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Gibbs states in statistical mechanics and dynamical systems

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Chapter 3

Gibbs Properties of Equilibrium States

Abstract: In this chapter, we consider the problem of equivalence of Gibbs states and equilibrium states for continuous potentials on full shift spaces $E^{\mathbb{Z}}$. Sinai, Bowen, Ruelle and others established equivalence under various assumptions on the potential ϕ. At the same time, it is known that every ergodic measure is an equilibrium state for some continuous potential. This means that the equivalence can occur only under some appropriate conditions on the potential function. In this chapter, we identify the necessary and sufficient conditions for the equivalence.

This chapter is based on M. Makhmudov, E. Verbitskiy, *"Gibbs Properties of Equilibrium States"*, arXiv:2503.15263.

3.1 Introduction

DLR Gibbs measures were introduced by Dobrushin (1968) and Lanford and Ruelle (1969) to describe the collective behaviour of a system composed of a large number of components, each governed by a local law. Soon after, the Gibbs measures found applications in other fields of science and various areas of mathematics. In particular, in the early 1970's, Sinai showed that natural invariant measures for hyperbolic dynamical systems are Gibbs measures. The original definition of Gibbs measures in statistical mechanics is somewhat cumbersome in the context of dynamical systems. For this reason, Bowen [3] provided a more suitable definition of Gibbs states from the dynamical systems perspective: a translation-invariant measure μ on $\Omega = E^{\mathbb{Z}}$, E is finite, is called *Gibbs in Bowen's sense*, or *Bowen-Gibbs*, for a continuous potential $\phi : \Omega \to \mathbb{R}$, if for some constants $C > 1$ and P, for all $n \in \mathbb{N}$ and every $\omega \in \Omega$

$$\frac{1}{C} \leq \frac{\mu\big(\{\tilde{\omega} \in \Omega : \tilde{\omega}_0^{n-1} = \omega_0^{n-1}\}\big)}{\exp\big(S_n\phi(\omega) - nP\big)} \leq C, \tag{3.1}$$

where $S_n\phi(\omega) = \sum_{k=0}^{n-1} \phi(S^k\omega)$ and $S : \Omega \to \Omega$ is the left shift on Ω. Subsequently, weaker versions of this notion were introduced. Namely, a translation-invariant measure μ on Ω is *weak Bowen-Gibbs* if μ satisfies

$$\frac{1}{C_n} \leq \frac{\mu(\{\tilde{\omega} \in \Omega : \tilde{\omega}_0^{n-1} = \omega_0^{n-1}\})}{e^{S_n\phi(\omega) - nP}} \leq C_n, \tag{3.2}$$

for some subexponential sequence $\{C_n\}$ of positive real numbers, i.e., $\log C_n = o(n)$.

Let us stress that Bowen's definition of Gibbs measures is actually a theorem in Statistical Mechanics. More specifically, if μ is a translation-invariant DLR-Gibbs measure (see Section 3.2 for the notion), then there exists a continuous function $\phi : \Omega \to \mathbb{R}$ and positive numbers $C_n = C_n(\phi)$ such that (3.2) holds. Therefore, we prefer to use the name of Gibbs measures for measures which are Gibbs in the DLR sense, and we refer to the measures satisfying (3.1) and (3.2) as Bowen-Gibbs and weak Bowen-Gibbs measures.

The notion of weak Gibbs states in Bowen's sense is also somewhat misleading, as it suggests some form of non-Gibbsianity and competes with a notion under the same name in Statistical Mechanics. As we will see below, weak Bowen-Gibbs measures can be bona-fide Gibbs measures in the DLR sense. We also note that there are examples of weak Gibbs measures in the DLR sense, which are weak Bowen-Gibbs as well [17].

Bowen's definition (3.1) and its weak form (3.2) are extremely convenient from the Dynamical Systems point of view. At the same time, using such definitions, one can, in principle, say very little about the conditional probabilities of the underlying measure, which is the classical approach to Gibbs measures in Statistical Mechanics. In fact, there exist Bowen-Gibbs measures that are not DLR-Gibbs measures [2, Subsection 5.2].

Another important notion is that of equilibrium states. A translation-invariant measure μ on $\Omega = E^{\mathbb{Z}}$ is called an *equilibrium state* for a (continuous) potential $\phi : \Omega \to \mathbb{R}$ if

$$h(\mu) + \int_{\Omega} \phi \, d\mu = P(\phi), \tag{3.3}$$

where $P(\phi)$ is the topological pressure of ϕ and $h(\mu)$ is the measure-theoretic entropy of μ. For expansive systems like those we consider in this chapter, equilibrium states always exist. It is easy to see that a weak Bowen-Gibbs state is also an equilibrium state for the same potential [19].

A fundamental result highlighting the breadth of the class of equilibrium states is the following: if $\mu_1, \ldots, \mu_k$ are some ergodic measures on Ω, then one can find a continuous potential $\phi \in C(\Omega)$ such that all these measures are equilibrium states for ϕ [7, 12, 22]. This remarkable generality suggests that equilibrium states can exhibit a wide range of behaviors, and in particular, one cannot expect them to possess any form of Gibbsianity in general. This leads to a natural question: under what conditions on ϕ are the equilibrium states Gibbs, either in the DLR or the Bowen sense? This question has a long history of research:

- For *Hölder continuous* potentials, Sinai proved that equilibrium states are Gibbs in the DLR sense [23, Theorem 1], and the Bowen-Gibbs property was established by Bowen [3]. Haydn extended Sinai's results to the non-symbolic setup [9, 10].

- Ruelle [21] (see also [14, Theorem 5.3.1]) studied the DLR Gibbsianity of the unique equilibrium states for potential ϕ with *summable variations*:

$$\sum_{n\geq 1} \mathtt{var}_n \phi < \infty, \quad \mathtt{var}_n \phi := \sup\left\{\phi(\omega) - \phi(\bar{\omega}) : \omega_j = \bar{\omega}_j,\ 0 \leq j \leq n-1\right\}.$$

  The Bowen-Gibbs property was treated by Keller [14, Theorem 5.2.4, (c)]

- Walters [25] considered potentials satisfying even a weaker condition:

$$\lim_{p\to\infty} \sup_{n\in\mathbb{N}} \mathtt{var}_{[-p,n+p]} S_{n+1}\phi = 0, \quad \text{here} \quad S_{n+1}\phi = \sum_{i=0}^{n} \phi \circ S^i. \tag{3.4}$$

  Walters showed that there exists a unique equilibrium state for potentials satisfying (3.4), and they have the so-called, $g-$measure property, which amounts to saying that the equilibrium state has continuous one-sided conditional probabilities. Combining this with the result of [2], one concludes the DLR Gibbsianity of the unique equilibrium state of a potential in the Walters class. In [11], Haydn and Ruelle extended this to a more general setup than the setup of shift spaces. In the same paper, Haydn and Ruelle also established the Bowen-Gibbs property of the unique equilibrium state for a potential satisfying the *Bowen condition*:

$$\sup_{n\in\mathbb{N}} \mathtt{var}_{[-n,n]} S_{n+1}\phi < +\infty,$$

which is slightly weaker than Walters' original condition.

- More recently, Pfister and Sullivan [20] established the weak Bowen-Gibbs property of equilibrium states for potentials ϕ with *summable oscillations*:

$$\sum_{i=-\infty}^{\infty} \delta_i\phi < +\infty, \quad \delta_i\phi := \sup\{\phi(\omega)-\phi(\bar{\omega}) : \omega_j = \bar{\omega}_j,\ j \neq i\}.$$

  Unlike the preceding conditions, the summable oscillations condition does not imply the uniqueness of the corresponding equilibrium states. However, the DLR Gibbs property of equilibrium states under summable oscillations has not been addressed.

