Gibbs states in statistical mechanics and dynamical systems
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Part II

Transfer operators for long-range potentials

Chapter 4

On an extension of a theorem by Ruelle to long-range potentials

Abstract: Ruelle's transfer operator plays an important role in understanding thermodynamic and probabilistic properties of dynamical systems. In this chapter, we develop a method of finding eigenfunctions of transfer operators based on comparing Gibbs measures on the half-line $\mathbb{Z}_+$ and the whole line $\mathbb{Z}$. For a rather broad class of potentials, including both the ferromagnetic and antiferromagnetic long-range Dyson potentials, we are able to establish the existence of integrable, but not necessarily continuous, eigenfunctions. For a subset thereof we prove that the eigenfunction is actually continuous.

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4.1 Introduction

One of the main issues of equilibrium statistical mechanics is to derive and describe the properties of the possible (global) states of a macroscopic system starting from the knowledge of the finite-volume (local) states of the system. In order to give a mathematical framework to this problem, Dobrushin, Lanford, and Ruelle developed the so-called DLR formalism in the second half of the last century.

Shortly after their introduction of Gibbs measures, Spitzer and Averintsev [3, 61] characterised Gibbs measures for short-range potentials in terms of measures having Markov properties. These results were then simplified and generalised, in various directions by Hammersley and Clifford [44], by Sullivan, Kozlov [49, 62], and by Geoffrey Grimmett in his first paper [41].

The novel DLR formalism was immediately adapted to the theory of Dynamical Systems by Sinai [58–60]. There are, however, two important differences between the models typically studied in Statistical Mechanics and those studied in Dynamical Systems. Firstly, in Dynamical Systems one is typically interested in one-dimensional systems, that is, systems with configuration spaces $E^{\mathbb{Z}}$, where E is the set of possible spin values and the spatial dimension represents time. The conditional probabilities in Dynamical Systems are typically "one-sided" (from past to future), those in Statistical Mechanics "two-sided" (from outside to inside). For short-range potentials this does not make much of a difference, although in general regularity properties between one-sided and two-sided conditional probabilities may differ [7, 8, 27, 28, 32]. Secondly, and perhaps more important, the natural description of dynamical systems often involves *half-line* configuration spaces $E^{\mathbb{Z}_+}$, rather than *whole-line* configuration spaces of the form $E^{\mathbb{Z}}$.

Already in his original papers, Sinai addressed these questions, [58–60]. He showed that Gibbs equilibrium states for exponentially decaying interactions are, in fact, equilibrium states for half-line potentials as well. The issue of half-line versus whole-line Gibbsianness is, therefore, as old as the theory of thermodynamic formalism in Dynamical Systems. For more recent results established in this area see, e.g., [7, 33, 64–67]. One of the most important Dynamical Systems tools used in the study of Gibbs equilibrium states is the so-called Ruelle's transfer operator [57, 64]. Various probabilistic properties of chaotic dynamical systems can be characterised in terms of transfer operators. In particular, a central issue is to characterise those potentials (interactions) for which transfer operators have positive continuous eigenfunctions that is, finding those potentials for which Ruelle's theorem holds. This question has been answered by different authors [30, 54, 55, 64, 66, 67] for different regularity classes of potentials. In this chapter, we answer it for potentials beyond these earlier studied classes.

This chapter is organised as follows:

- In Section 4.2, Section 4.3, and Section 4.4 we introduce the notions of thermodynamic formalism that are important for this chapter.
- In Section 4.5, we discuss the relationship between half-line and whole-line Gibbs measures and formulate the first part of the main results (Theorem 4.A and 4.B).
- In Section 4.6, we discuss when whole-line Gibbs measures are absolutely continuous with respect to the product of two half-line ones, and we formulate the second series of our main results (Theorem 4.C, 4.D and 4.E).
- In Section 4.7, the Dyson model, the main example of the chapter, is discussed. We then discuss what is the behaviour in other regimes of the phase transitions.
- Section 4.8 and Section 4.9 are dedicated to the proofs of our main results and the final remarks.

4.2 Basic notions: I. Specifications

The Dobrushin-Lanford-Ruelle definition of Gibbs states via specifications goes well beyond the standard lattices $\mathbb{Z}^d$. Consider the lattice system $\Omega = E^{\mathbb{L}}$, where $\mathbb{L}$ is an at most countable set (lattice) and E is a set of possible spin values. In this chapter, we will focus on finite E. We denote the Borel $\sigma-$algebra of the measurable subsets of Ω by $\mathscr{F}$. For a subset $\Lambda \subset \mathbb{L}$, we define $\mathscr{F}_\Lambda$ as the minimal σ-algebra that makes the maps $\omega \in \Omega \mapsto \omega_i \in E$, $i \in \Lambda$ measurable. Additionally, we let $\mathscr{T}$ represent the tail σ-algebra, defined by $\mathscr{T} := \cap_{\Lambda \Subset \mathbb{L}} \mathscr{F}_{\Lambda}$. The specification is a consistent family of probability kernels (conditional probabilities) indexed by finite subsets Λ of $\mathbb{L}$ denoted by $\Lambda \Subset \mathbb{L}$. The consistency condition is the requirement that $\gamma_\Lambda \gamma_\Delta = \gamma_\Lambda$ for all $\Delta \subset \Lambda \Subset \mathbb{L}$ [39, Chapter 1].

In the sequel, for $\Delta, \Lambda \subset \mathbb{L}$, we will denote the concatenation of strings $\xi_\Delta \in E^\Delta$, $\eta_\Lambda \in E^\Lambda$ by $\xi_\Delta \eta_\Lambda$, namely, $\xi_\Delta \eta_\Lambda$ is a string such that $(\xi_\Delta \eta_\Lambda)_i = \xi_i$ if $i \in \Delta$ and $(\xi_\Delta \eta_\Lambda)_i = \eta_i$ if $i \in \Lambda$. Given the specification $\gamma = \{\gamma_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ on $\Omega = E^{\mathbb{L}}$, we say that a probability measure μ on Ω is Gibbs for the specification γ or, equivalently, that μ is consistent with γ, if

$$\mu(\sigma_\Lambda | \sigma_{\Lambda^c}) = \gamma_\Lambda(\sigma_\Lambda \sigma_{\Lambda^c}) \quad \text{for } \mu\text{-a.a. } \sigma \in \Omega,$$

or, equivalently, if the DLR equations hold:

$$\int \gamma_\Lambda f \, d\mu = \int f \, d\mu,$$

for all $f \in L^1(\Omega,\mu)$, and every $\Lambda \Subset \mathbb{L}$, where

$$\gamma_\Lambda f(\sigma) = \sum_{\xi_\Lambda \in E^\Lambda} \gamma_\Lambda(\xi_\Lambda|\sigma_{\Lambda^c}) f(\xi_\Lambda \sigma_{\Lambda^c}),$$

note that for $\omega \in \Omega$ and $\Lambda \subset \mathbb{L}$, ω_{Λ^c} denotes the (infinite) string $\omega_{\mathbb{L}\setminus\Lambda}$. The set of all Gibbs measures for γ will be denoted by $\mathscr{G}(\Omega,\gamma)$. For any probability measure μ one can find at least one specification γ such that μ is Gibbs for γ [40]. However, useful and interesting specifications have additional properties such as finite energy (non-nullness) and quasi-locality (continuity). We now turn to two particular ways of defining specifications as used in Statistical Mechanics and Dynamical Systems.

4.2.1 Gibbs(ian) specifications in Statistical Mechanics

An interaction Φ is a family of functions $\{\Phi_\Lambda\}$, indexed by finite subsets $\Lambda \Subset \mathbb{L}$, such that each function Φ_Λ, depends only on values of σ in Λ, that is, with a slight abuse of notation, $\Phi_\Lambda(\sigma) = \Phi_\Lambda(\sigma_\Lambda)$. One needs to impose some additional summability conditions on the interaction Φ: Φ is said to be ***uniformly absolutely convergent*** (UAC) if for all $i \in \mathbb{L}$, $\sum_{i \in V \Subset \mathbb{L}} \|\Phi_V\|_\infty < \infty$. For an UAC interaction Φ, the specification (specification density) $\gamma^\Phi = \{\gamma^\Phi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ is defined as follows, for $\omega, \eta \in \Omega$,

$$\gamma^\Phi_\Lambda(\omega_\Lambda|\eta_{\Lambda^c}) := \frac{e^{-H^\Phi_\Lambda(\omega_\Lambda \eta_{\Lambda^c})}}{Z^\Phi_\Lambda(\eta)}, \tag{4.1}$$

where $H^\Phi_\Lambda(\omega) := \sum_{V \cap \Lambda \neq \emptyset} \Phi_V(\omega)$ is the Hamiltonian in the volume Λ, and Z^Φ_Λ is a normalization constant (the partition function), i.e., $Z^\Phi_\Lambda(\eta) := \sum_{\bar\omega_\Lambda \in E^\Lambda} e^{-H^\Phi_\Lambda(\bar\omega_\Lambda \eta_{\Lambda^c})}$.

It should be stressed that a Gibbsian specification γ^Φ is always *quasilocal* [5, 26, 39]. In the current setting, in which E is finite, this property is equivalent to the fact that for all $\Lambda \Subset \mathbb{L}$ and $\omega_\Lambda \in E^\Lambda$, $\gamma_\Lambda(\omega_\Lambda|\eta)$ is a continuous function of the boundary condition $\eta \in \Omega$. Another important property of the Gibbsian specifications is *non-nullness*, which means that for all volumes $\Lambda \Subset \mathbb{L}$, $\inf_{\eta,\omega \in \Omega} \gamma^\Phi_\Lambda(\omega_\Lambda|\eta_{\Lambda^c}) > 0$.

We denote the set of Gibbs states for the interaction Φ by $\mathscr{G}(\Omega,\Phi)$ (or $\mathscr{G}(\Phi)$). It is a convex set – in fact a simplex – which is always non-empty if, as is the case in this chapter, the spin space E is compact.

Depending on the symmetries of the lattice $\mathbb{L}$ and the spin space E, the interactions and specifications may also exhibit some symmetries. For example, if $\mathbb{L} = \mathbb{Z}$, then an interaction Φ on $X := E^{\mathbb{Z}}$ is called ***translation-invariant*** if for all $\Lambda \Subset \mathbb{Z}$, every $k \in \mathbb{Z}$ and $\omega \in X$, $\Phi_{\Lambda+k}(\omega) = \Phi_\Lambda(S^k(\omega))$, where $\Lambda + k := \{i + k : i \in \Lambda\}$

and $S : X \to X$ is left shift, i.e., for every $i \in \mathbb{Z}$ and $\omega \in X$, $(S\omega)_i = \omega_{i+1}$. Respectively, a specification γ on $X = E^{\mathbb{Z}}$ is called ***translation-invariant*** if for all $B \in \mathscr{F}$, $\Lambda \Subset \mathbb{Z}$, $k \in \mathbb{Z}$ and $\omega \in X$, $\gamma_{\Lambda+k}(B|\omega) = \gamma_\Lambda(S^k(B)|S^k(\omega))$. Translation-invariant interactions give rise to translation-invariant specifications. To some extent, the opposite statement is also true: Sullivan showed ([5, 62]) that for a quasilocal translation-invariant specification on $\mathbb{Z}$, one can find a translation-invariant interaction Φ such that $\gamma = \gamma^\Phi$. Recently, however, it was shown in [5] that this interaction is not necessarily uniformly absolutely convergent.

It should be noted that Gibbsian specifications can be uniquely recovered from a consistent family of single-site probability kernels (densities) $\{\gamma_{\{i\}} : i \in \mathbb{L}\}$ [33]. Therefore, it is sufficient to study only the single-site densities of a Gibbsian specification instead of studying *all* densities. Due to this fact, it is worth defining the single-site densities of a specification separately from the concept of specification as follows.

Definition 4.2.1. [31] *A collection* $\{\gamma_{\{i\}}\}_{i\in\mathbb{L}}$ *of positive functions* $\gamma_{\{i\}}(\cdot|\cdot) : E \times E^{\mathbb{L}\setminus\{i\}} \to (0,1)$ *is called the family of single-site densities of a specification if*

(i) $\sum_{a_i \in E} \gamma_{\{i\}}(a_i|\omega_{\{i\}^c}) = 1$ *for all* $\omega \in \Omega = E^{\mathbb{L}}$, $i \in \mathbb{L}$,

(ii) and for all $i, j \in \mathbb{L}$, $\alpha, \omega \in \Omega$ *the following holds*

$$\frac{\gamma_{\{i\}}(\alpha_i|\alpha_j\omega_{\{i,j\}^c})}{\sum_{\beta_{\{i,j\}}} \frac{\gamma_{\{j\}}(\beta_j|\beta_i\omega_{\{i,j\}^c})\gamma_{\{i\}}(\beta_i|\alpha_j\omega_{\{i,j\}^c})}{\gamma_{\{j\}}(\alpha_j|\beta_i\omega_{\{i,j\}^c})}} = \frac{\gamma_{\{j\}}(\alpha_j|\alpha_i\omega_{\{i,j\}^c})}{\sum_{\beta_{\{i,j\}}} \frac{\gamma_{\{i\}}(\beta_i|\beta_j\omega_{\{i,j\}^c})\gamma_{\{j\}}(\beta_j|\alpha_i\omega_{\{i,j\}^c})}{\gamma_{\{i\}}(\alpha_i|\beta_j\omega_{\{i,j\}^c})}}.$$

The following theorem signifies the importance of single-site densities of a specification, and it will be useful later.

Proposition 4.2.2. [33] *Let* $\{\gamma_{\{i\}}\}_{i\in\mathbb{L}}$ *be single-site densities of a specification. There is a unique non-null specification* γ *on* $(\Omega, \mathscr{F})$ *having* $\{\gamma_{\{i\}}\}_{i\in\mathbb{L}}$ *as its single-site densities. Furthermore,* γ *is quasilocal if and only if all functions in the collection* $\{\gamma_{\{i\}}\}_{i\in\mathbb{L}}$ *are continuous, and a probability measure* $\mu \in \mathscr{M}_1(\Omega, \mathscr{F})$ *is consistent with* γ *if and only if it is consistent with all single-site probability kernels* $\gamma_{\{i\}}$, $i \in \mathbb{L}$.

4.2.2 Gibbs(ian) specifications in Dynamical Systems and Transfer operators

As already mentioned above, in Dynamical Systems, the 'natural' lattice is the half-line $\mathbb{L} = \mathbb{Z}_+$. For an introductory treatment of some of these issues, see also [48]. Let $X_+ = E^{\mathbb{Z}_+}$ be the space of one-sided sequences $\omega = (\omega_n)_{n\geq 0}$ in alphabet E. We equip X_+ with the metric $d(\omega, \omega') = \sum_{n=0}^{\infty} \mathbb{I}[\omega_n \neq \omega'_n] 2^{-n}$. We also define the left-shift S on X_+ by $y = Sx$ where $y_i = x_{i+1}$ for all $i \geq 0$. Let $\phi : X_+ \to \mathbb{R}$ be a continuous function (potential). Following [16, 17, 64], we define the corresponding specification $\vec{\gamma}^{\phi} := \{\vec{\gamma}_n = \vec{\gamma}^{\phi}_{[0,n-1]},\ n \geq 1\}$, by

$$\vec{\gamma}_n(a_0^{n-1}|x_n^{\infty}) = \frac{\exp\big((S_n\phi)(a_0^{n-1}x_n^{\infty})\big)}{\sum_{\bar{a}_0^{n-1}} \exp\big((S_n\phi)(\bar{a}_0^{n-1}x_n^{\infty})\big)}, \quad \text{where } (S_n\phi)(x) = \sum_{k=0}^{n-1} \phi(S^k x). \tag{4.2}$$

This gives a family of probability kernels on finite intervals $[0, n-1]$ in $\mathbb{Z}_+$. However, the definition extends to general volumes $\Lambda \Subset \mathbb{Z}_+$ by

$$\vec{\gamma}_{\Lambda}(a_{\Lambda}|x_{\Lambda^c}) = \frac{\exp\left((S_{n+1}\phi)(a_{\Lambda}x_{\Lambda^c})\right)}{\sum_{\bar{a}_{\Lambda}} \exp\left((S_{n+1}\phi)(\bar{a}_{\Lambda}x_{\Lambda^c})\right)},\ a_{\Lambda} \in E^{\Lambda}, x \in X_+, \tag{4.3}$$

where $n = \max\Lambda$.

