Gibbs states in statistical mechanics and dynamical systems

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Part I

Gibbs Measures in Dynamical Systems

Chapter 2

On the Gibbs formalism in Dynamical Systems

Abstract: In this chapter, we review various notions of Gibbs measures, including DLR measures, G-measures, g-measures and compare them with each other. The chapter also aims to discuss the Kozlov-Sullivan characterisation of specifications in Equilibrium Statistical Mechanics within the framework of Dynamical Systems. In addition, on the whole-line shift space, we also demonstrate an interaction-independent variational principle.

Furthermore, we provide a necessary and sufficient criterion for a specification on a half-line shift space to ensure the existence of a generating translation-invariant UAC interaction, highlighting the lack of an analogous condition for whole-line shift spaces.

2.1 Introduction

The so-called Gibbs (DLR) formalism – the study of probability measures defined via their conditional probabilities, a perspective formalised in the notion of a Gibbs measure – was first initiated by Dobrushin, Lanford, and Ruelle to mathematically describe various phenomena in statistical mechanics, including phase transitions. The DLR formalism is developed based on the notion of *specifications*. Although it was formally established in the 1970s, the foundations of the theory of Gibbs measures trace back to the earlier works of Boltzmann and Gibbs. The classical Gibbs formalism starts with the well-known *Boltzmann factor*, which is a probability measure interpreted as the probability of a particular portion of a macroscopic system being in a certain energy state. This measure is connected to the interactions that define the model quantitatively. This is why the classical Gibbs formalism is developed based on the *interactions*. Subsequently, Kozlov [21] and Sullivan [28] established that the Gibbs formalism based on specifications is essentially equivalent to the one based on interactions, as every Gibbsian specification is generated by some UAC interaction. However, if one restricts oneself to the setup of translation-invariant measures, then one can see a subtle difference between those Gibbs formalisms. In fact, as shown in [2], not every translation-invariant Gibbs measure is prescribed by a translation-invariant uniformly absolutely continuous (UAC) interaction, even though it is always prescribed by a translation-invariant specification. This naturally calls for a reconstruction of classical statistical mechanics framed in terms of specifications. Moreover, the renormalisation group transformation of a Gibbs measure often results in a loss of the Gibbs property, meaning the transformed measure may lack a continuous version of its conditional probabilities. However, in more favourable cases, the transformed measure may retain an 'almost continuous' version of conditional probabilities, which frequently cannot be associated with any reasonable interaction. This challenge further emphasises the need to study Gibbs measures based on specifications alone. In this regard, considerable work has already been done, particularly in the context of generalised Gibbs measures, as explored in [10, 22, 24]. In particular, [22] establishes a version of the interaction-independent variational principle. Expanding on this, we present a new form of the interaction-independent variational principle, which is more suited to the framework of dynamical systems. We also discuss the Kozlov-Sullivan characterisation of the *translation-invariant* Gibbs measures. Furthermore, we also give an explicit example of a translation-invariant specification on the half-line $\mathbb{Z}_+$ which is not generated by any translation-invariant UAC interaction.

Later developments in both Statistical Mechanics and Dynamical Systems led to the definition of Gibbs measures independent of specifications or interactions, based instead on the concept of *families of multipliers* or *G-families*. The first of these alternative approaches was introduced to extend the notion of Gibbs measures to more general (semi)group actions beyond $\mathbb{Z}^d$ or $\mathbb{N}$. The idea of a G-family was inspired by Keane's work [20] and introduced into ergodic theory by Brown and Dooley [4], where it was used to study generalised Riesz products in harmonic analysis. In this chapter, we compare these alternative definitions of Gibbs measures with the classical definition based on specifications.

The chapter is structured as follows:

- In Section 2.2, we recall the different definitions of the Gibbs measures that appeared in Statistical Mechanics and Dynamical Systems literature and compare these definitions with the one based on the specifications.
- In Section 2.3, the Kozlov-Sullivan characterisation of the Gibbsian specifications on the full lattice $\mathbb{Z}^d$ will be discussed.
- In Section 2.4, we state the first set of our main results. There, we formulate an interaction-independent version of the variational principle, aligned with the spirit of dynamical systems.
- Section 2.5 focuses on the Gibbs formalism on the half-line $\mathbb{Z}_+$. In this section, we also compare the concept of G-measures, introduced by Brown and Dooley in the 1990s, with the DLR-Gibbs measures. Additionally, we provide an example of a translation-invariant Gibbsian specification on $\mathbb{Z}_+$ that is not generated by a translation-invariant UAC interaction.
- Section 2.6 is dedicated to the proofs of the main results of this chapter.

2.2 Preliminaries

2.2.1 The Gibbs measures in DLR formalism

The DLR formalism starts with the so-called *specification* – a given family of conditional probabilities. Since there is no a priori underlying probability measure, one should define the prescribing family of the conditional probabilities everywhere, not almost everywhere.

In this chapter, we consider the shift spaces in the form $\Omega := E^{\mathbb{L}}$, where E is a finite set equipped with the discrete topology and interpreted as the state-space of the system, and $\mathbb{L}$ is interpreted as the set of sites, which is at most a countable set. We denote the Borel $\sigma-$algebra in Ω by $\mathscr{F}$, and for $\Lambda \subset \mathbb{L}$, $\mathscr{F}_\Lambda$ denotes the minimal $\sigma-$algebra containing all the cylindrical $[\sigma_V]$ sets based on volume $V \Subset \Lambda$. $\mathscr{M}_1(\Omega)$ denotes the set of probability measures on Ω. A function $f : \Omega \to \mathbb{R}$ is called *local* if there exists a finite volume $\Lambda \Subset \mathbb{L}$ such that f is $\mathscr{F}_\Lambda-$measurable. Then a ***specification*** γ is a family of positive functions $\{\gamma_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ – the so-called *probability kernels* on Ω indexed by the finite volumes $\Lambda \Subset \mathbb{L}$ which has the following properties:

(P1) for each $\Lambda \Subset \mathbb{L}$ and every $B \in \mathscr{F}$, the function $(B|\omega) \in \mathscr{F} \times \Omega \xrightarrow{\gamma_\Lambda} [0,1]$ is measurable in $\mathscr{F}_{\Lambda^c}$ as a function of ω;

(P2) for each $\Lambda \Subset \mathbb{L}$ and all $\omega \in \Omega$, the function $\gamma_\Lambda(\cdot|\omega) : \mathscr{F} \to [0,1]$ is a probability measure on $\mathscr{F}$;

(P3) (properness) for each $\Lambda \Subset \mathbb{L}$ and every $B \in \mathscr{F}_{\Lambda^c}$ and $\omega \in \Omega$, $\gamma_\Lambda(B|\omega) = \mathbb{1}_B(\omega)$;

(P4) (consistency) for all $\Delta \subset \Lambda \Subset \mathbb{L}$,

$$\gamma_\Lambda \gamma_\Delta = \gamma_\Lambda, \tag{2.1}$$

where $\gamma_\Lambda \gamma_\Delta(B|\omega) := \int_\Omega \gamma_\Delta(B|\eta)\gamma_\Lambda(d\eta|\omega),\ B \in \mathscr{F},\ \omega \in \Omega.$

Definition 2.2.1. *A specification γ on Ω is called **quasilocal** if for any local function f and volume $\Lambda \Subset \mathbb{L}$, one has that $\gamma_\Lambda(f) \in C(\Omega)$, where for $\omega \in \Omega$, $\gamma_\Lambda(f)(\omega) := \int_\Omega f(\xi)\gamma_\Lambda(d\xi|\omega)$.*

Describing the specifications with their *specification densities* is often convenient. The ***density*** of a specification γ in the volume $\Lambda \Subset \mathbb{L}$ is $\gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) := \gamma_\Lambda([\omega]_\Lambda|\omega)$, here $[\omega]_\Lambda$ denotes the cylinderic set based on the volume Λ and the configuration ω, i.e., $[\omega]_\Lambda := \{\xi \in \Omega : \xi_\Lambda = \omega_\Lambda\}$.

Definition 2.2.2. *A specification γ is called **non-null** (or **positive**) if for all $\Lambda \Subset \mathbb{L}$,*

$$\inf_{\omega\in\Omega} \gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) > 0.$$

We call a specification ***Gibbsian*** if it is quasilocal and non-null at the same time. The consistency property (2.1) of specifications yields the following for specification densities:

$$\gamma_\Lambda(\eta_\Lambda|\omega_{\Lambda^c}) = \gamma_\Delta(\eta_\Delta|\eta_{\Lambda/\Delta}\omega_{\Lambda^c}) \sum_{\sigma_\Delta \in E^\Delta} \gamma_\Lambda(\sigma_\Delta \eta_{\Lambda/\Delta}|\omega_{\Lambda^c}),\ \forall \Delta \subset \Lambda \Subset \mathbb{L},\ \forall \omega, \eta \in \Omega. \tag{2.2}$$

The characterising property of the non-null specification densities is the so-called *bar moving property*, which we use frequently in the chapter; therefore, we state it below:

$$\frac{\gamma_\Delta(\beta_\Delta|\omega_{\Delta^c})}{\gamma_\Delta(\alpha_\Delta|\omega_{\Delta^c})} = \frac{\gamma_\Lambda(\beta_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})}{\gamma_\Lambda(\alpha_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})},\quad \Delta \subset \Lambda \Subset \mathbb{L},\quad \alpha, \beta, \omega \in \Omega. \tag{2.3}$$

Now we turn to the *consistent probability measures* with the specifications.

Definition 2.2.3. *A probability measure $\mu \in \mathscr{M}_1(\Omega)$ is **consistent** with a specification γ if for all $\Lambda \Subset \mathbb{L}$, the conditional probability $\mu(\cdot|\mathscr{F}_{\Lambda^c})$ is prescribed by the probability kernel γ_Λ, i.e., for all $B \in \mathscr{F}$, and μ−almost every $\omega \in \Omega$,*

$$\mu(B|\mathscr{F}_{\Lambda^c})(\omega) = \gamma_\Lambda(B|\omega).$$

This is equivalent to stating that the measure μ satisfies the DLR equations, i.e., for all $\Lambda \Subset \mathbb{L}$ and $B \in \mathscr{F}$,

$$\mu(B) = (\mu\gamma_\Lambda)(B) := \int_\Omega \gamma_\Lambda(B|\omega)\mu(d\omega). \tag{2.4}$$

The set of consistent measures with a specification γ is denoted by $\mathscr{G}(\Omega,\gamma)$.

It should be stressed that any probability measure $\mu \in \mathscr{M}_1(\Omega)$ is prescribed by some specification γ [17]. Conversely, in the general case, one can't guarantee the existence of a consistent probability measure with a given specification γ [16], i.e., it can be a case that $\mathscr{G}(\Omega,\gamma) = \emptyset$. Therefore, specifications without extra properties are not very interesting. In practice, the quasilocal specifications hold more significance. One of the important properties of the quasilocal specifications is that there always exists a consistent probability measure as long as the state-space E is compact, as is the case in this chapter. In fact, for any sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathscr{M}_1(\Omega)$ of probability measures, any weak* limiting point of the set $\{\tau_n\gamma_\Lambda : n \in \mathbb{N},\ \Lambda \Subset \mathbb{L}\}$ is consistent with γ. In particular, $\mathscr{G}(\Omega,\gamma)$ is a convex closed, thus compact subset of $\mathscr{M}_1(\Omega)$ in the weak* topology. Another crucial property of the quasilocal specifications is that one probability measure can not be consistent with two distinct quasilocal specifications. Thus it would be more interesting if a measure is consistent with a quasilocal or a Gibbsian specification.

