. \tag{5.15}$$

The sum in the right-hand side of (5.15) converges due to (5.8). Furthermore, $\bar{D}(\Phi)$ is invertible and $\bar{D}(\Phi)^{-1} = I_{\mathbb{Z}\times\mathbb{Z}} - \bar{C}(\Phi)$, where $I_{\mathbb{Z}\times\mathbb{Z}}$ denotes the identity matrix (operator) on $l^2(\mathbb{Z})$.
In general, one can show that the off-diagonal elements $\bar{D}(\Phi)$ decay as they get far away from the diagonal, i.e., $\bar{D}(\Phi)_{ij} = o(1)$ as $|i - j| \to \infty$. However, the decay rate largely depends on the interaction Φ. Now we discuss the decay rate in the case of Dyson interaction Φ defined by (5.6). For $i \neq j$, one can readily check that

$$\bar{C}(\Phi)_{ij} = \frac{2\beta \cdot \exp(-|h| + 2\beta\zeta(\alpha))}{|i - j|^{\alpha}} = O(|i - j|^{-\alpha}). \tag{5.16}$$

Thus by Jaffard's theorem ([14, Proposition 3] and [13, Theorem 1.1]), one has the same asymptotics for the off-diagonal elements of the inverse matrix $(I_{\mathbb{Z}\times\mathbb{Z}} - \bar{C}(\Phi))^{-1} = \bar{D}(\Phi)$, i.e., $\bar{D}(\Phi)_{ij} = O(|i-j|^{-\alpha})$. Thus there exists a constant $C_J > 0$ such that for all $i, j \in \mathbb{Z}$,

$$\bar{D}(\Phi)_{i,j} \leq \frac{C_J}{(1 + |i - j|)^{\alpha}}. \tag{5.17}$$

5.4 Construction of intermediate interactions

In this section, we recall the construction of the intermediate interaction from [5]. For the Dyson interaction Φ given by (5.6) and each $k \in \mathbb{Z}_+$, we construct intermediate interactions $\Psi^{(k)}$ in the following way. We will represent $\mathbb{Z} = -\mathbb{N} \cup \mathbb{Z}_+$. Consider the countable collection of finite subset of $\mathbb{Z}$ with endpoints in $-\mathbb{N}$ and $\mathbb{Z}_+$:

$$\mathscr{A} = \{\Lambda \Subset \mathbb{Z} : \min(\Lambda) < 0,\ \max\Lambda \geq 0\}.$$

Index elements of $\mathscr{A}$ in an order $\mathscr{A} = \{\Lambda_1, \Lambda_2, \ldots\}$ in a way that for every $N \in \mathbb{N}$, there exists $k_N \in \mathbb{N}$ such that

$$\sum_{i=1}^{k_N} \Phi_{\Lambda_i} = \sum_{\substack{\min V < 0 \leq \max V \\ V \subset [-N, N]}} \Phi_V. \tag{5.18}$$

Then define

$$\Psi_\Lambda^{(k)} = \begin{cases} \Phi_\Lambda, & \Lambda \notin \{\Lambda_i : i \geq k+1\}, \\ 0, & \Lambda \in \{\Lambda_i : i \geq k+1\}, \end{cases}$$

In other words, we first remove all Φ_Λ's with $\Lambda \in \mathscr{A}$ from Φ, and then add them one by one. Clearly, all the constructed interactions are UAC.

Remark 5.4.1. *1) Every $\Psi^{(k)}$ is a local (finite) perturbation of $\Psi^{(0)}$, and $\Psi^{(k)}$ tend to Φ as $k \to \infty$, in the sense that $\Psi_\Lambda^{(k)} \rightrightarrows \Phi_\Lambda$ for all $\Lambda \Subset \mathbb{Z}$.*

2) For any finite volume V, one can readily check that

$$\|H_V^{\Psi^{(k)}} - H_V^{\Phi}\|_\infty \leq \sum_{\substack{\Lambda_j \cap V \neq \emptyset \\ j \geq k}} \|\Phi_\Lambda\|_\infty \xrightarrow[k\to\infty]{} 0.$$

3) For specifications, it can also be concluded that $\gamma^{\Psi^{(k)}}$ converges to γ^Φ as $k \to \infty$, i.e., for all $B \in \mathscr{F}$ and $V \Subset \mathbb{Z}$,

$$\gamma_V^{\Psi^{(k)}}(B|\omega) \xrightarrow[k\to\infty]{} \gamma_V^{\Phi}(B|\omega) \ \text{ uniformly on the b.c. } \omega \in \Omega.$$

4) In addition, if $\nu^{(k)}$ is a Gibbs measure for $\Psi^{(k)}$, then by Theorem 4.17 in [9], any weak-limit point of the sequence $\{\nu^{(k)}\}_{k\geq 0}$ becomes a Gibbs measure for the potential Φ.*

Another important observation is the following: since we have constructed $\Psi^{(0)}$ from Φ by removing all the interactions between the left $-\mathbb{N}$ and the right $\mathbb{Z}_+$

half-lines, the corresponding specification $\gamma^{\Psi^{(0)}}$ becomes product type [9, Example 7.18]. More precisely, $\gamma^{\Psi^{(0)}} = \gamma^{\Phi^-} \times \gamma^{\Phi^+}$, where Φ^- and Φ^+ are the restrictions of Φ to the half-lines $-\mathbb{N}$ and $\mathbb{Z}_+$, respectively. Thus we have the following for the extreme Gibbs measures [9, Example 7.18]:

$$\mathrm{ex}\,\mathscr{G}(\gamma^{\Psi^{(0)}}) = \{\nu^- \times \nu^+ : \nu^- \in \mathrm{ex}\,\mathscr{G}(\gamma^{\Phi^-}),\ \ \nu^+ \in \mathrm{ex}\,\mathscr{G}(\gamma^{\Phi^+})\} \tag{5.19}$$

For $k \geq 0$, let $\nu^{(k)} \in \mathscr{G}(\Psi^{(k)})$. For any $k \geq 1$, we also consider the following function:

$$f^{(k)} = \frac{e^{-\sum_{i=1}^{k} \Phi_{\Lambda_i}}}{\int_\Omega e^{-\sum_{i=1}^{k} \Phi_{\Lambda_i}}\, d\,\nu^{(0)}}. \tag{5.20}$$

Now we state two important lemmas which will be used in the proof of Theorem 5.A.