In this chapter, we continue the long line of research on the Gibbsianity of equilibrium states, as discussed above, and also extend the main result of [2], where a similar question has been answered in the case of $g-$measures. In this chapter, we show that under a similar assumption on the potential ϕ, one can establish the Gibbs properties of equilibrium states and vice versa. This assumption on the regularity of the potential $\phi : \Omega \to \mathbb{R}$ is the *extensibility condition*, which requires that for all $a_0, b_0 \in E$ the sequence of functions

$$\rho_n^{a_0,b_0}(\omega) := \sum_{i=-n}^{n} \big(\phi\circ S^i(\omega_{-\infty}^{-1} b_0 \omega_1^{\infty}) - \phi\circ S^i(\omega_{-\infty}^{-1} a_0 \omega_1^{\infty})\big)$$

converges uniformly in $\omega \in \Omega$ as $n \to \infty$. The extensibility condition is not very restrictive. For example, it does not imply the uniqueness of the equilibrium states, unlike the results by Sinai, Bowen, Ruelle and Walters. Furthermore, the extensibility condition covers the previously treated classes, including the class of Hölder continuous potentials, potentials with summable variations, Walter's class, as well as the class of potentials with summable oscillations. However, the potentials in Bowen's class do not necessarily have the extensibility property [2, Section 5.5]. An important example of extensible potentials is the Dyson potential, $\phi^D(\omega) := h\omega_0 + \sum_{n=1}^{\infty} \frac{\beta\omega_0\omega_n}{n^\alpha}$, $\omega \in \{\pm1\}^{\mathbb{Z}}$, which has been extensively studied recently [6, 13, 18], where $h, \beta \in \mathbb{R}$ and $\alpha > 1$.

The following theorem, the first of our main results in this chapter, establishes the Gibbs properties of the equilibrium states of an extensible potential.

Theorem 3.A. *Suppose $\phi \in C(\Omega)$ has the extensibility property. Then any equilibrium state $\mu \in \mathscr{ES}(\phi)$ is*

(1) Gibbs in the Dobrushin-Lanford-Ruelle sense;

(2) weak Bowen-Gibbs relative to the potential ϕ.

The proof of Theorem 3.A is given in Section 3.5, and uses the following idea: for a given potential $\phi \in C(\Omega)$ satisfying the extensibility condition, we construct a natural two-sided Gibbsian specification (a consistent family of regular probability kernels) γ^{ϕ} on $\Omega = E^{\mathbb{Z}}$. Then we show that any translation-invariant DLR-Gibbs state ν for the specification γ^{ϕ} will be an equilibrium state for ϕ. Hence, the set of Gibbs states associated with the specification γ^{ϕ} is a subset of the set of equilibrium states for ϕ. Finally, if we take any equilibrium state $\tau \in \mathscr{ES}(\phi)$ and a DLR-Gibbs state $\mu \in \mathscr{G}_S(\gamma^{\phi})$, we will show that the relative entropy density rate $h(\tau|\mu)$ is zero. This allows us to use the classical variational principle. [15, Theorem 4.1] and conclude that $\tau \in \mathscr{G}_S(\gamma^{\phi})$ as well.

Our second main result is about the translation-invariant Gibbs measures, which in some sense is a converse of Theorem 3.A.

Theorem 3.B. *Assume μ is a translation-invariant DLR Gibbs measure on $\Omega = E^{\mathbb{Z}}$. Then μ is an equilibrium state for a potential with the extensibility property.*

We should note that if μ is a Gibbs measure for a translation-invariant *uniformly absolutely convergent* (UAC) interaction, then the claim is rather standard [16, Theorem 3.2]. However, as demonstrated in [1], not all translation-invariant Gibbs measures are compatible with a translation-invariant UAC interaction. Thus, Theorem 3.B generalises the result in [16, Theorem 3.2] to a broader setting, encompassing all translation-invariant Gibbs measures, including those that are not Gibbs for any translation-invariant UAC interaction.

The proof of Theorem 3.B is constructive and is also given in Section 3.5. In fact, we construct a natural one-sided potential ϕ_{γ} out of the Gibbsian specification γ for μ. Then we show that ϕ_{γ} is extensible, and this allows us to apply Theorem 3.A to ϕ_{γ}.

One might observe an analogy between our results and Sullivan's theorem [24, Theorem 1] in Statistical Mechanics. In Sullivan's theorem, the role of extensible potentials is played by the so-called $\mathscr{L}$-convergent interactions, a notion that is also syntactically similar to the notion of extensibility.

We also note that the statements of Theorems 3.A and 3.B, along with their proofs presented in this chapter, naturally extend to higher-dimensional lattices $\mathbb{Z}^d$ ordered lexicographically (see Proposition 2.4.5 and Proposition 2.6.3 in Chapter 2). Since there is no substantial difference in the proofs, we only focus on the one-dimensional lattice $\mathbb{Z}$.

The diagram on the right summarises the relationship between equilibrium states and various notions of Gibbs states.

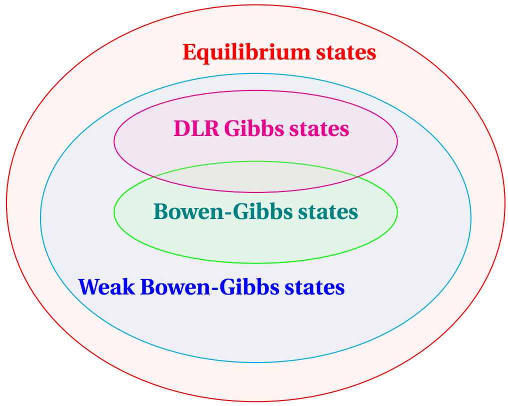

Figure 3.1

This chapter is organised as follows:

- In Section 3.2, we introduce the basic concepts in the DLR Gibbs formalism such as Gibbs measures, specifications and interactions. Here, we also recall some important results, such as the variational principle from the classical DLR Gibbs formalism.
- In Section 3.3, we discuss the motivation behind Bowen's definition of Gibbs states.
- In Section 3.4, we discuss the relationship between the extensible potentials and Gibbsian specifications.
- Section 3.5 is dedicated to the proofs of the main results in this chapter.

3.2 DLR Gibbs Formalism

The theory of Gibbs states, which is put forward by Dobrushin, Lanford, and Ruelle, is very flexible and allows one to define Gibbs states on very general lattice spaces $E^{\mathbb{L}}$, where E is a Polish space and $\mathbb{L}$ is a countable set. In the present chapter, we are primarily interested in probability measures on $\Omega = E^{\mathbb{Z}}$, E is finite, which are invariant under the left shift $S:\Omega\to\Omega$.

3.2.1 Specifications, Interactions, and Gibbs states in Statistical Mechanics

The standard Statistical Mechanics description of Gibbs states is rather different from the definitions of Bowen-Gibbs and weak Bowen-Gibbs measures. The principal point is the explicit description of the family of ***conditional expectations*** indexed by finite subsets Λ of $\mathbb{Z}$. More precisely, in Statistical Mechanics, one starts with a family of regular conditional expectations, which for $f:\Omega\to\mathbb{R}$, given by

$$\gamma_\Lambda(f|\omega) := \sum_{\xi_\Lambda\in E^\Lambda} \gamma_\Lambda(\xi_\Lambda|\omega_{\Lambda^c}) f(\xi_\Lambda\omega_{\Lambda^c}),$$

where

$$\gamma_\Lambda(\xi_\Lambda|\omega_{\Lambda^c}) = \frac{\exp\Big(-H_\Lambda(\xi_\Lambda\omega_{\Lambda^c})\Big)}{Z_\Lambda(\omega)}, \quad Z_\Lambda(\omega) = \sum_{\zeta_\Lambda \in E^\Lambda} \exp\Big(-H_\Lambda(\xi_\Lambda\omega_{\Lambda^c})\Big). \tag{3.5}$$