Definition 2.2.4. *A probability measure $\mu \in \mathscr{M}_1(\Omega)$ is called a **Gibbs measure** (state) if μ is prescribed by a Gibbsian specification, i.e., there exists a Gibbsian specification γ such that $\mu \in \mathscr{G}(\Omega,\gamma)$.*

Although there always exists a consistent measure with a quasilocal specification, in particular, a Gibbsian specification, it is often the case that there are multiple of such measures. In this case, one says that the underlying specification exhibits *phase transitions.*

2.2.2 The Classical Gibbs formalism

The classical Gibbs formalism starts with an ***interaction*** Φ - a family of *local functions* Φ_Λ indexed by $\Lambda \Subset \mathbb{L}$ on a configuration space $\Omega = E^{\mathbb{L}}$ such that for each Λ, Φ_Λ is $\mathscr{F}_\Lambda$−measurable. If the given interaction Φ is sufficiently regular, then one can associate a specification with it. More precisely, the interaction ***Hamiltonian*** H_Λ^Φ, a.k.a. *the interaction energy in a volume* $\Lambda \Subset \mathbb{L}$, is

$$H_\Lambda^\Phi(\omega) := \sum_{\substack{V \Subset \mathbb{L}, \\ V \cap \Lambda \neq \emptyset}} \Phi_V(\omega), \ \ \omega \in \Omega. \tag{2.5}$$

Without any condition on the interaction Φ, the sum in (2.5) does not need to converge. In literature, depending on the mode of the convergence of the sum in (2.5), different classes of interactions arise.

(1) An interaction Φ is called ***uniformly convergent*** if the sum (2.5) converges uniformly on $\omega \in \Omega$, i.e., for any cofinal sequence $\{V_n\}_{n\in\mathbb{N}}$ of the finite volumes, the finite sum

$$\sum_{\substack{V \subset V_n \\ V \cap \Lambda \neq \emptyset}} \Phi_V(\omega)$$

converges to $H_\Lambda^\Phi(\omega)$ uniformly in $\omega \in \Omega$ as $n \to \infty$;

(2) An interaction Φ is ***uniformly absolutely convergent*** *(UAC)* if for all $i \in \mathbb{L}$,

$$\sum_{i \in V \Subset \mathbb{L}} \sup_{\omega \in \Omega} |\Phi_V(\omega)| < \infty.$$

(3) Fix a configuration $\theta \in \Omega$. An interaction Φ is called ***relatively absolutely convergent*** if for all $j \in \mathbb{L}$, the sum

$$\sum_{j \in V \Subset \mathbb{L}} |\Phi_V(\omega) - \Phi_V(\omega_j \theta_{\{j\}^c})|$$

converges uniformly in $\omega \in \Omega$. It should be mentioned that the concept of relatively absolute convergence of an interaction is independent of the choice of the configuration θ.

(4) An interaction Φ on Ω is called ***variation-summable*** if for all $i \in \mathbb{L}$,

$$\sum_{i \in V \Subset \mathbb{L}} \delta_i \Phi_V < \infty,$$

where $\delta_i f$ denotes the *oscillation/variance* of the function $f : \Omega \to \mathbb{R}$ at the coordinate i: $\delta_i f := \sup\{f(\xi) - f(\eta) : \xi_{\mathbb{L}\setminus\{i\}} = \eta_{\mathbb{L}\setminus\{i\}}\}$.

Among the above convergence classes of interactions, the class of UAC interactions is the smallest, and the class of relatively convergent potentials is the largest one. Note that the first two summability modes guarantee, for all $\Lambda \Subset \mathbb{L}$, the existence of a continuous Hamiltonian H_Λ^Φ defined by (2.5). Then for all $\Lambda \Subset \mathbb{L}$, one can define the *Boltzmann factor* γ_Λ^Φ for Φ by

$$\gamma_\Lambda^\Phi(\eta_\Lambda|\omega_{\Lambda^c}) := \frac{e^{-H_\Lambda^\Phi(\eta_\Lambda\omega_{\Lambda^c})}}{\sum\limits_{\bar\eta_\Lambda\in E^\Lambda} e^{-H_\Lambda^\Phi(\bar\eta_\Lambda\omega_{\Lambda^c})}}, \quad \eta_\Lambda\in E^\Lambda,\ \omega\in\Omega. \tag{2.6}$$

One can readily check that the family $\{\gamma_\Lambda^\Phi\}_{\Lambda\Subset\mathbb{L}}$ comprises *specification-densities.* The associated specification γ^Φ can be recovered from the Boltzmann factor γ_Λ^Φ by

$$\gamma_\Lambda^\Phi(B|\omega) := \sum_{\eta_\Lambda\in E^\Lambda} \gamma_\Lambda^\Phi(\eta_\Lambda|\omega_{\Lambda^c})\delta_{\eta_\Lambda\omega_{\Lambda^c}}(B),\ \ B\in\mathscr{F}, \omega\in\Omega.$$

It is clear that the specification γ^Φ is non-null and quasilocal, if the interaction Φ is uniformly convergent.
One easily notices that the relative absolute convergence or variation-summability of an interaction Φ is not sufficient to guarantee the convergence of the Hamiltonian (2.5), therefore, in these classes of interactions, one may not have a defined Hamiltonian. Despite this, there is another way of associating a specification with an interaction. Note that if one writes (2.6) in the following form,

$$\begin{aligned}\gamma_\Lambda^\Phi(\xi_\Lambda|\eta_{\Lambda^c}) &= \frac{1}{\sum\limits_{\bar\xi_\Lambda\in E^\Lambda} \exp\Big(H_\Lambda^\Phi(\xi_\Lambda\eta_{\Lambda^c}) - H_\Lambda^\Phi(\bar\xi_\Lambda\eta_{\Lambda^c})\Big)} \\ &= \frac{1}{\sum\limits_{\bar\xi_\Lambda\in E^\Lambda} \exp\Big(\sum\limits_{V\cap\Lambda\neq\emptyset}\Big[\Phi_V(\xi_\Lambda\eta_{\Lambda^c}) - \Phi_V(\bar\xi_\Lambda\eta_{\Lambda^c})\Big]\Big)}\end{aligned} \tag{2.7}$$

then it becomes apparent that one does not really need the convergence of the sum $H_\Lambda = \sum\limits_{\substack{V\Subset\mathbb{L}\\ V\cap\Lambda\neq\emptyset}} \Phi_V$ to associate a specification γ^Φ with an interaction Φ. In fact, if the interaction Φ is relatively absolutely convergent, then the equation (2.7) makes sense; therefore, γ^Φ is well-defined.

Once the relationship between specifications and interactions is established by either (2.5) or (2.7), the following question arises: how generic is the class of specifications generated by interactions? In fact, this question has been studied by Kozlov, Sullivan and Grimmett and others [1, 2, 18, 21, 28]. Here we only mention a theorem by Kozlov which answers the question, and will discuss the question and Kozlov's method in greater detail in Section 2.3.

Theorem 2.2.5 (Kozlov's theorem)**.** [2, 12, 16, 21] *Every Gibbsian specification is generated by a UAC interaction. In other words, for any Gibbsian specification γ, there exists a UAC interaction Φ on Ω such that $\gamma = \gamma^\Phi$.*

Notably, the Kozlov theorem applies to any countable set $\mathbb{L}$, and this is the elegance of the theorem.

2.2.3 The parameterization of specifications with cocycles

We start this subsection by introducing the *asymptotic equivalence relation*—also known as the *Gibbs relation* or *homoclinic relation*—denoted by $\mathfrak{T}(\Omega) \subset \Omega \times \Omega$. We say that two-configurations $\omega, \omega' \in \Omega = E^{\mathbb{L}}$ are asymptotically equivalent/homoclinic, i.e., $(\omega, \omega') \in \mathfrak{T}(\Omega)$, if there exists $\Lambda \Subset \mathbb{L}$ such that $\omega_{\Lambda^c} = \omega'_{\Lambda^c}$. For each fixed $\Delta \Subset \mathbb{L}$ one can introduce a subequivalence relation $\mathfrak{T}_\Delta(\Omega)$ by $(\omega, \omega') \in \mathfrak{T}_\Delta(\Omega) \Leftrightarrow \omega_{\Delta^c} = \omega'_{\Delta^c}$. Then it is clear that $\mathfrak{T}(\Omega) = \bigcup_{\Delta \Subset \mathbb{L}} \mathfrak{T}_\Delta(\Omega)$. Note that each $\mathfrak{T}_\Delta(\Omega) \Subset \Omega \times \Omega$ is closed, thus a compact subset of the product space $\Omega \times \Omega$. One can introduce a relevant topology on $\mathfrak{T}(\Omega)$ as follows: $\mathscr{O} \subset \mathfrak{T}(\Omega)$ is open if and only if $\mathscr{O} = \bigcup_{\Delta \Subset \mathbb{L}} \mathscr{O}_\Delta$, where $\mathscr{O}_\Delta$ is an open subset of $\mathfrak{T}_\Delta(\Omega) \subset \Omega \times \Omega$ in the induced product topology on $\Omega \times \Omega$. Note that this topology in $\mathfrak{T}(\Omega)$ is strictly finer than the induced topology on $\mathfrak{T}(\Omega)$ by $\Omega \times \Omega$. Nevertheless, these two topologies in $\mathfrak{T}(\Omega)$ induce the same Borel sigma-algebra. It should also be stressed that a function $f : \mathfrak{T}(\Omega) \to \mathbb{R}$ is continuous iff for all $\Delta \Subset \mathbb{L}$, the restriction $f|_{\mathfrak{T}_\Delta(\Omega)}$ is continuous in the induced topology by $\Omega \times \Omega$.

Now we turn to cocycles on the asymptotic equivalence relation $\mathfrak{T}(\Omega)$. A map $\rho : \mathfrak{T}(\Omega) \to \mathbb{R}$ is called a *cocycle on* Ω (or $\mathfrak{T}(\Omega)$–*cocycle*) if for all $(\omega, \eta), (\eta, \xi) \in \mathfrak{T}(\Omega)$,

$$\rho(\omega, \eta) + \rho(\eta, \xi) = \rho(\omega, \xi). \tag{2.8}$$

If ρ is a cocycle, it can be readily checked that for all $(\omega, \xi) \in \mathfrak{T}(\Omega)$, $\rho(\omega, \omega) = 0$ and $\rho(\omega, \xi) = -\rho(\xi, \omega)$. Another important observation is that there is a one-to-one correspondence between the non-null specifications on Ω and measurable cocycles on Ω. In fact, let ρ be a measurable cocycle on $\mathfrak{T}(\Omega)$, then one can associate a specification (density) γ with ρ by

$$\gamma^\rho_\Lambda(\eta_\Lambda | \eta_{\Lambda^c}) := \frac{1}{\sum\limits_{\omega_\Lambda \in E^\Lambda} e^{-\rho(\omega_\Lambda \eta_{\Lambda^c}, \eta)}}, \quad \Lambda \Subset \mathbb{L},\ \eta \in \Omega. \tag{2.9}$$

One can easily verify that (2.9) is indeed a specification density. In fact, by the cocycle condition (2.8), one has

$$\begin{aligned}
\sum_{\xi_\Lambda \in E^\Lambda} \gamma^\rho_\Lambda(\xi_\Lambda | \eta_{\Lambda^c}) &= \sum_{\xi_\Lambda \in E^\Lambda} \frac{e^{-\rho(\xi_\Lambda \eta_{\Lambda^c}, \eta)}}{\sum\limits_{\omega_\Lambda \in E^\Lambda} e^{-\rho(\omega_\Lambda \eta_{\Lambda^c}, \xi_\Lambda \eta_{\Lambda^c}) - \rho(\xi_\Lambda \eta_{\Lambda^c}, \eta)}} \\
&= \frac{\sum\limits_{\xi_\Lambda \in E^\Lambda} e^{-\rho(\xi_\Lambda \eta_{\Lambda^c}, \eta)}}{\sum\limits_{\omega_\Lambda \in E^\Lambda} e^{-\rho(\omega_\Lambda \eta_{\Lambda^c}, \eta)}} \\
&= 1
\end{aligned} \tag{2.10}$$