For $f \in C(X_+, \mathbb{R})$, one has

$$\vec{\gamma}_n(f)(x) = \frac{\sum_{\bar{a}_0^{n-1}} \exp\big((S_n\phi)(\bar{a}_0^{n-1}x_n^{\infty})\big) f(\bar{a}_0^{n-1}x_n^{\infty})}{\sum_{\bar{a}_0^{n-1}} \exp\big((S_n\phi)(\bar{a}_0^{n-1}x_n^{\infty})\big)}. \tag{4.4}$$

It turns out that $\vec{\gamma}_n(f)$ can naturally be expressed in terms of the Ruelle-Perron-Frobenius transfer operator. This is the operator $\mathscr{L}_{\phi}$ acting on the space of continuous functions $C(X_+, \mathbb{R})$ as

$$\mathscr{L}_{\phi} f(x) = \sum_{y \in S^{-1}x} e^{\phi(y)} f(y) = \sum_{a \in E} e^{\phi(ax)} f(ax), \tag{4.5}$$

where the configuration ax is obtained by the concatenation of the letter a and the configuration x. Thus, for any $n \geq 1$,

$$\mathscr{L}_{\phi}^n f(x) = \sum_{a_0^{n-1} \in E^n} e^{S_n\phi(a_0^{n-1}x)} f(a_0^{n-1}x), \text{ and hence, } \vec{\gamma}_n(f)(x) = \frac{\mathscr{L}_{\phi}^n f(S^n x)}{\mathscr{L}_{\phi}^n \mathbf{1}(S^n x)}.$$

One readily checks that the family of probability kernels $\{\vec{\gamma}_n\}$ has the standard properties of specifications; most importantly, the consistency condition

$$\vec{\gamma}_m(\vec{\gamma}_n(f)) = \vec{\gamma}_n(\vec{\gamma}_m(f)) = \vec{\gamma}_m(f)$$

for all $f \in C(X_+, \mathbb{R})$ and every $m \geq n \geq 1$, and this particular property can be readily validated using properties of transfer operators [64, Theorem 2.1].

The set of all Gibbs states on X_+ for potential ϕ, i.e., the set of measures consistent with the specification $\vec{\gamma}^{\phi}$, will be denoted by $\mathscr{G}(X_+, \phi)$. The set of Gibbs measures $\mathscr{G}(X_+, \phi)$ is a closed convex set and the extremal points of $\mathscr{G}(X_+, \phi)$ are tail-trivial.

Transfer operators allow for a dual view on Gibbs measures $\mathscr{G}(X_+,\phi)$. Define the dual operator $\mathscr{L}_\phi^*$, acting on the space of measures $\mathscr{M}(X_+)$ by

$$\int f\, d(\mathscr{L}_\phi^* \nu) = \int \mathscr{L}_\phi f\, d\nu \quad \text{for all } f \in C(X_+, \mathbb{R}).$$

It is well-known that there exists at least one eigenprobability ν on X_+ for the maximal eigenvalue $\lambda = e^{P(\phi)}$, i.e.,

$$\mathscr{L}_\phi^* \nu = e^{P(\phi)} \nu,$$

where $P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X_+} \mathscr{L}^n \mathbf{1}(x)$ is the so-called topological pressure of ϕ. Combining the results of [64, Corollary 2.3] and [17, Theorem 4.8], we can conclude that the sets of probability eigenmeasures of $\mathscr{L}_\phi^*$ and the Gibbs states for ϕ on X_+ coincide:

$$\nu \in \mathscr{G}(X_+,\phi) \text{ if and only if } \mathscr{L}_\phi^* \nu = \lambda \nu.$$

There is an interesting phenomenon in the theory of Gibbs measures on one-sided (half-line) symbolic spaces, which has no direct analogue in the two-sided (whole-line) context. Let us say that a continuous potential $\phi : X_+ \to \mathbb{R}$ is quasi-normalized if $\mathscr{L}_\phi \mathbf{1}$ is a constant function on X_+. It turns out that ϕ is quasi-normalized if and only if all Gibbs measures μ in $\mathscr{G}(X_+,\phi)$ are translation invariant, i.e., $\mu = \mu \circ S^{-1}$ [65, Theorem 1.2 (iii)].

4.2.3 Relation between Gibbsian specifications

Specifications discussed above are defined on different spaces: $X = E^{\mathbb{Z}}$ vs $X_+ = E^{\mathbb{Z}_+}$, as well as, in different terms: namely, the interaction Φ vs the potential ϕ. What is the relation between these classes of specifications?

For a given interaction Φ, the potential ϕ should be interpreted as minus the contribution to the energy from (the neighborhood of) the origin [57, Section 3.2], [26, Section 2.4.5]. In fact there are multiple possibilities to define relevant ϕ, e.g.,

$$\phi(\omega) := -\sum_{0\in V\Subset\mathbb{Z}} \frac{1}{|V|} \Phi_V(\omega_V), \tag{4.6}$$

or,

$$\phi(\omega) := -\sum_{0\in V\Subset\mathbb{Z}_+} \Phi_V(\omega_V). \tag{4.7}$$

In the setup of Thermodynamic Formalism, the second choice is particularly convenient, and it will be used in this chapter.

It is worth mentioning that for every $\phi \in C(E^{\mathbb{Z}_+}, \mathbb{R})$ there exists a translation-invariant interaction Φ^ϕ on $X = E^{\mathbb{Z}}$ satisfying (4.7) such that

$$\sum_{0 \in V \Subset \mathbb{Z}} \frac{1}{|V|} \|\Phi_V^\phi\|_\infty < \infty,$$

nonetheless, such Φ^ϕ does not need to be unique [57]. The reciprocal property would be more interesting, namely that for each $\phi \in C(X_+)$ there would exist a translation-invariant UAC interaction Φ on X satisfying (4.7). Unfortunately, there exist counterexamples showing this to be false (c.f. Proposition 4.6.2 and Section 4.9).

4.3 Basic notions II: The Ruelle theorem and Equilibrium states

Gibbs measures are eigenmeasures of the duals of transfer operators corresponding to the maximal eigenvalue. What can we say about the eigenfunctions of transfer operators? Ruelle established the first result for smooth potentials [55, 57]: if $\phi : X_+ \to \mathbb{R}$ is Hölder continuous, then the transfer operator $\mathscr{L}_\phi$ has a positive continuous eigenfunction h; $\mathscr{L}_\phi h = \lambda h$, with $\lambda = e^{P(\phi)}$. The existence of continuous eigenfunctions has been established for larger classes of smooth potentials: for example, by Walters for potentials with summable variations ([66, 67]). For less regular potentials – the so-called Bowen class – Walters established the existence of bounded measurable eigenfunctions.

There is an interesting principal relation between the Gibbs measures $\mathscr{G}(X_+, \phi)$, eigenfunctions of transfer operators, and translation invariant equilibrium states.

Proposition 4.3.1. [51, Theorem 1] *Consider a continuous potential $\phi \in C(X_+)$, and suppose $\nu \in \mathscr{G}(X_+, \phi)$ or, equivalently, $\mathscr{L}_\phi^* \nu = e^{P(\phi)} \nu$. Then*

1) *If there exists a non-negative eigenfunction $h \in L^1(X_+, \nu)$ of the transfer operator $\mathscr{L}_\phi$ with $\mathscr{L}_\phi h = e^{P(\phi)} h$, then $\mu = h \cdot \nu$ (i.e., $d\mu = h\, d\nu$) is a translation invariant equilibrium state of ϕ: namely, μ is the measure satisfying the variational principle*

$$h(S,\mu) + \int \phi\, d\mu = \sup_{\rho \in \mathscr{M}_1(X_+, S)} \Big[h(S,\rho) + \int \phi\, d\rho \Big] =: P(\phi).$$

where $h(S, \cdot)$ is the Kolmogorov-Sinai entropy, the supremum is taken over the set of all translation invariant probability measures on X_+, and $P(\phi)$ is the (topological) pressure of ϕ.

2) *If there exists a translation invariant measure* $\mu \in \mathscr{M}_1(X_+, S)$ *such that* $\mu \ll \nu$*, then* μ *is an equilibrium state for* ϕ *and the Radon-Nikodym derivative* $h = \frac{d\mu}{d\nu}$ *is the eigenfunction of* $\mathscr{L}_\phi$ *with* $\mathscr{L}_\phi h = e^{P(\phi)} h$.

The proof of Proposition 4.3.1 follows standard arguments under the assumption that the eigenfunction (Radon-Nikodym density) is continuous [64]. While the transfer operator $\mathscr{L}_\phi$ is typically considered as an operator acting on continuous functions, it can be readily extended to $L^1(\nu)$. The standard proof is then adapted to integrable functions in a straightforward fashion.

By the above proposition, the condition that the transfer operator $\mathscr{L}_\phi$ has an eigenfunction for the maximal eigenvalue $\lambda = e^{P(\phi)}$ is equivalent to the existence of an equilibrium state for ϕ on X_+ having this eigenfunction as Radon-Nikodym derivative with respect to some Gibbs state $\nu \in \mathscr{G}(X_+, \phi)$.

We should note that the approach based on Proposition 4.3.1 has already been used in at least two particular cases: for Dyson potentials by Johansson, Öberg, Pollicott [47], and for product-type potentials by Cioletti, Denker, Lopes, Stadlbauer [15]. Let us now recall these results. The Dyson potential ϕ^D is given by

$$\phi^D(x) := \beta \sum_{n=1}^{\infty} \frac{x_0 x_n}{n^\alpha}, \quad x \in \{-1, 1\}^{\mathbb{Z}_+}, \tag{4.8}$$

where $\beta \geq 0$ is the inverse temperature and $\alpha > 1$ is the model parameter. The Dyson potential ϕ^D originates from the standard Dyson interaction Φ, by means of (4.7),

$$\Phi_\Lambda(\omega) := \begin{cases} -\dfrac{\beta \omega_i \omega_j}{|i-j|^\alpha}, & \text{if } \Lambda = \{i, j\} \subset \mathbb{Z},\ i \neq j; \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

If $\alpha > 2$, ϕ^D has summable variations, and, hence, the Ruelle-Walters theorem applies, and the transfer operator has a unique positive continuous eigenfunction [67]. For $\alpha \in (1, 2]$, in complete analogy to the classical (whole-line) Dyson model on $\{-1, 1\}^{\mathbb{Z}}$, phase transitions – the existence of multiple Gibbs measures – occur [46], and no general result in Dynamical Systems applies. The main result of [47] reads

Theorem 4.3.2. [47] *For* $\alpha \in \left(\frac{3}{2}, 2\right]$ *and all sufficiently small*[1] $\beta \in [0, +\infty)$ *there exists a positive continuous eigenfunction of the Perron-Frobenius transfer operator* $\mathscr{L}_{\phi^D}$.

[1] This result has been improved by the authors since we first submitted our paper [29]. Their result now reads that the continuous eigenfunction exists for all supercritical temperatures.

Remark 4.3.3. *Theorem 4.3.2 holds for all $\beta < \beta_c^1$, where β_c^1 is the critical value for a certain long-range Bernoulli percolation*[1]*. For further details, see Section 4.9 and* [47].

In [15], the authors introduced a product-type potential $\phi^P : \{-1,1\}^{\mathbb{Z}_+} \to \mathbb{R}$, with

$$\phi^P(x) := \beta \sum_{n=1}^{\infty} \frac{x_n}{n^\alpha}, \tag{4.10}$$

where again $\beta \geq 0$ and $\alpha > 1$. As above, ϕ^P has a summable variation for $\alpha > 2$, and thus the standard theory applies [67]. For each β and $\alpha > 1$, there is a unique Gibbs state ν which has the product form $\nu = \prod_{n=0}^{\infty} \lambda_n$, with $\lambda_n(1) = p_n = \frac{\exp(\beta \sum_{i=1}^n i^{-\alpha})}{2\cosh(\beta \sum_{i=1}^n i^{-\alpha})}$, while the unique equilibrium state for ϕ^P is the Bernoulli measure μ on $\{-1,1\}^{\mathbb{Z}_+}$ with $\mu([1]_0) = \lim_{n\to\infty} p_n = \frac{e^{\beta\zeta(\alpha)}}{2\cosh(\beta\zeta(\alpha))}$.

Theorem 4.3.4. [15]

(i) For $\alpha > 3/2$, the equilibrium state μ^P is absolutely continuous with respect to the Gibbs state ν^P, and thus the Perron-Frobenius transfer operator $\mathscr{L}_{\phi^{Pt}}$ has an eigenfunction $h^P := \frac{d\mu^P}{d\nu^P} \in L^1(X_+, \nu^P)$. The density h^P is continuous for $\alpha > 2$, and essentially discontinuous if $\alpha \leq 2$.

(ii) If $1 < \alpha \leq 3/2$, then μ^P and ν^P are singular measures, and therefore, the transfer operator $\mathscr{L}_{\phi^P}$ does not have an eigenfunction in $L^1(X_+, \nu^P)$.

4.4 Basic notions III: Equivalence of interactions. Dobrushin uniqueness condition

4.4.1 Equivalent specifications

We start this section with the notion of equivalent Gibbsian specifications; we follow closely the book by Georgii [39, Chapter 7].

Definition 4.4.1. *Two Gibbsian specifications $\gamma, \tilde{\gamma}$ are called equivalent, denoted by $\gamma \simeq \tilde{\gamma}$ if there exists a constant $C > 1$ such that*

$$c^{-1}\gamma_\Lambda(A\,|\,\cdot) \leq \tilde{\gamma}_\Lambda(A\,|\,\cdot) \leq c\,\gamma_\Lambda(A\,|\,\cdot)$$

for all $\Lambda \in \mathbb{Z}$ and $A \in \mathscr{F}$.

In particular, if Φ and Ψ are both UAC interactions, and $\gamma = \gamma^{\Phi}$, $\tilde{\gamma} = \gamma^{\Psi}$ are the corresponding Gibbsian specifications, then $\gamma^{\Phi} \simeq \gamma^{\Psi}$ if

$$\sup_{\Lambda \Subset \mathbb{Z}} \left\| H_{\Lambda}^{\Phi} - H_{\Lambda}^{\Psi} \right\| < \infty. \tag{4.11}$$

The sufficient condition (4.11) certainly holds if the collection of sets

$$\{\Lambda \Subset \mathbb{Z} : \Phi_{\Lambda}(\cdot) \neq \Psi_{\Lambda}(\cdot)\}$$

is finite. In this case, we say that Ψ is a ***finite perturbation*** of Φ, and vice versa.

The following theorem summarizes the properties of sets of Gibbs measures of two equivalent specifications.

Theorem 4.4.2. [39, Theorem 7.3] *Let γ and $\tilde{\gamma}$ be two equivalent specifications. Then $\mathscr{G}(\gamma) \neq \emptyset$ if and only if $\mathscr{G}(\tilde{\gamma}) \neq \emptyset$, and in this case there is an affine bijection $\mu \longleftrightarrow \tilde{\mu}$ between $\mathscr{G}(\gamma)$ and $\mathscr{G}(\tilde{\gamma})$ such that $\mu = \tilde{\mu}$ on $\mathscr{T}$. In particular, $|\mathrm{ex}\mathscr{G}(\gamma)| = |\mathrm{ex}\mathscr{G}(\tilde{\gamma})|$.*

4.4.2 Dobrushin uniqueness condition and its corollaries

This is one of the most general criteria for the uniqueness of Gibbs states. We discuss it in the framework of a general countable set $\mathbb{L}$ of sites and a configuration space $\Omega := E^{\mathbb{L}}$. Consider a uniformly absolutely convergent (UAC) interaction $\Phi = \{\Phi_{\Lambda}(\cdot) : \Lambda \Subset \mathbb{L}\}$ on Ω and let γ^{Φ} be the corresponding Gibbsian specification. For any sites $i, j \in \mathbb{L}$, define

$$C(\gamma^{\Phi})_{i,j} := \sup_{\eta_{\mathbb{L}\setminus\{j\}} = \overline{\eta}_{\mathbb{L}\setminus\{j\}}} \|\gamma_{\{i\}}^{\Phi}(\cdot|\eta) - \gamma_{\{i\}}^{\Phi}(\cdot|\overline{\eta})\|_{\infty},$$

where $\|\cdot\|_{\infty}$ is the supremum norm on $\mathscr{M}(\Omega)$ defined by $\|\tau\|_{\infty} := \sup_{B \in \mathscr{B}(\Omega)} |\tau(B)|$ for any finite signed Borel measure τ. The infinite matrix $C(\gamma^{\Phi}) := (C(\gamma^{\Phi})_{i,j})_{i,j \in \mathbb{L}}$ is called the Dobrushin interdependence matrix.