In order to guarantee the tower property of the conditional expectations, one needs to assume *consistency* of γ_Λ's: $\gamma_\Lambda = \gamma_\Lambda \circ \gamma_V$ for $V \subset \Lambda$, where for $f : \Omega \to \mathbb{R}$ measurable and $\omega \in \Omega$, $(\gamma_\Lambda \circ \gamma_V)(f|\omega) := \int_\Omega \gamma_V(f|\eta)\gamma_\Lambda(d\eta|\omega)$. The latter is ensured if the functions $H_\Lambda : \Omega \to \mathbb{R}$ – called a *Hamiltonian* in Λ– are of a rather special form:

$$H_\Lambda(\omega) = \sum_{\substack{V \Subset \mathbb{Z} \\ V \cap \Lambda \neq \varnothing}} \Phi_V(\omega_V), \tag{3.6}$$

here the summation is taken over all finite subsets of $\mathbb{Z}$ (denoted by $\Subset \mathbb{Z}$) which have non-empty intersection with Λ. Here $\Phi = \{\Phi_V, V \Subset \mathbb{Z}\}$ is called an ***interaction*** and each function $\Phi_V : \Omega \to \mathbb{R}$ is ***local***, meaning that the value of $\Phi_V(\omega)$ depends only on the values of ω within V, hence, we write $\Phi_V(\omega_V)$. In order for these expressions to make sense, one needs to assume a suitable form of summability in (3.6). The standard and sufficient assumption is ***uniform absolute convergence*** (UAC): for all $i \in \mathbb{Z}$,

$$\sum_{i \in V \Subset \mathbb{Z}} \|\Phi_V\|_\infty = \sum_{i \in V \Subset \mathbb{Z}} \sup_{\omega\in\Omega} |\Phi_V(\omega_V)| < \infty.$$

If Φ is an UAC interaction, the corresponding *specification* $\gamma = (\gamma_\Lambda)_{\Lambda \Subset \mathbb{Z}}$ defined by (3.5), is

- *non-null*: for all Λ, $\inf_\omega \gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) > 0$,
- *continuous*: for every Λ, $\omega \mapsto \gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c})$ is continuous.

In statistical mechanics, the second property is often referred to as quasi-locality. An important property of positive specifications, which will be used in the proofs, is the so-called *bar moving property*:

$$\frac{\gamma_\Delta(\xi_\Delta|\omega_{\Delta^c})}{\gamma_\Delta(\zeta_\Delta|\omega_{\Delta^c})} = \frac{\gamma_\Lambda(\xi_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})}{\gamma_\Lambda(\zeta_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})}, \tag{3.7}$$

for all $\xi, \zeta, \omega \in \Omega$ and every $\Delta \subset \Lambda \Subset \mathbb{Z}$. The bar moving property is equivalent to the consistency condition of specifications.

A non-null continuous specification $\gamma = \{\gamma_\Lambda\}$ is called ***Gibbsian***.

Definition 3.2.1. *Suppose $\gamma = \{\gamma_\Lambda\}$ is a Gibbsian specification on Ω. The measure μ is called Gibbs for γ, denoted by $\mu \in \mathscr{G}(\gamma)$, if for every $\Lambda \Subset \mathbb{Z}$,*

$$\mu(\omega_\Lambda|\omega_{\Lambda^c}) = \gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}), \ \text{for } \mu - a.e. \ \omega \in \Omega,$$

equivalently, if the DLR equations hold: for every $f \in C(\Omega)$ and $\Lambda \Subset \mathbb{Z}$,

$$\int_\Omega \gamma_\Lambda(f|\omega)\mu(d\omega) = \int_\Omega f(\omega)\mu(d\omega).$$

For any Gibbsian specification γ, the set of corresponding Gibbs measures $\mathscr{G}(\gamma)$ is a non-empty convex set. In case, $\mathscr{G}(\gamma)$ consists of multiple measures, one says that γ exhibits phase transitions.

3.2.2 Translation-invariance and the Variational Principle

The modern approach to Gibbsian formalism is to think about Gibbsian specifications γ in an ***interaction independent*** fashion. The reason for this is that a measure μ can be consistent with at most one Gibbsian specification, while there are infinitely many interactions Φ giving rise to the same Gibbsian specification. In fact, it was proven by Kozlov that for any Gibbsian specification γ there exists a (in fact many) UAC interaction Φ such that $\gamma = \gamma^\Phi$. However, such a representation is not always possible in a way that respects translation invariance. It is shown in [1] that there exists a Gibbsian specification γ on $\{0,1\}^\mathbb{Z}$ that is *translation-invariant* – meaning $\gamma_{\Lambda+1} = \gamma_\Lambda \circ S$ for every $\Lambda \Subset \mathbb{Z}$ – but can not be associated with any *translation-invariant UAC interaction* Φ, where $\Phi_\Lambda = \Phi_\Lambda \circ S$ for all $\Lambda \Subset \mathbb{Z}$, via (3.5). Nevertheless, many important results in Statistical Mechanics, including the *variational principle*, can be formulated independently of interactions.

Theorem 3.2.2. [15] *Let γ be a translation-invariant Gibbsian specification and $\mu \in \mathscr{G}_S(\gamma)$. Then for all $\tau \in \mathscr{M}_{1,S}(\Omega)$, the specific relative entropy $h(\tau|\mu)$ exists and*

$$h(\tau|\mu) = 0 \iff \tau \in \mathscr{G}_S(\gamma).$$

In Chapter 2, we proved some technical lemmas about translation-invariant specifications, which would be useful for us. For $n \in \mathbb{N}$, we set $\Lambda_n := [-n, n] \cap \mathbb{Z}$. We fix a letter $\mathfrak{a}$ in the finite alphabet E and we will use the notation $\mathbf{a}$, to denote the constant configuration consisting of $\mathfrak{a}$'s.

Lemma 3.2.3. *Suppose γ is a translation-invariant Gibbsian specification on Ω. Then*

(i)

$$\sup_{\sigma,\omega,\eta\in\Omega} \left|\log\Big(\frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\omega_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\omega_{[0,n]^c})} \cdot \frac{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})}{\gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})}\Big)\right| = o(n). \tag{3.8}$$

(ii) the sequence $\left\{-\frac{1}{|\Lambda_n|}\log\gamma_{\Lambda_n}(\mathbf{a}_{\Lambda_n}|\omega_{\Lambda_n^c})\right\}_{n\in\mathbb{N}}$ *converges uniformly in* $\omega\in\Omega$ *to a constant which we denote by* $P^{\mathbf{a}}(\gamma)$*.*

3.3 Bowen's property of Gibbs measures

By comparing the definitions of Gibbs measures in Statistical Mechanics and that of Bowen, it is immediately clear why Bowen's definition is so attractive and popular for dynamicists: it captures the most important, from the Dynamical Systems point of view, properties of Gibbsian states – uniform estimates on measures of cylindric sets in terms of ergodic averages of the potential function. In fact, whether the measure has the Bowen property or the weak Bowen property, rarely makes any difference in Dynamical Systems: the subexponential bound is as good as the uniform bound in practically any computations.

Nevertheless, Bowen was fully aware that his definition of Gibbs states is not the same as in Statistical Mechanics: *"In statistical mechanics, Gibbs states are not defined by the above theorem. We have ignored many subtleties that come up in more complicated systems"*, [3, page 6]. To introduce the definition, Bowen was motivated by an example ([3, page 5]) of a translation-invariant pair interaction Φ, $\Phi_V\not\equiv 0$ only if $V=\{k\}$ or $V=\{k,n\}$, $k,n\in\mathbb{Z}$, satisfying a strong summability condition

$$||\Phi_{\{0\}}||_\infty+\frac{1}{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}|n|\cdot||\Phi_{\{0,n\}}||_\infty<\infty. \tag{3.9}$$

The above condition is a special case of a well-known uniqueness condition in thermodynamic formalism [8, 22]:

$$\sum_{0\in V\Subset\mathbb{Z}}\frac{\mathtt{diam}(V)}{|V|}\cdot||\Phi_V||_\infty<\infty. \tag{3.10}$$

Let us now discuss the Bowen-Gibbs and the weak Bowen-Gibbs properties of DLR Gibbs states.