Lemma 5.4.2. [5, Theorem 6.5] *If the interaction Φ satisfies the Dobrushin uniqueness condition (5.8), so do all interactions $\Psi^{(k)}$. Furthermore, $\nu^{(k)}$ and $\nu^{(0)}$ are equivalent with*

$$\frac{d\,\nu^{(k)}}{d\,\nu^{(0)}} = \frac{e^{-\sum_{i=1}^{k} \Phi_{\Lambda_i}}}{\int_X e^{-\sum_{i=1}^{k} \Phi_{\Lambda_i}}\, d\,\nu^{(0)}} \quad , \quad \frac{d\,\nu^{(0)}}{d\,\nu^{(k)}} = \frac{e^{\sum_{i=1}^{k} \Phi_{\Lambda_i}}}{\int_X e^{\sum_{i=1}^{k} \Phi_{\Lambda_i}}\, d\,\nu^{(k)}}\,.$$

Lemma 5.4.2 further implies that for any $k \geq 1$,

$$\frac{d\,\nu^{(k)}}{d\,\nu^{(k-1)}} = \frac{e^{-\Phi_{\Lambda_k}}}{\int e^{-\Phi_{\Lambda_k}}\, d\,\nu^{(k-1)}}. \tag{5.21}$$

Lemma 5.4.3. [5, Theorem D] *Assume that Φ satisfies the Dobrushin uniqueness condition (5.8). Suppose the family $\{f^{(k)}\}_{k\in\mathbb{N}}$ is uniformly integrable in $L^1(\nu^{(0)})$. Then the weak* limit point of the sequence $\{\nu^{(k)}\}$ is a Gibbs measure for Φ and absolutely continuous with respect to $\nu^{(0)}$.*

Proofs of both Lemma 5.4.2 and Lemma 5.4.3 can be found in [5].

We end this section by noting that if $h \geq 0$, then the Griffiths–Kelly–Sherman (GKS) inequality applies to the Gibbs measures $\nu^{(k)}$ and μ [10–12, 16]. Namely, for every $\tau \in \{\mu, \nu^{(k)}, k \geq 0\}$ and all $A, B \Subset \mathbb{Z}$, one has

$$\int_\Omega \sigma_A\, d\tau \geq 0 \tag{5.22}$$

and

$$\int_\Omega \sigma_A \sigma_B\, d\tau \geq \int_\Omega \sigma_A\, d\tau \cdot \int_\Omega \sigma_B\, d\tau, \tag{5.23}$$

where $\sigma_A := \prod_{i\in A}\sigma_i$ and for $i\in\mathbb{Z}$, $\omega\in X\mapsto\sigma_i(\omega)=\omega_i$. The GKS inequalities also imply the following inequality for $\tau\in\{\mu, \nu^{(k)},\ k\geq 0\}$: for every $A, B \Subset \mathbb{Z}$ and $t\geq 0$,

$$\int_\Omega \sigma_B e^{t\sigma_A} d\tau \geq \int_\Omega \sigma_B d\tau \cdot \int_\Omega e^{t\sigma_A} d\tau. \tag{5.24}$$

5.5 Proof of Theorem 5.A

Part (i): Firstly, note that the uniqueness of Gibbs measures for Φ and $\Psi^{(0)}$ easily follows from Theorem 5.3.3. Thus by (5.19), the restriction Φ^+ of Φ to the half-line $\mathbb{Z}_+$ also has a unique Gibbs measure ν^+. Furthermore, Theorem 4.8 in [3] implies that the transfer operator $\mathscr{L}_\phi$ has a unique eigenprobability ν, and $\nu=\nu^+$ (see also Subsection 2.2 in [5]).

Part (ii): We start the proof of the second part by noting that the positive and negative fields are related by the global-spin flip transformation $\mathscr{S}: X_+\to X_+$ defined by $\mathscr{S}(x)_n=-x_n$ for all $n\in\mathbb{Z}_+$. In fact, for the Dyson potential ϕ given by (5.3), one has that

$$\phi\circ\mathscr{S} = -h\sigma_0 + \beta\sum_{n=1}^{\infty}\frac{\sigma_0\sigma_n}{n^\alpha}. \tag{5.25}$$

For any continuous potential $\psi\in C(X_+)$ one can easily check that

$$\left(\mathscr{L}_{\psi\circ\mathscr{S}}\,\mathfrak{f}\right)\circ\mathscr{S} = \mathscr{L}_\psi(\mathfrak{f}\circ\mathscr{S}),\ \ \mathfrak{f}\in\mathbb{R}^{X_+}. \tag{5.26}$$

Thus, the topological pressures of ψ and $\psi\circ\mathscr{S}$ coincide, i.e., $p(\psi)=p(\psi\circ\mathscr{S})$ and $\nu\in\mathscr{M}_1(X_+)$ is an eigenprobability of $\mathscr{L}^*_\psi$ if and only if $\nu\circ\mathscr{S}$ is an eigenprobability of $\mathscr{L}^*_{\psi\circ\mathscr{S}}$. Furthermore, $\mathfrak{h}$ is an eigenfunction of $\mathscr{L}_\psi$ corresponding to $\lambda=e^{p(\psi)}$ if and only if $\mathfrak{h}\circ\mathscr{S}$ is an eigenfunction of $\mathscr{L}_{\psi\circ\mathscr{S}}$ corresponding to λ.

By the above argument, we may assume without loss of generality that the external field h is positive for the remainder of the proof.

We below show that the unique Gibbs measure μ for Φ is absolutely continuous with respect to $\nu^{(0)}$. Then this yields that the restriction μ_+ of μ to $\mathbb{Z}_+$ is absolutely continuous with respect to ν and the Radon-Nikdoym density $f_+ := \dfrac{d\mu_+}{d\nu}$ is given by

$$f_+(\omega_0^\infty) = \int_{X_-} f(\xi_{-\infty}^{-1}\omega_0^\infty)\,\nu_-(d\xi), \tag{5.27}$$

where $f=\dfrac{d\mu}{d\nu^{(0)}}$. Then Proposition 3.1 in [5] also yields that the Radon-Nikodym density $\dfrac{d\mu_+}{d\nu}$ is an eigenfunction of the transfer operator $\mathscr{L}_\phi$ corresponding to $\lambda=e^{P(\phi)}$.

Note that for the Dyson interaction Φ, one has, for all $i \in \mathbb{N}$, $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$,

$$\delta_k(\Phi_{\{-i,j\}}) = \begin{cases} 0, & k \notin \{-i, j\}; \\ \dfrac{2\beta}{(i+j)^\alpha}, & k \in \{-i, j\}. \end{cases}$$

Thus for all $i \in \mathbb{N}$, $j \in \mathbb{Z}_+$, $||\underline{\delta}(\Phi_{\{-i,j\}})||_2^2 = \dfrac{8\beta^2}{(i+j)^{2\alpha}}$. Therefore, $\sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} ||\underline{\delta}(\Phi_{\{-i,j\}})||_2^2 =$

$\sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} \dfrac{8\beta^2}{(i+j)^{2\alpha}} < \infty.$

Consider any $\Lambda_k \in \mathscr{A}$ and $n \in \mathbb{N}$. It follows from (5.21) that

$$\int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n)} = \frac{-J_{\Lambda_k} \int_\Omega \sigma_{\Lambda_k} e^{J_{\Lambda_n}\sigma_{\Lambda_n}} d\,\nu^{(n-1)}}{\int_\Omega e^{J_{\Lambda_n}\sigma_{\Lambda_n}} d\,\nu^{(n-1)}},$$

here for $\Lambda \Subset \mathbb{Z}_+$, $J_\Lambda := h$ if $\Lambda = \{i\}$ and $J_\Lambda := \dfrac{\beta}{|i-j|^\alpha}$ if $\Lambda = \{i, j\}$. Then by (5.24),

$$\int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n)} \le \frac{\int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n-1)} \cdot \int_\Omega e^{J_{\Lambda_n}\sigma_{\Lambda_n}} d\,\nu^{(n-1)}}{\int_\Omega e^{J_{\Lambda_n}\sigma_{\Lambda_n}} d\,\nu^{(n-1)}} = \int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n-1)}. \tag{5.28}$$