Definition 4.4.3. [18, 37, 39] *The specification γ^{Φ} satisfies the **Dobrushin uniqueness (contraction) condition** if*

$$c(\gamma^{\Phi}) := \sup_{i \in \mathbb{L}} \sum_{j \in \mathbb{L}} C(\gamma^{\Phi})_{i,j} < 1. \tag{4.12}$$

The Dobrushin uniqueness condition admits a slightly stronger — and easy to check— form:

Proposition 4.4.4. [39, Proposition 8.8] *Let $\mathbb{L}$ be any countable set, and suppose a UAC interaction $\Phi = \{\Phi_\Lambda(\cdot) : \Lambda \Subset \mathbb{L}\}$ is such that*

$$\bar{c}(\Phi) := \frac{1}{2} \sup_{i\in\mathbb{L}} \sum_{\Lambda \ni i} (|\Lambda| - 1)\,\delta(\Phi_\Lambda) < 1, \tag{4.13}$$

where $\delta(f) := \sup\{|f(\xi) - f(\eta)| : \xi, \eta \in E^{\mathbb{L}}\}$ is the variation of $f : E^{\mathbb{L}} \to \mathbb{R}$. Then γ^Φ satisfies the Dobrushin uniqueness condition.

The proof of Proposition 4.4.4 boils down to showing that for all i, j, $i \neq j$,

$$C(\gamma^\Phi)_{ij} \leq \frac{1}{2} \sum_{\{i,j\}\subset\Lambda} \delta(\Phi_\Lambda) =: \bar{C}(\Phi)_{ij},$$

and hence $c(\gamma^\Phi) = \sup_i \sum_j C(\gamma^\Phi)_{ij} \leq \sup_i \sum_j \bar{C}(\Phi)_{ij} = \bar{c}(\Phi)$. Notice that the non-negative matrix $\bar{C}(\Phi) := (\bar{C}(\Phi)_{i,j})_{i,j\in\mathbb{L}}$ is symmetric.

Note that by transitioning to condition (4.13), we are reinforcing the primary condition of this chapter, which is the Dobrushin uniqueness condition (4.12), but with a particular purpose. The condition (4.13) is stable under a *perturbation* of the underlying model/interaction Φ. Indeed, let $\Psi = \{\Psi_V\}_{V\Subset\mathbb{L}}$ be an interaction such that for $V \Subset \mathbb{L}$, either $\Psi_V = \Phi_V$ or $\Psi_V = 0$, then it is straightforward to check that $\bar{c}(\Psi) \leq \bar{c}(\Phi)$. Thus Ψ inherits the Dobrushin uniqueness condition from Φ as long as Φ satisfies (4.13).

The crucial property of the Dobrushin uniqueness condition is that it provides the uniqueness of the compatible probability measures with the specification γ^Φ. In fact, we have the following theorem.

Theorem 4.4.5. [18][37, Section 6.5][39, Chapter 8] *If γ^Φ satisfies the Dobrushin uniqueness condition (4.12), then $\left|\mathcal{G}(\gamma^\Phi)\right| \leq 1$.*

If E is compact, as is the case in this chapter, there is always at least one Gibbs state; hence, the inequality becomes an equality.

The validity of Dobrushin's criterion yields two important properties of the unique Gibbs state: concentration inequalities and explicit bounds on the decay of correlations.

The first property involves the coefficient $\bar{c}(\Phi)$, and it provides the tail bounds for the unique Gibbs measure. Let

$$\delta_k F := \sup\left\{F(\xi) - F(\eta) : \xi_j = \eta_j,\ j \in \mathbb{L} \setminus \{k\}\right\} \tag{4.14}$$

denote the oscillation of a local function $F : \Omega \to \mathbb{R}$ at a site $k \in \mathbb{L}$ and $\underline{\delta}(F) = (\delta_k F)_{k\in\mathbb{L}}$ is the oscillation vector, where $\Omega = E^{\mathbb{L}}$.

Theorem 4.4.6. [50] *Suppose Φ is a UAC interaction satisfying (4.13) and let μ_Φ be its unique Gibbs measure. Set*

$$D := \frac{4}{(1-\bar{c}(\Phi))^2}\ . \tag{4.15}$$

Then, for all $t > 0$ and every continuous function F on Ω, one has

$$\mu_\Phi\Big(\Big\{\omega\in\Omega:\ F(\omega)-\int_\Omega F\,d\mu_\Phi\ \geq\ t\Big\}\Big)\ \leq\ e^{-\frac{2t^2}{D\|\underline{\delta}(F)\|_2^2}}, \tag{4.16}$$

where $\|\underline{\delta}(F)\|_2^2 := \sum_{k\in\mathbb{L}}(\delta_k F)^2$.

It is a well-known fact that (4.16) implies that F is *sub-Gaussian* and μ_Φ has the *moment concentration bounds* as stated in the following theorem ([63, Proposition 2.5.2]).

Theorem 4.4.7. [63] *Assume that a probability measure μ_Φ satisfies (4.16) with a constant $D = D(\mu_\Phi) > 0$. Then:*

(i) μ_Φ satisfies a ***Gaussian Concentration Bound*** *with the constant D, i.e., for any continuous function F on $\Omega = E^{\mathbb{L}}$, one has*

$$\int_\Omega e^{F-\int_\Omega F\,d\mu_\Phi}\,d\mu_\Phi \leq e^{D\|\underline{\delta}(F)\|_2^2}. \tag{4.17}$$

(ii) for all $m\in\mathbb{N}$ and any continuous function F on Ω, one has

$$\int_\Omega\Big|F-\int_\Omega F\,d\mu_\Phi\Big|^m d\mu_\Phi \leq \Big(\frac{D\|\underline{\delta}(F)\|_2^2}{2}\Big)^{\frac{m}{2}} m\Gamma\Big(\frac{m}{2}\Big), \tag{4.18}$$

where Γ is Euler's gamma function.

We present the second property in the particular setup $\mathbb{L}=\mathbb{Z}$, and it involves the $\mathbb{Z}\times\mathbb{Z}$ matrix

$$D(\gamma^\Phi) = \sum_{n=0}^{\infty} C(\gamma^\Phi)^n\ . \tag{4.19}$$

The sum of the $\mathbb{Z}\times\mathbb{Z}$ matrices in the right-hand side converges due to the Dobrushin condition (4.12).

Proposition 4.4.8. [36, 39] *Consider a UAC interaction Φ on $X = E^{\mathbb{Z}}$.*

(i) Assume the specification γ^Φ satisfies the Dobrushin condition (4.12) and let μ be its unique Gibbs measure. Then, for all $f, g \in C(X)$ and $i\in\mathbb{Z}$,

$$\Big|cov_\mu(f, g\circ S^i)\Big| \leq \frac{1}{4}\sum_{k,j\in\mathbb{Z}} D(\gamma^\Phi)_{jk}\cdot\delta_k f\cdot\delta_{j-i}\,g. \tag{4.20}$$

(ii) *Suppose* Φ *satisfies (4.13) and define the non-negative symmetric* $\mathbb{Z}\times\mathbb{Z}$*-matrix by*

$$\bar{D}(\Phi) := \sum_{n\geq 0} \bar{C}(\Phi)^n \,. \tag{4.21}$$

Then,

$$\sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \bar{D}(\Phi)_{ij} \leq \frac{1}{1-\bar{c}(\Phi)}. \tag{4.22}$$

4.5 Main results I: From half-line to whole-line specifications and measures

4.5.1 From half-line to whole-line specifications

The main results of this chapter are grouped into two parts. In the first part, we consider a half-line potential $\phi : X_+ \to \mathbb{R}$ satisfying minor technical assumptions and we identify the natural translation-invariant specification on $X = E^{\mathbb{Z}}$ which extends $\overrightarrow{\gamma}^{\phi}$. In the second part we provide sufficient conditions for the Gibbs/equilibrium state on X_+ to be absolutely continuous/equivalent with respect to the corresponding half-line Gibbs state ν. Then Proposition 3.1 allows us to conclude that the transfer operator admits an eigenfunction.
First, we consider the issue of whether there is a whole-line specification naturally associated with a half-line specification $\overrightarrow{\gamma}^{\phi}$. It turns out that under a very mild condition on ϕ we obtain an affirmative answer.

Definition 4.5.1. *We say that a continuous potential* $\phi : X_+ \to \mathbb{R}$ *satisfies the* ***extensibility*** *condition if for all* $a_0, b_0 \in E$ *the sequence*

$$F_n^{a_0,b_0}(x) := S_{n+1}\phi(x_{-n}^{-1} b_0 x_1^{\infty}) - S_{n+1}\phi(x_{-n}^{-1} a_0 x_1^{\infty}) = \sum_{i=0}^{n} \big(\phi(x_{-i}^{-1} b_0 x_1^{\infty}) - \phi(x_{-i}^{-1} a_0 x_1^{\infty})\big)$$

converges uniformly on $x \in X$ *as* $n \to \infty$.

This condition has first appeared in [7] in connection to a related question of whether g-measures are also Gibbs. In terms of the extensibility condition, we can reformulate the main result of [7] as follows.

Theorem 4.5.2. [7] *Let* μ *be a* $g-$*measure for a continuous* $g-$*function* $g : X_+ \to (0,1)$, *i.e.,* μ *is translation-invariant and for* $\mu-$*almost all* $x \in X_+$, $\mu(x_0|\mathscr{F}_{[1,\infty)})(x) = g(x)$. *Then* μ *(the natural extension) is a Gibbs measure on* X *if and only if* $\log g$ *satisfies the extensibility condition.*

In [7], the authors identified several sufficient conditions for ϕ to satisfy the extensibility condition, such as the Walters condition and the so-called Good Future condition. Note that a function $\phi \in C(X_+, \mathbb{R})$ satisfies the Walters condition and the Good Future condition if $\lim_{p\to\infty} \sup_{n\geq 1} v_{n+p}(S_n\phi) = 0$ and $\sum_{k=1}^{\infty} \delta_k \phi < \infty$, respectively. Here

$$v_k(\varphi) := \sup_{x_0^{k-1}=y_0^{k-1}} |\varphi(x) - \varphi(y)| \tag{4.23}$$

is the k^{th}–variation (or the oscillation in the volume $[0, k-1]$) of a function $\varphi : X_+ \to \mathbb{R}$ and

$$\delta_k \varphi \equiv \sup_{x\in X_+, a_k, b_k \in E} \left|\varphi(x_0^{k-1} a_k x_{k+1}^{\infty}) - \varphi(x_0^{k-1} b_k x_{k+1}^{\infty})\right|. \tag{4.24}$$

is the oscillation of φ at the site k. Interesting examples of potentials satisfying the extensibility condition are the Dyson potential (4.8) and the product-type potential (4.10). Note that both potentials are in the Walters class if and only if $\alpha > 2$. However, both satisfy the Good Future condition, and thus the extensibility condition as well, for all admissible values of the parameter α, since $\delta_n(\phi^D) = \mathcal{O}(n^{-\alpha})$ and $\delta_n(\phi^P) = \mathcal{O}(n^{-\alpha})$.

Gibbsianness for potentials satisfying the extensibility condition stems from the fact that they lead to a natural whole-line translation-invariant specification. Indeed, if $\phi \in C(X_+, \mathbb{R})$ satisfies the extensibility condition the limits

$$\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\{i\}}(\sigma_i|\omega_{\{i\}^c}) := \lim_{p\to\infty} \frac{e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\sigma_i\omega_{i+1}^{\infty})}}{\sum_{\bar{\omega}_i} e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\bar{\omega}_i\omega_{i+1}^{\infty})}} \tag{4.25}$$

are well defined for all $i \in \mathbb{Z}$. In turn, they lead to a full specification.

Proposition 4.5.3. *There is a unique translation-invariant quasilocal non-null specification* $\overset{\rightleftharpoons}{\gamma}{}^{\phi}$ *on X such that the functions (4.25) become its single-site densities.*

Proof. By Proposition 4.2.2, it is sufficient to check that $\{\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\{i\}}\}_{i\in\mathbb{Z}}$ satisfy conditions (i) and (ii) of Definition 4.2.1. Clearly, $\{\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\{i\}}\}_{i\in\mathbb{Z}}$ satisfies the first condition In order to check the second condition, consider arbitrary $i, j \in \mathbb{Z}$ with $i < j$, and $p \in \mathbb{N}$ such that $i, j \gg -p$. Then a straightforward computation shows that

$$\frac{e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\alpha_i\omega_{i+1}^{j-1}\alpha_j\omega_{j+1}^{\infty})}}{\sum_{\beta_{i,j}} e^{S_{j+p+1}\phi(\omega_{-p}^{i-1}\beta_i\omega_{i+1}^{j-1}\beta_j\omega_{j+1}^{\infty}) + S_{i+p+1}\phi(\omega_{-p}^{i-1}\beta_i\omega_{i+1}^{j-1}\alpha_j\omega_{j+1}^{\infty}) - S_{j+p+1}\phi(\omega_{-p}^{i-1}\beta_i\omega_{i+1}^{j-1}\alpha_j\omega_{j+1}^{\infty})}} =$$

$$= \frac{e^{S_{j+p+1}\phi(\omega_{-p}^{i-1}\alpha_i\omega_{i+1}^{j-1}\alpha_j\omega_{j+1}^{\infty})}}{\sum_{\beta_{i,j}} e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\beta_i\omega_{i+1}^{j-1}\beta_j\omega_{j+1}^{\infty}) + S_{j+p+1}\phi(\omega_{-p}^{i-1}\alpha_i\omega_{i+1}^{j-1}\beta_j\omega_{j+1}^{\infty}) - S_{i+p+1}\phi(\omega_{-p}^{i-1}\alpha_i\omega_{i+1}^{j-1}\beta_j\omega_{j+1}^{\infty})}}.$$

The limit $p \to \infty$ of this identity yields condition (ii).

Finally, note that for all $i \in \mathbb{Z}$,

$$\overset{\rightleftharpoons}{\gamma}^{\phi}_{\{0\}}((S^i\sigma)_0|S^i(\omega)_{\{0\}^c}) = \lim_{p\to\infty} \frac{\exp(S_{p+1}\phi(\omega_{i-p}^{i-1}\sigma_i\omega_{i+1}^{\infty}))}{\sum_{\bar\sigma_i}\exp(S_{p+1}\phi(\omega_{i-p}^{i-1}\bar\sigma_i\omega_{i+1}^{\infty}))} = \overset{\rightleftharpoons}{\gamma}^{\phi}_{\{i\}}(\sigma_i|\omega_{\{i\}^c})$$

and hence $\overset{\rightleftharpoons}{\gamma}^{\phi}$ is a translation-invariant specification. □

Example 4.5.4. *Let us illustrate the construction in the previous proof with the Dyson potential ϕ defined in (4.8). In this case a straightforward computation shows that for all $p \in \mathbb{N}$, $\beta \ge 0$, $\sigma_0 \in \{-1,1\}$ and $\omega \in \{-1,1\}^{\mathbb{Z}}$,*

$$\frac{\exp\Big(S_{p+1}\phi(\omega_{-p}^{-1}\sigma_0\omega_1^{\infty})\Big)}{\exp\big(S_{p+1}\phi(\omega_{-p}^{-1}\bar\sigma_0\omega_1^{\infty})\big)} = \frac{\exp\Big(\beta\sum_{k=-p}^{-1}\frac{\sigma_0\omega_k}{|k|^\alpha}+\beta\sum_{k=1}^{+\infty}\frac{\sigma_0\omega_k}{|k|^\alpha}\Big)}{\exp\Big(\beta\sum_{k=-p}^{-1}\frac{\bar\sigma_0\omega_k}{|k|^\alpha}+\beta\sum_{k=1}^{+\infty}\frac{\bar\sigma_0\omega_k}{|k|^\alpha}\Big)}.$$

Thus

$$\frac{\exp\Big(S_{p+1}\phi(\omega_{-p}^{-1}\sigma_0\omega_1^{\infty})\Big)}{\exp\big(S_{p+1}\phi(\omega_{-p}^{-1}\sigma_0\omega_1^{\infty})\big)+\exp\big(S_{p+1}\phi(\omega_{-p}^{-1}\bar\sigma_0\omega_1^{\infty})\big)} \to \gamma_{\{0\}}(\sigma_0|\omega_{-\infty}^{-1},\omega_1^{\infty})$$

as $p \to \infty$, where

$$\gamma_{\{0\}}(\sigma_0|\omega_{-\infty}^{-1},\omega_1^{\infty}) = \frac{\exp\Big(\beta\sum_{k=1}^{+\infty}\frac{\sigma_0(\omega_k+\omega_{-k})}{k^\alpha}\Big)}{\exp\Big(\beta\sum_{k=1}^{+\infty}\frac{\sigma_0(\omega_k+\omega_{-k})}{k^\alpha}\Big)+\exp\Big(\beta\sum_{k=1}^{+\infty}\frac{\bar\sigma_0(\omega_k+\omega_{-k})}{k^\alpha}\Big)}.$$

4.5.2 Whole-line Gibbs measures as limits of half-line ones

Once the correspondence between $\vec{\gamma}^{\phi}$ and $\overset{\rightleftharpoons}{\gamma}^{\phi}$ is established, it is interesting to understand the relation between $\mathscr{G}(X_+,\vec{\gamma}^{\phi})\big(=\mathscr{G}(X_+,\phi)\big)$ and $\mathscr{G}(X,\overset{\rightleftharpoons}{\gamma}^{\phi})$.