Theorem 3.3.1. *Suppose* $\Phi=\{\Phi_V\}_{V\Subset\mathbb{Z}}$ *is a translation-invariant UAC interaction and let* $\phi=-\sum_{0\in V\Subset\mathbb{Z}_+}\Phi_V$*. Then there exists a sequence* $\{C_n\}$ *with* $n^{-1}\log C_n\to 0$*, such that for every translation-invariant Gibbs measure* μ *for* Φ*, for all* n *and* $\omega\in\Omega$*, one has*

$$\frac{1}{C_n}\le\frac{\mu\big(\{\tilde\omega\in\Omega:\ \tilde\omega_0^{n-1}=\omega_0^{n-1}\}\big)}{\exp\big(S_n\phi(\omega)-nP(\phi)\big)}\le C_n.$$

If, furthermore, the interaction Φ *satisfies a stronger summability condition (3.10), then there exists a unique Gibbs measure* μ *for* Φ*, and for some* $C>1$*, every* $n\ge 1$

and all $\omega \in \Omega$,

$$\frac{1}{C} \le \frac{\mu(\{\tilde{\omega} \in \Omega : \ \tilde{\omega}_0^{n-1} = \omega_0^{n-1}\})}{\exp\bigl(S_n\phi(\omega) - nP(\phi)\bigr)} \le C. \tag{3.11}$$

Let us sketch the proof of this theorem using known results in Statistical Mechanics. The first claim is standard [8, Theorem 15.23]. Applying the DLR equations to the indicator function of the cylinder set $[\sigma_0^{n-1}]$, one concludes that

$$\mu([\sigma_0^{n-1}]) = \int \frac{\exp\bigl(-H_{\Lambda_n}(\sigma_{\Lambda_n}\eta_{\Lambda_n^c})\bigr)}{Z_{\Lambda_n}(\eta)}\mu(d\eta).$$

The sequence of functions $\dfrac{1}{|\Lambda_n|}\log Z_{\Lambda_n}(\eta)$ converges to the pressure $P(\Phi) = P(\phi)$ uniformly in η as $n \to \infty$ [8, Theorem 15.30, part (a)]. Thus $\log Z_{\Lambda_n}(\eta) = \exp(|\Lambda_n|P + o(n))$.

To study the numerator, we use the estimate (15.25) in [8]:

$$\sup_{\sigma,\eta\in\Omega}\Bigl|\sum_{i\in\Lambda_n}\phi\circ S^i(\sigma) + H_{\Lambda_n}(\sigma_{\Lambda_n}\eta_{\Lambda_n^c})\Bigr| \le \sum_{i\in\Lambda_n}\sum_{\substack{V\ni i\\ V\not\subset\Lambda_n}}\|\Phi_V\|_\infty, \tag{3.12}$$

and the fact that the uniformly absolute convergence of the interaction Φ ensures that

$$\sum_{i\in\Lambda_n}\sum_{\substack{V\ni i\\ V\not\subset\Lambda_n}}\|\Phi_V\|_\infty = o(|\Lambda_n|).$$

The uniqueness of the Gibbs measures under (3.10) follows from Theorem 8.39 in [8] (see also Comment 8.41 and equation (8.42) in [8]). Note that (3.10) implies that $\sum_{i\in\Lambda_n}\sum_{\substack{V\ni i\\ V\not\subset\Lambda_n}}\|\Phi_V\|_\infty$ remains bounded as $n \to \infty$, in fact, for all $n \in \mathbb{N}$, one has that

$$\sum_{i\in\Lambda_n}\sum_{\substack{V\ni i\\ V\not\subset\Lambda_n}}\|\Phi_V\|_\infty \le \sum_{0\in V\Subset\mathbb{Z}_+}\mathtt{diam}(V)\cdot\|\Phi_V\|_\infty = \sum_{0\in V\Subset\mathbb{Z}}\frac{\mathtt{diam}(V)}{|V|}\cdot\|\Phi_V\|_\infty =: D.$$

This together with (3.12) yields that

$$\sup_{\sigma,\eta\in\Omega}\Bigl|\sum_{i\in\Lambda_n}\phi\circ S^i(\sigma) - \sum_{i\in\Lambda_n}\phi\circ S^i(\sigma_{\Lambda_n}\eta_{\Lambda_n^c})\Bigr| \le 2D, \tag{3.13}$$

i.e., ϕ satisfies Bowen's condition [27]. Then (3.11) follows from Theorem 4.6 in [27] and from the fact that the translation-invariant Gibbs measures for Φ are equilibrium states for ϕ [8, Theorem 15.39].

Remark 3.3.2. *Note that it is not a coincidence that the condition (3.10) implying the Bowen property, also implies uniqueness. Indeed, suppose μ and ν are two ergodic measures on Ω with the Bowen-Gibbs property for some continuous potential ϕ. Then the Bowen-Gibbs property implies that*

$$\frac{1}{C} \le \frac{\mu([\omega_0^n])}{\nu([\omega_0^n])} \le C$$

for all n and every $\omega \in \Omega$, and hence the measures μ and ν are equivalent, and thus are equal.

3.4 Potentials, Cocycles, and Specifications

An alternative, more dynamical approach to DLR-Gibbs measures, also known as the Ruelle-Capocaccia approach, was introduced in [4]. However, as demonstrated in the original work [4], for lattice systems, which is the setting in this chapter, the Ruelle-Capocaccia definition of Gibbs measures coincides with the specification-based definition given in this chapter. Keller's book [14, Chapter 5] provides an excellent summary of Gibbs measures following the Ruelle-Capocaccia approach under the assumption that the underlying potential has summable variations. Below we explore how this approach extends to cases where the potential lacks summable variations but still has the extensibility property.

Recall that the extensibility condition requires that for all $\omega \in \Omega$ and every $a, \tilde{a} \in E$, the sequence of functions

$$\rho_n^{a,\tilde{a}}(\omega) = \sum_{i=-n}^{n} \left[\phi(S^i\omega^a) - \phi(S^i\omega^{\tilde{a}})\right], \quad n \ge 0, \tag{3.14}$$

converges uniformly as $n \to \infty$. Here, we use the notation $\omega^a = (\omega_k^a)_{k\in\mathbb{Z}}$ is given by

$$\omega_k^a = \begin{cases} a, & k = 0, \\ \omega_k, & k \neq 0. \end{cases}$$

Therefore, we can define a continuous function $\rho_n^{a,\tilde{a}}(\omega) : \Omega \to \mathbb{R}$,

$$\rho(\omega^a, \omega^{\tilde{a}}) = \lim_{n\to\infty} \rho_n^{a,\tilde{a}}(\omega).$$

Proposition 3.4.1. *Suppose ϕ satisfies the extensibility condition, then for any pair $\xi, \eta \in \Omega$, such that the set $\{k \in \mathbb{Z} : \xi_k \neq \eta_k\}$ is finite, the sequence of continuous functions*

$$\rho_n(\xi, \eta) = \sum_{i=-n}^{n} \left[\phi(S^i\xi) - \phi(S^i\eta)\right], \quad n \ge 0, \tag{3.15}$$

converges. Furthermore,

(1) *the limiting function* $\rho(\xi,\eta) = \lim_n \rho_n(\xi,\eta)$ *is a* ***cocycle****, i.e., for every* $\xi,\eta,\zeta \in \Omega$ *with* $\xi_i = \eta_i = \zeta_i$ *for all* i *with* $|i| \gg 1$*, one has*

$$\rho(\xi,\zeta) = \rho(\xi,\eta) + \rho(\eta,\zeta); \tag{3.16}$$

(2) ρ *is translation-invariant in the sense that for every pair* (ξ,η) *with* $\{k \in \mathbb{Z} : \xi_k \neq \eta_k\}$ *finite,*

$$\rho(\xi,\eta) = \rho(S\xi, S\eta). \tag{3.17}$$

(3) *for every* $\Lambda \Subset \mathbb{Z}$ *and* $\eta_\Lambda, \zeta_\Lambda \in E^\Lambda$, $\rho(\eta_\Lambda \xi_{\mathbb{Z}\setminus\Lambda}, \zeta_\Lambda \xi_{\mathbb{Z}\setminus\Lambda})$ *is a continuous function of* ξ.