By a direct calculation, it can be confirmed that the cocycle condition implies the consistency condition (2.2) as well. In fact, for all $\Delta \subset \Lambda \Subset \mathbb{L}$ and $\eta \in \Omega$,

$$\begin{aligned}
\gamma_\Delta^\rho(\eta_\Delta|\eta_{\Delta^c})\sum_{\sigma_\Delta}\gamma_\Lambda^\rho(\sigma_\Delta\eta_{\Lambda\setminus\Delta}|\eta_{\Lambda^c}) &= \sum_{\sigma_\Delta}\Big(\sum_{\xi_\Delta,\omega_\Lambda} e^{-\rho(\xi_\Delta\eta_{\Delta^c},\eta)-\rho(\omega_\Lambda\eta_{\Lambda^c},\sigma_\Delta\eta_{\Delta^c})}\Big)^{-1} \\
&= \sum_{\sigma_\Delta}\Big(\sum_{\xi_\Delta,\omega_\Lambda} e^{-\rho(\omega_\Lambda\eta_{\Lambda^c},\eta)-\rho(\xi_\Delta\eta_{\Delta^c},\sigma_\Delta\eta_{\Delta^c})}\Big)^{-1} \\
&= \sum_{\sigma_\Delta}\Big(\sum_{\omega_\Lambda} e^{-\rho(\omega_\Lambda\eta_{\Lambda^c},\eta)}\Big)^{-1}\Big(\sum_{\xi_\Delta} e^{-\rho(\xi_\Delta\eta_{\Delta^c},\sigma_\Delta\eta_{\Delta^c})}\Big)^{-1} \\
&= \Big(\sum_{\omega_\Lambda} e^{-\rho(\omega_\Lambda\eta_{\Lambda^c},\eta)}\Big)^{-1}\sum_{\sigma_\Delta}\Big(\sum_{\xi_\Delta} e^{-\rho(\xi_\Delta\eta_{\Delta^c},\sigma_\Delta\eta_{\Delta^c})}\Big)^{-1} \\
&= \gamma_\Lambda^\rho(\eta_\Lambda|\eta_{\Lambda^c})\sum_{\sigma_\Delta}\gamma_\Delta^\rho(\sigma_\Delta|\eta_{\Delta^c}) \\
&\overset{(2.10)}{=} \gamma_\Lambda^\rho(\eta_\Lambda|\eta_{\Lambda^c}).
\end{aligned} \tag{2.11}$$

The other way around, let γ be a non-null specification on the configuration space Ω, then one can associate a cocycle ρ^γ with γ by the following formula: for $(\omega,\xi) \in \mathfrak{T}_\Lambda(\Omega)$,

$$\rho^\gamma(\omega,\xi) := \log\gamma_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) - \log\gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}). \tag{2.12}$$

Note that ρ^γ is well-defined due to the bar moving property (2.3), and it is also clear that ρ^γ is measurable and satisfies (2.8).

Remark 2.2.6. *It is easy to see that (2.9) associates a Gibbsian (quasilocal) specification to a continuous cocycle, and (2.12) associates a continuous cocycle to a Gibbsian specification. Hence, there is a one-to-one correspondence between Gibbsian specifications and continuous cocycles on* Ω.

Now let the specification γ be generated by an interaction Φ, i.e., $\gamma := \gamma^\Phi$. Then the corresponding cocycle ρ^Φ has the following form:

$$\rho^\Phi(\omega,\omega') = \sum_{V\Subset\mathbb{L}}\big(\Phi_V(\omega)-\Phi_V(\omega')\big),\ \ (\omega,\omega') \in \mathfrak{T}(\Omega). \tag{2.13}$$

It is clear that if the sum in (2.5) converges for all $\omega \in \Omega$, then ρ^Φ is well-defined and for $(\omega,\omega') \in \mathfrak{T}_\Lambda(\Omega)$, one has

$$\rho^\Phi(\omega,\omega') = H_\Lambda^\Phi(\omega) - H_\Lambda^\Phi(\omega'),$$

and hence one can restore (2.7) for the specification γ^Φ. However, another important observation is that the convergence of the sum in (2.13) for all $(\omega,\omega') \in \mathfrak{T}(\Omega)$ does not imply the convergence of the sum in (2.5).

2.2.4 Capocaccia's definition of Gibbs measures

In 1976, Capocaccia extended the notion of the Gibbs measure to the general setup of the actions of the group $\mathbb{Z}^d$. This subsection aims to recall her definition of the particular setup of the full shift space $\Omega = E^{\mathbb{Z}^d}$, and we also observe that for this specific setup, Capocaccia's definition agrees with the definition of Gibbs measures in DLR formalism (see Definition 2.2.4.)

In her original paper, Capocaccia showed that any Gibbs measure in the classical DLR sense is also Gibbs in the sense of [5]. In this subsection, we prove the opposite.

Capocaccia's definition involves the notions of *conjugating homeomorphism* and a *family of multipliers*. The ***conjugating homeomorphism*** between asymptotically equivalent points $(x, y) \in \mathfrak{T}(\Omega)$ is a homeomorphism φ defined on some open neighbourhood $\mathcal{O}$ of x and taking values in Ω such that $\varphi(x) = y$ and there exists $\Lambda \Subset \mathbb{Z}^d$ satisfying, $(x', \varphi(x')) \in \mathfrak{T}_\Lambda(\Omega)$ for all $x' \in \mathcal{O}$. It should be noted that for every open subset $x \in \mathcal{O}' \subset \mathcal{O}$, $\varphi' := \varphi|_{\mathcal{O}'}$ has the same properties as $(\mathcal{O}, \varphi)$.

Remark 2.2.7. *Note that for any asymptotic pair $(x, y) \in \mathfrak{T}_\Lambda(\Omega)$, there is a canonical conjugating homeomorphism $\tilde{\varphi} : [x]_\Lambda \to [y]_\Lambda$ defined by*

$$x' \in [x]_\Lambda \overset{\tilde{\varphi}}{\mapsto} y_\Lambda x'_{\Lambda^c}. \tag{2.14}$$

Thus if $(\mathcal{O}, \varphi)$ is another conjugating homeomorphism for the pair (x, y), then by the second part of Theorem 1 in [5], there is a cylindrical set $[x]_V$ contained in $[x]_\Lambda \cap \mathcal{O}$ such that $\varphi|_{[x]_V} = \tilde{\varphi}|_{[x]_V}$. Therefore, without loss of generality, we can always assume that the conjugating homeomorphism φ has a canonical form (2.14) and the open set $\mathcal{O}$ containing x is actually a cylindrical set.

Definition 2.2.8. *A **family of multipliers** is a family $f := (f_{\mathcal{O},\varphi})$ of positive continuous functions $f_{\mathcal{O},\varphi}$ defined on $\mathcal{O}$ indexed by the conjugating homeomorphisms φ on open sets $\mathcal{O}$ such that*

(i) if $\mathcal{O}' \subset \mathcal{O}$ and $\varphi' = \varphi|_{\mathcal{O}'}$ then $f_{\mathcal{O}',\varphi'} = f_{\mathcal{O},\varphi}|_{\mathcal{O}'}$;

(ii) if $\mathcal{O} \subset \mathcal{O}' \cap (\varphi')^{-1}(\mathcal{O}'')$, then

$$f_{\mathcal{O},\varphi} = \left(f_{\mathcal{O}',\varphi'}|_{\mathcal{O}}\right) \cdot \left(f_{\mathcal{O}'',\varphi''} \circ \varphi'|_{\mathcal{O}}\right). \tag{2.15}$$

*A family $f = (f_{\mathcal{O},\varphi})$ of multipliers is called **translation-invariant** if*

(iii) for every $i \in \mathbb{Z}^d$ and all $(\mathcal{O}, \varphi)$,

$$f_{S_i(\mathcal{O}), S_i \circ \varphi \circ S_{-i}} = f_{\mathcal{O},\varphi} \circ S_{-i}, \tag{2.16}$$

where S_i is the shift by $i \in \mathbb{Z}^d$, i.e., $\omega \in \Omega \mapsto S_i \omega = (\omega_{j+i})_{j \in \mathbb{Z}^d}$

Below, we define the concept of Gibbs measure for a family of multipliers.

Definition 2.2.9. *A* ***Gibbs measure*** *for a family of multipliers* $f = (f_{\mathcal{O},\varphi})$ *is a probability measure* $\mu \in \mathcal{M}_1(\Omega)$ *such that for any pair* $(\mathcal{O},\varphi)$*, one has*

$$\varphi_*(f_{\mathcal{O},\varphi} \cdot \mu|_{\mathcal{O}}) = \mu|_{\varphi(\mathcal{O})}, \tag{2.17}$$

where φ_* *denotes the pushforward of the map* $\varphi : \mathcal{O} \to \Omega$ *and for the Borel set* $B \in \mathcal{F}$, $\mu|_B$ *is the restriction of the measure* μ *to the set* B*, i.e., for* $\tilde{B} \in \mathcal{F}$, $\mu|_B(\tilde{B}) = \mu(B \cap \tilde{B})$.

Now we show that if a probability measure is Gibbs in the sense of the above definition, then it is also Gibbs for an appropriate specification in the sense of Definition 2.2.4. In fact, if μ is a Gibbs measure for a family of multipliers $f = (f_{\mathcal{O},\varphi})$, consider a cocycle ρ^f defined as follows: let $(x, y) \in \mathfrak{T}(\Omega)$ and $(\mathcal{O},\varphi)$ be a conjugating homeomorphism corresponding to this pair (x, y), then put

$$\rho^f(x, y) := -\log f_{\mathcal{O},\varphi}(x). \tag{2.18}$$

Then, using (2.15), one can check that ρ^f satisfies the cocycle equation (2.8). Indeed, consider a triplet $(x, y), (y, z), (x, z) \in \mathfrak{T}(\Omega)$ of asymptotic pairs, assume the conjugating homeomorphisms $(\mathcal{O}', \varphi')$, $(\mathcal{O}'', \varphi'')$ and $(\mathcal{O}, \varphi)$ with $\mathcal{O} \subset \mathcal{O}' \cap (\varphi')^{-1}(\mathcal{O}'')$ correspond to these pairs. Then

$$\begin{aligned}
\rho^f(x, y) + \rho^f(y, z) &= -\log\Big[f_{\mathcal{O}',\varphi'}(x) \cdot f_{\mathcal{O}'',\varphi''}(y)\Big] \\
&= -\log\Big[f_{\mathcal{O}',\varphi'}(x) \cdot f_{\mathcal{O}'',\varphi''}(\varphi'(x))\Big] \\
&\overset{(2.15)}{=} -\log f_{\mathcal{O},\varphi}(x) \\
&= \rho^f(x, z).
\end{aligned}$$

The continuity of the cocycle ρ^f is clear from its definition (2.18). Note that if the family f of multipliers is translation-invariant, then the associated cocycle ρ^f is translation-invariant in the sense

$$\rho^f(x, y) = \rho^f(S_i x, S_i y), \ \ (x, y) \in \mathfrak{T}(\Omega), \ i \in \mathbb{Z}^d$$

(see also Section 2.3). In fact, let $(x, y) \in \mathfrak{T}(\Omega)$ and assume $(\mathcal{O},\varphi)$ is a corresponding conjugating homeomorphism, then for any $i \in \mathbb{Z}^d$, one has

$$\begin{aligned}
\rho^f(S_i x, S_i y) &= \rho^f(S_i x, S_i \varphi(x)) \\
&= \rho^f(S_i x, S_i \varphi S_{-i}(S_i x)) \\
&= -\log f_{S_i\mathcal{O}, S_i\varphi S_{-i}}(S_i x) \\
&\overset{(2.16)}{=} -\log f_{\mathcal{O},\varphi} \circ S_{-i}(S_i x) \\
&= \rho^f(x, y).
\end{aligned}$$

With a similar argument, one can also show the opposite; namely, if the cocycle ρ^f is translation-invariant, so is the family f of multipliers.