Suppose ν is a Gibbs probability measure on X_+ for some potential ϕ satisfying the extensibility condition, and consider an arbitrary measure ρ on $X_- = E^{-\mathbb{N}}$(e.g. a uniform Bernoulli measure), and consider the measure $\mu_0 = \rho \times \nu$ on $X = X_- \times X_+$.

For any $n \ge 0$, let

$$\mu_n = \mu_0 \circ S^{-n},$$

This is a sequence of Borel probability measures $\{\mu_n\}_{n\ge0}$ on the compact metric space X, and hence it has weak*-converging subsequences $\{\mu_{n_k}\}$. It turns out that their limits are bona fide whole-line Gibbs measures.

Theorem 4.A. *Suppose ϕ satisfies the extensibility condition and $\nu \in \mathcal{G}(X_+, \phi)$. Consider $\mu_0 := \rho \times \nu$, where ρ is any probability measure on X_-. Assume that a subsequence $\{\mu_{n_k} = \mu_0 \circ S^{-n_k}\}_{k \geq 0}$ converges to a probability measure μ in the weak* topology as $k \to \infty$. Then for μ-almost all x:*

$$\mu(x_0|x_{-\infty}^{-1}, x_1^{\infty}) = \overrightleftharpoons{\gamma}^{\phi}_{\{0\}}(x_0|x_{-\infty}^{-1}, x_1^{\infty}) = \lim_{n\to\infty} \frac{\exp\left(S_{n+1}\phi(x_{-n}^{-1}x_0x_1^{\infty})\right)}{\sum_{\bar{a}_0}\exp\left(S_{n+1}\phi(x_{-n}^{-1}\bar{a}_0x_1^{\infty})\right)}.$$

Hence μ is a whole-line Gibbs measure for the whole-line specification defined by the kernels $\overrightleftharpoons{\gamma}^{\phi}_{\{0\}}$.

Now we turn to translation invariant measures.

Theorem 4.B. *Assume the conditions of Theorem 4.A, let μ be a weak* limit point of the following sequence of measures*

$$\tilde{\mu}_n := \frac{1}{n}\sum_{i=0}^{n-1}\mu_0 \circ S^{-i}.$$

Then μ is a translation-invariant Gibbs measure for $\overrightleftharpoons{\gamma}^{\phi}$, i.e., $\mu \in \mathcal{G}_S(\overrightleftharpoons{\gamma}^{\phi}) := \mathcal{G}(\overrightleftharpoons{\gamma}^{\phi}) \cap \mathcal{M}_1(X, S)$.

We shall give the proofs of Theorem 4.A and Theorem 4.B in Section 4.8.

4.6 Main results II: Eigenfunctions of transfer operators and absolute continuity of Gibbs measures

As mentioned in the introduction, we can show the existence of an eigenfunction if we can show that the equilibrium state μ is absolutely continuous with respect to the half-line non-translation invariant Gibbs measure ν. This approach has recently been used by Johansson, Öberg and Pollicott [47] to show Theorem 4.3.2. We also use this approach to prove the following theorem.

Theorem 4.C. *Let ϕ be the Dyson potential (4.8). Suppose $\alpha > 1$ and $\beta > 0$ is sufficiently small. Then,*

(i) the half-line Dyson model ϕ on $X_+ = \{\pm 1\}^{\mathbb{Z}_+}$ admits a unique equilibrium state μ_+ and Gibbs state ν;

(ii) for all $\alpha > 1$, μ_+ is equivalent to ν, i.e., $\mu_+ \ll \nu$ and $\nu \ll \mu_+$, and thus the Perron-Frobenius transfer operator $\mathcal{L}_\phi$ has an eigenfunction in $L^1(X_+, \nu)$;

(iii) *furthermore, if* $\alpha > \frac{3}{2}$, *there exists a continuous version of the Radon-Nikodym density* $\frac{d\mu_+}{d\nu}$, *and thus the Perron-Frobenius transfer operator* $\mathscr{L}_\phi$ *has a continuous eigenfunction.*

Remark 4.6.1. *In fact, we prove Theorem 4.C under the Dobrushin uniqueness condition (c.f. (4.12)-(4.13)). Therefore, the parameter* $\beta \geq 0$ *should be sufficiently small so that the corresponding whole-line Dyson interaction on* $X = \{\pm 1\}^{\mathbb{Z}}$ *satisfies the Dobrushin uniqueness condition (4.13).*

We will show a much more general statement (Theorem 4.E below), from which one can easily obtain the first two parts of Theorem 4.C. Our method of proof of statement (iii) differs from that of [47], and can be generalised to other models. The proof of Theorem 4.C will be presented in Section 4.8.

4.6.1 From $C(X_+)$ to the space of interactions on X

It is clear from Section 4.5 that every (half-line) translation-invariant UAC interaction $\vec{\Phi}$ (i.e., $\vec{\Phi}_\Lambda \circ S = \vec{\Phi}_{\Lambda+1}$) on X_+, yields a potential $\phi \in C(X_+)$ with $\phi = -\sum_{0 \in V \Subset \mathbb{Z}_+} \vec{\Phi}_V$, such that $\gamma^\phi = \gamma^{\vec{\Phi}}$.

In the opposite direction, it is known that every potential ϕ yields an equivalent interaction $\vec{\Phi}$ on X_+ [57] which in its turn can be extended by translations to an interaction Φ on X. This extension, however, is only guaranteed to belong to the so-called $\mathscr{B}_0(X)$ class —-i.e., is such that $\sum_{0 \in V \Subset \mathbb{Z}_+} \frac{1}{|V|} \|\Phi_V\|_\infty < \infty$— but it may fail to be UAC.

The determination of necessary and sufficient conditions for ϕ to yield a UAC potential Φ on X is an open problem that we do not address here. Rather, in the sequel we determine a class of potentials ϕ that admit a translation invariant UAC interaction Φ on $\mathbb{Z}$ satisfying

$$\phi = -\sum_{0 \in V \Subset \mathbb{Z}_+} \Phi_V. \tag{4.26}$$

The following proposition shows that potentials in this class satisfy the extensibility condition.

Proposition 4.6.2. *Let* $\phi \in C(X_+)$ *such that there exists a translation invariant UAC interaction* Φ *on* $\mathbb{Z}$ *satisfying* $\phi = -\sum_{0 \in V \Subset \mathbb{Z}_+} \Phi_V$. *Then* ϕ *satisfies the extensibility condition. Furthermore,* $\overset{\rightleftharpoons}{\gamma}^\phi = \gamma^\Phi$.

Proof. Note that

$$S_{n+1}\phi = -\sum_{V\cap[0,n]\neq\emptyset;\ V\Subset\mathbb{Z}_+} \Phi_V.$$

Thus since the interaction is translation-invariant ($\Phi\circ S^{-n} = \Phi_{V-n}$)

$$S_{n+1}\phi\circ S^{-n} = -\sum_{V\cap[-n,0]\neq\emptyset;\ V\Subset[-n,\infty]} \Phi_V.$$

(The LHS of the above equation should be considered on X, otherwise, S^{-n} is not defined on X_+.)

Pick any $\xi,\eta\in X$, such that $\xi_j=\eta_j$ if $j\neq 0$. Then

$$\begin{aligned} S_{n+1}\phi(\xi_{-n}^{\infty}) - S_{n+1}\phi(\eta_{-n}^{\infty}) &= \sum_{V\cap[-n,0]\neq\emptyset;\ V\Subset[-n,\infty]} (\Phi_V(\eta)-\Phi_V(\xi)) \\ &= \sum_{0\in V\Subset[-n,\infty]} (\Phi_V(\eta)-\Phi_V(\xi)). \end{aligned}$$

Since Φ is UAC, $\sum_{0\in V\Subset[-n,\infty]}[\Phi_V(\xi)-\Phi_V(\eta)]$ converges uniformly to $H^{\Phi}_{\{0\}}(\xi) - H^{\Phi}_{\{0\}}(\eta) = \sum_{0\in V\Subset\mathbb{Z}}(\Phi_V(\xi)-\Phi_V(\eta))$ as $n\to\infty$. Thus it is also clear that $\overset{\rightleftharpoons}{\gamma}^{\phi} = \gamma^{\Phi}$. □

Example 4.6.3. *Consider the Dyson potential ϕ (c.f. (4.8)) with the state space $E=\{-1,1\}$, and the standard (whole-line) Dyson interaction Φ on $\mathbb{Z}$ (c.f. (4.9)). Then it is easy to see that $\phi = -\sum_{0\in\Lambda\Subset\mathbb{Z}_+}\Phi_\Lambda$.*

4.6.2 Decoupling across the origin

To every UAC interaction Φ on $\mathbb{Z}$ we can associate a sequence $\Psi^{(k)}$ of interactions obtained by removing all bonds linking $-\mathbb{N}$ and $\mathbb{Z}_+$ and adding them one at a time. Formally, we consider the family

$$\mathscr{A} = \left\{\Lambda\Subset\mathbb{Z} : \min(\Lambda)<0,\ \max\Lambda\geq 0\right\}.$$

indexed according to some arbitrary order: $\mathscr{A}=\{\Lambda_1,\Lambda_2,\ldots\}$. Then define, for each $k\in\mathbb{Z}_+$,

$$\Psi^{(k)}_\Lambda = \begin{cases} 0, & \Lambda\in\{\Lambda_i : i\geq k+1\}, \\ \Phi_\Lambda, & \text{otherwise.}\end{cases} \tag{4.27}$$

In particular, $\Psi^{(0)}$ has no interaction between the left and right half lines. Clearly, all the constructed interactions are UAC and in addition have the following properties:

Remark 4.6.4. *1) Every $\Psi^{(k)}$ in (4.27) is a local (finite) perturbation of $\Psi^{(0)}$. Moreover, the sequence $\Psi^{(k)}$ tends to Φ as $k\to\infty$, in the sense that $\Psi^{(k)}_\Lambda \rightrightarrows \Phi_\Lambda$ for all $\Lambda \Subset \mathbb{Z}$.*

2) For any finite volume V,

$$||H_V^{\Psi^{(k)}} - H_V^{\Phi}||_\infty \le \sum_{\substack{\Lambda_j\cap V\neq\emptyset\\ j\ge k}} ||\Phi_{\Lambda_j}||_\infty \xrightarrow[k\to\infty]{} 0.$$

3) The specifications $\gamma^{\Psi^{(k)}}$ converge to γ^Φ as $k\to\infty$. More precisely, for all $B\in\mathscr{F}$ and $V\Subset\mathbb{Z}$,

$$\gamma_V^{\Psi^{(k)}}(B|\omega) \xrightarrow[k\to\infty]{} \gamma_V^{\Phi}(B|\omega) \text{ uniformly in the boundary conditions } \omega\in X.$$

4) In addition, if $\nu^{(k)}$ is a Gibbs measure for $\Psi^{(k)}$, then by Lemma 4.2, any weak-limit point, μ of the sequence $\{\nu^{(k)}\}_{k\ge0}$ is a Gibbs measure for the potential Φ.*

Another important observation applies to the interaction $\Psi^{(0)}$. As it is constructed from Φ by removing all interaction between $-\mathbb{N}$ and $\mathbb{Z}_+$, the corresponding specification $\gamma^{\Psi^{(0)}}$ is of product type [39, Example 7.18]: $\gamma^{\Psi^{(0)}} = \gamma^{\Phi^-}\times\gamma^{\Phi^+}$, where Φ^- and Φ^+ are the restrictions of Φ respectively to the negative and positive half-lines. Thus, the extreme Gibbs measures also factorize [39, Example 7.18]:

$$\mathrm{ex}\,\mathscr{G}(\gamma^{\Psi^{(0)}}) = \{\nu_-\times\nu_+ : \nu_-\in \mathrm{ex}\,\mathscr{G}(X_-,\gamma^{\Phi^-}),\ \ \nu_+\in\mathrm{ex}\,\mathscr{G}(X_+,\gamma^{\Phi^+})\}. \tag{4.28}$$

4.6.3 Absolute continuity

If $\Psi^{(0)}$ does not exhibit phase transitions, i.e., has a unique Gibbs measure, then by Theorem 4.4.2, all the interactions $\Psi^{(k)}$, $k\ge1$, do not exhibit phase transitions as well. Let us denote by $\nu^{(k)}$ the unique Gibbs state for $\Psi^{(k)}$.

Theorem 4.6.5. *If the interaction Φ satisfies the Dobrushin uniqueness criterion (4.13), so do all interactions $\Psi^{(k)}$ and, furthermore, $\nu^{(k)}$ and $\nu^{(0)}$ are equivalent with*

$$\frac{d\nu^{(k)}}{d\nu^{(0)}} = \frac{e^{-\sum_{i=1}^k\Phi_{\Lambda_i}}}{\int_X e^{-\sum_{i=1}^k\Phi_{\Lambda_i}}\,d\nu^{(0)}}\quad,\quad \frac{d\nu^{(0)}}{d\nu^{(k)}} = \frac{e^{\sum_{i=1}^k\Phi_{\Lambda_i}}}{\int_X e^{\sum_{i=1}^k\Phi_{\Lambda_i}}\,d\nu^{(k)}}. \tag{4.29}$$

Proof. The hereditary character of Dobrushin's criterion follows from the obvious fact that, for all $k\in\mathbb{Z}_+$, $\bar c(\Psi^{(k)})\le\bar c(\Phi)$.

The proof of (4.29) follows, telescopically, from the partial Radon-Nikodym derivatives

$$\frac{d\,\nu^{(k)}}{d\,\nu^{(k-1)}} = \frac{e^{-\Phi_{\Lambda_k}}}{\int e^{-\Phi_{\Lambda_k}}\,d\,\nu^{(k-1)}} \quad , \quad \frac{d\,\nu^{(k-1)}}{d\,\nu^{(k)}} = \frac{e^{\Phi_{\Lambda_k}}}{\int e^{\Phi_{\Lambda_k}}\,d\,\nu^{(k)}}\,, \tag{4.30}$$

which is a particular case of the following elementary lemma. □

Lemma 4.6.6. *Let Ψ be a UAC interaction that does not exhibit phase transitions. Consider a perturbed interaction of the form $\bar\Psi = \Psi + \mathfrak{P}$ with $\mathfrak{P}$ a finite interaction supported on $A \Subset \mathbb{Z}$. If $\{\nu\} = \mathscr{G}(\Psi)$ and $\bar\nu \in \mathscr{G}(\bar\Psi)$, then $\bar\nu \ll \nu$, and*

$$\frac{d\,\bar\nu}{d\,\nu} = \frac{e^{-H_A^{\mathfrak{P}}}}{\int_X e^{-H_A^{\mathfrak{P}}}\,d\,\nu}. \tag{4.31}$$

Proof. First, note that by Theorem 4.4.2, the model $\bar\Psi$ does not exhibit phase transitions since it is a finite perturbation of Ψ. Thus $\{\bar\nu\} = \mathscr{G}(\bar\Psi)$. By uniqueness of the Gibbs state, we have that, for both interactions, the Gibbs measures are achieved through limits

$$\gamma_\Lambda(\cdot) \xrightarrow[\Lambda\uparrow\mathbb{Z}]{*} \nu \text{ and } \bar\gamma_\Lambda(\cdot) \xrightarrow[\Lambda\uparrow\mathbb{Z}]{*} \bar\nu \tag{4.32}$$

where $\gamma := \gamma^{\Psi}$ and $\bar\gamma := \gamma^{\bar\Psi}$ are the Gibbsian kernels corresponding to the interactions Ψ and $\bar\Psi$ and free boundary conditions (that is, considering only bonds within Λ). Consider any $V \subset \Lambda \Subset \mathbb{Z}$. As $H_\Lambda^{\bar\Psi} = H_A^{\mathfrak{P}} + H_\Lambda^{\Psi}$, for any cylindrical event $[\sigma_V]$, we have that

$$\bar\gamma_\Lambda(\sigma_V) = \frac{\sum_{\xi_\Lambda} \mathbf{1}_{\sigma_V}\, e^{-H_A^{\mathfrak{P}}(\xi_A)}\, e^{-H_\Lambda^{\Psi}(\xi_\Lambda)}}{\sum_{\xi_\Lambda} e^{-H_A^{\mathfrak{P}}(\xi_A)}\, e^{-H_\Lambda^{\Psi}(\xi_\Lambda)}}.$$