Note that $\nu^{(n)}$ converges to μ as $n \to \infty$ in the weak star topology. Thus one can obtain from (5.28) that for all $\Lambda_k \in \mathscr{A}$, $n \in \mathbb{N}$,

$$\int_\Omega \Phi_{\Lambda_k} d\,\nu^{(0)} \ge \int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n)} \ge \lim_{n\to\infty} \int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n)} = \int_\Omega \Phi_{\Lambda_k} d\mu. \tag{5.29}$$

For all $k \in \mathbb{N}$, denote $W_k := \sum_{i=1}^{k} \Phi_{\Lambda_i}$. Then we aim to prove

$$\sup_{k\ge 0} \int_X f^{(k)} \log f^{(k)} d\,\nu^{(0)} < \infty. \tag{5.30}$$

Hence by applying de la Vallée Poussin's theorem to the family $\{f^{(k)} : k \in \mathbb{N}\}$ and to the function $t \in (0, +\infty) \mapsto t \log t$, one concludes that the family $\{f^{(k)} : k \in \mathbb{N}\}$ is uniformly integrable in $L^1(\nu^{(0)})$. Then Theorem 5.4.3 imply that the unique limit point μ is absolutely continuous with respect to $\nu^{(0)}$.

Note that for every $k \in \mathbb{N}$,

$$\int_X f^{(k)} \log f^{(k)} d\,\nu^{(0)} = \int_X -W_k d\,\nu^{(k)} - \log \int_X e^{-W_k} d\,\nu^{(0)}. \tag{5.31}$$

Here one can easily check that

$$\sup_{k\in\mathbb{N}} \int_X e^{-W_k}\, d\,\nu^{(0)} = +\infty.$$

Under conditions of our main theorem, we argue that

$$\sup_{k\in\mathbb{N}} \int_X -W_k\, d\,\nu^{(k)} = +\infty.$$

Therefore, the method of the proof of Theorem E in [5] does not work here. We instead prove the following two claims:

Claim 1:

$$\sup_{k\in\mathbb{N}} \left| \int_X W_k\, d\,\nu^{(k)} - \int_X W_k\, d\mu \right| < \infty;$$

Claim 2:

$$\sup_{k\in\mathbb{N}} \left| -\log \int_X e^{-W_k}\, d\,\nu^{(0)} - \int_X W_k\, d\mu \right| < \infty.$$

One can easily see from (5.31) that Claim 1 and Claim 2 indeed imply (5.30).

Note that for all $k \in \mathbb{N}$, $\bar{c}(\Psi^{(0)}) \leq \bar{c}(\Psi^{(k)}) \leq \bar{c}(\Phi)$. Therefore, the Dobrushin uniqueness condition $\bar{c}(\Phi) < 1$ is inherited by all the intermediate interactions. Applying the first part of Theorem 4.4.7 we see that the (only) measure $\mu \in \mathscr{G}(\Phi)$ and all the intermediate measures $\nu^{(k)}$, $k \geq 0$, satisfy the Gaussian Concentration Bound with the same constant $D := \dfrac{4}{(1-\bar{c}(\Phi))^2}$. This implies that, for all $k \in \mathbb{N}$,

$$\int_X e^{-\Phi_{\Lambda_k}}\, d\,\nu^{(k-1)} \leq e^{D\|\underline{\delta}(\Phi_{\Lambda_k})\|_2^2} e^{-\int_X \Phi_{\Lambda_k}\, d\,\nu^{(k-1)}}. \tag{5.32}$$

We combine this inequality with (5.21) to iterate

$$\begin{aligned}
&\int_X e^{-(\Phi_{\Lambda_k}+\Phi_{\Lambda_{k-1}})}\, d\,\nu^{(k-2))} \\
&= \int_X e^{-\Phi_{\Lambda_k}}\, d\,\nu^{(k-1))} \int_X e^{-\Phi_{\Lambda_{k-1}}}\, d\,\nu^{(k-2))} \\
&\leq e^{D(\|\underline{\delta}(\Phi_{\Lambda_k})\|_2^2 + \|\underline{\delta}(\Phi_{\Lambda_{k-1}})\|_2^2)} \cdot e^{-(\int_X \Phi_{\Lambda_k}\, d\,\nu^{(k-1)} + \int_X \Phi_{\Lambda_{k-1}}\, d\,\nu^{(k-2)})}.
\end{aligned} \tag{5.33}$$

By induction this yields

$$\int_X e^{-\sum_{i=1}^k \Phi_{\Lambda_i}}\, d\,\nu^{(0)} \leq e^{D\sum_{i=1}^k \|\underline{\delta}(\Phi_{\Lambda_i})\|_2^2} \cdot e^{-\sum_{i=1}^k \int_X \Phi_{\Lambda_i}\, d\,\nu^{(i-1)}}. \tag{5.34}$$

Similarly, one can also obtain the lower bound:

$$\int_X e^{-\sum_{i=1}^k \Phi_{\Lambda_i}} d\,\nu^{(0)} \geq e^{-D\sum_{i=1}^k ||\underline{\delta}(\Phi_{\Lambda_i})||_2^2} \cdot e^{-\sum_{i=1}^k \int_X \Phi_{\Lambda_i} d\,\nu^{(i-1)}}. \tag{5.35}$$

Hence for all $k \in \mathbb{N}$,

$$-C_1 - \sum_{i=1}^k \int_X \Phi_{\Lambda_i} d\,\nu^{(i-1)} \leq \log \int_X e^{-\sum_{i=1}^k \Phi_{\Lambda_i}} d\,\nu^{(0)} \leq C_1 - \sum_{i=1}^k \int_X \Phi_{\Lambda_i} d\,\nu^{(i-1)}, \tag{5.36}$$

where $C_1 := D \cdot \sum_{i=1}^{\infty} ||\underline{\delta}(\Phi_{\Lambda_i})||_2^2 = \sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} \frac{8\beta^2}{(i+j)^{2\alpha}} < \infty$. Thus instead of Claim 2, it suffices to show the following:

$$\sup_{k\in\mathbb{N}} \left| \sum_{i=1}^k \int_X \Phi_{\Lambda_i} d\,\nu^{(i-1)} - \int_X W_k \, d\mu \right| < \infty. \tag{5.37}$$

Thus, in the light of (5.29), Claim 1 and Claim 2 are implied by the following:

Claim 3:

$$\sup_{k\in\mathbb{N}} \left| \int_X W_k \, d\,\nu^{(0)} - \int_X W_k \, d\mu \right| < \infty. \tag{5.38}$$