Now consider the Gibbsian specification γ^f associated with ρ^f via (2.9). Now we aim to show $\mu \in \mathscr{G}(\gamma^f)$. Pick any configuration $x \in \Omega$ and a finite volume $\Lambda \Subset \mathbb{Z}^d$. Then by Remark 2.2.7, one can choose a finite volume V with $\Lambda \subset V$ such that for all $y \in \Omega$ with $y_{\Lambda^c} = x_{\Lambda^c}$, the conjugating homeomorphism $(\mathscr{O}, \varphi)$ corresponding to the pair (x, y) has a canonical form (2.14) in the cylinder $[x_V]$. Now take a bounded measurable test function $g : [x]_V \to \mathbb{R}$. Then by (2.17), for all y with $y_{\Lambda^c} = x_{\Lambda^c}$, one has that

$$\int_\Omega \frac{g(y_V x'_{V^c})}{\gamma_V^f(x_V|x'_{V^c})}\Big(\gamma_V^f(y_V|x'_{V^c})\mathbb{1}_{[x]_V}(x') - \gamma_V^f(x_V|x'_{V^c})\mathbb{1}_{[y]_V}(x')\Big)\mu(dx') = 0, \quad (2.19)$$

hence

$$\int_\Omega \frac{g(y_V x'_{V^c})}{\gamma_V^f(x_V|x'_{V^c})}\Big(\gamma_V^f(y_V|x'_{V^c})\mu(x_V|x'_{V^c}) - \gamma_V^f(x_V|x'_{V^c})\mu(y_V|x'_{V^c})\Big)\mu(dx') = 0. \quad (2.20)$$

Thus, since g is chosen arbitrarily,

$$\gamma_V^f(y_V|x'_{V^c})\mu(x_V|x'_{V^c}) = \gamma_V^f(x_V|x'_{V^c})\mu(y_V|x'_{V^c}), \quad \mu-\text{a.e. } x' \in \Omega. \quad (2.21)$$

The bar moving property (2.3) and the above equation yield that

$$\gamma_\Lambda^f(y_\Lambda|x'_{\Lambda^c})\mu(x_\Lambda|x'_{\Lambda^c}) = \gamma_\Lambda^f(x_\Lambda|x'_{\Lambda^c})\mu(y_\Lambda|x'_{\Lambda^c}), \quad \mu-\text{a.e. } x' \in \Omega. \quad (2.22)$$

Then by summing up both sides of the above equation against $y_\Lambda \in E^\Lambda$, one gets that

$$\mu(x_\Lambda|x'_{\Lambda^c}) = \gamma_\Lambda^f(x_\Lambda|x'_{\Lambda^c}), \quad \mu-\text{a.e. } x' \in \Omega. \quad (2.23)$$

Thus μ is indeed consistent with the specification γ^f.

2.2.5 Equilibrium states

The notion of *equilibrium state* is developed in parallel with the notion of Gibbs measure and plays an important role both in Mathematical Statistical Mechanics and Dynamical Systems. In fact, one can see the concept of the equilibrium state as a counterpart of the concept of Gibbs states in the sense that the Gibbs states are defined in terms of "local rules" (the DLR equations), but the equilibrium states are defined in terms of a "global rule" (the variational equation). Note that a measure $\mu \in \mathscr{M}_1(\Omega)$ is translation-invariant if for all $i \in \mathbb{Z}^d$, $\mu = \mu \circ S_i^{-1}$. We denote the set of translation-invariant probability measures on Ω by $\mathscr{M}_{1,S}(\Omega)$.

Definition 2.2.10. *Let $\varphi : \Omega \to \mathbb{R}$ be a continuous function on the shift space $\Omega = E^{\mathbb{Z}^d}$. A translation-invariant probability measure $\mu \in \mathscr{M}_{1,S}(\Omega)$ is called an* ***equilibrium state*** *for φ if*

$$h(\mu)+\int_{\Omega}\varphi\, d\mu=\sup\left\{h(\tau)+\int_{\Omega}\varphi\, d\tau : \tau\in\mathscr{M}_{1,S}(\Omega)\right\}, \tag{2.24}$$

where $h(\mu)$ is the measure-theoretic entropy of the measure μ. We denote the set of equilibrium states for a function φ by $\mathscr{E}\mathscr{S}(\varphi)$.

Note that any continuous function φ on Ω has at least one equilibrium state. There is an interesting relationship between the equilibrium states and Gibbs measures known as the *variational principle*, and we shall discuss it in detail in Section 2.4.

2.3 On the Kozlov-Sullivan characterisation of the Gibbsian specifications on $\mathbb{Z}^d$

In this section, we discuss Kozlov's theorem, as presented in Theorem 2.2.5, and Kozlov's regrouping method, which is key to proving the Kozlov theorem.

2.3.1 Translation-invariant generators of Gibbsian specifications

Recall that a measure $\mu \in \mathscr{M}_1(\Omega)$ is translation-invariant if for all $i \in \mathbb{Z}^d$, $\mu = \mu \circ S_i^{-1}$. An interaction $\Phi = (\Phi_V)_{V \Subset \mathbb{Z}^d}$ is called translation-invariant if for all $\Lambda \Subset \mathbb{Z}^d$ and $i \in \mathbb{Z}^d$, one has

$$\Phi_\Lambda \circ S_i = \Phi_{\Lambda+i}. \tag{2.25}$$

Similarly, a specification γ on Ω is *translation-invariant* if for all $\Lambda \Subset \mathbb{Z}$, $i \in \mathbb{Z}^d$, and every $B \in \mathscr{F}$ $\omega \in \Omega$,

$$\gamma_{\Lambda+i}(B|\omega)=\gamma_\Lambda(S_i(B)|S_i\omega). \tag{2.26}$$

Note that a translation-invariant uniform-summable (or relatively uniformly convergent) interaction gives rise to a translation-invariant Gibbsian specification γ^Φ. Surprisingly, the opposite of this statement is not true: a non-translation-invariant interaction may also give rise to a translation-invariant specification (see the interaction in (2.32)). Nonetheless, a translation-invariant non-null specification γ always generates a *translation-invariant cocycle* ρ^γ i.e., for all $(\omega,\bar{\omega}) \in \mathfrak{T}(\Omega)$ and $i \in \mathbb{Z}^d$,

$$\rho^\gamma(\omega,\bar{\omega})=\rho^\gamma(S_i\omega,S_i\bar{\omega}) \tag{2.27}$$

and vice versa.

In the context of Gibbs measures, a translation-invariant quasilocal specification always admits a translation-invariant Gibbs measure, and vice versa. That is, a quasilocal specification prescribing a translation-invariant measure must itself be translation-invariant. However, this statement is not true for the interactions, in fact, a translation-invariant measure can be a Gibbs measure for a non-translation-invariant interaction (see the figure below)

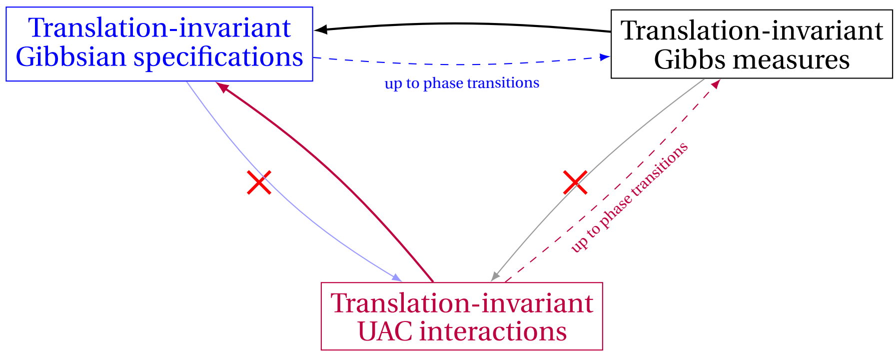

Figure 2.1

By Kozlov's theorem (Theorem 2.2.5), it is a well-known fact that any Gibbsian specification is generated by some UAC interaction on Ω. However, one may ask if every translation-invariant Gibbsian specification is also generated by a translation-invariant UAC interaction. Unfortunately, the answer to this question is not affirmative in general [2].

Theorem 2.3.1. [2] *There exists a translation-invariant Gibbsian specification on* $\Omega = \{0,1\}^{\mathbb{Z}}$ *which is not generated by any translation-invariant UAC interaction.*

The proof of Theorem 2.3.1 is not constructive but based on showing the non-surjectivity of certain bounded operators. Therefore, the specification mentioned in Theorem 2.3.1 is not explicit. It is worth noting that in Subsection 2.5.5, we consider a similar question on the one-sided full shift space $X_+ = E^{\mathbb{Z}_+}$, and prove an analogue of Theorem 2.3.1; however, the example that we provide is explicit.
The mismatch issue between translation-invariant Gibbs measures and translation-invariant UAC interactions raised by Theorem 2.3.1 can be resolved by two approaches:

(i) relaxing the UAC condition for the generating interaction;

(ii) imposing stronger regularity conditions on the specification than the quasilocality.

Below, we discuss these approaches in order.

One way of solving the issue by the first approach is done by Sullivan.

Theorem 2.3.2 (Sullivan's theorem)**.** [2,28] *Let γ be a translation-invariant Gibbsian specification on Ω. Then there exists a translation-invariant variation-summable interaction Ψ such that $\gamma = \gamma^{\Psi}$.*

In his original paper [21], Kozlov suggests a condition which solves the issue with the second approach. Denote the r^{th}–variation of the single-site density $\gamma_{\{0\}}$ of a Gibbsian specification γ on $E^{\mathbb{Z}^d}$ by $v(r)$, i.e.,

$$v(r) := \sup\{\gamma_{\{0\}}(\xi_0|\xi_{\{0\}^c}) - \gamma_{\{0\}}(\eta_0|\eta_{\{0\}^c}) \mid \xi_i = \eta_i,\ \forall i \in [-r,r]^d \cap \mathbb{Z}^d\}, \tag{2.28}$$

then the following is one of the results in [21].

Theorem 2.3.3. *Assume a translation-invariant Gibbsian specification γ on $\Omega = E^{\mathbb{Z}^d}$ has summable-variations in the sense that*

$$\sum_{r=1}^{\infty} r^{d-1} v(r) < \infty. \tag{2.29}$$

Then the specification γ is generated by a translation-invariant UAC interaction.

The proof of Theorem 2.3.3 is constructive and relies on the same method as the proof of Theorem 2.2.5. One can easily notice that Theorem 2.3.3 applies to the finite-range specifications since $v(r) = 0$ after some $r \in \mathbb{N}$. However, if it comes to the infinite-range specifications, (2.29) is quite a restrictive condition; for example, in dimension one, it prevents the underlying specification from the phase transitions.

Theorem 2.3.4. *Let γ be a translation-invariant Gibbsian specification on $\Omega = E^{\mathbb{Z}}$ satisfying (2.29). Then there is a unique Gibbs measure compatible with γ, i.e., $\#\mathscr{G}(\Omega,\gamma) = 1$.*

For completeness, we shall provide a proof of the above theorem in Section 2.6.

2.3.2 The challenges of extending Kozlov's method to the translation-invariant setup

In his original paper [21], Kozlov proved two theorems – Theorem 2.2.5 and Theorem 2.3.3 above – on the generating interactions of Gibbsian specifications. The idea of proofs of both theorems is based on a regrouping of the bonds in Grimmett's vacuum interaction [18]. The regrouping procedure in the proof of Kozlov's first theorem (Theorem 2.2.5 above), which we refer to as *Kozlov's first way of regrouping*, does not adhere to any order, algebraic, or graph structure in $\mathbb{Z}^d$.

Consequently, it produces an interaction that is uniformly absolutely convergent (UAC) but does not align with the structural properties of $\mathbb{Z}^d$. In contrast, the regrouping procedure in his second theorem (Theorem 2.3.3 above), referred to as *Kozlov's second way of regrouping*, respects the order and graph structure of the lattice $\mathbb{Z}^d$. As a result, while it produces an interaction with good algebraic properties, it may have very bad summability properties. Below, we demonstrate these disadvantages of Kozlov's regrouping method in a special example of the specification of the Dyson model.