Dividing top and bottom by $\sum_{\xi_\Lambda} e^{-H_\Lambda^{\Psi}(\xi_\Lambda)}$ we get

$$\bar\gamma_\Lambda(\sigma_V) = \frac{\gamma_\Lambda\big(\mathbf{1}_{\sigma_V}\, e^{-H_A^{\mathfrak{P}}}\big)}{\gamma_\Lambda\big(e^{-H_A^{\mathfrak{P}}}\big)}.$$

Taking limits over Λ and using (4.32) we obtain

$$\bar\nu(\mathbf{1}_{\sigma_V}) = \nu(\mathbf{1}_{\sigma_V}\, f)$$

where f is, precisely, the right-hand side of (4.31). This concludes the proof because cylindrical events uniquely determine the measures. □

Now we turn to the restrictions $\nu_+^{(k)}$ of measures $\nu^{(k)}$ to the half-line $\mathbb{Z}_+$, i.e., for $B \in \mathscr{F}_+$, $\nu_+^{(k)}(B) := \nu^{(k)}(X_- \times B)$. Note that for the cylindrical sets $[\sigma_\Lambda]$, $\Lambda \Subset \mathbb{Z}_+$, one has $\nu_+^{(k)}([\sigma_\Lambda]) = \nu^{(k)}([\sigma_\Lambda])$. Similarly, one can define the restrictions to the left half-line $-\mathbb{N}$. If the model $\Psi^{(0)}$ does not exhibit phase transitions, then $\nu^{(0)}$ is a product measure, i.e., $\nu^{(0)} = \nu_-^{(0)} \times \nu_+^{(0)}$ (c.f. (4.28)), where $\nu_-^{(0)}$ is the unique Gibbs measure for the interaction Φ^- on X_-, and $\nu_+^{(0)}$ is the unique Gibbs measure for Φ^+ on X_+. If this is the case, then one can compute the Radon-Nikodym density $f_+^{(k)} := \dfrac{d\nu_+^{(k)}}{d\nu_+^{(0)}}$, in fact, for all $\sigma \in X_+$, $k \in \mathbb{Z}_+$,

$$f_+^{(k)}(\sigma) = \frac{\int_{X_-} e^{-\sum_{j=1}^{k} \Phi_{\Lambda_j}(\xi,\sigma)} \nu_-^{(0)}(d\xi)}{\int_{X_+}\int_{X_-} e^{-\sum_{j=1}^{k} \Phi_{\Lambda_j}(\xi,\zeta)} \nu_-^{(0)}(d\xi)\nu_+^{(0)}(d\zeta)}. \tag{4.33}$$

By Theorem 4.6.5, all the measures $\nu^{(k)}$, $k \geq 0$ are equivalent to $\nu^{(0)}$. However, it is not clear whether the weak*-limit points of the sequence $\{\nu^{(k)}\}_{k\in\mathbb{Z}_+}$ are absolutely continuous with respect to $\nu^{(0)}$ or not. The following theorem provides sufficient conditions.

Theorem 4.D. *Assume that Φ satisfies the Dobrushin uniqueness condition (4.13). Suppose the family $\{f^{(k)}\}_{k\in\mathbb{N}}$ is uniformly integrable in $L^1(\nu^{(0)})$. Then the weak* limit point of the sequence $\{\nu^{(k)}\}$ is a Gibbs measure for Φ and absolutely continuous with respect to $\nu^{(0)}$.*

The next theorem is the main theorem of this section and it provides sufficient conditions for uniform integrability of the family $\{f^{(k)}\}_{k\in\mathbb{N}}$, and thus absolute continuity of the weak* limit points with respect to $\nu^{(0)}$.

Theorem 4.E. *Assume the following*

1) *the interaction Φ satisfies the Dobrushin uniqueness condition (4.13);*
2) $\displaystyle\sum_{k=1}^{\infty} \|\underline{\delta}(\Phi_{\Lambda_k})\|_2^2 < \infty$;
3) $\displaystyle\sum_{k=1}^{\infty} \rho_k < \infty$, *where* $\displaystyle\rho_k := \sup_{n\in\mathbb{N}} \left| \int_X \Phi_{\Lambda_k} d\nu^{(n)} \right|$.

Then $\Psi^{(0)}$ does not exhibit phase transitions, and $\{f^{(k)} : k \in \mathbb{N}\}$ is uniformly integrable in $L^1(\nu^{(0)})$.

Remark 4.6.7. *Note that the third, summability condition in Theorem 4.E is important. To illustrate this, consider the product-type potential ϕ^P defined by (4.10).*

One can readily verify that the potential ϕ^P coincides with the half-line mean energy at 0 *— that is, $\phi^P = -\sum_{0\in V\Subset\mathbb{Z}_+} \Phi_V$ — for the translation-invariant UAC interaction:*

$$\Phi_\Lambda(\omega) = \begin{cases} -\dfrac{\beta\omega_j}{|i-j|^\alpha}, & \Lambda = \{i,j\}\subset\mathbb{Z},\ j>i, \\ 0, & \textit{otherwise}\,. \end{cases}$$

This interaction Φ, which is not spin-flip invariant, satisfies the first and second conditions of Theorem 4.E for β sufficiently small. In fact, for all $\beta \geq 0$, both Dobrushin interdependence matrices $C(\gamma^\Phi)$ and $C(\gamma^{\phi^P})$ are zero matrices, thus, both specifications (not the interactions) satisfy the condition (4.12) for all β's. However, Φ does not satisfy the last condition of the theorem. Indeed, as the unique Gibbs measure μ for Φ is Bernoulli with

$$\mu([1]_0) = \frac{e^{\beta\zeta(\alpha)}}{2\cosh(\beta\zeta(\alpha))},$$

one has

$$\int_X \sigma_0\, d\mu = \tanh(\beta\zeta(\alpha)) > 0.$$

Therefore, for all $i\in\mathbb{N}$, $j\in\mathbb{Z}_+$,

$$\rho_{-i,j} \geq \left|\int_X \Phi_{\{-i,j\}}\, d\mu\right| = \frac{\beta}{(i+j)^\alpha}\tanh(\beta\zeta(\alpha)),$$

and thus the sum $\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\rho_{-i,j}$ diverges, where $\rho_{-i,j} := \sup_{n\in\mathbb{N}}\left|\int_X \Phi_{\{-i,j\}}\, d\nu^{(n)}\right|$. Hence, Theorem 4.E does not apply to this particular model. Moreover, as asserted in the second part of Theorem 4.3.4, for $1<\alpha\leq\frac{3}{2}$, the conclusion of Theorem 4.E fails to hold for this model.

The combination of Theorems 4.E and 4.D implies that if Φ satisfies the conditions of the former, the unique Gibbs measure $\mu\in\mathscr{G}(\Phi)$ is absolutely continuous with respect to $\nu^{(0)}$. Note that, as a consequence, Theorem 4.E can not be true for the previous interaction if $1<\alpha\leq 3/2$. Indeed, the second part of Theorem 4.3.4 directly implies that the measures μ and $\nu^{(0)}$ are singular for those values of α.

It is natural to ask whether, reciprocally, $\nu^{(0)}$ is also absolutely continuous with respect to μ. The answer is affirmative modulo conditions comparable to those of Theorem 4.E. The argument resorts also to a sequence of interactions but this time obtained by removing one by one the bonds in the volumes in $\mathscr{A} = \{\Lambda\Subset\mathbb{Z}:$

$\min\Lambda < 0, \max\Lambda \geq 0\}$ instead of adding them to $\Psi^{(0)}$. More precisely, at each step $k \in \mathbb{Z}_+$, we construct a new interaction $\Phi^{(k)}$ as follows:

$$\Phi_\Lambda^{(k)} = \begin{cases} 0, & \Lambda \in \{\Lambda_i : 1 \leq i \leq k\}, \\ \Phi_\Lambda, & \text{otherwise.} \end{cases}$$

Then as previously, for all $k \in \mathbb{Z}_+$, the matrix $\bar{C}(\Phi^{(k)})$ is dominated by $\bar{C}(\Phi)$, thus if Φ satisfies the Dobrushin uniqueness condition (4.13) so do all the interactions $\Phi^{(k)}$. Furthermore, Remark 4.6.4 still remains valid interchanging Φ and $\Psi^{(0)}$. Thus the sequence $\mu^{(k)}$ of unique Gibbs measures for each $\Phi^{(k)}$ converges in weak* sense to a Gibbs measure μ for $\Psi^{(0)}$. In addition, by Lemma 4.6.6, all the measures $\mu^{(k)}$ are equivalent to μ and their Radon-Nikodym derivatives are given by

$$f_\mu^{(k)} := \frac{d\mu^{(k)}}{d\mu} = \frac{e^{\sum_{i=1}^k \Phi_{\Lambda_i}}}{\int_X e^{\sum_{i=1}^k \Phi_{\Lambda_i}} d\mu}.$$

To conclude, we present the following analogue of Theorem 4.E which can be proven in a similar way.

Theorem 4.F. *Assume that*

1) *the interaction Φ satisfies (4.13);*

2) $\sum_{k=1}^{\infty} ||\underline{\delta}(\Phi_{\Lambda_k})||_2^2 < \infty$*;*

3) $\sum_{k=1}^{\infty} \rho_k^\mu < \infty$*, where* $\rho_k^\mu := \sup_{n\in\mathbb{N}} \left| \int_X \Phi_{\Lambda_k} d\mu^{(n)} \right|$.

Then $\Psi^{(0)}$ does not exhibit phase transitions, and $\{f_\mu^{(k)} : k \in \mathbb{N}\}$ is uniformly integrable in $L^1(\mu)$. In particular, $\nu^{(0)}$ is absolutely continuous with respect to μ.

We postpone the proofs of Theorem 4.D and Theorem 4.E until Section 4.8.

4.7 Application: Dyson model

This was our motivating example. Let us recall that the Dyson potential ϕ is defined on the half-line configuration space $X_+ = \{-1, +1\}^{\mathbb{Z}_+}$ as

$$\phi(x) = \beta \sum_{n=1}^{\infty} \frac{x_0 x_n}{n^\alpha}, \quad x = (x_0, x_1, \ldots) \in X_+,$$

for some $\alpha > 1$. As mentioned in Example 4.6.3, this potential is related to the whole-line Dyson model Φ, defined in (4.9), by $\phi = -\sum_{0 \in V \Subset \mathbb{Z}_+} \Phi_V$, and thus $\overset{\rightleftharpoons}{\gamma}^{\phi} = \gamma^{\Phi}$. Hence by Theorem 4.A, for all $\nu \in \mathscr{G}(\phi)$, any weak* accumulation point of the sequence $\{\nu \circ S^{-n}\}$ is a restriction of some $\mu \in \mathscr{G}(\Phi)$ to $\mathbb{Z}_+$.

The Dyson potential satisfies the Good Future condition because its oscillations (4.24) $\delta_n(\phi) = \mathscr{O}(n^{-\alpha})$ are summable. As a consequence, it satisfies the extensibility condition. For $\alpha > 2$, furthermore, its variations $v_n(S_n\phi) := \sup\{S_n\phi(x) - S_n\phi(y) : x_0^{n-1} = y_0^{n-1}\}$ are also summable and, therefore, the standard theory applies [67]. The case of $\alpha \in (1,2]$ is significantly more subtle, and its theory is less developed. Its most recent advance is (Theorem 4.3.2 above), obtained by Johansson, Öberg and Pollicott [47] through the random-cluster representation for the whole-line Dyson model Φ (c.f. (4.9)).

Now, we recall some important properties of the Dyson model and its phase diagram. It is clear that the interaction Φ is translation and spin-flip invariant. One of the most interesting properties of the associated Gibbs measures is that all the measures in $\mathscr{G}(\Phi)$ are translation-invariant for all the values of the parameters α and β. However, this statement is not true as regards spin-flip invariance. In fact, there is only one spin-flip invariant Gibbs measure in $\mathscr{G}(\Phi)$ which is the cause of the phase transitions for high β's. By applying FKG inequalities, it can be shown that the (weak*) limits $\mu^+ := \lim_{\Lambda \uparrow \mathbb{Z}} \gamma_\Lambda^\Phi(\cdot|+)$ and $\mu^- := \lim_{\Lambda \uparrow \mathbb{Z}} \gamma_\Lambda^\Phi(\cdot|-)$ exist, and both are extremal. In fact, μ^- and μ^+ are the only extremal elements of $\mathscr{G}(\Phi)$ since they stochastically dominate all other Gibbs measures of the model i.e. $\int_X f\,d\mu^- \le \int_X f\,d\mu \le \int_X f\,d\mu^+$ for any Gibbs state $\mu \in \mathscr{G}(\Phi)$ and for any non-decreasing function $f \in C(X)$. The phase diagram of this model is in many ways similar to the phase diagram of the two-dimensional nearest-neighbour Ising model, in fact, we have the following result.

Theorem 4.7.1. [23, 38, 56] *For all $\alpha \in (1,2]$, there exists a critical temperature $\beta_c(\alpha) \in (0,+\infty)$ such that there is no phase transition for all $\beta \in [0,\beta_c(\alpha))$ (i.e. $\mu^+ = \mu^-$), and there is a phase transition for all $\beta \in (\beta_c(\alpha),+\infty)$ (i.e. $\mu^+ \neq \mu^-$). Furthermore, for all the values of β, $\mathscr{G}(\Phi) = [\mu^-;\mu^+]$.*

Note that if $\alpha > 2$, then for all the values of the inverse tempereature $\beta > 0$, Φ does not exhibit phase transitions, i.e., $|\mathscr{G}(\Phi)| = 1$.

In [46], the authors proved that the phase diagram of the half-line Dyson model ϕ is similar to the phase diagram of the whole-line Dyson model. In fact, they showed that for all $\alpha \in (1,2]$, there exists β_c^+, such that the for all $\beta \in (0,\beta_c^+)$, there exists a unique half-line Gibbs state for ϕ, and for $\beta > \beta_c^+$, there exist multiple Gibbs states. The authors also conjectured that the critical values β_c^+ and β_c of the half and whole-line Dyson models are, in fact, equal $\beta_c^+ = \beta_c$.

4.8 Proofs of the main results

4.8.1 Proofs of Theorems 4.A and 4.B

We shall use the following two simple lemmas in the proof of Theorem 4.A.