By the GKS inequality (c.f., 5.29), we, in fact, have that $\int_X \Phi_{\Lambda_i} d\,\nu^{(0)} - \int_X \Phi_{\Lambda_i} d\mu \geq 0$, hence the sequence $\{\int_X W_k \, d\,\nu^{(0)} - \int_X W_k \, d\mu\}_k$ increases with k. Thus instead of considering all W_k's, it is enough to consider the subsequence $\{W_{[N]}\}_{N\in\mathbb{N}}$s, where

$$W_{[N]}(\omega) := \sum_{i=1}^N \sum_{j=0}^N \Phi_{\{-i,j\}}(\omega) = \sum_{i=1}^N \sum_{j=0}^N - \frac{\beta\,\omega_{-i}\,\omega_j}{(i+j)^{\alpha}}.$$

Then

$$\begin{aligned}\int_X W_{[N]} d\nu^{(0)} - \int_X W_{[N]} d\mu &= \sum_{i=1}^{N}\sum_{j=0}^{N}\int_X -\frac{\beta\omega_{-i}\omega_j}{(i+j)^\alpha}\nu^{(0)}(d\omega) \\ &- \sum_{i=1}^{N}\sum_{j=0}^{N}\int_X -\frac{\beta\omega_{-i}\omega_j}{(i+j)^\alpha}\mu(d\omega) \\ &= \sum_{i=1}^{N}\sum_{j=0}^{N}\frac{-\beta\nu_-(\sigma_{-i})\cdot\nu_+(\sigma_j)}{(i+j)^\alpha} \\ &- \sum_{i=1}^{N}\sum_{j=0}^{N}\frac{-\beta\,\mathrm{cov}_\mu(\sigma_{-i},\sigma_j)-\beta\mu(\sigma_0)^2}{(i+j)^\alpha} \\ &= \sum_{i=1}^{N}\sum_{j=0}^{N}\frac{\beta\,\mathrm{cov}_\mu(\sigma_{-i},\sigma_j)}{(i+j)^\alpha} \qquad (5.39) \\ &+ \beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{\mu(\sigma_0)^2-\nu_-(\sigma_{-i})\cdot\nu_+(\sigma_j)}{(i+j)^\alpha}, \qquad (5.40)\end{aligned}$$

here we used the fact that $\nu^0 = \nu_- \times \nu_+$. By (5.14) and (5.17), we know that in the Dobrushin uniqueness region (5.39) is bounded, i.e.,

$$\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{\beta|\mathrm{cov}_\mu(\sigma_{-i},\sigma_j)|}{(i+j)^\alpha} < \infty.$$

Therefore, to show

$$\sup_{N\in\mathbb{N}}\Big(\int_X W_{[N]} d\nu^{(0)} - \int_X W_{[N]} d\mu\Big) < \infty \tag{5.41}$$

it is enough to show that

$$\beta\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{|\mu(\sigma_0)^2-\nu_-(\sigma_{-i})\cdot\nu_+(\sigma_j)|}{(i+j)^\alpha} < \infty. \tag{5.42}$$

By Theorem 4.A in Chapter 4, we have that $\nu_+(\sigma_j)\xrightarrow[j\to\infty]{}\mu(\sigma_0)$, similarly, one can also argue that $\nu_-(\sigma_{-i})\xrightarrow[i\to\infty]{}\mu(\sigma_0)$. For $n\in\mathbb{Z}$, denote

$$t_n := |\nu^{(0)}(\sigma_n) - \mu(\sigma_0|. \tag{5.43}$$

Then (5.40) can be rewritten in the following form

$$\beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{|\mu(\sigma_0)^2-\nu_-(\sigma_{-i})\cdot\nu_+(\sigma_j)|}{(i+j)^\alpha} = \beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{|\mu(\sigma_0)t_j+\mu(\sigma_0)t_{-i}-t_{-i}t_j|}{(i+j)^\alpha}. \tag{5.44}$$

Now we show that there exists a constant $C_2 = C_2(\alpha,\beta,h) > 0$ such that for every $n \in \mathbb{Z}$,

$$t_n \le \frac{C_2}{(|n|+1)^{\alpha-1}}. \tag{5.45}$$

In fact, we prove the above inequality for positive n's and the case of negative n's can be treated similarly. Fix $j \in \mathbb{N}$. Then by Theorem 5.3.5, one has that

$$t_j = \left| \int_X \sigma_j \, d\mu - \int_X \sigma_j \, d\nu^{(0)} \right| \le 2(\bar{D} b)_j, \tag{5.46}$$

where $\bar{D} = \bar{D}(\Phi)$ which is given by (5.15) and $b = (b_s)_{s\in\mathbb{Z}}$ is given by the following:

$$b_s := \sup_{\omega\in X} \|\gamma^{\Phi}_{\{s\}}(\cdot|\omega_{\{s\}^c}) - \gamma^{\Psi^{(0)}}_{\{s\}}(\cdot|\omega_{\{s\}^c})\|_\infty. \tag{5.47}$$

For any $\xi, \omega \in X$, one has that

$$\begin{aligned}
&\gamma^{\Phi}_{\{s\}}(\xi_s|\omega_{\{s\}^c}) - \gamma^{\Psi^{(0)}}_{\{s\}}(\xi_s|\omega_{\{s\}^c}) \\
&= \frac{e^{-H^{\Phi}_s(\xi_s\omega_{\{s\}^c}) - H^{\Psi^{(0)}}_s(\xi_s\omega_{\{s\}^c})} \sum_{\eta_s} \left[e^{H^{\Psi^{(0)}}_s(\xi_s\omega_{\{s\}^c}) - H^{\Psi^{(0)}}_s(\eta_s\omega_{\{s\}^c})} - e^{H^{\Phi}_s(\xi_s\omega_{\{s\}^c}) - H^{\Phi}_s(\eta_s\omega_{\{s\}^c})} \right]}{\left(\sum_{\eta_s} e^{-H^{\Phi}_s(\eta_s\omega_{\{s\}^c})} \right) \cdot \left(\sum_{\eta_s} e^{-H^{\Phi^{(0)}}_s(\eta_s\omega_{\{s\}^c})} \right)}.
\end{aligned}$$

Thus since

$$\sup_{\xi,\eta,\omega} \left| H^{\Phi}_s(\xi_s\omega_{\{s\}^c}) - H^{\Phi}_s(\eta_s\omega_{\{s\}^c}) - \left(H^{\Psi^{(0)}}_s(\xi_s\omega_{\{s\}^c}) - H^{\Psi^{(0)}}_s(\eta_s\omega_{\{s\}^c}) \right) \right| = O\Big(\frac{1}{(|s|+1)^{\alpha-1}} \Big)$$

$$\sup_{\omega\in X} \|\gamma^{\Phi}_{\{s\}}(\cdot|\omega_{\{s\}^c}) - \gamma^{\Psi^{(0)}}_{\{s\}}(\cdot|\omega_{\{s\}^c})\|_\infty = O\Big(\frac{1}{(|s|+1)^{\alpha-1}} \Big). \tag{5.48}$$