Recall that the single-site density of Dyson's specification γ^D at a site $i \in \mathbb{Z}$ is defined by

$$\gamma^D_{\{i\}}(\omega_i|\omega_{\mathbb{Z}\setminus\{i\}}) := \frac{\exp(\beta\sum_{k=1}^{\infty}\frac{\omega_i(\omega_{i+k}+\omega_{i-k})}{k^\alpha})}{\exp(\beta\sum_{k=1}^{\infty}\frac{\omega_{i+k}+\omega_{i-k}}{k^\alpha})+\exp(-\beta\sum_{k=1}^{\infty}\frac{\omega_{i+k}+\omega_{i-k}}{k^\alpha})},\ \forall\omega\in\Omega, \tag{2.30}$$

where $\alpha \in (1,2]$ is the decay rate of the coupling constants and $\beta \geq 0$ is the inverse temperature. Note that the associated specification γ^D is uniquely recovered from these single-site densities [13].

The vacuum interaction for the specification γ^D corresponding to the vacuum configuration $\mathbf{-} := -1_{\mathbb{Z}}$ is given by the the following lattice-gas interaction Φ^-:

$$\Phi^-_\Lambda(\omega) := \begin{cases} -\dfrac{\beta(1+\omega_i)(1+\omega_j)}{|i-j|^\alpha}, & \text{if } \Lambda=\{i,j\}\subset\mathbb{Z},\ i\neq j;\\ 2\beta\zeta(\alpha)(1+\omega_i), & \text{if } \Lambda=\{i\}\subset\mathbb{Z};\\ 0, & \text{otherwise,}\end{cases} \tag{2.31}$$

where ζ is the Riemann zeta function.

If we enumerate the elements in $\mathbb{Z}$ by $\{\ell_1,\ell_2,\ell_3,\ldots\}$, where for $k\in\mathbb{N}$, $\ell_{2k-1} := k-1$ and $\ell_{2k} := -k$, and define for $(k,i)\in\mathbb{N}^2$, $L^k_{2^i} := \{\ell_j : k\leq j\leq k+2^i\}$, $\mathscr{S}^k_0 := \emptyset$ and

$$\mathscr{S}^k_i := \{B\subset L^k_{2^i} : \ell_k\in B\}\setminus\mathscr{S}^k_{i-1},$$

then Kozlov's first way of regrouping produces the following interaction for γ^D:

$$\Phi^{KR1}_\Lambda := \begin{cases} 0, & \text{if } \Lambda\neq L^k_{2^i} \text{ for all } (k,i)\in\mathbb{N}^2;\\ \displaystyle\sum_{B\in\mathscr{S}^k_i}\Phi^-_B, & \text{if } \Lambda = L^k_{2^i}, \text{ for some } (k,i)\in\mathbb{N}^2.\end{cases} \tag{2.32}$$

By the construction, the interaction $\Phi^{KR1} = (\Phi^{KR1}_\Lambda)_{\Lambda\Subset\mathbb{Z}}$ generates the specification γ^D and is UAC. However, Φ^{KR1} is not translation-invariant, because

$$\Phi^{KR1}_{\{-1,0,1\}}(\omega) = \beta(1+\omega_0)(2\zeta(\alpha)-2-\omega_{-1}-\omega_1)$$

and for all $k\in\mathbb{Z}\setminus\{0\}$, one has $\Phi^{KR1}_{\{k-1,k,k+1\}}\equiv 0$.

Kozlov's second way of regrouping produces the following interaction for the specification γ^D:

$$\Phi_\Lambda^{KR2} := \begin{cases} \sum\limits_{\ell \le j \le \ell+2} \Phi^-_{\{\ell,j\}}, & \text{if } \Lambda = [\ell, \ell+2] \cap \mathbb{Z},\ \ell \in \mathbb{Z}; \\ \sum\limits_{\ell+2^{k-1} < j \le \ell+2^k} \Phi^-_{\{\ell,j\}}, & \text{if } \Lambda = [\ell, \ell+2^k] \cap \mathbb{Z},\ \ell \in \mathbb{Z},\ k \in \mathbb{N} \setminus \{1\}; \\ 0, & \text{otherwise,} \end{cases} \tag{2.33}$$

This time, by the construction, the produced interaction Φ^{KR2} is translation-invariant, but not UAC. In fact, for $k \ge 2$, one has the following for $\|\Phi^{KR2}_{[0,2^k]}\|_\infty$:

$$\|\Phi^{KR2}_{[0,2^k]}\|_\infty = \sum_{i=2^{k-1}+1}^{2^k} \frac{4\beta}{i^\alpha}, \tag{2.34}$$

thus for all $k \ge 2$,

$$\|\Phi^{KR2}_{[0,2^k]}\|_\infty \ge 2\beta \cdot 2^{(1-\alpha)k}. \tag{2.35}$$

Hence for $\alpha \in (1,2]$, one has

$$\sum_{0 \in V \Subset \mathbb{Z}} \|\Phi_V^{KR2}\|_\infty \;\ge\; \sum_{k=2}^{\infty} (2^k+1) \|\Phi^{KR2}_{[0,2^k]}\|_\infty \;\ge\; 2\beta \cdot \sum_{k=2}^{\infty} 2^{(2-\alpha)k} \;=\; \infty. \tag{2.36}$$

Thus Φ^{KM2} is not a UAC interaction.

Remark 2.3.5. *Although Kozlov's method does not yield interactions for the specification of the Dyson model having desired algebraic and summability properties, Grimmett's vacuum interaction* Φ^- *given by (2.31) already possesses these properties. However, Grimmett's vacuum interaction does not always have good summability properties and, in fact, it often lacks the UAC property.*

2.4 Main Results I: An Interaction-Independent Variational-Principle on $\mathbb{Z}^d$ and its Consequences

2.4.1 An Interaction-Independent Variational Principle

In probability theory, there are several notions that aim to measure the discrepancy between probability measures. One such concept is *relative entropy*, which plays a key role in both information theory and statistics and is also important in the formulation of the variational principle in Mathematical Statistical Mechanics.

Consider probability measures $\tau, \mu \in \mathscr{M}_1(\Omega)$ with full support, i.e., for any cylindrical set $[\sigma_\Lambda]$, $\sigma \in \Omega$, $\Lambda \Subset \mathbb{Z}^d$, one has $\tau([\sigma_\Lambda]) > 0$ and $\mu([\sigma_\Lambda]) > 0$. For a finite volume $\Lambda \Subset \mathbb{Z}^d$, consider the *marginals* τ_Λ and μ_Λ of probability measures τ and μ which are finite-volume measures on the sub-sigma algebra $\mathscr{F}_\Lambda$. By the full-support condition, one has $\tau_\Lambda \ll \mu_\Lambda$ and vice versa. Then the *relative entropy* $H(\tau_\Lambda|\mu_\Lambda)$ *of* τ_Λ *relative to* μ_Λ is

$$\begin{aligned} H(\tau_\Lambda|\mu_\Lambda) &:= \int_\Omega \log \frac{d\tau_\Lambda}{d\mu_\Lambda} d\tau_\Lambda \\ &= \sum_{\sigma_\Lambda \in E^\Lambda} \tau([\sigma_\Lambda]) \log \frac{\tau([\sigma_\Lambda])}{\mu([\sigma_\Lambda])}. \end{aligned} \tag{2.37}$$

By a direct application of the Jensen inequality, one can show that $H(\tau_\Lambda|\mu_\Lambda) \geq 0$. However, for the infinite volume measures τ and μ, it is often the case that $H(\tau|\mu) = +\infty$ which is not a quite useful information in comparing τ to μ. Nevertheless, in the case of infinite volume measures, the *specific relative entropy* – which can be interpreted as the relative entropy per site – gives some information. Recall that the ***specific relative entropy*** $h(\tau|\mu)$ *of* τ *relative to* μ is defined by

$$h(\tau|\mu) := \lim_{n\to\infty} \frac{1}{\Lambda_n} H(\tau_{\Lambda_n}|\mu_{\Lambda_n}) \tag{2.38}$$

provided the limit exists.

Now consider a translation-invariant UAC interaction Φ on Ω. Then one has the following, the so-called *variational principle* for translation-invariant Gibbs measures for the interaction Φ. In order to stress the difference, we state two versions of the variational principle in separate statements. Yet, these versions are equivalent, at least in the case where the specification is generated by a translation-invariant UAC interaction.

First, for the UAC interaction Φ, define a continuous function (potential) by

$$u_\Phi := -\sum_{0 \in V \Subset \mathbb{Z}^d} \frac{1}{|V|} \Phi_V$$

which is interpreted in Statistical Mechanics as the ***energy contribution from*** *(the neighbourhood of)* ***the origin***.

Theorem 2.4.1. [16, Chapter 15]

(VP1) *Let* $\mu \in \mathscr{G}_S(\Phi)$ *and* $\tau \in \mathscr{M}_{1,S}(\Omega)$, *then the* ***specific relative entropy*** $h(\tau|\mu)$ *of* τ *with respect to* μ *exists and*

$$h(\tau|\mu) = 0 \iff \tau \in \mathscr{G}_S(\Phi). \tag{2.39}$$

(VP2) *The translation-invariant Gibbs measures for the interaction Φ are exactly the equilibrium states for the potential u_Φ and vice versa, i.e., $\mathscr{G}_S(\Phi) = \mathscr{E}\mathscr{S}(u_\Phi)$. In other words, for a translation-invariant measure $\mu \in \mathscr{M}_{1,S}(\Omega)$, one has*

$$h(\mu) + \int_\Omega u_\Phi \, d\mu = \sup\left\{ h(\tau) + \int_\Omega u_\Phi \, d\tau : \tau \in \mathscr{M}_{1,S}(\Omega) \right\} \iff \mu \in \mathscr{G}_S(\Phi). \tag{2.40}$$

Remark 2.4.2. *In* [25], *Pfister proves that for any translation-invariant probability measure $\tau \in \mathscr{M}_{1,S}(\Omega)$ and every translation-invariant measure $\mu \in \mathscr{M}_{1,S}(\Omega)$ which is* ***asymptotically decoupled from above****, the relative entropy $h(\tau|\mu)$ exists. Note that a measure $\nu \in \mathscr{M}_1(\Omega)$ is asymptotically decoupled from above if there exist $g : \mathbb{N} \to \mathbb{N}$ and $c : \mathbb{N} \to [0, \infty)$ with $\lim_{n\to\infty} \frac{g(n)}{n} = 0$ and $\lim_{n\to\infty} \frac{c(n)}{|\Lambda_n|} = 0$ such that for every $i \in \mathbb{Z}^d$, $n \in \mathbb{N}$, $A \in \mathscr{F}_{\Lambda_n + i}$ and all $B \in \mathscr{F}_{i+\Lambda_{n+g(n)}}$, one has*

$$\nu(A \cap B) \le e^{c(n)} \nu(A) \nu(B). \tag{2.41}$$

Any Gibbs measure, in the sense of Definition 2.2.4, is asymptotically decoupled from both above and below, namely together with the upper bound (2.41), the following lower bound also holds for ν:

$$e^{-c(n)} \nu(A) \nu(B) \le \nu(A \cap B). \tag{2.42}$$

In the first statement of Theorem 2.4.1, all participating quantities and notions are independent of the form of interaction Φ. Therefore, it is reasonable to expect a generalisation of the statement to the Gibbsian specifications. In the light of Theorem 2.3.1, such a generalisation would be strictly stronger than the first statement of Theorem 2.4.1. In [22], the authors proved such a generalisation.

Theorem 2.4.3. [22] *Let γ be a translation-invariant Gibbsian specification and $\mu \in \mathscr{G}_S(\gamma)$. Then for all $\tau \in \mathscr{M}_{1,S}(\Omega)$, the specific relative entropy $h(\tau|\mu)$ exists and*

$$h(\tau|\mu) = 0 \iff \tau \in \mathscr{G}_S(\gamma).$$

The purpose of this section is to state the VP2 in terms of specifications. To do so, first, we have to extend certain interaction-dependent thermodynamic quantities such as the notion of *the contribution to energy from the origin* to the setup of specifications.