Lemma 4.8.1. [39, Remark 5.10] *Let γ be a specification on X and $\mu \in \mathscr{G}(\gamma)$. Then $\mu \circ S^{-1}$ is consistent with the specification $\gamma^{(1)}$, where*

$$\gamma^{(1)}_{\Lambda}(B|\omega) := \gamma_{\Lambda+1}(S^{-1}(B)|S^{-1}(\omega)),\ \forall B \in \mathscr{F},\ \omega \in X,\ \forall \Lambda \Subset \mathbb{Z}.$$

Lemma 4.8.2. [39, Theorem 4.17] *Suppose γ and $\gamma^{(n)}$, $n \geq 1$ are specifications on $X = E^{\mathbb{Z}}$. Assume that $\gamma^{(n)}$ converge uniformly to γ as $n \to \infty$, in the sense that for all $\Lambda \Subset \mathbb{Z}$ and all $\sigma \in X$,*

$$\gamma_{\Lambda}(\sigma_{\Lambda}|\omega_{\Lambda^c}) = \lim_{n\to\infty} \gamma^{(n)}_{\Lambda}(\sigma_{\Lambda}|\omega_{\Lambda^c})$$

uniformly in the boundary condition $\omega \in X$. Take $\mu^{(n)} \in \mathscr{G}(\gamma^{(n)})$, and assume that the sequence $\mu^{(n)}$ converges to some $\mu \in \mathscr{M}_1(X)$ in the weak topology. Then $\mu \in \mathscr{G}(\gamma)$.*

Proof of Theorem 4.A. We, first, prove the theorem in the case where ρ is the uniform Bernoulli measure on X_-. Let us consider a family of functions $\gamma^{(0)}_{\{n\}} : E \times E^{\mathbb{Z}\setminus\{n\}} \to (0,1)$ given by:

$$\gamma^{(0)}_{\{n\}}(a_n|x_{-\infty}^{n-1}, x_{n+1}^{\infty}) := \begin{cases} \overrightarrow{\gamma}^{\phi}_{\{n\}}(a_n|x_{-\infty}^{n-1}, x_{n+1}^{\infty}), & n \geq 0; \\ 1/|E|, & n < 0 \end{cases} \tag{4.34}$$

with

$$\overrightarrow{\gamma}^{\phi}_{\{n\}}(a_n|x_{-\infty}^{n-1}, x_{n+1}^{\infty}) = \frac{e^{S_{n+1}\phi(x_0^{n-1} a_n x_{n+1}^{\infty})}}{\sum_{\bar{a}_n} e^{S_{n+1}\phi(x_0^{n-1} \bar{a}_n x_{n+1}^{\infty})}} \tag{4.35}$$

are the single-site kernels for the half-line specification $\overrightarrow{\gamma}^{\phi}$. [To simplify we adopt the convention $x_m^n = \emptyset$ if $n < m$.] It can be easily checked that it is a family of single-site densities of a specification, therefore, by Proposition 4.2.2 there is a non-null quasilocal specification $\gamma^{(0)}$ on $(X, \mathscr{F})$ having $\{\gamma^{(0)}_{\{n\}}\}_{n\in\mathbb{Z}}$ as its single-site densities. Note that $\gamma^{(0)}$ is not translation-invariant. We claim that $\mu_0 \in \mathscr{G}(\gamma^{(0)})$. To show this, it is enough, by Proposition 4.2.2, to check that $\mu_0\big(\gamma^{(0)}_{\{n\}}(F)\big) = \mu_0(F)$ for each local cylindrical F. That is, we must show that

$$[\rho \times \nu]\big(\gamma^{(0)}_{\{n\}}(\mathbb{1}_{b_k^r})\big) = [\rho \times \nu]\big(\mathbb{1}_{b_k^r}\big) \tag{4.36}$$

for all integer $k \le r$ and n and all $b_k^r \in E^{r-k+1}$. A quick inspection shows that the only non-trivial case is $k \le n \le r$, $n \ge 0$. In this case

$$[\rho \times \nu]\big(\gamma^{(0)}_{\{n\}}(\mathbb{1}_{b_k^r})\big) = \rho\big(\mathbb{1}_{b^0_{k\wedge(-1)}}\big)\, \nu\big(\overrightarrow{\gamma}_{\{n\}}(\mathbb{1}_{b^r_{k\vee 0}})\big), \tag{4.37}$$

and, as ν is a Gibbs measure for ϕ, we conclude that

$$[\rho \times \nu]\big(\gamma^{(0)}_{\{n\}}(\mathbb{1}_{b_k^r})\big) = \rho\big(\mathbb{1}_{b^0_{k\wedge(-1)}}\big)\, \nu\big(\mathbb{1}_{b^r_{k\vee 0}}\big), \tag{4.38}$$

proving (4.36).

The proof is concluded by invoking the previous lemmas. By Lemma 4.8.1, for all $p \in \mathbb{N}$, the measure $\mu_p = \mu_0 \circ S^{-p}$ is consistent with $\gamma^{(p)}$, where

$$\gamma^{(p)}_\Lambda(B|\omega) := \gamma^{(0)}_{\Lambda+p}(S^{-p}(B)|S^{-p}(\omega)),\ \forall B \in \mathscr{F},\ \omega \in X,\ \forall \Lambda \Subset \mathbb{Z}.$$

Note that the single-site density functions of $\gamma^{(p)}$ can be calculated explicitly, namely, for all $\sigma, \omega \in X$,

$$\gamma^{(p)}_{\{i\}}(\sigma_i|\omega_{\{i\}^c}) := \begin{cases} \dfrac{e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\sigma_i\omega_{i+1}^\infty)}}{\sum_{\bar\omega_i} e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\bar\omega_i\omega_{i+1}^\infty)}}, & i \ge -p; \\ 1/|E|, & i < -p. \end{cases} \tag{4.39}$$

Thus $\overset{\rightleftharpoons}{\gamma}^{\phi}$ is the uniform limit of the sequence of specifications $\{\gamma^{(p)}\}_p$. Thus by Lemma 4.8.2, we obtain that $\mu \in \mathscr{G}(\overset{\rightleftharpoons}{\gamma}^{\phi})$. Hence, we establish the theorem in the case where ρ is the uniform Bernoulli measure.

Now let $\tilde\rho \in \mathscr{M}_1(X_-)$ be an arbitrary measure, and let ρ once again denote the uniform Bernoulli measure on X_-. Then for any local function $g : X \to \mathbb{R}$, $g \circ S^n$ becomes $F_{\mathbb{Z}_+}$ measurable, for sufficiently large $n \ge 1$. Thus by Fubini's theorem, for sufficiently large n,

$$\int_X g \circ S^n\, d(\tilde\rho \times \nu) = \int_{X_+} g \circ S^n\, d\nu = \int_X g \circ S^n\, d(\rho \times \nu).$$

Hence, one concludes that a subsequence $\{(\tilde\rho \times \nu) \circ S^{-n_k}\}_{k\ge1}$ converges in the weak* topology if and only if $\{(\rho \times \nu) \circ S^{-n_k}\}_{k\ge1}$ converges and the limiting points coincide. □

The following lemma will be useful in the proof of Theorem 4.B.

Lemma 4.8.3. *Under the conditions of Theorem 4.A, for all cylindrical sets $C \subset X$, and all volumes $\Lambda \Subset \mathbb{Z}$,*

$$\lim_{n\to\infty} \int_X \Big[\overset{\rightleftharpoons}{\gamma}^{\phi}_\Lambda(C|x) - \mathbb{1}_C(x)\Big]\mu_n(dx) = 0\,.$$

Proof. We will show that

$$\liminf_{n\to\infty}\int_X \Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_n(dx)=0\,; \tag{4.40}$$

the analogous result for the lim sup can be shown similarly. Take any subsequence $\{n_k\}$ such that

$$\lim_{k\to\infty}\int_X \Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_{n_k}(dx)=\liminf_{n\to\infty}\int_X \Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_n(dx).$$

By compactness of $\mathscr{M}_1(X)$ the subsequence $\{\mu_{n_k}\}_k$ converges in the weak*-topology. If μ is its limit, then $\mu\in\mathscr{G}(\overset{\rightleftharpoons}{\gamma}{}^{\phi})$ by Theorem 4.A. Thus,

$$\lim_{k\to\infty}\int_X \Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_{n_k}(dx)=\int_X \Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu(dx)=0\,.$$

□

Proof of Theorem 4.B. The proof of translation-invariance of μ is standard. Thus it is enough to check the consistency, i.e., $\mu\in\mathscr{G}(\overset{\rightleftharpoons}{\gamma}{}^{\phi})$. Let $\mu=\lim_k\tilde{\mu}_{n_k}$, and take any cylindrical event C. Then, the weak convergence implies that

$$\int_X \overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)\tilde{\mu}_{n_k}(dx)\xrightarrow[k\to\infty]{}(\mu\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda})(C). \tag{4.41}$$

On the other hand, the Stolz-Cesaro theorem and Lemma 4.8.3 yield that

$$\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int_X \Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_i(dx)\xrightarrow[k\to\infty]{}0\,.$$

Thus,

$$\int_X \overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)\tilde{\mu}_{n_k}(dx)=\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int_X \Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_i(dx)+\tilde{\mu}_{n_k}(C)\xrightarrow[k\to\infty]{}\mu(C),$$

□

4.8.2 Proofs of Theorem 4.D, 4.E and 4.C

Proof of Theorem 4.D. By Theorem 4.6.5, each measure $\nu^{(k)}$ is absolutely continuous with respect to $\nu^{(0)}$ with $f^{(k)}=\dfrac{d\,\nu^{(k)}}{d\,\nu^{(0)}}$. Furthermore, by Theorem 4.4.2, $\nu^{(k)}$

is the unique Gibbs measure for $\Psi^{(k)}$ for all $k \geq 0$. Let μ^* be a weak* limit of a subsequence $\{\nu^{(k_s)}\}_{s\in\mathbb{N}}$. By Lemma 4.8.2, $\mu^* \in \mathscr{G}(\Phi)$ (c.f. Remark 4.6.4). By the weak star convergence, we have that for all $g_0 \in C(X)$,

$$\int_X g_0 d\nu^{(k_s)} = \int_X g_0 f^{(k_s)} d\nu^{(0)} \xrightarrow[s\to\infty]{} \int_X g_0 d\mu^*. \tag{4.42}$$

Since the family $\{f^{(k)} : k \in \mathbb{Z}_+\}$ is uniformly integrable, it is relatively weakly compact in $L^1(\nu^{(0)})$ by the Dunford-Pettis theorem. Therefore, there exists a weak limit point $f \in L^1(\nu^{(0)})$ of the sequence $\{f^{(k_s)}\}_{s\in\mathbb{N}}$. Without loss of generality, assume that $f^{(k_s)} \xrightarrow[s\to\infty]{} f$. Thus for all $g \in L^\infty(X, \nu^{(0)})$,

$$\int_X g f^{(k_s)} d\nu^{(0)} \xrightarrow[s\to\infty]{} \int_X g f d\nu^{(0)}. \tag{4.43}$$

By combining (4.42) and (4.43), we conclude that for all $g_0 \in C(X)$,

$$\int_X g_0 d\mu^* = \int_X g_0 f d\nu^{(0)}.$$

□

Proof of Theorem 4.E. For all $k \in \mathbb{N}$, denote $W_k := \sum_{i=1}^{k} \Phi_{\Lambda_i}$. Our argument relies on two claims:

Claim 1:

$$\sup_{k\geq 0} \left| \int_X -W_k d\nu^{(k)} \right| < \infty .$$

Claim 2:

$$\sup_{k\geq 0} \int_X e^{-W_k} d\nu^{(0)} < \infty.$$

Then, as

$$\int_X f^{(k)} \log f^{(k)} d\nu^{(0)} = \int_X -W_k d\nu^{(k)} - \log \int_X e^{-W_k} d\nu^{(0)},$$

these claims imply that

$$\sup_{k\geq 0} \int_X f^{(k)} \log f^{(k)} d\nu^{(0)} < \infty. \tag{4.44}$$

Hence by applying de la Vallée Poussin's theorem to the family $\{f^{(k)} : k \in \mathbb{N}\}$ and to the function $t \in (0, +\infty) \mapsto t \log t$, one concludes that the family $\{f^{(k)} : k \in \mathbb{N}\}$ is uniformly integrable in $L^1(\nu^{(0)})$.

The proof of Claim 1 is immediate:

$$\left|\int_X -W_k\, d\,\nu^{(k)}\right| \leq \sum_{i=1}^{k}\left|\int_X \Phi_{\Lambda_i}\, d\,\nu^{(k)}\right| \leq \sum_{i=1}^{k}\rho_i \leq \sum_{i=1}^{\infty}\rho_i < \infty\,. \tag{4.45}$$

The proof of Claim 2 relies on the Gaussian concentration bounds. Note that for all $k \in \mathbb{N}$, $\bar{c}(\Psi^{(0)}) \leq \bar{c}(\Psi^{(k)}) \leq \bar{c}(\Phi)$. Therefore, the Dobrushin uniqueness condition $\bar{c}(\Phi) < 1$ is inherited by all the intermediate interactions. Applying the first part of Theorem 4.4.7 we see that the (only) measure $\mu \in \mathscr{G}(\Phi)$ and all the intermediate measures $\nu^{(k)}$, $k \geq 0$, satisfy the Gaussian Concentration Bound with the same constant $D := \dfrac{4}{(1-\bar{c}(\Phi))^2}$. This implies that, for all $k \in \mathbb{N}$,

$$\int_X e^{-\Phi_{\Lambda_k}}\, d\,\nu^{(k-1)} \leq e^{D\|\underline{\delta}(\Phi_{\Lambda_k})\|_2^2}\, e^{-\int_X \Phi_{\Lambda_k}\, d\,\nu^{(k-1)}}. \tag{4.46}$$

We combine this inequality with (4.30) to iterate

$$\begin{aligned}
&\int_X e^{-(\Phi_{\Lambda_k}+\Phi_{\Lambda_{k-1}})} d\,\nu^{(k-2)} \\
&= \int_X e^{-\Phi_{\Lambda_k}}\, d\,\nu^{(k-1))} \int_X e^{-\Phi_{\Lambda_{k-1}}}\, d\,\nu^{(k-2)} \\
&\leq e^{D(\|\underline{\delta}(\Phi_{\Lambda_k})\|_2^2+\|\underline{\delta}(\Phi_{\Lambda_{k-1}})\|_2^2)} \cdot e^{-(\int_X \Phi_{\Lambda_k}\, d\,\nu^{(k-1)}+\int_X \Phi_{\Lambda_{k-1}}\, d\,\nu^{(k-2)})}.
\end{aligned} \tag{4.47}$$

By induction this yields

$$\int_X e^{-\sum_{i=1}^k \Phi_{\Lambda_i}}\, d\,\nu^{(0)} \leq e^{D\sum_{i=1}^k \|\underline{\delta}(\Phi_{\Lambda_i})\|_2^2} \cdot e^{-\sum_{i=1}^k \int_X \Phi_{\Lambda_i}\, d\,\nu^{(i-1)}}. \tag{4.48}$$

Thus, by using (4.17) and (4.45), we have that

$$\int_X e^{-W_k}\, d\,\nu^{(0)} \leq e^{D\sum_{i=1}^k \|\underline{\delta}(\Phi_{\Lambda_i})\|_2^2} \cdot e^{\sum_{i=1}^k \rho_i} \tag{4.49}$$

for all $k \in \mathbb{N}$, and

$$\sup_{k\in\mathbb{N}} \int_X e^{-W_k}\, d\,\nu^{(0)} \leq e^{D\sum_{i=1}^\infty \|\underline{\delta}(\Phi_{\Lambda_i})\|_2^2} \cdot e^{\sum_{i=1}^\infty \rho_i} < \infty. \tag{4.50}$$

This proves Claim 2 and, hence, concludes the proof of the theorem. □

Proof of Theorem 4.C.

Part (i): Its proof is rather straightforward. We choose $\beta > 0$, so that the resulting Φ satisfies the Dobrushin uniqueness condition $\bar{c}(\Phi) < 1$. This condition is inherited by $\Psi^{(0)}$, hence both potentials have a unique Gibbs state. Furthermore, as direct products of Gibbs measures for the restricted interactions Φ^- and Φ^+ are Gibbs measures for $\Psi^{(0)}$, neither Φ^- nor Φ^+ may exhibit phase transitions (c.f. Equation (4.28)). In particular, $\gamma^{\phi} = \gamma^{\Phi^+}$ admits only one Gibbs state.

Part (ii): We just have to verify the hypotheses of Theorem 4.E; the absolute continuity follows then from Theorem 4.D. We chose $0 < \beta \leq \beta_{DU}$ with

$$\beta_{DU} = \big(2\sum_{i=1}^{\infty}\frac{1}{i^{\alpha}}\big)^{-1} = \frac{1}{2}\zeta(\alpha)^{-1} . \tag{4.51}$$

Hence, by Proposition 4.4.4, for all $\beta \in (0, \beta_{DU})$, Φ satisfies the Dobrushin uniqueness condition $\bar{c}(\Phi) < 1$. *Hypothesis 1):* Consequence of the Dobrushin uniqueness criterion.