Proof. We carry out the proof in two steps.
First Step: Let $\xi,\eta \in \Omega$ such that for some $k \in \mathbb{Z}$, $\xi_{\mathbb{Z}\setminus\{k\}} = \eta_{\mathbb{Z}\setminus\{k\}}$. Without loss of generality, let $k > 0$. Then

$$\begin{aligned}
\sum_{i=-n}^{n} [\phi \circ S^{i-k}(S^k\xi) - \phi \circ S^{i-k}(S^k\eta)] &= \sum_{i=-n-k}^{n-k} [\phi \circ S^{i}(S^k\xi) - \phi \circ S^{i}(S^k\eta)] \\
&= \sum_{i=-n}^{n} [\phi \circ S^{i}(S^k\xi) - \phi \circ S^{i}(S^k\eta)] \qquad (3.18) \\
&- \sum_{i=n-k+1}^{n} [\phi \circ S^{i}(S^k\xi) - \phi \circ S^{i}(S^k\eta)] \qquad (3.19) \\
&+ \sum_{i=-n-k}^{-n-1} [\phi \circ S^{i}(S^k\xi) - \phi \circ S^{i}(S^k\eta)]. \qquad (3.20)
\end{aligned}$$

Since $(S^k\xi)_{\mathbb{Z}\setminus\{0\}} = (S^k\eta)_{\mathbb{Z}\setminus\{0\}}$, the extensibility property of ϕ yields that the sum in (3.18) converges uniformly to $\rho^\phi(S^k\xi, S^k\eta)$ as $n \to \infty$. Note that for any $i \in \mathbb{Z}$,

$$|\phi \circ S^i(S^k\xi) - \phi \circ S^i(S^k\eta)| \le \delta_i \phi \le \mathtt{var}_{(-|i|,\,|i|)}\phi \xrightarrow[|i|\to\infty]{} 0.$$

Thus for (3.19) and (3.20), one has that

$$\Big|\sum_{i=n-k+1}^{n} [\phi \circ S^{-i}(S^k\xi) - \phi \circ S^{-i}(S^k\eta)]\Big| \le k \cdot \mathtt{var}_{(k-n,\,n-k)}\phi$$

and

$$\Big|\sum_{i=-n-k}^{-n-1} [\phi \circ S^{-i}(S^k\xi) - \phi \circ S^{-i}(S^k\eta)]\Big| \le k \cdot \mathtt{var}_{(k-n,\,n-k)}\phi.$$

Thus since k is fixed and $\mathtt{var}_{(k-n,\,n-k)}\phi \xrightarrow[n\to\infty]{} 0$, both sums in (3.19) and (3.20) converge uniformly to 0 as $n \to \infty$. Therefore, ρ is defined at (ξ,η) and $\rho(\xi,\eta) = \rho(S^k\xi, S^k\eta)$.

Second Step: Let $\xi,\eta\in\Omega$ such that for some $\Lambda\Subset\mathbb{Z}$, $\xi_{\mathbb{Z}\setminus\Lambda}=\eta_{\mathbb{Z}\setminus\Lambda}$. Then there exists $m\in\mathbb{N}$ such that $\Lambda\subset[-m,m]\cap\mathbb{Z}$. Then for $n>m$, one has

$$\sum_{i=-n}^{n}[\phi\circ S^i(\xi)-\phi\circ S^i(\eta)]=\sum_{j=-m}^{m}\sum_{i=-n}^{n}[\phi\circ S^i(\xi_{-\infty}^{-m-1}\eta_{-m}^{j-1}\xi_j^{\infty})-\phi\circ S^i(\xi_{-\infty}^{-m-1}\eta_{-m}^{j}\xi_{j+1}^{\infty})].$$

By the first step, the sum over i on the RHS of the last equation above converges uniformly for each j as $n\to\infty$.
We now address statements (1)-(3). Claim (1) follows directly from the definition of ρ. Equations (3.16) and (3.17) easily follow from the first and second steps discussed above. The continuity of the map $\xi\mapsto\rho(\eta_\Lambda\xi_{\mathbb{Z}\setminus\Lambda},\zeta_\Lambda\xi_{\mathbb{Z}\setminus\Lambda})$ for all $\Lambda\Subset\mathbb{Z}$ and $\eta,\zeta\in\Omega$ is a consequence of the uniform convergence of (3.15).

□

Now, we shall discuss how to associate a Gibbsian specification with an extensible potential and vice versa. We have the following theorem.

Theorem 3.4.2. *(i) Suppose $\phi:\Omega\to\mathbb{R}$ is a continuous function with extensibility property, then $\gamma^\phi=(\gamma_\Lambda^\phi)_{\Lambda\Subset\mathbb{Z}}$ given by*

$$\gamma_\Lambda^\phi(\omega_\Lambda|\omega_{\Lambda^c})=\Big(\sum_{\xi_\Lambda\in E^\Lambda}e^{\rho^\phi(\xi_\Lambda\omega_{\mathbb{Z}\setminus\Lambda},\,\omega)}\Big)^{-1},\quad\omega\in\Omega \tag{3.21}$$

is a translation invariant Gibbsian specification.

(ii) Suppose $\gamma=(\gamma_\Lambda)_{\Lambda\Subset\mathbb{Z}}$ is a translation invariant Gibbsian specification, then

$$\phi_\gamma(\omega)=\log\frac{\gamma_{\{0\}}(\omega_0|\mathbf{a}_{-\infty}^{-1}\omega_1^\infty)}{\gamma_{\{0\}}(\mathbf{a}_0|\mathbf{a}_{-\infty}^{-1}\omega_1^\infty)},\quad\omega\in\Omega \tag{3.22}$$

is a continuous function with extensibility property such that $\gamma^{\phi_\gamma}=\gamma$.

Proof. ***(i):*** This part is proven in Subsection 2.2.3.

(ii): The second part has been established in greater generality in Proposition 2.4.5; here, we consider the particular case where ρ is the Dirac measure $\delta_{\mathbf{a}}$ and $d=1$. Owing to its concreteness and reduced level of abstraction, we provide a separate proof for this case.

Pick any $a_0,b_0\in E$ and $\omega\in\Omega$. Since ϕ_γ is a one-sided function, one can check that for any $n\in\mathbb{N}$,

$$\rho_n^{a_0,b_0}(\omega)-\rho_0^{a_0,b_0}(\omega)=\sum_{i=1}^{n}\Big[\phi_\gamma\circ S^{-i}(b_0\omega_{\{0\}^c})-\phi_\gamma\circ S^{-i}(a_0\omega_{\{0\}^c})\Big] \tag{3.23}$$

and for the right-hand side of the above equation, one has

$$RHS = \sum_{i=1}^{n} \log \frac{\gamma_{\{0\}}((\omega_{-i})_0|\mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1} b_0 \omega_1^{\infty})_1^{\infty})}{\gamma_{\{0\}}(\mathbf{a}_0|\mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1} b_0 \omega_1^{\infty})_1^{\infty})} \cdot \frac{\gamma_{\{0\}}(\mathbf{a}_0|\mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1} a_0 \omega_1^{\infty})_1^{\infty})}{\gamma_{\{0\}}((\omega_{-i})_0|\mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1} a_0 \omega_1^{\infty})_1^{\infty})}.$$