Hence there exists $C_3 > 0$ such that for all $s \in \mathbb{Z}$,

$$b_s \le \frac{C_3}{(|s|+1)^{\alpha-1}}. \tag{5.49}$$

Then (5.46) yields that

$$t_j \le 2C_3 \sum_{s\in\mathbb{Z}} \frac{\bar{D}_{j,s}}{(|s|+1)^{\alpha-1}}. \tag{5.50}$$

Hence by using (5.17), we obtain that

$$t_j \le 2C_3 C_J \sum_{s\in\mathbb{Z}} \frac{1}{(1+|s-j|)^{\alpha}(|s|+1)^{\alpha-1}}. \tag{5.51}$$

The right-hand side of the above inequality can be written as

$$\begin{aligned}
\sum_{s\in\mathbb{Z}} \frac{1}{(1+|s-j|)^{\alpha}(|s|+1)^{\alpha-1}} &= \sum_{s=1}^{\infty} \frac{1}{(1+s+j)^{\alpha}(1+s)^{\alpha-1}} + \frac{1}{(1+j)^{\alpha}} \\
&+ \sum_{s=1}^{j-1} \frac{1}{(1+j-s)^{\alpha}(s+1)^{\alpha-1}} + \frac{1}{(1+j)^{\alpha-1}} \\
&+ \sum_{s=j+1}^{\infty} \frac{1}{(1+s-j)^{\alpha}(s+1)^{\alpha-1}} \\
&\le \sum_{s=1}^{\infty} \frac{1}{(s+j)^{\alpha}} + \frac{1}{j^{\alpha}} + \sum_{s=1}^{\lfloor j/2 \rfloor} \frac{1}{(j-s)^{\alpha-1}s^{\alpha}} \qquad (5.52) \\
&+ \sum_{s=\lceil j/2 \rceil}^{j-1} \frac{1}{(j-s)^{\alpha-1}s^{\alpha}} + \frac{1}{j^{\alpha-1}} + \frac{\zeta(\alpha)}{j^{\alpha-1}},
\end{aligned}$$

here $\lfloor t \rfloor$ and $\lceil t \rceil$ denote respectively the floor and ceiling of $t \in \mathbb{R}$. Note that

$$\sum_{s=1}^{\lfloor j/2 \rfloor} \frac{1}{(j-s)^{\alpha-1}s^{\alpha}} + \sum_{s=\lceil j/2 \rceil}^{j-1} \frac{1}{(j-s)^{\alpha-1}s^{\alpha}} \le \frac{2^{\alpha-1}}{j^{\alpha-1}} \sum_{s=1}^{\infty} \frac{1}{s^{\alpha}} + \frac{2^{\alpha}}{j^{\alpha}} \sum_{s=1}^{j} \frac{1}{s^{\alpha-1}}$$

and by the Stolz-Cesaro theorem

$$\sum_{s=1}^{j} \frac{1}{s^{\alpha-1}} = o(j^{-1}),$$

consequently,

$$\sum_{s=1}^{j-1} \frac{1}{(1+j-s)^{\alpha}(s+1)^{\alpha-1}} = O(j^{1-\alpha}). \qquad (5.53)$$

By combining (5.52) and (5.53), one obtains that

$$\sum_{s\in\mathbb{Z}} \frac{1}{(1+|s-j|)^{\alpha}(|s|+1)^{\alpha-1}} = O(j^{1-\alpha}). \qquad (5.54)$$

Hence, in light of (5.51), we conclude that (5.45) holds in the case of $n > 0$.

Using (5.45), we can now estimate (5.44) from the above. In fact, one has

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=0}^{N} \frac{|\mu(\sigma_0)t_j + \mu(\sigma_0)t_{-i} - t_{-i}t_j|}{(i+j)^{\alpha}} \\
&\qquad\qquad \le \sum_{i=1}^{N}\sum_{j=0}^{N} \frac{|\mu(\sigma_0)|t_j}{(i+j)^{\alpha}} + \sum_{i=1}^{N}\sum_{j=0}^{N} \frac{|\mu(\sigma_0)|t_{-i}}{(i+j)^{\alpha}} + \sum_{i=1}^{N}\sum_{j=0}^{N} \frac{t_{-i}t_j}{(i+j)^{\alpha}}.
\end{aligned}$$

Now we only show that the first term remains bounded as $N \to \infty$, and the boundedness of the other terms can be shown similarly. In fact, there exists $C_4 > 0$ such that for all N,

$$\sum_{j=0}^{N}\sum_{i=1}^{N}\frac{t_j}{(i+j)^\alpha} \le \sum_{j=0}^{N}\sum_{i=1}^{\infty}\frac{t_j}{(i+j)^\alpha} \le C_4\sum_{j=1}^{N}\frac{t_j}{j^{\alpha-1}}.$$

Thus by (5.45) and by taking into account the fact that $\alpha > \frac{3}{2}$, one immediately concludes that for all $N \in \mathbb{N}$,

$$\sum_{j=0}^{N}\sum_{i=1}^{N}\frac{t_j}{(i+j)^\alpha} \le C_2C_4\sum_{j=1}^{N}\frac{1}{j^{2\alpha-2}} \le C_2C_4\sum_{j=1}^{\infty}\frac{1}{j^{2\alpha-2}} < \infty. \tag{5.55}$$

Part (iii): By the first part, we can also conclude that the entire sequence $\{f^{(k)}\}_{k\in\mathbb{N}}$ converges to $f = \frac{d\mu}{d\,\nu^{(0)}}$ as $k \to \infty$ in the weak topology in $L^1(\nu^{(0)})$. This follows from the fact that a limit point of the sequence $\{f^{(k)}\}_{k\in\mathbb{N}}$ should be a Radon-Nikodym density of some $\mu^* \in \mathscr{G}(\Phi)$, but $\mathscr{G}(\Phi) = \{\mu\}$, therefore, $\frac{d\mu}{d\,\nu^{(0)}}$ is the only limit point. Thus

$$f = \lim_{k\to\infty}\frac{e^{-W_k}}{\int_X e^{-W_k}d\,\nu^{(0)}}, \tag{5.56}$$

here the limit is understood in the weak topology in $L^1(\nu^{(0)})$.

Note that the weak convergence of $\{f^{(k)}\}$ to f in the weak topology in $L^1(\nu^{(0)})$ also implies the weak convergence of $f_+^{(k)} := \int_{X_-} f^{(k)}d\,\nu_-$ to f_+ in the weak topology in $L^1(\nu)$. In fact, for any bounded $g \in \mathscr{F}_{\mathbb{Z}_+}$, by taking into account the fact that $\nu^{(0)} = \nu_- \times \nu$, one has

$$\begin{aligned}
\int_{X_+}(f_+^{(k)} - f_+)g\,d\,\nu &= \int_{X_+}\Big[\int_{X_-}(f^{(k)} - f)d\,\nu_-\Big]g\,d\,\nu \\
&= \int_{X_+}\int_{X_-} g(f^{(k)} - f)d\,\nu_- d\,\nu \\
&= \int_X g(f^{(k)} - f)d\,\nu^{(0)} \xrightarrow[k\to\infty]{} 0.
\end{aligned}$$

Hence we obtain an analog of (5.56) for f_+, i.e.,

$$f_+ = \lim_{k\to\infty}\frac{\int_{X_-} e^{-W_k}d\,\nu_-}{\int_X e^{-W_k}d\,\nu^{(0)}}. \tag{5.57}$$

Here the above limit should be again understood in an appropriate topology.