Hypothesis 2): For all $i \in \mathbb{N}$, $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$,

$$\delta_k(\Phi_{\{-i,j\}}) = \begin{cases} 0, & k \notin \{-i, j\}; \\ \dfrac{2\beta}{(i+j)^{\alpha}}, & k \in \{-i, j\}. \end{cases}$$

Thus for all $i \in \mathbb{N}$, $j \in \mathbb{Z}_+$, Therefore,

$$\sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} \|\underline{\delta}(\Phi_{\{-i,j\}})\|_2^2 = \sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} \frac{8\beta^2}{(i+j)^{2\alpha}} < \infty .$$

Hypothesis 3): We shall use inequalities (4.20) and (4.22). We use the notation introduced in Section 4.4.2. Note that for all $k \geq 0$ we have the componentwise domination

$$C(\gamma^{\Psi^{(k)}})_{i,j} \leq \bar{C}(\Phi)_{i,j} \quad , \quad D(\gamma^{\Psi^{(k)}})_{i,j} \leq \bar{D}(\Phi)_{i,j} . \tag{4.52}$$

Thus since $\bar{c}(\Phi) < 1$ all the specifications $\gamma^{\Psi^{(k)}}$ satisfy (4.12) (c.f. Section 4.4.2). Applying (4.20) to the measures $\nu^{(k)}$, $k \geq 0$ and using (4.52), we see that

$$\sup_{k\geq 0}\left|\int_X \sigma_{-m}\cdot\sigma_0\circ S^n\, d\,\nu^{(k)}\right| \leq \frac{1}{4}\sum_{r,j\in\mathbb{Z}} \bar{D}(\Phi)_{rj}\cdot\delta_r\sigma_{-m}\cdot\delta_{n-j}\sigma_0. \tag{4.53}$$

Hence for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}_+$,

$$(m+n)^{\alpha}\rho_{-m,n} \leq \frac{1}{4}\sum_{r,j\in\mathbb{Z}} \bar{D}(\Phi)_{rj}\cdot\delta_r\sigma_{-m}\cdot\delta_{n-j}\sigma_0. \tag{4.54}$$

Summing and applying inequality (4.22), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(m+n)^{\alpha}\rho_{-m,n} &\le \frac{1}{4}\sum_{r,j,n\in\mathbb{Z}}\bar{D}(\Phi)_{rj}\cdot\delta_r\sigma_{-m}\cdot\delta_{n-j}\sigma_0 \\ &= \frac{1}{2}\sum_{r\in\mathbb{Z}}\Big[\delta_r\sigma_{-m}\cdot\sum_{j\in\mathbb{Z}}\bar{D}(\Phi)_{rj}\Big] \\ &\overset{(4.22)}{\le} \frac{1}{1-\bar{c}(\Phi)}.\end{aligned} \tag{4.55}$$

As a consequence, $m^{\alpha}\cdot\sum_{n=0}^{\infty}\rho_{-m,n}\le(1-\bar{c}(\Phi))^{-1}$ for all $m\in\mathbb{N}$, which implies

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\rho_{-m,n}\le\frac{1}{1-\bar{c}(\Phi)}\sum_{m=1}^{\infty}\frac{1}{m^{\alpha}}<\infty. \tag{4.56}$$

This concludes the verification of the hypotheses of Theorem 4.E, which, together with Theorem 4.D imply that the unique limit point μ of the sequence $\{\nu^{(k)}\}_{k\in\mathbb{Z}_+}$ is absolutely continuous with respect to $\nu^{(0)}$. As $\nu^{(0)}=(\nu^{(0)})^-\times(\nu^{(0)})^+$ and $(\nu^{(0)})^+$ coincides with the unique Gibbs state ν of ϕ, it follows that $\mu\ll\nu^{(0)}$ implies that $\mu^+\ll\nu$.

The proof of $\nu\ll\mu^+$ is analogous but using Theorem 4.F instead of Theorem 4.E.

Part (iii): This part follows from an application of the Arzelà–Ascoli theorem. Before proceeding with the proof, we fix an enumeration $\{\Lambda_1,\Lambda_2,\dots\}$ of the set $\mathscr{A}=\{\Lambda\Subset\mathbb{Z}:\min\Lambda<0\le\max\Lambda\}$ such that, for every $N\in\mathbb{N}$, there exists $k_N\in\mathbb{N}$ satisfying

$$\sum_{i=1}^{k_N}\Phi_{\Lambda_i}=\sum_{\substack{\min V<0\le\max V\\ V\subset[-N,N]}}\Phi_V. \tag{4.57}$$

Let us denote ξ the configurations in $X_-=\{-1,1\}^{-\mathbb{N}}$ and σ those in $X_+=\{-1,1\}^{\mathbb{Z}_+}$. For all $N\in\mathbb{N}$, and $(\xi,\sigma)\in X_-\times X_+$, define

$$W_N(\xi,\sigma):=\sum_{i=1}^{N}\sum_{j=0}^{N}-\frac{\beta\xi_{-i}\sigma_j}{(i+j)^{\alpha}}$$

and

$$f_+^{[N]}(\sigma):=\frac{d\nu_+^{[N]}}{d\nu_+^{(0)}}(\sigma)=\frac{\int_{X_-}e^{-W_N(\xi,\sigma)}\nu_-^{(0)}(d\xi)}{\int_{X_+}\int_{X_-}e^{-W_N(\xi,\zeta)}\nu_-^{(0)}(d\xi)\nu_+^{(0)}(d\zeta)}. \tag{4.58}$$

The sequence $\{f_+^{[N]}\}_{N\in\mathbb{N}}$ is a subsequence of the sequence $\{f_+^{(k)}\}_{k\in\mathbb{Z}_+}$ defined in (4.33), since $f_+^{[N]}=f_+^{k_N}$. All $f_+^{[N]}$ are local functions on X_+, thus continuous. We claim that it is enough to prove that the family $\{f_+^{[N]} : N \in \mathbb{N}\}$ is relatively compact in $C(X_+)$. Indeed, if this is true, there exists a function $f_+ \in C(X_+)$ and a subsequence $\{f_+^{[N_k]}\}_{k\in\mathbb{N}}$ such that $f_+^{[N_k]} \rightrightarrows f_+$ as $k \to \infty$. Thus, by the argument presented in the proof of Theorem 4.D, f_+ is the Radon-Nikodym density of μ_+ with respect to ν, and

$$\mathscr{L}_\phi f_+(x) = e^{P(S,\phi)} f_+(x)$$

for all for $x \in X_+$ [in principle, the identity holds for ν–almost all $x \in X_+$, but ν is fully supported]. Hence f_+ is the continuous eigenfunction of the transfer operator $\mathscr{L}_\phi$ corresponding to the largest eigenvalue. Note that we can also conclude from this argument that the entire sequence $\{f_+^{[N]}\}_{N\in\mathbb{N}}$ converges in the uniform topology.

To conclude, we turn to the proof of the relative compactness of the family $\{f_+^{[N]} : N \in \mathbb{N}\}$. By the Arzela-Ascoli theorem, it is enough to show that this family is uniformly bounded and equicontinuous. These properties are proven separately.

Uniform boundedness: As

$$\bar{c}(\Phi^-) = \frac{1}{2}\sup_{i\in-\mathbb{N}} \sum_{i\in V\Subset-\mathbb{N}} (|V|-1)\delta(\Phi_V) \le \frac{1}{2}\sup_{i\in\mathbb{Z}} \sum_{i\in V\Subset\mathbb{Z}} (|V|-1)\delta(\Phi_V) = \bar{c}(\Phi) < 1\,,$$

the interaction Φ^- on $-\mathbb{N}$ satisfies the Dobrushin uniqueness condition, therefore, the unique Gibbs measure $\nu_- \in \mathscr{G}(\Phi^-)$ satisfies the Gaussian Concentration Bound (Theorem 4.4.7) with the constant $D = 4(1-\bar{c}(\Phi))^{-2}$. Fix any $\sigma \in X_+$ and consider $W_N(\xi,\sigma)$ as a function of $\xi \in X_-$. Clearly, it is a local function, thus by the first part of Theorem 4.4.7, for all $\kappa \in \mathbb{R}$,

$$\int_{X_-} e^{\kappa W_N(\xi,\sigma)}\nu_-(d\xi) \le e^{D\kappa^2\|\underline{\delta}(W_N(\cdot,\sigma))\|_2^2} \cdot e^{\kappa\int_{X_-} W_N(\xi,\sigma)\nu_-(d\xi)}. \tag{4.59}$$

First, note that the interaction Φ^- is invariant under the global spin-flip transformation, therefore, so is the unique Gibbs measure ν_-. Thus for all $N \in \mathbb{N}$ and $\sigma \in X_+$, $\int_{X_-} W_N(\xi,\sigma)\nu_-(d\xi) = 0$. Second, for all $k \in \mathbb{N}$,

$$\delta_{-k}(W_N(\cdot,\sigma)) = 2\beta\Big|\sum_{j=0}^{N}\frac{\sigma_j}{(k+j)^\alpha}\Big| \le 2\beta\sum_{j=0}^{N}\frac{1}{(k+j)^\alpha}.$$

Hence if $\alpha > \frac{3}{2}$, for all $N \in \mathbb{N}$, $\sigma \in X_+$,

$$\|\underline{\delta}(W_N(\cdot,\sigma))\|_2^2 \le 4\beta^2\sum_{k=1}^{\infty}\Big(\sum_{j=k}^{\infty}\frac{1}{j^\alpha}\Big)^2 =: 4\beta^2 C_1(\alpha) < \infty. \tag{4.60}$$

Then (4.59) implies that for all $N \in \mathbb{N}$, $\sigma \in X_+$, one has

$$\int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \le e^{4D\kappa^2\beta^2 C_1(\alpha)}. \tag{4.61}$$

Changing $\kappa \to -\kappa$ in (4.61) we also obtain

$$\int_{X_-} e^{-\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \le e^{4D\kappa^2\beta^2 C_1(\alpha)}. \tag{4.62}$$

Thus by applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} 1 &= \left[\int_{X_-} e^{\kappa W_N(\xi,\sigma)/2} e^{-\kappa W_N(\xi,\sigma)/2} \nu_-(d\xi)\right]^2 \\ &\le \int_{X_-} e^{-\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \\ &\le e^{4D\kappa^2\beta^2 C_1(\alpha)} \cdot \int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \end{aligned}$$

which yields the lower bound

$$e^{-4D\kappa^2\beta^2 C_1(\alpha)} \le \int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \tag{4.63}$$

for all $\sigma \in X_+$, $N \in \mathbb{N}$. Putting (4.61) and (4.63) together yields the bounds

$$e^{-4D\kappa^2\beta^2 C_1(\alpha)} \le \int_{X_+}\int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi)\nu_+(d\sigma) \le e^{4D\kappa^2\beta^2 C_1(\alpha)} \tag{4.64}$$

which implies that the family $\{f_+^{[N]} : N \in \mathbb{N}\}$ is uniformly bounded from above and below:

$$e^{-8D\beta^2 C_1(\alpha)} \le f_+^{[N]}(\sigma) \le e^{8D\beta^2 C_1(\alpha)}. \tag{4.65}$$

Equicontinuity: As the denominator $\int_X e^{-W_N} d\nu^{(0)}$ is uniformly bounded from above and below as shown in (4.64), it is enough to show that the family

$$\left\{\int_{X_-} e^{-W_N(\xi,\cdot)} \nu_-(d\xi) : N \in \mathbb{N}\right\}$$

is equicontinuous. Consider $n \in \mathbb{N}$ and configurations $\sigma, \tilde\sigma \in X_+$ such that $\sigma_0^{n-1} = \tilde\sigma_0^{n-1}$. Then

$$\left|\int_{X_-} \left[e^{-W_N(\xi,\sigma)} - e^{-W_N(\xi,\tilde\sigma)}\right] \nu_-(d\xi)\right| \le \int_{X_-} e^{-W_N(\xi,\sigma)} \cdot \left|e^{W_N(\xi,\sigma)-W_N(\xi,\tilde\sigma)} - 1\right| \nu_-(d\xi). \tag{4.66}$$

Thus, by the Cauchy-Schwarz inequality,

$$RHS \leq \Big(\int_{X_-} e^{-2W_N(\xi,\sigma)} \nu_-(d\xi) \Big)^{\frac{1}{2}} \Big(\int_{X_-} \Big[e^{W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})} - 1 \Big]^2 \nu_-(d\xi) \Big)^{\frac{1}{2}} \tag{4.67}$$

and (4.61) yields that

$$\Big| \int_{X_-} \Big[e^{-W_N(\xi,\sigma)} - e^{-W_N(\xi,\tilde{\sigma})} \Big] \nu_-(d\xi) \Big| \leq C_2 \Big(\int_{X_-} \Big[e^{W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})} - 1 \Big]^2 \nu_-(d\xi) \Big)^{\frac{1}{2}} \tag{4.68}$$

with $C_2 := e^{8D\beta^2 C_1(\alpha)}$. We will bound the last integral by bounding the exponent.

Note that

$$W_N(\xi,\sigma) - W_N(\xi,\tilde{\sigma}) = -\beta \sum_{j=n}^{N} (\sigma_j - \tilde{\sigma}_j) \sum_{i=1}^{N} \frac{\xi_{-i}}{(i+j)^\alpha} \tag{4.69}$$

and, thus, for all $k \in \mathbb{N}$,

$$\delta_{-k}(W_N(\cdot,\sigma) - W_N(\cdot,\tilde{\sigma})) = 2\beta \Big| \sum_{j=n}^{N} \frac{\sigma_j - \tilde{\sigma}_j}{(k+j)^\alpha} \Big| \leq 4\beta \sum_{j=n}^{N} \frac{1}{(k+j)^\alpha} \,. \tag{4.70}$$

Hence, for sufficiently large n and $N > n$,

$$||\underline{\delta}(W_N(\cdot,\sigma) - W_N(\cdot,\tilde{\sigma}))||_2^2 \leq 16\beta^2 \sum_{k=1}^{\infty} \Big(\sum_{j=n}^{N} \frac{1}{(k+j)^\alpha} \Big)^2 \leq 32\beta^2 \sum_{k=n+1}^{\infty} \frac{1}{k^{2(\alpha-1)}} =: u_n. \tag{4.71}$$

with

$$\lim_{n\to\infty} u_n = 0 \text{ if and only if } \alpha > \frac{3}{2} \,.$$

The spin-flip invariance of ν_- implies that for all $N \in \mathbb{N}$,

$$\int_{X_-} [W_N(\xi,\sigma) - W_N(\xi,\tilde{\sigma})] \, \nu_-(d\xi) = 0 \,.$$

Then since Φ^- satisfies (4.13), we have, from the second part of Theorem 4.4.7, that for all $m \in \mathbb{N}$,

$$\begin{aligned} \int_{X_-} \Big| W_N(\xi,\sigma) - W_N(\xi,\tilde{\sigma}) \Big|^m \nu_-(d\xi) \quad &\leq \quad \Big(\frac{D||\underline{\delta}(W_N(\cdot,\sigma) - W_N(\cdot,\tilde{\sigma}))||_2^2}{2} \Big)^{\frac{m}{2}} m\Gamma\Big(\frac{m}{2}\Big) \\ &= \quad m\, v_n^m \, \Gamma\Big(\frac{m}{2}\Big) \end{aligned} \tag{4.72}$$

with

$$v_n := \Big(\frac{D\, u_n}{2} \Big)^{\frac{1}{2}} \,.$$

We conclude by expanding the square in the right-hand side in (4.68):

Expanding the square in the right-hand side of (4.68),

$$\begin{aligned}
&\int_{X_-}\left[e^{W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})}-1\right]^2\nu_-(d\xi)\\
&= \int_{X_-}\left[e^{2[W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})]}-2\,e^{W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})}+1\right]\nu_-(d\xi)\\
&\le 1+\sum_{m=1}^{\infty}\frac{2^m+2}{m!}\int_{X_-}\left|W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})\right|^m\nu_-(d\xi). \qquad (4.73)
\end{aligned}$$

Through elementary analysis one can show that

$$\sum_{m=1}^{\infty}\frac{2^m+2}{m!}m\,v_n^m\,\Gamma\Big(\frac{m}{2}\Big)\le 4v_n+3e^{v_n^2}+2e^{4v_n^2}-5. \qquad (4.74)$$

Therefore, we obtain from (4.68) and (4.72),

$$\left|\int_{X_-}\left[e^{-W_N(\xi,\sigma)}-e^{-W_N(\xi,\tilde{\sigma})}\right]\nu_-(d\xi)\right|\le C_2(4v_n+3e^{v_n^2}+2e^{4v_n^2}-5)^{\frac{1}{2}}. \qquad (4.75)$$

Since $\lim_{n\to\infty} v_n = 0$, we conclude that the family $\left\{\int_{X_-} e^{-W_N(\xi,\cdot)}\nu_-(d\xi) : N\in\mathbb{N}\right\}$ is indeed equicontinuous. □

4.9 Final Remarks and Future Directions

The main results of the present work rely on two major assumptions: uniqueness of the Gibbs measure and validity of the Gaussian concentration inequalities for that measure. We informally refer to the combination of these two conditions as strong uniqueness. Strong uniqueness holds for a wide class of Gibbs interactions (potentials).

However, we would like to end with a discussion of one particular model – the Dyson model, which served as the primary motivation for the present work. The picture below (Figure 4.1) summarises the current state of the art in the eigenfunction problem for the Dyson potential with the parameters $\alpha > 1$, $\beta > 0$.