Then, by applying the bar moving property, one gets

$$\begin{aligned}\rho_n^{a_0,b_0}(\omega) - \rho_0^{a_0,b_0}(\omega) = \sum_{i=1}^{n} \log \frac{\gamma_{\{0,i\}}((\omega_{-i})_0 (b_0)_i|\mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}{\gamma_{\{0,i\}}(\mathbf{a}_0 (b_0)_i|\mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})} \\ + \sum_{i=1}^{n} \log \frac{\gamma_{\{0,i\}}(\mathbf{a}_0 (a_0)_i|\mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}{\gamma_{\{0,i\}}((\omega_{-i})_0 (a_0)_i|\mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}\end{aligned}$$

and thus, by applying the bar moving property once more,

$$\begin{aligned}\rho_n^{a_0,b_0} - \rho_0^{a_0,b_0} = \sum_{i=1}^{n} \log \frac{\gamma_{\{i\}}((b_0)_i|\mathbf{a}_{-\mathbb{N}}(\omega_{-i}^{-1})_0^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}{\gamma_{\{i\}}((a_0)_i|\mathbf{a}_{-\mathbb{N}}(\omega_{-i}^{-1})_0^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})} \\ + \sum_{i=1}^{n} \log \frac{\gamma_{\{i\}}((a_0)_i|\mathbf{a}_{\mathbb{Z}_-}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}{\gamma_{\{i\}}((b_0)_i|\mathbf{a}_{\mathbb{Z}_-}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})},\end{aligned}$$

here $\mathbb{Z}_-$ denotes $-\mathbb{N} \cup \{0\}$. Hence, by the translation-invariance of γ,

$$\begin{aligned}\rho_n^{a_0,b_0}(\omega) - \rho_0^{a_0,b_0}(\omega) &= \sum_{i=1}^{n} \log \frac{\gamma_{\{0\}}(b_0|\mathbf{a}_{-\infty}^{-i-1}\,\omega_{-i}^{-1}\,\omega_1^{\infty})}{\gamma_{\{0\}}(a_0|\mathbf{a}_{-\infty}^{-i-1}\,\omega_{-i}^{-1}\,\omega_1^{\infty})} \cdot \frac{\gamma_{\{0\}}(a_0|\mathbf{a}_{-\infty}^{-i}\,\omega_{-i+1}^{-1}\,\omega_1^{\infty})}{\gamma_{\{0\}}(b_0|\mathbf{a}_{-\infty}^{-i}\,\omega_{-i+1}^{-1}\,\omega_1^{\infty})} \\ &= \sum_{i=1}^{n} \log \frac{\gamma_{\{0\}}(b_0|\mathbf{a}_{-\infty}^{-i-1}\,\omega_{-i}^{-1}\,\omega_1^{\infty})}{\gamma_{\{0\}}(a_0|\mathbf{a}_{-\infty}^{-i-1}\,\omega_{-i}^{-1}\,\omega_1^{\infty})} - \sum_{i=0}^{n-1} \log \frac{\gamma_{\{0\}}(b_0|\mathbf{a}_{-\infty}^{-i-1}\,\omega_{-i}^{-1}\,\omega_1^{\infty})}{\gamma_{\{0\}}(a_0|\mathbf{a}_{-\infty}^{-i-1}\,\omega_{-i}^{-1}\,\omega_1^{\infty})} \\ &= \log \frac{\gamma_{\{0\}}(b_0|\mathbf{a}_{-\infty}^{-n-1}\,\omega_{-n}^{-1}\,\omega_1^{\infty})}{\gamma_{\{0\}}(a_0|\mathbf{a}_{-\infty}^{-n-1}\,\omega_{-n}^{-1}\,\omega_1^{\infty})} - \log \frac{\gamma_{\{0\}}(b_0|\mathbf{a}_{-\infty}^{-1}\,\omega_1^{\infty})}{\gamma_{\{0\}}(a_0|\mathbf{a}_{-\infty}^{-1}\,\omega_1^{\infty})}.\end{aligned}$$

Thus

$$\rho_n^{a_0,b_0}(\omega) = \log \frac{\gamma_{\{0\}}(b_0|\mathbf{a}_{-\infty}^{-n-1}\,\omega_{-n}^{-1}\,\omega_1^{\infty})}{\gamma_{\{0\}}(a_0|\mathbf{a}_{-\infty}^{-n-1}\,\omega_{-n}^{-1}\,\omega_1^{\infty})} \xrightarrow[\substack{n\to\infty \\ \omega\in\Omega}]{} \log \frac{\gamma_{\{0\}}(b_0|\omega_{-\infty}^{-1}\,\omega_1^{\infty})}{\gamma_{\{0\}}(a_0|\omega_{-\infty}^{-1}\,\omega_1^{\infty})}. \tag{3.24}$$

The last limit also shows that $\gamma^{\phi_\gamma} = \gamma$. □

3.5 Proofs

The proofs of Theorem 3.A and Theorem 3.B will be based on two lemmas. Our first lemma states the following.

Lemma 3.5.1. *Let $\phi \in C(\Omega)$ be an extensible function and γ^{ϕ} be the associated Gibbsian specification. Then the one-sided extensible function $\phi_{\gamma^{\phi}} \in C(\Omega_+)$ is weakly cohomologous to ϕ, i.e., there exists $C \in \mathbb{R}$ such that for all $\tau \in \mathcal{M}_{1,S}(\Omega)$,*

$$\int_{\Omega} \phi_{\gamma^{\phi}}\, d\tau = \int_{\Omega} \phi\, d\tau + C. \tag{3.25}$$

Remark 3.5.2. *The second part of Theorem 3.4.2 implies that the chain $\gamma \to \phi_{\gamma} \to \gamma^{\phi_{\gamma}}$ is closed, i.e., $\gamma = \gamma^{\phi_{\gamma}}$. For an extensible function ϕ, the diagram $\phi \to \gamma^{\phi} \to \phi_{\gamma^{\phi}}$ is, in general, not closed, i.e., it is not always true that $\phi = \phi_{\gamma^{\phi}}$.*

Proof. Note that $\phi_{\gamma^{\phi}}$ is a half-line function. For any $\omega_{\mathbb{Z}_+} \in \Omega_+$, one can easily check the following:

$$\phi_{\gamma^{\phi}}(\omega_{\mathbb{Z}_+}) = \lim_{n\to\infty} \sum_{i=-n}^{n} \left[\phi \circ S^i(\mathbf{a}_{-\infty}^{-1}\omega_0^{\infty}) - \phi \circ S^i(\mathbf{a}_{-\infty}^{0}\omega_1^{\infty})\right] \tag{3.26}$$

and the above limit is uniform on $\omega_{\mathbb{Z}_+} \in \Omega_+$. For any $\tau \in \mathcal{M}_{1,S}(\Omega_+)$ and $n \in \mathbb{N}$, denote

$$I_n(\tau) \;:=\; \int_{\Omega} \sum_{i=-n}^{n} \left[\phi \circ S^i(\mathbf{a}_{-\infty}^{-1}\omega_0^{\infty}) - \phi \circ S^i(\mathbf{a}_{-\infty}^{0}\omega_1^{\infty})\right]\tau(d\omega).$$

For any $\ell_1, \ell_2 \in \mathbb{Z} \cup \{-\infty, \infty\}$ with $\ell_1 \le \ell_2$, define a transformation

$$\Theta_{[\ell_1,\ell_2]}(\omega)_i := \begin{cases} \omega_i, & \text{if } i \notin [\ell_1,\ell_2]; \\ \mathfrak{a}, & \text{if } i \in [\ell_1,\ell_2]. \end{cases} \tag{3.27}$$

If $\ell_1 \not\le \ell_2$, then $[\ell_1,\ell_2] = \emptyset$, therefore, for all $\omega \in \Omega$, we set $\Theta_{[\ell_1 \ell_2]}(\omega) = \omega$. Note that the families $\{S_\ell : \ell \in \mathbb{Z}\}$ and $\{\Theta_{[\ell_1,\ell_2]} : \ell_1,\ell_2 \in \mathbb{Z}, \ell_1 \le \ell_2\}$ of the transformations on Ω have the following commutativity-type property: for any $\ell_{1,2,3} \in \mathbb{Z}$ with $\ell_1 \le \ell_2$,

$$S_{\ell_3} \circ \Theta_{[\ell_1,\ell_2]} = \Theta_{[\ell_1-\ell_3,\ell_2-\ell_3]} \circ S_{\ell_3}. \tag{3.28}$$

Thus $I_n(\tau)$ is written in terms of the transformations Θ as follows:

$$I_n(\tau) = \sum_{i=-n}^{n} \int_{\Omega} \phi \circ \Theta_{(-\infty,-i-1]} \circ S^i(\omega)\tau(d\omega) - \sum_{i=-n}^{n} \int_{\Omega} \phi \circ \Theta_{(-\infty,-i]} \circ S^i(\omega)\tau(d\omega) \tag{3.29}$$