Now assume the transfer operator $\mathscr{L}_\phi$ has a continuous eigenfunction $\tilde{f}$, or in other words, the Radon-Nikodym derivative $\frac{d\mu_+}{d\nu}$ has a continuous version $\tilde{f}$. Then for ν–a.e. $x \in X_+$, $\tilde{f}(x) = f_+(x)$, and $\tilde{f}$, as a function on compact space X_+, is bounded. Hence $f_+ \in L^\infty(\nu)$, therefore,

$$\sup\left\{\frac{1}{\nu([y_\Lambda])}\int_{[y_\Lambda]} f_+ \, d\nu : y \in X_+,\ \Lambda \Subset \mathbb{Z}_+\right\} \le \|f_+\|_{L^\infty(\nu)} < \infty. \tag{5.58}$$

Hence to prove the second part of Theorem 5.A, it is enough to show that the supremum in (5.58) is infinite. In fact, below we show that

$$\sup_{n\in\mathbb{N}} \frac{1}{\nu([1_0^n])}\int_{[1_0^n]} f_+ d\nu = +\infty. \tag{5.59}$$

In other words, we show that $1_{\mathbb{Z}_+} = (+1_i)_{i\in\mathbb{Z}_+} \in X_+$ is an *essential discontinuity point* of the Radon-Nikdoym density $f_+ = \frac{d\mu_+}{dd\nu}$. To do so, fix $n \in \mathbb{N}$ and consider $g = \mathbb{1}_{[1_0^n]}$. By (5.57), one has that

$$\int_{X_+} f_+ \mathbb{1}_{[1_0^n]} d\nu = \lim_{k\to\infty} \int_{[1_0^n]} \frac{\int_{X_-} e^{-W_k(\xi,\eta)}\nu_-(d\xi)}{\int_{X_+}\int_{X_-} e^{-W_k(\xi,\bar{\eta})}\nu_-(d\xi)\nu(d\bar{\eta})}\nu(d\eta). \tag{5.60}$$

Thus, since $\{W_{[N]}\}_{N\in\mathbb{N}}$ (for the definition, see the proof of the first part) is a subsequence of $\{W_k\}_{k\in\mathbb{N}}$,

$$\int_{X_+} f_+ \mathbb{1}_{[1_0^n]} d\nu = \lim_{N\to\infty} \int_{[1_0^n]} \frac{\int_{X_-} e^{-W_{[N]}(\xi,\eta)}\nu_-(d\xi)}{\int_{X_+}\int_{X_-} e^{-W_{[N]}(\xi,\bar{\eta})}\nu_-(d\xi)\nu(d\bar{\eta})}\nu(d\eta). \tag{5.61}$$

Now fix $N \in \mathbb{N}$ and $\eta \in X_+$, consider $W_{[N]}$ as a function of ξ. Clearly, it is a local function, thus by the first part of Theorem 5.3.4, for all $\kappa \in \mathbb{R}$,

$$\int_{X_-} e^{\kappa[W_{[N]}(\xi,\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)]}\nu_-(d\xi) \le e^{D\kappa^2\|\underline{\delta}(W_{[N]}(\cdot,\eta))\|_2^2}, \tag{5.62}$$

where $D = 4(1-\bar{c}(\Phi))^{-2}$. By the Cauchy-Schwarz inequality, one can also obtain a lower bound for the integral, in fact,

$$e^{-D\kappa^2\|\underline{\delta}(W_{[N]}(\cdot,\eta))\|_2^2} \le \int_{X_-} e^{\kappa[W_{[N]}(\xi,\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)]}\nu_-(d\xi). \tag{5.63}$$

For all $s \in \mathbb{N}$,

$$\delta_{-s}(W_{[N]}(\cdot,\eta)) = 2\beta \left| \sum_{j=0}^{N} \frac{\eta_j}{(s+j)^\alpha} \right| \leq 2\beta \sum_{j=0}^{N} \frac{1}{(s+j)^\alpha}.$$

Hence for $\alpha > \frac{3}{2}$,

$$||\underline{\delta}(W_{[N]}(\cdot,\eta))||_2^2 \leq 4\beta^2 \sum_{s=1}^{\infty} \Big(\sum_{j=s}^{\infty} \frac{1}{j^\alpha} \Big)^2 =: 4\beta^2 C_1(\alpha) < \infty.$$

Hence one obtains from (5.62) and (5.63) that

$$C_5^{-1} \cdot e^{-\int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)} \leq \int_{X_-} e^{-W_{[N]}(\xi,\eta)} \nu_-(d\xi) \leq C_5 \cdot e^{-\int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)}, \quad (5.64)$$

where $C_5 := e^{4D\beta^2 C_1(\alpha)}$.

By applying the Gaussian Concentration Bounds to $W_{[N]}$ as a function of both ξ and η, one can obtain an analogue of (5.64) for $\nu^{(0)}$. In fact, by using the Cauchy-Schwarz inequality, one can easily check that

$$e^{-D||\underline{\delta}(W_{[N]})||_2^2} \cdot e^{\int_X -W_{[N]} d\nu^{(0)}} \leq \int_X e^{-W_{[N]}} d\nu^{(0)} \leq e^{D||\underline{\delta}(W_{[N]})||_2^2} \cdot e^{\int_X -W_{[N]} d\nu^{(0)}}. \quad (5.65)$$

For any $s \in \mathbb{Z}$,

$$\delta_s(W_{[N]}) \leq \frac{8\beta}{(|s|+1)^\alpha} + 2\beta \sum_{i=|s|+1}^{\infty} \frac{1}{i^\alpha} \leq \frac{\beta C_6}{(|s|+1)^{\alpha-1}}, \quad (5.66)$$

where $C_6 > 0$ is only dependent on α. Hence, since $\alpha > 3/2$,

$$||\underline{\delta}(W_{[N]})||_2^2 = \sum_{s\in\mathbb{Z}} (\delta_s(W_{[N]}))^2 \leq (\beta C_6)^2 \sum_{s\in\mathbb{Z}} \frac{1}{(|s|+1)^{2\alpha-2}} =: C_7(\alpha,\beta) < \infty. \quad (5.67)$$