Henceforth, $\le$ denotes the lexicographic order in the lattice $\mathbb{Z}^d$, and for $i, j \in \mathbb{Z}^d$ with $i \le j$, $[i, j] := \{k \in \mathbb{Z}^d : i \le k \le j\}$. Open and half-open intervals are also defined analogously. For $n \in \mathbb{N}$, Λ_n denotes the volume $[-n, n]^d \cap \mathbb{Z}^d$. Consider a *translation-invariant* probability measure $\rho \in \mathscr{M}_{1,S}(\Omega)$, and define a function interpreted as the ***contribution to the energy from the origin*** by

$$u_\gamma^\rho(\omega) := \int_\Omega \log \frac{\gamma_{\{\mathbf{0}\}}(\omega_\mathbf{0}|\theta_{<\mathbf{0}}\omega_{>\mathbf{0}})}{\gamma_{\{\mathbf{0}\}}(\theta_\mathbf{0}|\theta_{<\mathbf{0}}\omega_{>\mathbf{0}})}\rho(d\theta), \quad \omega \in \Omega. \tag{2.43}$$

Remark 2.4.4. *Note that the notion of the contribution to the energy from the origin that we have introduced above generalise the corresponding notion in Statistical Mechanics. This can be illustrated through the concept of physical equivalence. In fact, if a translation-invariant Gibbsian specification γ is generated by a translation-invariant UAC interaction Φ, then u_γ^ρ is physically equivalent to $u_\Phi = -\sum_{0\in V\Subset\mathbb{Z}^d} \frac{1}{|V|}\Phi_V$, i.e., there exists a constant $C \in \mathbb{R}$ such that for all $\tau \in \mathscr{M}_{1,S}(\Omega)$, one has $\int_\Omega u_\gamma^\rho\, d\tau = \int_\Omega u_\Phi\, d\tau + C$. In particular, all functions u_γ^ρ parameterised by the translation-invariant probability measures $\rho \in \mathscr{M}_{1,S}(\Omega)$ are physically equivalent to each other. It should also be noted that if the reference measure ρ is a Dirac measure δ_+ where $+$ is a constant configuration with $+_i = + \in E$ for all $i \in \mathbb{Z}^d$, then (2.43) provides the following function:*

$$u_\gamma^+(\omega) := \log \frac{\gamma_{\{\mathbf{0}\}}(\omega_\mathbf{0}|+_{<\mathbf{0}}\,\omega_{>\mathbf{0}})}{\gamma_{\{\mathbf{0}\}}(+_\mathbf{0}|+_{<\mathbf{0}}\,\omega_{>\mathbf{0}})}, \quad \omega \in \Omega. \tag{2.44}$$

The following theorem is a natural generalisation of the second part (VP2) of Theorem 2.4.1 to the setup of specifications.

Theorem 2.A. *Assume γ is a translation-invariant Gibbs specification on Ω. Then the translation-invariant Gibbs measures for γ are exactly the equilibrium states for the potential u_γ^ρ and vice versa, i.e., $\mathscr{G}_S(\gamma) = \mathscr{E}\mathscr{S}(u_\gamma^\rho)$. In other words,*

$$h(\mu) + \int_\Omega u_\gamma^\rho\, d\mu = \sup\left\{h(\tau) + \int_\Omega u_\gamma^\rho\, d\tau : \tau \in \mathscr{M}_{1,S}(\Omega)\right\} \quad \Longleftrightarrow \quad \mu \in \mathscr{G}_S(\gamma). \tag{2.45}$$

We shall present a proof of Theorem 2.A in Section 2.6. It should be noted that our proof of the theorem yields, as a byproduct, the following formula for the specific relative entropy $h(\tau|\mu)$ of a translation measure τ with respect to a Gibbs measure μ.

Corollary A.1. *Let μ be a translation-invariant Gibbs measure prescribed by a Gibbsian specification γ and τ be a translation-invariant measure on Ω. Then for the specific relative entropy $h(\tau|\mu)$, one has*

$$h(\tau|\mu) \;=\; -h(\tau) - \lim_{n\to\infty} \frac{1}{|\Lambda_n|}\int_\Omega \log \gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})\tau(d\omega). \tag{2.46}$$

So far, we have not established any regularity properties of the function u_γ^ρ, which is associated with a translation-invariant Gibbsian specification γ via (2.43). We now demonstrate that u_γ^ρ possesses the *extensibility property*. Recall that a continuous potential $\phi : \Omega \to \mathbb{R}$ satisfies the **extensibility** condition if for all $a, b \in E$ the sequence

$$F_n^{b,a}(\omega) := \sum_{i\in\Lambda_n} \big(\phi \circ S_i(\omega^b) - \phi \circ S_i(\omega^a)\big)$$

converges uniformly on $\omega \in \Omega$ as $n \to \infty$, here the configuration ω^a is given by

$$\omega_k^a = \begin{cases} a, & k = 0, \\ \omega_k, & k \in \mathbb{Z}^d \setminus \{0\}. \end{cases}$$

Proposition 2.4.5. *Let γ be a translation-invariant Gibbsian specification on Ω. Then u_γ^ρ satisfies the extensibility condition.*

Proof. Pick any $a, b \in E$ and $\omega \in \Omega$. One can check that for any $n \in \mathbb{N}$,

$$\begin{aligned} F_n^{b,a}(\omega) &= \sum_{i\in\Lambda_n} \left[u_\gamma^\rho \circ S_i(\omega^b) - u_\gamma^\rho \circ S_i(\omega^a) \right] \\ &= \sum_{\substack{i\in\Lambda_n \\ i<0}} \int_\Omega \log \frac{\gamma_{\{0\}}((\omega_i)_0 | \theta_{<0}(\omega_{(i,0)} b_0 \omega_{>0})_{>0})}{\gamma_{\{0\}}(\theta_0 | \theta_{<0}(\omega_{(i,0)} b_0 \omega_{>0})_{>0})} \rho(d\theta) \\ &\quad - \sum_{\substack{i\in\Lambda_n \\ i<0}} \int_\Omega \log \frac{\gamma_{\{0\}}((\omega_i)_0 | \theta_{<0}(\omega_{(i,0)} a_0 \omega_{>0})_{>0})}{\gamma_{\{0\}}(\theta_0 | \theta_{<0}(\omega_{(i,0)} a_0 \omega_{>0})_{>0})} \rho(d\theta) + F_0^{b,a}(\omega). \end{aligned}$$

Then, by applying the bar moving property, one gets

$$\begin{aligned} F_n^{b,a}(\omega) - F_0^{b,a}(\omega) &= \sum_{\substack{i\in\Lambda_n \\ i<0}} \int_\Omega \log \frac{\gamma_{\{0,-i\}}\Big((\omega_i)_0 (b_0)_{-i} | \theta_{<0}(\omega_{(i,0)} \omega_{>0})_{\substack{>0 \\ \neq -i}}\Big)}{\gamma_{\{0,-i\}}\Big(\theta_0 (b_0)_{-i} | \theta_{<0}(\omega_{(i,0)} \omega_{>0})_{\substack{>0 \\ \neq -i}}\Big)} \rho(d\theta) \\ &\quad + \sum_{\substack{i\in\Lambda_n \\ i<0}} \int_\Omega \log \frac{\gamma_{\{0,-i\}}\Big(\theta_0 (a_0)_{-i} | \theta_{<0}(\omega_{(i,0)} \omega_{>0})_{\substack{>0 \\ \neq -i}}\Big)}{\gamma_{\{0,-i\}}\Big((\omega_i)_0 (a_0)_{-i} | \theta_{<0}(\omega_{(i,0)} \omega_{>0})_{\substack{>0 \\ \neq -i}}\Big)} \rho(d\theta), \end{aligned}$$

and thus

$$F_n^{b,a}(\omega) - F_0^{b,a}(\omega) =$$

$$\sum_{\substack{i\in\Lambda_n \\ i<0}} \int_\Omega \log \frac{\gamma_{\{-i\}}((b_0)_{-i} | \theta_{<0}(\omega_{[i,0)} \omega_{>0})_{\substack{\geq 0 \\ \neq -i}})}{\gamma_{\{-i\}}((a_0)_{-i} | \theta_{<0}(\omega_{[i,0)} \omega_{>0})_{\substack{\geq 0 \\ \neq -i}})} \cdot \frac{\gamma_{\{-i\}}((a_0)_{-i} | \theta_{\leq 0}(\omega_{(i,0)} \omega_{>0})_{\substack{>0 \\ \neq -i}})}{\gamma_{\{-i\}}((b_0)_{-i} | \theta_{\leq 0}(\omega_{(i,0)} \omega_{>0})_{\substack{>0 \\ \neq -i}})} \rho(d\theta).$$

Hence by the translation-invariance of γ and ρ,

$$
\begin{aligned}
&F_n^{b,a}(\omega)-F_0^{b,a}(\omega)=\\
&=\sum_{\substack{i\in\Lambda_n\\ i<0}}\int\limits_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{<i}\,\omega_{[i,0)}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{<i}\,\omega_{[i,0)}\omega_{>0})}\rho(d\theta)-\sum_{\substack{i\in\Lambda_n\\ i<0}}\int\limits_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{\le i}\,\omega_{(i,0)}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{\le i}\,\omega_{(i,0)}\omega_{>0})}\rho(d\theta)\\
&=\int\limits_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{<\min\Lambda_n}\,\omega_{[\min\Lambda_n,0)}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{<\min\Lambda_n}\,\omega_{[\min\Lambda_n,0)}\omega_{>0})}\rho(d\theta)-\int\limits_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{<0}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{<0}\omega_{>0})}\rho(d\theta).
\end{aligned}
$$

Thus

$$
F_n^{b,a}(\omega)=\int\limits_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{<\min\Lambda_n}\,\omega_{[\min\Lambda_n,0)}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{<\min\Lambda_n}\,\omega_{[\min\Lambda_n,0)}\omega_{>0})}\rho(d\theta)\ \underset{\substack{n\to\infty\\ \omega\in\Omega}}{\rightrightarrows}\ \log\frac{\gamma_{\{0\}}(b_0|\omega_{\mathbb{Z}^d\setminus\{0\}})}{\gamma_{\{0\}}(a_0|\omega_{\mathbb{Z}^d\setminus\{0\}})}. \tag{2.47}
$$

□

2.4.2 Erasure entropies

In the rest of this section, we restrict ourselves to the one-dimensional setup, i.e., $d=1$.

For the measure-theoretic entropy of a translation-invariant measure τ on the shift space $\Omega=E^{\mathbb{Z}}$, one can prove that

$$
h(\tau)=-\int_\Omega \log\tau(\omega_0|\omega_{>0})\tau(d\omega). \tag{2.48}
$$

One can generalise the above formula for any finite set $\Lambda\Subset\mathbb{Z}_+$ by

$$
h(\tau)=-\frac{1}{|\Lambda|}\int_\Omega \log\tau(\omega_\Lambda|\omega_{\mathbb{Z}_+\setminus\Lambda})\tau(d\omega). \tag{2.49}
$$

These formulas are valid if one conditions on the *past* instead of the *future*. It is then natural to ask whether these formulas remain valid if the one-sided conditioning is replaced with two-sided conditioning. In fact, by substituting the one-sided conditioning with two-sided conditioning in (2.48), one obtains the ***erasure entropy*** $h^-(\tau)$ of the measure τ [29], namely,

$$
h^-(\tau)=-\int_\Omega \log\tau(\omega_0|\omega_{\{0\}^c})\tau(d\omega). \tag{2.50}
$$

It should be stressed that a similar formula to (2.49) does not hold for the erasure entropies; therefore, for a finite volume $\Lambda\Subset\mathbb{Z}$, the *erasure entropy of* τ *in* Λ is defined in [11] as

$$
h_\Lambda^-(\tau):=-\int_\Omega \log\tau(\omega_\Lambda|\omega_{\Lambda^c})\tau(d\omega). \tag{2.51}
$$

The erasure entropy is always less than, and generally not equal to, the measure-theoretic entropy. However, in the limit, one can expect the equality of these two notions as the following theorem states.