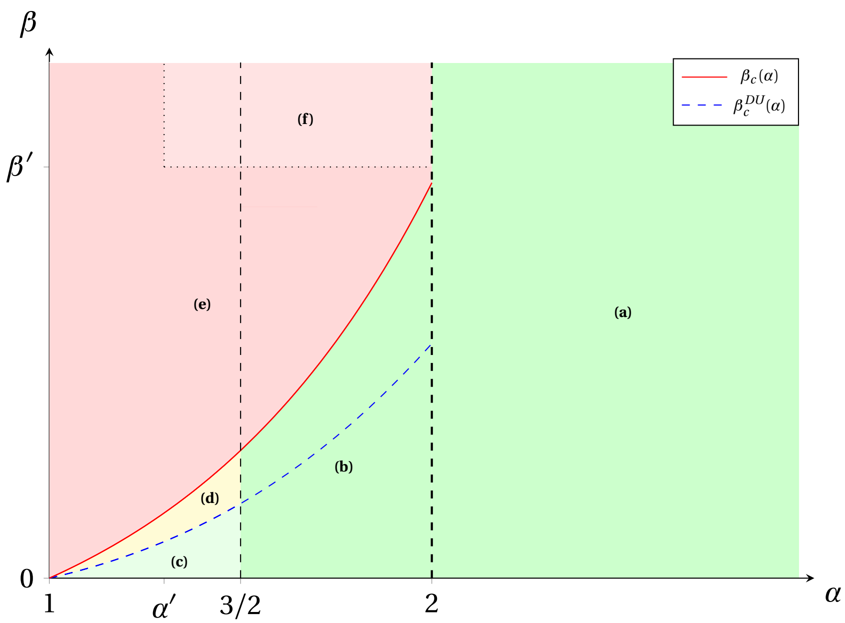

Figure 4.1: Eigenfunctions for the Dyson model across the phase diagram [2]

(a) For $\alpha > 2$, at all temperatures, the Dyson potential has summable variation, and hence the classical results of Walters [66, 67] allow one to conclude that the transfer operator admits a continuous eigenfunction with summable variation. Note also that for $\alpha > 2$ we also have strong uniqueness: uniqueness is due to Bowen [9], and the Gaussian Concentration Bounds have been established in [12, 13]. Thus our results are also applicable, although summable variation of the unique continuous positive eigenfunction requires a separate argument.

(b) For $\alpha \in (3/2, 2]$ existence of a continuous eigenfunction was first proved by Johansson, Öberg, and Pollicott for sufficiently small β and then extended to the whole subcritical region [47]. The proof in [47] relies on rather specific properties of the Dyson model, e.g., representation of the model via the random cluster model, which typically restricts one to ferromagnetic models [22, 45] We strongly believe that the Gaussian concentration inequality, and hence, the method used in this chapter can be extended to the whole uniqueness region as well (See the Appendix).

[2] The monotonicity of the function $\alpha \mapsto \beta_c(\alpha)$ follows from the GKS inequality [37], while the monotonicity of the function $\alpha \mapsto \beta_c^{DU}(\alpha)$ follows from (4.51)

(c) In this region, we have established the existence of an integrable eigenfunction. (c.f. the second part of Theorem 4.C). We conjecture that the result is sharp: for $\alpha \le 3/2$, the transfer operator does not have a continuous eigenfunction.

(d) We conjecture that the result obtained for the region (c) must hold for all β's below the critical value $\beta_c(\alpha)$.

(e) In the supercritical regime: $\alpha > 1$ and $\beta > \beta_c(\alpha)$, we believe that transfer operators do not admit integrable eigenfunctions.

(f) The only result which applies to the supercritical phase is [8], where it has been shown that in a particular region, both pure phases of the Dyson model Φ are not $g-$measures. That immediately implies that the transfer operator does not have a continuous eigenfunction; otherwise, the normalized function $g = \dfrac{h \cdot e^{\phi}}{\lambda \cdot h \circ S}$ would become a *g–function* for all the phases, in particular, the pure phases of the model.

To summarize the picture, we conjecture that for the Dyson potential, transfer operator does not have an eigenfunction in the supercritical regime, and does have an eigenfunction in the subcritical regime. The smoothness of the eigenfunction is varying with α: from summable variation for $\alpha > 2$, to continuous for $\alpha \in (3/2, 2]$, to L^1 but not continuous for $\alpha \le 3/2$.

The key to establishing properties of transfer operators for the Dyson potential is, in our opinion, a proper understanding of the probabilistic properties of the left-right interaction energy function:

$$W(\xi,\sigma) = \sum_{i<0\le j} -\frac{\beta \xi_i \sigma_j}{(j-i)^{\alpha}}, \quad \xi \in X_-, \sigma \in X_+.$$

For example, in the case $\alpha > 2$, the results follow almost immediately from the simple observation that the interaction energy function is uniformly bounded

$$\sup_{\xi\in X_-,\sigma\in X_+} |W(\xi,\sigma)| = \beta \sum_{i<0\le j} \frac{1}{(j-i)^{\alpha}} < \infty,$$

and all expressions for densities above automatically lead to continuous functions. The next interesting 'critical value' $\alpha = 3/2$ also appears quite naturally: The left-right interaction energies $W(\xi,\sigma)$ for a fixed $\sigma \in X_+$, but ν_--random configurations $\xi \in X_-$, can be represented as

$$W(\xi,\sigma) = \sum_{i<0} Z_i, \quad Z_i = \xi_i \Big(-\sum_{j\ge 0} \frac{\sigma_j}{(j-i)^{\alpha}} \Big).$$

The variances $\mathsf{var}(Z_i) = \mathcal{O}(|i|^{-2(\alpha-1)})$ become summable for $\alpha > 3/2$ [c.f. (4.60)]. Hence, assuming weak correlations, the condition $\alpha > 3/2$ corresponds to the almost sure existence of the left-right interaction energy for random conditions on the left half-line interacting with any (in particular, the all plus or all minus) configuration on the right half-line. The concentration inequality can be interpreted as the rigorous transcription of this observation, and the Dobrushin condition as the guarantor of weak correlations.

We strongly believe that the analysis of the left-right interaction energy function can be extended to the whole subcritical regime $\beta < \beta_c(\alpha)$.

We finish the discussion with two interesting questions. As customary in dynamical systems, we study continuous potentials $\phi \in C(X_+)$. In Section 4.6.1, however, we switch to the language of Statistical Mechanics and assume that the potential can be represented as $\phi = -\sum_{0\in V\Subset\mathbb{Z}_+} \Phi_V$ for some translation invariant UAC interaction Φ on Z, c.f. [57]. This is clearly the case for the Dyson potential. In general, however, we do not know any reasonable description of the class of such potentials ϕ.

Finally, in the opposite direction, under which conditions can a Gibbsian specification γ, on the half-lattice $\mathbb{Z}_+$, be represented as $\gamma = \gamma^{\phi}$ for some $\phi \in C(X_+)$? One possible approach to finding such representations would be extending the Kozlov-Sullivan characterisation on $\mathbb{Z}$ to the half-line $\mathbb{Z}_+$ [5].

4.10 Appendix: Eigenfunctions for the near-critical ferromagnetic Dyson model

The appendix aims to present an extension of the results of Chapter 4 to the entire uniqueness region of the ferromagnetic Dyson model and sketch its proof by building on recent advances by Bauerschmidt and Dagallier [6], as well as a fundamental result by Duminil-Copin and Tassion [21].

In [6], the authors established *log-Sobolev inequalities* for the finite volume Gibbs measures of ferromagnetic Ising models, building on newly obtained correlation inequality by Ding, Song, and Sun [19].

Definition 4.10.1. *A measure τ on $X_{\mathscr{S}} := \{\pm 1\}^{\mathscr{S}}$, where $\mathscr{S} \subseteq \mathbb{Z}$ is a finite or infinite subset, satisfies the* ***log-Sobolev inequality*** *(LSI) if there exists a constant $D = D(\tau) > 0$ such that, for every local function $f : X_{\mathscr{S}} \to \mathbb{R}$,*

$$\mathtt{Ent}_{\tau}(f^2) \leq 2D \int_{X_{\mathscr{S}}} \sum_{i\in\mathscr{S}} (f(\omega) - f(\omega^{(i)}))^2 \tau(d\omega), \tag{4.76}$$

This appendix is based on M. Makhmudov, *"Concentration inequalities and Transfer operators for supercritical Dyson models"*, arXiv:2508.01703.

where $\omega^{(i)}$ denotes the configuration obtained from ω by flipping the spin at site i and keeping all other spins unchanged, i.e.,

$$\omega_j^{(i)} := \begin{cases} \omega_j, & i \neq j; \\ -\omega_i, & i = j, \end{cases}$$

and for a non-negative local function $\tilde{f} : X_{\mathscr{S}} \to \mathbb{R}$, the entropy of $\tilde{f}$ with respect to τ is defined as $\mathtt{Ent}_\tau(\tilde{f}) := \int_{X_{\mathscr{S}}} \tilde{f} \log \tilde{f} \, d\tau - \int_{X_{\mathscr{S}}} \tilde{f} \, d\tau \cdot \log \int_{X_{\mathscr{S}}} \tilde{f} \, d\tau$*. The smallest such constant D is called the log-Sobolev constant for τ.*

Bauerschmidt and Dagallier studied ferromagnetic Ising models in a general setting on finite lattices, under suitable assumptions on the model's coupling matrix. Below, we recall their result in the absence of external fields. Specifically, consider a probability measure τ on $X_V := \{\pm 1\}^V$, where V is finite, defined by

$$\tau^\beta(\{\omega_V\}) = \frac{e^{-\frac{\beta}{2}(\omega_V, A\omega_V)}}{\sum_{\omega_V \in X_V} e^{-\frac{\beta}{2}(\omega_V, A\omega_V)}}, \ \ \omega_V \in X_V,$$

where A is a symmetric $V \times V$ matrix and $\beta \geq 0$. Then, under suitable conditions on the coupling matrix A, the measure τ^β satisfies the log-Sobolev inequality (4.76) with a constant

$$D \leq \frac{1}{4} + \frac{\beta}{2} e^{2\beta \chi_{\beta,V}}, \tag{4.77}$$

where $\chi_{\beta,V}$ denotes the *susceptibility* of τ^β, defined as

$$\chi_{\beta,V} := \sup_{j \in V} \sum_{i \in V} \int_{X_V} \omega_i \omega_j \, \tau^\beta(d\omega). \tag{4.78}$$

The following are the conditions that the coupling matrix A must satisfy for (4.77) to hold:

(A1) Off-diagonal entries of A are non-positive, i.e., $A_{ij} \leq 0$ for $i \neq j$;

(A2) A is positive definite;

(A3) The spectral radius $\rho(A)$ of A is bounded by 1.

The coupling matrices of the Dyson model (4.9) satisfy condition (A1), but generally fail to satisfy (A2) and (A3). By carefully modifying the coupling matrices of the Dyson model to fulfill conditions (A2) and (A3), one can ensure that the log-Sobolev inequalities hold for the finite-volume Gibbs measures of the Dyson model with free boundary conditions throughout the entire uniqueness region.

However, to obtain meaningful results for infinite-volume measures, it is necessary to control the log-Sobolev constants across different volumes V, since the right-hand side of (4.77) depends on V. Thanks to the second GKS inequality [37], for every supercritical $\beta < \beta_c(\alpha) := \beta_c(\Phi)$, the following bound holds:

$$\chi_{\beta,V} \leq \sum_{i\in\mathbb{Z}} \int_X \omega_0\omega_i\,\mu(d\omega) =: \chi_\beta, \tag{4.79}$$

where μ denotes the unique Gibbs measure of Φ. In 2018, Duminil-Copin and Tassion [21] proved the finiteness of the infinite-volume susceptibility for all $\beta < \beta_c(\alpha)$. As a consequence, one obtains a uniform log-Sobolev constant $D_0 := \frac{1}{4} + \frac{\beta}{2}e^{2\beta\chi_\beta}$ valid for all finite-volume Gibbs measures $\mu_V^\emptyset$ of Φ. This, in turn, implies that the infinite-volume Gibbs measure μ satisfies the log-Sobolev inequality with constant $D(\mu) \leq D_0$. Analogously, one can establish the log-Sobolev inequalities with the same constant $D_0 = \frac{1}{4} + \frac{\beta}{2}e^{2\beta\chi_\beta}$ for the Gibbs measures corresponding to the intermediate interactions $\{\Psi^{(k)}\}_{k\in\mathbb{Z}_+}$. This can be achieved through the following two steps:

Step (1): First, we establish the uniqueness of the Gibbs measures for the interactions $\Psi^{(k)}$, $k \in \mathbb{Z}_+$, if $\beta < \beta_c(\alpha)$. In fact, by using the GKS inequalities, one can show that $\mu_{(+)} = \mu_{(-)}$ implies $\nu_{(-)}^{(k)} = \nu_{(+)}^{(k)}$ for every $k \in \mathbb{Z}_+$, where $\mu_{(\pm)}$ and $\nu_{(\pm)}^{(k)}$ denote the plus and minus phases of Φ and $\Psi^{(k)}$, respectively. In light of the FKG inequalities, this yields that $\beta_c(\alpha) \leq \beta_c^{(k)}(\alpha) := \beta_c(\Psi^{(k)})$.

Step (2): For $\beta < \beta_c(\alpha)$, one can apply the same technique used for the finite-volume Gibbs measures of Φ to show that the finite-volume Gibbs measure $\nu_V^{(k),\emptyset}$ of $\Psi^{(k)}$ in a volume $V \Subset \mathbb{Z}$ satisfies the LSI with a constant not exceeding $\frac{1}{4} + \frac{\beta}{2}e^{2\beta\chi_{\beta,V}^{(k)}}$, where $\chi_{\beta,V}^{(k)} := \sup_{j\in V}\sum_{i\in V}\nu_V^{(k),\emptyset}(\sigma_i\sigma_j)$. By applying the GKS inequalities again, it follows for every $k \in \mathbb{Z}_+$ and $V \Subset \mathbb{Z}$ that $\chi_{\beta,V}^{(k)} \leq \chi_\beta$. Therefore, by passing to the limit as $V \uparrow \mathbb{Z}$, one obtains that the unique infinite-volume Gibbs measure $\nu^{(k)}$ of $\Psi^{(k)}$ satisfies the LSI with the constant $D_0 = \frac{1}{4} + \frac{\beta}{2}e^{2\beta\chi_\beta}$.

In the next stage, we obtain the Gaussian Concentration Bound (4.17) from the LSI using the Herbst argument [4]. In fact, the following statement is valid.

Proposition 4.10.2. *Assume $\tau \in \mathcal{M}_1(X)$ is a unique Gibbs measure for an interaction $\Psi = (\Psi_V)_{V\Subset\mathbb{Z}}$ with*

$$\sup_{i\in\mathbb{Z}} \sum_{\substack{V\ni i\\ V\Subset\mathbb{Z}}} \|\Psi_V\|_\infty < \infty. \tag{4.80}$$

If τ satisfies the log Sobolev inequality with a constant $D > 0$, then τ has the Gaussian concentration bounds (4.17) with the constant $\frac{D(M+1)}{2}$*, where* $M := \exp\left(2\sup_{i\in\mathbb{Z}}\sum_{V\ni i}\|\Psi_V\|_\infty\right)$

By applying Proposition 4.10.2, along with the techniques used in the proofs of Theorems 4.D, 4.E, and 4.C, we have established the theorem below.

Theorem 4.G. *Let Φ be the ferromagnetic Dyson interaction (4.9), and let ϕ be the corresponding Dyson potential (4.8). Then the following statements hold.*

(i) For every $\alpha \in (1,2]$, we have $\beta_c(\alpha) \le \beta_c^+(\alpha)$, where $\beta_c^+(\alpha)$ is the critical temperature for ϕ. (See Section 4.7 for details.)

(ii) For all $\alpha \in (1,2]$ and for every $\beta \in [0,\beta_c(\alpha))$, the GCB (4.17) holds for the unique Gibbs measure μ of Φ and the unique half-line Gibbs measure ν_+ of ϕ with the constant $D = \frac{(1+2\beta e^{2\beta\chi_\beta})(1+e^{4\beta\zeta(\alpha)})}{8}$*, where χ_β is the susceptibility of the Dyson model at β (c.f., (4.79)) and* $\zeta(\alpha) := \sum_{n=1}^{\infty} n^{-\alpha}$.

(iii) For each $\alpha \in (1,2]$ and all $\beta \in [0,\beta_c(\alpha))$, the restriction $\mu|_{X_+}$ of μ to X_+ is equivalent to ν_+, i.e., $\mu|_{X_+} \ll \nu_+$ and $\nu_+ \ll \mu|_{X_+}$. In particular, $\mathscr{L}_\phi$ admits an integrable eigenfunction $\frac{d\mu|_{X_+}}{d\nu_+} \in L^1(\nu_+)$ *corresponding to its spectral radius.*

(iv) If $\alpha \in \left(\frac{3}{2},2\right]$, then for all $\beta \in [0,\beta_c(\alpha))$, there exists a continuous version of the Radon-Nikodym derivative $\frac{d\mu|_{X_+}}{d\nu_+}$*. Hence, $\mathscr{L}_\phi$ has a continuous principal eigenfunction.*

Remark 4.10.3. *The fourth part of Theorem 4.G has also been established by Johansson, Öberg, and Pollicott by using the random cluster representation of the ferromagnetic Ising models in combination with the concentration inequality established specifically for the random cluster models in* [45].

A detailed proof of Theorem 4.G, along with further developments based on the log-Sobolev approach, can be found in [52].

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