Then, by the translation-invariance of measure τ,

$$I_n(\tau) \;=\; \sum_{i=-n}^{n} \int_{\Omega} \phi \circ \Theta_{(-\infty,-i-1]} d\tau - \sum_{i=-n}^{n} \int_{\Omega} \phi \circ \Theta_{(-\infty,-i]} d\tau \tag{3.30}$$

$$= \int_{\Omega} \phi \circ \Theta_{(-\infty,-n-1]} d\tau - \int_{\Omega} \phi \circ \Theta_{(-\infty,n]} d\tau. \tag{3.31}$$

By continuity of ϕ, the sequences $\{\phi\circ\Theta_{(-\infty,-n-1]}(\omega)\}_{n\in\mathbb{Z}_+}$ and $\{\phi\circ\Theta_{(-\infty,n]}(\omega)\}_{n\in\mathbb{Z}_+}$ converge uniformly in $\omega\in\Omega$ to $\phi(\omega)$ and $\phi(\mathbf{a})$, respectively, as $n\to\infty$. Thus one concludes that

$$\lim_{n\to\infty} I_n(\tau)=\int_\Omega \phi\, d\tau-\phi(\mathbf{a}). \tag{3.32}$$

Since the limit in (3.26) is uniform, we conclude from (3.32) that

$$\int_\Omega \phi_{\gamma^\phi}(\omega)\tau(d\omega)=\int_\Omega \phi(\omega)\tau(d\omega)-\phi(\mathbf{a}). \tag{3.33}$$

□

Now, we formulate the second lemma.

Lemma 3.5.3. *Let γ be a translation-invariant Gibbsian specification and ϕ_γ be the associated extensible function. Then*

(i) Every translation-invariant Gibbs state $\mu\in\mathscr{G}_S(\gamma)$ is weak Bowen-Gibbs with respect to ϕ_γ, i.e., μ satisfies (3.2);

(ii) The set of translation-invariant Gibbs states for γ coincides with the set of the equilibrium states for ϕ_γ, i.e., $\mathscr{G}_S(\gamma)=\mathscr{ES}(\phi_\gamma)$.

Proof. ***(i):*** Now we shall prove that any translation-invariant DLR Gibbs measure μ prescribed by a specification γ is weak Bowen-Gibbs relative to the potential ϕ_γ.

The first part of Lemma 3.2.3 and translation-invariance of the specification γ imply

$$\lim_{n\to\infty} -\frac{1}{n}\log\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\omega_{[0,n]^c})=P^{\mathbf{a}}(\gamma) \tag{3.34}$$

and the convergence is uniform in $\omega\in\Omega$.

Now consider a configuration $\sigma\in\Omega_+$ and a cylindric set $[\sigma_0^n]$, then by the DLR equations,

$$\begin{aligned}\mu([\sigma_0^n]) &= \int_X \gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})\mu(d\eta)\\ &= \int_X \frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})}\cdot\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})\mu(d\eta). \end{aligned} \tag{3.35}$$

By translation-invariance of the specification γ, Lemma 3.2.3 yields that

$$\mu([\sigma_0^n]) = \int_X \frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})}\cdot e^{-nP^{\mathbf{a}}(\gamma)+o(n)}\mu(d\eta),$$

here the error factor $o(n)$ is independent on η and only depends on n. Therefore,

$$\mu([\sigma_0^n]) = e^{-nP^{\mathbf{a}}(\gamma)+o(n)} \int_X \frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})}\mu(d\eta), \tag{3.36}$$

Hence, by taking into account (3.8), we obtain that

$$\mu([\sigma_0^n]) = e^{-nP^{\mathbf{a}}(\gamma)} \cdot \frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\mathbf{a}_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\mathbf{a}_{[0,n]^c})} \cdot e^{o(n)}. \tag{3.37}$$

Using the bar moving property (3.7), we have that

$$\begin{aligned}\frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\mathbf{a}_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\mathbf{a}_{[0,n]^c})} &= \prod_{i=0}^{n} \frac{\gamma_{[0,n]}(\mathbf{a}_{[0,i)}\sigma_{[i,n]}|\mathbf{a}_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,i]}\sigma_{(i,n]}|\mathbf{a}_{[0,n]^c})} \overset{(3.7)}{=} \prod_{i=0}^{n} \frac{\gamma_{\{i\}}(\sigma_i|\sigma_{(i,n]}\mathbf{a}_{[i,n]^c})}{\gamma_{\{i\}}(\mathbf{a}_i|\sigma_{(i,n]}\mathbf{a}_{[i,n]^c})} \\ &= \prod_{i=0}^{n} e^{\phi_\gamma\circ S^i(\sigma_{[0,n]}\mathbf{a}_{[0,n]^c})}. \end{aligned} \tag{3.38}$$

Note that in the last equation of (3.38), we used the fact that ϕ_γ is independent of the components in the negative half-line $-\mathbb{N}$. Combining (3.38) with (3.37), we get

$$\mu([\sigma_0^n]) = \frac{e^{S_{n+1}\phi_\gamma(\sigma_{[0,n]}\mathbf{a}_{[0,n]^c})}}{e^{nP^{\mathbf{a}}(\gamma)}} \cdot e^{o(n)}. \tag{3.39}$$

Note that

$$\mathtt{var}_n(S_{n+1}\phi_\gamma) \le \sum_{k=0}^{n} \mathtt{var}_k(\phi_\gamma) \tag{3.40}$$

and since ϕ_γ is continuous, $\mathtt{var}_k(\phi_\gamma) \to 0$ as $k \to \infty$. Thus $\mathtt{var}_n(S_{n+1}\phi_\gamma) = o(n)$, and hence we obtain from (3.39) that

$$\mu([\sigma_0^n]) = \frac{e^{S_{n+1}\phi_\gamma(\sigma)}}{e^{nP^{\mathbf{a}}(\gamma)}} e^{o(n)}. \tag{3.41}$$

We note that the weak Bowen-Gibbs property (3.41) implies that $P^{\mathbf{a}}(\gamma) = P(\phi_\gamma)$, where $P(\phi_\gamma)$ is the topological pressure of ϕ_γ.

(ii): Now take any $\tau \in \mathscr{M}_{1,S}(\Omega)$ and $\mu \in \mathscr{G}_S(\gamma)$. The relative entropy $H_n(\tau|\mu)$ is given by

$$\begin{aligned} H_n(\tau|\mu) &= \sum_{a_0^n \in E^{n+1}} \tau([a_0^n]) \log \frac{\tau([a_0^n])}{\mu([a_0^n])} \\ &= \sum_{a_0^n \in E^{n+1}} \tau([a_0^n]) \log \tau([a_0^n]) - \sum_{a_0^n \in E^{n+1}} \tau([a_0^n]) \log \mu([a_0^n]) \end{aligned} \tag{3.42}$$

For the first sum in (3.42), one has that

$$\frac{1}{n}\sum_{a_0^n\in E^{n+1}}\tau([a_0^n])\log\tau([a_0^n])\xrightarrow[n\to\infty]{}-h(\tau). \tag{3.43}$$

For the second sum, by inserting (3.40) and using (3.41), one has

$$\begin{aligned}\sum_{a_0^n\in E^{n+1}}\tau([a_0^n])\log\mu([a_0^n]) &= \sum_{a_0^n\in E^{n+1}}\tau([a_0^n])(S_{n+1}\phi_\gamma-(n+1)P^{\mathbf{a}}(\gamma)+o(n)\\ &= (n+1)\int_\Omega\phi_\gamma\,d\tau-(n+1)P^{\mathbf{a}}(\gamma)+o(n). \end{aligned}\tag{3.44}$$

By combining (3.43) and (3.44), and since $P^{\mathbf{a}}(\gamma)=P(\phi_\gamma)$, one concludes that the relative entropy density rate $h(\tau|\mu)$ indeed exists and

$$h(\tau|\mu)=\lim_{n\to\infty}\frac{1}{n}H_n(\tau|\mu)=-h(\tau)-\int_\Omega\phi_\gamma\,d\tau+P(\phi_\gamma). \tag{3.45}$$

Thus the Variational Principle (Theorem 3.2.2) yields that τ is a Gibbs state for γ if and only if τ is an equilibrium state for ϕ_γ, i.e., $\mathscr{G}_S(\gamma)=\mathscr{ES}(\phi_\gamma)$.