Theorem 2.4.6. [11] *Assume τ be a translation-invariant Gibbs measure for a translation-invariant UAC interaction Φ, i.e., $\tau \in \mathscr{G}_S(\Phi)$. Then*

$$\lim_{\Lambda\uparrow\mathbb{Z}} \frac{1}{|\Lambda|} h^-_\Lambda(\tau) = h(\tau). \tag{2.52}$$

The following corollary of Theorem 2.A enables us to extend the above result to any translation-invariant Gibbs measure.

Corollary 2.4.7. *Let τ be a translation-invariant Gibbs measure on $\Omega = E^{\mathbb{Z}}$. Then*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} h^-_{\Lambda_n}(\tau) = h(\tau). \tag{2.53}$$

2.5 Main Results II: Gibbs formalism on $\mathbb{Z}_+$

2.5.1 DLR-Gibbs formalism on $\mathbb{Z}_+$

In this section, we work on the half-line $\mathbb{Z}_+$, denoting the configuration space $E^{\mathbb{Z}_+}$ by X_+, with the left-shift (or translation) map S acting on X_+ as follows: for $x \in X_+$, and all $i \in \mathbb{Z}_+$, $(Sx)_i = x_{i+1}$.

Let $\vec{\gamma}$ be a non-null specification on X_+. Set

$$\tilde{g}_0(x) := \vec{\gamma}_{\{0\}}(x_0|x_1^\infty), \;\; x \in X_+, \tag{2.54}$$

and for all $n \geq 1$, consider a function $\tilde{g}_n : X_+ \to [0,1]$ given by

$$\tilde{g}_n(x) := \frac{\vec{\gamma}_{[0,n]}(x_0^n|x_{n+1}^\infty)}{\vec{\gamma}_{[0,n-1]}(x_0^{n-1}|x_n^\infty)}, \;\; x \in X_+. \tag{2.55}$$

Note that by the construction, for all $n \geq 0$, one has

$$\vec{\gamma}_{[0,n]}(x_0^n|x_{n+1}^\infty) = \prod_{k=0}^{n} \tilde{g}_k(x), \;\; x \in X_+. \tag{2.56}$$

The functions $\{\tilde{g}_n, \; n \in \mathbb{Z}_+\}$ has the following properties:

Proposition 2.5.1. *(1) For all $n \geq 0$, $\tilde{g}_n$ is a positive function, and $\tilde{g}_n \in \mathscr{F}_{[n,\infty)}$, i.e., $\tilde{g}_n$ is independent of the first $n-1$ coordinates.*

(2) For all $n \geq 0$, and $x \in X_+$, $\sum_{y_n \in E} \tilde{g}_n(y_n x_{n+1}^\infty) = 1$.

Proof. (1) Positivity follows from the non-nullness of the specification $\vec{\gamma}$. The independence from the first $n-1$ coordinates follows from the bar moving property (2.3).

(2) Firstly, since $\vec{\gamma}$ is a specification, for all $x \in X_+$, $\sum_{\bar{x}_0 \in E} \tilde{g}_0(y_0 x_1^\infty) = 1$. Fix $n \geq 1$, and assume that the statement is correct for all $k < n$. Then, since $\vec{\gamma}$ is a specification, by the first part of the proposition and from (2.56), for all $x \in X_+$, one has

$$
\begin{aligned}
1 &= \sum_{y_0^n \in E^{n+1}} \prod_{k=0}^{n} \tilde{g}_k(y_k^n x_{n+1}^\infty) \\
&= \sum_{y_n \in E} \tilde{g}_n(y_n x_{n+1}^\infty) \sum_{y_{n-1} \in E} \tilde{g}_{n-1}(y_{n-1}^n x_{n+1}^\infty) \cdots \sum_{y_0 \in E} \tilde{g}_0(y_0^n x_{n+1}^\infty) \\
&= \sum_{y_n \in E} \tilde{g}_n(y_n x_{n+1}^\infty)
\end{aligned}
$$

□

Remark 2.5.2. *The first part of Proposition 2.5.1 yields that for all $n \in \mathbb{Z}_+$, there exists a positive measurable function g_n on X_+ such that $\tilde{g}_n = g_n \circ S^n$. Then the second part of Proposition 2.5.1 is equivalent to that for all $n \geq 0$, g_n is a g—function* [20] *on X_+, i.e., for all $x \in X_+$,* $\sum_{a \in E} g_n(ax) = 1$.

In terms of g_n's, (2.56) reads

$$
\vec{\gamma}_{[0,n]}(x_0^n | x_{n+1}^\infty) = \prod_{k=0}^{n} g_k \circ S^k(x) = \exp\Big(\sum_{k=0}^{n} \log g_k \circ S^k(x)\Big), \quad x \in X_+. \tag{2.57}
$$

Thus, a non-null specification on X_+ uniquely determines a sequence of g-functions. Now consider the opposite situation: assume that a sequence $\bar{\varphi} := \{\varphi_k : k \in \mathbb{Z}_+\}$ of Borel functions on X_+ is given. For all $\Lambda \Subset \mathbb{Z}_+$, define

$$
\vec{\gamma}_\Lambda^{\bar{\varphi}}(y_\Lambda | x_{\Lambda^c}) := \frac{\exp(\sum_{k=0}^{n} \varphi_k \circ S^k(y_\Lambda x_{\Lambda^c}))}{\sum_{\bar{y}_\Lambda \in E^\Lambda} \exp(\sum_{k=0}^{n} \varphi_k \circ S^k(\bar{y}_\Lambda x_{\Lambda^c}))}, \quad y, x \in X_+, \tag{2.58}
$$

where $n := \max \Lambda$. Then one can readily check that the family $\vec{\gamma}^{\bar{\varphi}} := (\vec{\gamma}_\Lambda^{\bar{\varphi}})_{\Lambda \Subset \mathbb{Z}_+}$ is indeed specification densities, therefore, one can associate a non-null specification with the sequence $\bar{\varphi}$. Then we conclude that there is an association between the non-null specifications on X_+ and the *generalized Birkhoff sums* $S_n \bar{\varphi} := \sum_{i=0}^{n-1} \phi_i \circ S^i$, $n \in \mathbb{N}$, where each φ_i is a Borel function. However, this association is not one-to-one, in fact, different generalized Birkhoff sums might give rise to the same specification which leads to the notion of *physical equivalence of potentials* [10]. In terms of cocycles on $\mathbb{Z}_+$, this means that any measurable (continuous) cocycle ρ on X_+ is given by

$$\rho(\omega,\omega')=\sum_{k=0}^{\infty}[\varphi_k\circ S^k(\omega)-\varphi_k\circ S^k(\omega')],\ (\omega,\omega')\in\mathfrak{T}(X_+), \tag{2.59}$$

where each ϕ_k is a measurable (continuous) function on X_+, and vise versa, i.e., for any sequence $\{\varphi_k\}_{k\in\mathbb{Z}_+}$ of measurable (continuous) functions $\varphi_k : X_+\to\mathbb{R},\ k\geq 0$, (2.59) defines a measurable (continuous) cocycle on $\mathfrak{T}(X_+)$.

Now consider a measurable (continuous) cocycle ρ on X_+. We define *the base of the cocycle* ρ by the following formula:

$$\rho_0(\omega,\omega')=\rho(\omega,\omega')-\rho(S\omega,S\omega'),\ (\omega,\omega')\in\mathfrak{T}(X_+). \tag{2.60}$$

Then it is clear that ρ_0 is also a measurable cocycle on $\mathfrak{T}(X_+)$, and it is continuous if ρ is continuous. Furthermore, one has the following important identity for ρ:

$$\rho(\omega,\omega')=\sum_{k=0}^{\infty}\rho_0(S^k\omega,S^k\omega'),\ (\omega,\omega')\in\mathfrak{T}(X_+). \tag{2.61}$$

2.5.2 Interactions on the lattice $\mathbb{Z}_+$

Now we turn to the interaction on $X_+=E^{\mathbb{Z}_+}$. By following the concept of translation-invariant interactions on $\mathbb{Z}$, we define such a notion for the interactions on $\mathbb{Z}_+$. Note that we also use the term one-sided interaction for the interactions on X_+. We again define the *translation-invariance of a one-sided interaction* $\vec{\Phi}$ with (2.25), but this time we only assume that Λ runs over the finite subsets of $\mathbb{Z}_+$, i.e., for all $\Lambda\Subset\mathbb{Z}_+$, the following holds true

$$\vec{\Phi}_\Lambda\circ S=\vec{\Phi}_{\Lambda+1}. \tag{2.62}$$

Note that the restriction of a translation-invariant interaction Φ on $\Omega=E^{\mathbb{Z}}$ to X_+ remains translation-invariant. Simultaneously, any left-shift invariant interaction $\vec{\Phi}$ on X_+ can be extended to Ω by translation, namely, the translated interaction Φ is defined for the volumes $\Lambda\Subset\mathbb{Z}$ with $\min\Lambda<0$ by $\Phi_\Lambda:=\vec{\Phi}_{\Lambda-\min\Lambda}\circ S^{\min\Lambda}$ (note that S is invertible on Ω). Thus there is a one-to-one correspondence between the translation-invariant interactions on Ω and X_+. It should also be mentioned that one-sided translation-invariant interaction $\vec{\Phi}$ does not give rise to a "translation-invariant" specification $\gamma^{\vec{\Phi}}$ on $\mathbb{Z}_+$ in a sense that the associated cocycle $\rho^{\vec{\Phi}}$ satisfies $\rho^{\vec{\Phi}}(\omega,\omega')=\rho^{\vec{\Phi}}(S\omega,S\omega')$ (note that we define *translation-invariant specifications* in Subsection 2.5.5 in a slightly different way).

We should note that there is a one-to-one correspondence between the translation -invariant UAC interactions on X_+ and the continuous potentials in $C(X_+)$. In fact, any translation-invariant UAC interaction Φ on X_+ can be associated with a continuous potential $\phi \in C(X_+)$ by the following formula:

$$\phi = -\sum_{0 \in V \Subset \mathbb{Z}_+} \vec{\Phi}_V. \tag{2.63}$$

Then, in fact, one can check that for all $n \in \mathbb{N}$,

$$S_n\phi = \sum_{i=0}^{n-1} \phi \circ S^i = -H^{\vec{\Phi}}_{[0,n-1]} \tag{2.64}$$

where $H^{\vec{\Phi}}_{[0,n-1]}$ is the Hamiltonian in the volume V, i.e., $H^{\vec{\Phi}}_{[0,n-1]} = \sum_{\substack{V \Subset \mathbb{Z}_+ \\ V \cap [0,n-1] \neq \emptyset}} \vec{\Phi}_V$.

Hence $\vec{\gamma}^{\phi} = \gamma^{\vec{\Phi}}$, in other words, the one-sided interaction $\vec{\Phi}$ and the function ϕ are the same from the viewpoint of Thermodynamic Formalism.

The opposite of the above observation is also true. More precisely, as the following proposition states, with any continuous function ϕ, one can associate an (in fact, many) translation-invariant one-sided UAC interaction $\vec{\Phi}$ satisfying (2.63).