Then one obtains from (5.65) that

$$C_8^{-1} \cdot e^{\int_X -W_{[N]} d\nu^{(0)}} \leq \int_X e^{-W_{[N]}} d\nu^{(0)} \leq C_8 \cdot e^{\int_X -W_{[N]} d\nu^{(0)}}, \quad (5.68)$$

here $C_8 = C_8(\alpha,\beta,h) := D \cdot C_7(\alpha,\beta)$. By combining (5.61), (5.64) and (5.68), one concludes that for any $N \in \mathbb{N}$ and $\eta \in X_+$,

$$C_9^{-1} \cdot e^{\int_X W_{[N]} d\nu^{(0)} - \int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)} \leq \frac{\int_{X_-} e^{-W_{[N]}(\xi,\eta)} \nu_-(d\xi)}{\int_X e^{-W_{[N]}} d\nu^{(0)}} \quad (5.69)$$

$$\leq C_9 \cdot e^{\int_X W_{[N]} d\nu^{(0)} - \int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)}, \quad (5.70)$$

here $C_9 = C_9(\alpha,\beta,h) := \max\{C_5, C_8\}$. Hence, by Jensen's inequality,

$$\frac{1}{\nu([1_0^n])}\int\limits_{[1_0^n]} \frac{\int\limits_{X_-} e^{-W_{[N]}(\xi,\eta)}\,\nu_-(d\xi)}{\int\limits_X e^{-W_{[N]}}\,d\,\nu^{(0)}}\,\nu(d\eta) \geq \frac{C_9^{-1}}{\nu([1_0^n])}\cdot\int\limits_{[1_0^n]} e^{\int\limits_X W_{[N]}\,d\,\nu^{(0)} - \int\limits_{X_-} W_{[N]}(\xi,\eta)\,\nu_-(d\xi)}\,\nu(d\eta)$$
$$\geq C_9^{-1}\cdot\exp\left(\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}\left[\int_X W_{[N]}(\xi,\bar\eta)\,\nu^{(0)}(d\xi,d\bar\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\,\nu_-(d\xi)\right]\nu(d\eta)\right) \tag{5.71}$$

Note that for any $\eta\in X_+$, one has

$$\int_X W_{[N]}(\xi,\bar\eta)\,\nu^{(0)}(d\xi,d\bar\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\,\nu_-(d\xi) = \beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{\nu_-(\sigma_{-i})[\eta_j-\nu(\sigma_j)]}{(i+j)^\alpha} \tag{5.72}$$

Hence

$$\begin{aligned}\int_{[1_0^n]}\left[\int_X W_{[N]}(\xi,\bar\eta)\,\nu^{(0)}(d\xi,d\bar\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\,\nu_-(d\xi)\right]\nu(d\eta)\\ = \beta\int_{[1_0^n]}\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{\nu_-(\sigma_{-i})[\eta_j-\nu(\sigma_j)]}{(i+j)^\alpha}\,\nu(d\eta)\\ = \beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{\nu_-(\sigma_{-i})}{(i+j)^\alpha}\int_{[1_0^n]}[\sigma_j-\nu(\sigma_j)]\,d\,\nu.\end{aligned} \tag{5.73}$$

Then since both σ_j and $\mathbb{1}_{[1_0^n]}$ are non-decreasing functions and ν is a positively correlated measure, by the FKG inequality [7], for any $j\in\mathbb{Z}_+$,

$$\int_{[1_0^n]}[\sigma_j-\nu(\sigma_j)]\,d\,\nu = \int_{X_+}\sigma_j\mathbb{1}_{[1_0^n]} - \int_{X_+}\sigma_j\,d\,\nu\int_{X_+}\mathbb{1}_{[1_0^n]}\,d\,\nu \geq 0. \tag{5.74}$$

Furthermore, for $1\leq j\leq n$, one has

$$\int_{[1_0^n]}[\sigma_j-\nu(\sigma_j)]\,d\,\nu = \nu([1_0^n])(1-\nu(\sigma_j)). \tag{5.75}$$

We also note that by (5.22), for all $i\in\mathbb{N}$,

$$\nu_-(\sigma_{-i})\geq 0. \tag{5.76}$$

By combining these arguments,

$$\beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{\nu_-(\sigma_{-i})}{(i+j)^\alpha}\int_{[1_0^n]}[\sigma_j-\nu(\sigma_j)]d\nu \geq \beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})}{(i+j)^\alpha}\int_{[1_0^n]}[\sigma_j-\nu(\sigma_j)]d\nu$$
$$=\beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})}{(i+j)^\alpha}\nu([1_0^n])(1-\nu(\sigma_j)). \quad (5.77)$$

Hence by (5.73),

$$\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}\Big[\int_X W_{[N]}(\xi,\bar\eta)\nu^{(0)}(d\xi,d\bar\eta)-\int_{X_-}W_{[N]}(\xi,\eta)\nu_-(d\xi)\Big]\nu(d\eta)$$
$$\geq\frac{\beta}{\nu([1_0^n])}\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))\nu([1_0^n])}{(i+j)^\alpha}$$
$$=\beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))}{(i+j)^\alpha} \quad (5.78)$$

Then (5.71) and (5.78) yields that for any $N\in\mathbb{N}$,

$$\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}\frac{\int_{X_-}e^{-W_{[N]}(\xi,\eta)}\nu_-(d\xi)}{\int_X e^{-W_{[N]}}d\nu^{(0)}}\nu(d\eta)\geq C_9^{-1}\cdot\exp\Big(\beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))}{(i+j)^\alpha}\Big) \quad (5.79)$$

Hence (5.61) implies

$$\begin{aligned}\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}f_+d\nu &\geq \lim_{N\to\infty}C_9^{-1}\cdot\exp\Big(\beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))}{(i+j)^\alpha}\Big)\\ &= C_9^{-1}\cdot\exp\Big(\beta\sum_{j=0}^{n}\sum_{i=1}^{\infty}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))}{(i+j)^\alpha}\Big). \quad (5.80)\end{aligned}$$

It follows from (5.45) that $\lim_{i\to\infty}\nu_-(\sigma_{-i})=\lim_{j\to\infty}\nu(\sigma_j)=\mu(\sigma_0)>0$. Thus

$$\tilde\kappa:=\sup_{j\in\mathbb{Z}_+}\nu(\sigma_j)<1$$

and there exists $R\in\mathbb{N}$ which depends only on $\nu^{(0)}=\nu_-\times\nu$ and μ (thus R only depends on the model parameters α,β,h) such that $\nu_-(\sigma_{-i})\geq\frac{\mu(\sigma_0)}{2}$ for all $i\geq R$. Hence we obtain from (5.80) that

$$\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}f_+d\nu\geq C_9^{-1}\cdot\exp\Big(\frac{\beta(1-\tilde\kappa)\mu(\sigma_0)}{2}\sum_{j=0}^{n}\sum_{i=R}^{\infty}\frac{1}{(i+j)^\alpha}\Big). \quad (5.81)$$