□

Proof of Theorem 3.A. One can easily see that weakly cohomologous potentials have the same equilibrium states because the functionals $\tau\in\mathscr{M}_{1,S}(\Omega)\mapsto h(\tau)+\int_\Omega\phi\,d\tau$ and $\tau\in\mathscr{M}_{1,S}(\Omega)\mapsto h(\tau)+\int_\Omega\phi_{\gamma^\phi}\,d\tau$ differ by only a constant $P(\phi_{\gamma^\phi})-P(\phi)$. Furthermore, the weak cohomology between ϕ_{γ^ϕ} and ϕ also yields the following [7, Proposition 2.34]:

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n-1}\left[\phi_{\gamma^\phi}-\phi-P(\phi_{\gamma^\phi})+P(\phi)\right]\circ S^i\right\|_\infty=0. \tag{3.46}$$

Then the first part of Theorem 3.A follows from Lemma 3.5.1 and the second part of Lemma 3.5.3 since the weak cohomologous potentials have the same set of equilibrium states, i.e., $\mathscr{ES}(\phi)=\mathscr{ES}(\phi_{\gamma^\phi})$. The second part of Theorem 3.A also follows from Lemma 3.5.1 and Lemma 3.5.3. In fact, by applying the first and second parts of Lemma 3.5.3 to $\gamma=\gamma^\phi$, one obtains from Lemma 3.5.1 that for any equilibrium state $\mu\in\mathscr{ES}(\phi)$,

$$\mu([\sigma_0^{n-1}])=\frac{e^{(S_n\phi_{\gamma^\phi})(\sigma)}}{e^{nP(\phi_{\gamma^\phi})}}e^{o(n)},\ \ \sigma\in\Omega, \tag{3.47}$$

and by (3.46), one has that $S_n(\phi-P(\phi))=S_n(\phi_{\gamma^\phi}-P(\phi_{\gamma^\phi}))+o(n)$. Hence one immediately concludes the weak Bowen-Gibbs property of μ with respect to the potential ϕ.

□

Proof of Theorem 3.B. It is easy to see that the second part of Lemma 3.5.3 implies Theorem 3.B. □

Bibliography

[1] Sebastián Barbieri, Ricardo Gómez, Brian Marcus, Tom Meyerovitch, and Siamak Taati, *Gibbsian representations of continuous specifications: the theorems of Kozlov and Sullivan revisited*, Comm. Math. Phys. **382** (2021), no. 2, 1111–1164, DOI 10.1007/s00220-021-03979-2. MR4227169

[2] Steven Berghout, Roberto Fernández, and Evgeny Verbitskiy, *On the relation between Gibbs and g-measures*, Ergodic Theory Dynam. Systems **39** (2019), no. 12, 3224–3249, DOI 10.1017/etds.2018.13. MR4027547

[3] Rufus Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Second revised edition, Lecture Notes in Mathematics, vol. 470, Springer-Verlag, Berlin, 2008. With a preface by David Ruelle; Edited by Jean-René Chazottes. MR2423393

[4] D. Capocaccia, *A definition of Gibbs state for a compact set with Z^{ν} action*, Comm. Math. Phys. **48** (1976), no. 1, 85–88. MR415675

[5] R. L. Dobrushin, *Gibbsian random fields for lattice systems with pairwise interactions*, Funkcional. Anal. i Priložen. **2** (1968), no. 4, 31–43 (Russian). MR250630

[6] Aernout C. D. van Enter, Roberto Fernández, Mirmukhsin Makhmudov, and Evgeny Verbitskiy, *On an extension of a theorem by Ruelle to long-range potentials* (2024).

[7] Aernout C. D. van Enter, Roberto Fernández, and Alan D. Sokal, *Regularity properties and pathologies of position-space renormalization-group transformations: scope and limitations of Gibbsian theory*, J. Statist. Phys. **72** (1993), no. 5-6, 879–1167, DOI 10.1007/BF01048183. MR1241537

[8] Hans-Otto Georgii, *Gibbs measures and phase transitions*, De Gruyter Studies in Mathematics, vol. 9, Walter de Gruyter & Co., Berlin, 1988. MR956646

[9] Nicolai T. A. Haydn, *On Gibbs and equilibrium states*, Ergodic Theory Dynam. Systems **7** (1987), no. 1, 119–132, DOI 10.1017/S0143385700003849. MR886374

[10] ———, *Classification of Gibbs' states on Smale spaces and one-dimensional lattice systems*, Nonlinearity **7** (1994), no. 2, 345–366. MR1267693

[11] N. T. A. Haydn and D. Ruelle, *Equivalence of Gibbs and equilibrium states for homeomorphisms satisfying expansiveness and specification*, Comm. Math. Phys. **148** (1992), no. 1, 155–167. MR1178139

[12] Robert B. Israel, *Convexity in the theory of lattice gases*, Princeton Series in Physics, Princeton University Press, Princeton, NJ, 1979. With an introduction by Arthur S. Wightman. MR517873

[13] Anders Johansson, Anders Öberg, and Mark Pollicott, *Continuous eigenfunctions of the transfer operator for the Dyson model*, Arxiv 2304.04202 (2023).

[14] Gerhard Keller, *Equilibrium states in ergodic theory*, London Mathematical Society Student Texts, vol. 42, Cambridge University Press, Cambridge, 1998. MR1618769

[15] Christof Külske, Arnaud Le Ny, and Frank Redig, *Relative entropy and variational properties of generalized Gibbsian measures*, The Annals of Probability **32** (2004), no. 2, 1691–1726, DOI 10.1214/009117904000000342.

[16] O. E. Lanford III and D. Ruelle, *Observables at infinity and states with short range correlations in statistical mechanics*, Comm. Math. Phys. **13** (1969), 194–215. MR256687

[17] Christian Maes, Frank Redig, Floris Takens, Annelies van Moffaert, and Evgeny Verbitski, *Intermittency and weak Gibbs states*, Nonlinearity **13** (2000), no. 5, 1681–1698, DOI 10.1088/0951-7715/13/5/314. MR1781814

[18] Mirmukhsin Makhmudov, *The Eigenfunctions of the Transfer Operator for the Dyson model in a field* (2025).

[19] C.-E. Pfister and W. G. Sullivan, *Weak Gibbs measures and large deviations*, Nonlinearity **31** (2018), no. 1, 49–53, DOI 10.1088/1361-6544/aa99a3. MR3746632

[20] ———, *Asymptotic decoupling and weak Gibbs measures for finite alphabet shift spaces*, Nonlinearity **33** (2020), no. 9, 4799–4817, DOI 10.1088/1361-6544/ab8fb7. MR4135096

[21] D. Ruelle, *Statistical mechanics of a one-dimensional lattice gas*, Comm. Math. Phys. **9** (1968), 267–278. MR234697

[22] David Ruelle, *Thermodynamic formalism*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004. The mathematical structures of equilibrium statistical mechanics. MR2129258

[23] Yakov.G. Sinai, *Gibbs measures in Ergodic Theory*, Russian Math. Surveys **27:4** (1972), 21–69.

[24] Wayne G. Sullivan, *Potentials for almost Markovian random fields*, Comm. Math. Phys. **33** (1973), 61–74. MR410987

[25] Peter Walters, *Invariant measures and equilibrium states for some mappings which expand distances*, Trans. Amer. Math. Soc. **236** (1978), 121–153, DOI 10.2307/1997777. MR466493

[26] ———, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982. MR648108

[27] ———, *Convergence of the Ruelle operator for a function satisfying Bowen's condition*, Trans. Amer. Math. Soc. **353** (2001), no. 1, 327–347.