One can readily check that the sum on the right-hand side of (5.81) diverges as $n \to \infty$. In fact, there exists $C_{10} = C_{10}(\alpha) \in (0,1)$ such that $\displaystyle\sum_{i=R}^{\infty} \frac{1}{(i+j)^{\alpha}} \geq C_{10}(R+j)^{1-\alpha}$, therefore,

$$\sum_{j=0}^{\infty}\sum_{i=R}^{\infty} \frac{1}{(i+j)^{\alpha}} \geq C_{10} \sum_{j=0}^{\infty} \frac{1}{(R+j)^{\alpha-1}} = \infty.$$

Hence, one indeed concludes that

$$\sup_{n\in\mathbb{N}} \frac{1}{\nu([1_0^n])} \int_{[1_0^n]} f_+ d\nu = +\infty.$$

$\square$

Remark 5.5.1. *The statements of Theorem 5.A remain valid for the Dyson interaction $\hat{\Phi} = (\hat{\Phi}_{\Lambda})_{\Lambda \Subset \mathbb{Z}}$ with inhomogeneous external fields:*

$$\hat{\Phi}_{\Lambda}(\omega) := \begin{cases} \dfrac{-\beta\omega_i\omega_j}{|i-j|^{\alpha}}, & \text{if } \Lambda = \{i,j\},\ i \neq j; \\ h_i\omega_i, & \text{if } \Lambda = \{i\}; \\ 0, & \text{otherwise,} \end{cases} \tag{5.82}$$

as long as the fields $h_i \in \mathbb{R}$ are sufficiently strong. Specifically, if $\alpha \in \left(\frac{3}{2}, 2\right]$, $\beta \geq 0$, and

$$\inf_{i\in\mathbb{Z}} |h_i| \geq 2\beta\zeta(\alpha) + \log(4\beta\zeta(\alpha)),$$

then the interactions $\hat{\Phi}$ and $\hat{\Psi}^{(0)}$ admit unique Gibbs measures $\mu \in \mathscr{G}(\hat{\Phi})$ and $\nu^{(0)} \in \mathscr{G}(\hat{\Psi}^{(0)})$, and we have $\mu \ll \nu^{(0)}$. Moreover, the Radon-Nikodym density of the restriction $\mu|_{X_+}$ with respect to $\nu^{(0)}|_{X_+}$ does not have a continuous version. However, in this general (inhomogeneous) case, we cannot always associate a potential to $\hat{\Phi}$ via the formula $\displaystyle\sum_{0\in V\Subset\mathbb{Z}_+} \hat{\Phi}_V$, due to the lack of translation invariance of $\hat{\Phi}$. We note that $\hat{\Phi}$ is translation-invariant if and only if $h_i \in \mathbb{R}$ is constant over $i \in \mathbb{Z}$.

Bibliography

[1] Rodrigo Bissacot, Eric Endo, Aernout C.D. van Enter, and Arnaud Le Ny, *Entropic repulsion and lack of the g-measure property for Dyson models*, Comm.Math.Phys. **363** (2018), 767–788.

[2] L Cioletti, M Denker, O Lopes, and M Stadlbauer, *Spectral properties of the Ruelle operator for product-type potentials on shift spaces*, J. London Math. Soc. **95** (2017), no. 2, 684–704.

[3] Leandro Cioletti, Artur O. Lopes, and Manuel Stadlbauer, *Ruelle Operator for Continuous Potentials and DLR-Gibbs Measures*, Discrete and Continuous Dynamical Systems - Series A **40** (2020), no. 8, 4625–4652.

[4] Dyson Freeman, *Existence of a Phase-Transition in a One-Dimensional Ising ferromagnet*, Commun. Math. Phys. **12** (1969), 91–107.

[5] Aernout C. D. van Enter, Roberto Fernández, Mirmukhsin Makhmudov, and Evgeny Verbitskiy, *On an extension of a theorem by Ruelle to long-range potentials* (2024).

[6] Hans Föllmer, *A covariance estimate for Gibbs measures*, J. Functional Analysis **46** (1982), no. 3, 387–395, DOI 10.1016/0022-1236(82)90053-2. MR661878

[7] S. Friedli and Y. Velenik, *Statistical mechanics of lattice systems*, Cambridge University Press, Cambridge, 2018. A concrete mathematical introduction. MR3752129

[8] Frohlich Jurg and Spencer Thomas, *The Phase Transition in the One-Dimensional Ising Model with $1/r^2$ Interaction Energy*, Commun. Math. Phys. **84** (1982), 87–101.

[9] Hans-Otto Georgii, *Gibbs measures and phase transitions*, De Gruyter Studies in Mathematics, vol. 9, Walter de Gruyter & Co., Berlin, 1988. MR956646

[10] J. Ginibre, *General formulation of Griffiths' inequalities*, Comm. Math. Phys. **16** (1970), 310–328. MR269252

[11] R.B. Griffiths, *Correlations in Ising Ferromagnets. I*, J. Math. Phys. **8** (1966), 478—483, DOI 10.1063/1.1705219.

[12] ———, *Rigorous Results for Ising Ferromagnets of Arbitrary Spin*, J. Math. Phys. **10** (1968), 1559–1565, DOI 10.1063/1.1665005.

[13] Karlheinz Gröchenig and Andreas Klotz, *Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices*, Constr. Approx. **32** (2010), no. 3, 429–466, DOI 10.1007/s00365-010-9101-z. MR2726441

[14] S. Jaffard, *Propriétés des matrices "bien localisées" près de leur diagonale et quelques applications*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **7** (1990), no. 5, 461–476, DOI 10.1016/S0294-1449(16)30287-6 (French, with English summary). MR1138533

[15] Anders Johansson, Anders Öberg, and Mark Pollicott, *Continuous eigenfunctions of the transfer operator for the Dyson model*, Arxiv 2304.04202 (2023).

[16] D.G. Kelly and S. Sherman, *General Griffiths' Inequalities on Correlations in Ising Ferromagnets*, J. Math. Phys. **9** (1967), 466—484, DOI 10.1063/1.1664600.

[17] Christof Külske, *Concentration inequalities for functions of Gibbs fields with application to diffraction and random Gibbs measures*, Comm. Math. Phys. **239** (2003), no. 1-2, 29–51, DOI 10.1007/s00220-003-0841-5. MR1997114

[18] David Ruelle, *Thermodynamic formalism*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004. The mathematical structures of equilibrium statistical mechanics. MR2129258

[19] Roman Vershynin, *High-dimensional probability*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 47, Cambridge University Press, Cambridge, 2018. An introduction with applications in data science; With a foreword by Sara van de Geer. MR3837109

[20] Peter Walters, *Ruelle's operator and g−measures*, Trans. Amer. Math. Soc. **214** (1975), 375–387.

[21] ______, *Invariant measures and equilibrium states for some mappings which expand distances*, Trans. Amer. Math. Soc. **236** (1978), 121–153.

[22] ______, *Convergence of the Ruelle operator for a function satisfying Bowen's condition*, Trans. Amer. Math. Soc. **353** (2001), no. 1, 327–347.

[23] ______, *Regularity conditions and Bernoulli properties of equilibrium states and g-measures*, J. London Math. Soc. (2) **71** (2005), no. 2, 379–396, DOI 10.1112/S0024610704006076. MR2122435