Proposition 2.5.3. *For any potential $\phi \in C(X_+)$, there exists a translation-invariant UAC interaction $\vec{\Phi}$ on X_+ such that $\phi = -\sum_{0 \in V \Subset \mathbb{Z}_+} \vec{\Phi}_V$.*

Proof. It is an immediate application of the Stone-Weirstrass theorem to show that for the potential ϕ, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of local functions on X_+ such that $\phi = \sum_{n\in\mathbb{N}} f_n$ and $\sum_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. Let Λ_n denote the finite volume on which the local function f_n is based. We define an interaction $\vec{\Phi}$ on X_+ as follows: $\vec{\Phi}_\Lambda = -f_n$ if Λ is a translation of Λ_n, otherwise, $\vec{\Phi}_\Lambda = 0$. Then by the construction, $\vec{\Phi}$ is translation-invariant interaction on X_+ and satisfies the conditions of the proposition. □

Remark 2.5.4. *By Theorem 2.2.5 and using the arguments in Subsection 2.5.1, we summarise the correspondence between the UAC interactions and the Birkhoff sums on X_+ in the following box:*

2.5.3 G-measures and their relationship with the DLR-Gibbs measures

So far, we have not needed an algebraic structure in the state space E and the configuration space X_+. Now, we identify the state space E with the finite cyclic

(General) UAC interactions	$\Longleftrightarrow$	**Generalized Birkhoff sums**
Translation-invariant UAC interactions	$\Longleftrightarrow$	**Birkhoff sums**

group $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z} \simeq \{0,1,\cdots,l-1\}$, where $l = |E|$. For the configurations $\omega, \omega' \in X_+$, we denote the coordinate-wise addition by $\omega+\omega'$, i.e., $\omega+\omega' = (\omega_i+\omega'_i)_{i\in\mathbb{Z}_+}$. Then, clearly, X_+ becomes a compact Abelian group. We denote the direct sum $\bigoplus_{\mathbb{Z}_+} E$ by Γ. Note that $\Gamma \subset X_+$, and $\omega \in X_+$ is in Γ iff there exists $\Delta \Subset \mathbb{Z}_+$ such that $\omega_i = 0$ for all $i \in \Delta^c$. For each $\Delta \Subset \mathbb{Z}_+$, one can define a subgroup Γ_Δ of Γ by $\Gamma_\Delta := \{\omega \in \Gamma : \omega_i = 0,\ i \in \Delta^c\}$. Then it is clear that $\Gamma = \bigcup_{\Delta\Subset\mathbb{Z}_+} \Gamma_\Delta$.

Now consider a family $G := (G_\Lambda)_{\Lambda\Subset\mathbb{Z}_+}$ of nonnegative Borel functions $G_\Lambda : X_+ \to [0,1]$. G is called *compatible* if for all $\Delta \subset \Lambda \Subset \mathbb{Z}_+$,

$$G_\Lambda(x+\omega)G_\Delta(\omega) = G_\Lambda(\omega)G_\Delta(x+\omega), \;\; x \in \Gamma_\Delta, \omega \in X_+, \tag{2.65}$$

and *normalized* if for all $\Lambda \Subset \mathbb{Z}_+$,

$$\sum_{\xi_\Lambda\in E^\Lambda} G_\Lambda(\xi_\Lambda \eta_{\Lambda^c}) = 1. \tag{2.66}$$

We call a compatible normalized family $G = (G_\Lambda)_{\Lambda\Subset\mathbb{Z}_+}$ a ***G-family*** [4].

Note that each $x \in \Gamma$ defines a map $x : X_+ \to X_+$ by $\omega \mapsto x(\omega) = x+\omega$. Then let $x_* : \mathscr{M}_1(X_+) \to \mathscr{M}_1(X_+)$ denote the pushforward of x. For a probability measure $\nu \in \mathscr{M}_1(X_+)$ and a finite volume $\Lambda \Subset \mathbb{Z}_+$, define

$$\nu_\Lambda := \frac{1}{l^{|\Lambda|}} \sum_{x\in\Gamma_\Lambda} x_*\nu. \tag{2.67}$$

Then it is clear that $\nu \ll \nu_\Lambda$ since $e_*\nu = \nu$ where e is the neutral element of the group Γ.

Definition 2.5.5. [4] *A probability measure $\nu \in \mathscr{M}_1(X_+)$ is called a **G-measure**, if there exists a G-family $G = (G_\Lambda)_{\Lambda\Subset\mathbb{Z}_+}$ such that for all $\Lambda \Subset \mathbb{Z}_+$,*

$$\frac{d\nu}{d\nu_\Lambda} = l^{|\Lambda|}\, G_\Lambda. \tag{2.68}$$

*A $G-$measure ν is called **g-measure** if for the associated $G-$family, one has*

$$\frac{G_{[0,n+1]\cap\mathbb{Z}_+}}{G_{[0,n]\cap\mathbb{Z}_+}} = G_{\{0\}} \circ S^{n+1}, \;\; n \in \mathbb{Z}_+. \tag{2.69}$$

The $g-$measures have applications in harmonic analysis, in particular, in the theory of Riesz products [20]. Inspired by these applications of g-measures, Brown and Dooley developed G-formalism by extending the concept of g-measures and utilising the formalism of Riesz products [4]. However, the following simple proposition and Theorem 2.B demonstrate that the notions of G-family and G-measure are equivalent to the notions of specification and DLR-Gibbs measures on X_+.

Proposition 2.5.6. *(i) Let $G=(G_V)_{V\Subset\mathbb{Z}_+}$ be a G-family, then $\gamma=(\gamma_V)_{V\Subset\mathbb{Z}_+}$ defined by*

$$\gamma_V(\xi_V|\eta_{V^c}):=G(\xi_V\eta_{V^c}),\ \ V\Subset\mathbb{Z}_+,\ \xi_V\in E^V,\ \eta\in X_+, \tag{2.70}$$

is a family of specification densities.

(ii) Let γ be a specification on X_+, then

$$G_V(x):=\gamma_V(x_V|x_{V^c}),\ \ x\in X_+, V\Subset\mathbb{Z}_+ \tag{2.71}$$

is a G-family.

Proof. "*(i)*" By the normalization condition (2.66), for all $\eta\in X_+$, $\sum\limits_{\xi_V\in E^V}\gamma_V(\xi_V|\eta_{V^c})=1$. Then for $\Delta\subset\Lambda\Subset\mathbb{L}$, $\omega,\eta\in X_+$, by the compatibility condition (2.65),

$$\begin{aligned}\gamma_\Delta(\eta_\Delta|\eta_{\Lambda/\Delta}\omega_{\Lambda^c})\sum_{\sigma_\Delta\in E^\Delta}\gamma_\Lambda(\sigma_\Delta\eta_{\Lambda/\Delta}|\omega_{\Lambda^c})=G_\Delta(\eta_\Lambda\omega_{\Lambda^c})&\sum_{\sigma_\Delta}G_\Lambda(\sigma_\Delta\eta_{\Lambda\setminus\Delta}\omega_{\Lambda^c})\\ &\overset{(2.65)}{=}\sum_{\sigma_\Delta}\Big(G_\Delta(\sigma_\Delta\eta_{\Lambda\setminus\Delta}\omega_{\Lambda^c})G_\Lambda(\eta_\Lambda\omega_{\Lambda^c})\Big)\\ &\overset{(2.66)}{=}G_\Lambda(\eta_\Lambda\omega_{\Lambda^c})\\ &=\gamma_\Lambda(\eta_\Lambda|\omega_{\Lambda^c}).\end{aligned}$$

"(ii)" Clearly, G is a normalized family. To prove compatibility, take any $x_\Delta\in E^\Delta$ and $\omega\in X_+$, then

$$\begin{aligned}G_\Lambda(\omega)G_\Delta(x_\Delta\omega_{\Delta^c})&=\gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c})\gamma_\Delta(x_\Delta|\omega_{\Delta^c})\\ &\overset{(2.2)}{=}\gamma_\Delta(x_\Delta|\omega_{\Delta^c})\gamma_\Delta(\omega_\Delta|\omega_{\Delta^c})\sum_{\sigma_\Delta}\gamma_\Lambda(\sigma_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})\\ &\overset{(2.2)}{=}\gamma_\Delta(\omega_\Delta|\omega_{\Delta^c})\gamma_\Lambda(x_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})\\ &=G_\Delta(\omega)G_\Lambda(x_\Delta\omega_{\Delta^c}).\end{aligned}$$

□

Remark 2.5.7. *The above proposition establishes a one-to-one correspondence between specifications and G−families on X_+. Furthermore, a G-family is positive (continuous) if and only if the associated specification is non-null (qusilocal).*

Theorem 2.B. *Let G be a family and γ be the associated specification. Then μ is a G-measure for G if and only if $\mu \in \mathscr{G}(\gamma)$.*

We postpone the proof of Theorem 2.B until Section 2.6.

As we have already seen in this subsection and Subsection 2.2.3, the notions of non-null specification, positive G-family and cocycle are equivalent in the half-line setup. Due to this fact, we have been able to derive equations for the associated DLR-Gibbs measures in terms of specifications and G-families. By the next theorem, we derive such an equation for the associated DLR-Gibbs measures in terms of the cocycles.

Theorem 2.5.8. *Let G be a positive G-family on X_+ and ρ be the associated measurable cocycle (c.f., (2.12) and (2.70)). Then ν is a G-measure for G, or equivalently, ν is a DLR-Gibbs measure for the corresponding specification $\gamma = \gamma^\rho$ (c.f., (2.9)), if and only if for all $y \in \Gamma$, $y_*\nu \ll \nu$, and the following equations are satisfied:*

$$\frac{d(y_*\nu)}{d\nu}(\omega) = \exp\rho(\omega, \omega - y), \quad \nu - \text{ a.e.}, \omega \in X_+. \tag{2.72}$$

Proof. ***"if":*** Assume that for all $y \in \Gamma$, $y_*\nu \ll \nu$ and (2.72) is satisfied. Then from (2.9) and (2.71), for all $\Lambda \Subset \mathbb{Z}_+$ and for $\nu-$ almost every $\omega \in X_+$,

$$\frac{d\nu_\Lambda}{d\nu}(\omega) = \frac{1}{l^{|\Lambda|}} \sum_{x\in\Gamma_\Lambda} \frac{d(x_*\nu)}{d\nu}(\omega) = \frac{1}{l^{|\Lambda|}} \sum_{x\in\Gamma_\Lambda} e^{\rho(\omega,\omega-x)} = \frac{1}{l^{|\Lambda|}} \sum_{\xi_\Lambda\in E^\Lambda} e^{-\rho(\xi_\Lambda\omega_{\Lambda^c},\,\omega)}$$

$$= \frac{1}{l^{|\Lambda|}G_\Lambda(\omega)}.$$

"only if": Let ν be a G-measure for G, and take any $y \in \Gamma$, and let $y \in \Gamma_\Lambda$. Consider the measure $y_*\nu$. It can be checked that

$$(y_*\nu)_\Lambda = \frac{1}{l^{|\Lambda|}} \sum_{x\in\Gamma_\Lambda} x_*(y_*\nu) = \frac{1}{l^{|\Lambda|}} \sum_{x\in\Gamma_\Lambda} (x+y)_*\nu = \nu_\Lambda.$$

Clearly, $y_*\nu \ll \nu_\Lambda$ and let $f := \dfrac{d(y_*\nu)}{d\nu_\Lambda}$. Then for all $B \in \mathscr{F}$, one has $\nu(B - y) = \int_B f\, d\nu_\Lambda$, and hence

$$\nu(B) = \int_{X_+} f \cdot \mathbb{1}_{B+y}\, d\nu_\Lambda = \int_{X_+} f \circ y \cdot \mathbb{1}_B\, d(\nu_\Lambda \circ y). \tag{2.73}$$

Note that $\nu_\Lambda \circ y = \frac{1}{l^{|\Lambda|}} \sum_{x\in\Gamma_\Lambda} (\nu \circ x) \circ y = \frac{1}{l^{|\Lambda|}} \sum_{x\in\Gamma_\Lambda} \nu \circ (x+y) = \nu_\Lambda$. Hence (2.73) yields $\dfrac{d\nu}{d\nu_\Lambda} = f \circ y$, ν_Λ- a.e. Thus since ν is a G-measure for G, (2.68) implies that for

ν_Λ—a.e. $\omega \in X_+$, $f(\omega) = l^{|\Lambda|} G_\Lambda(\omega - y)$, therefore,

$$\frac{d(y_* \nu)}{d\nu_\Lambda} = l^{|\Lambda|} G_\Lambda \circ (-y), \quad \nu_\Lambda\text{-a.e. .} \tag{2.74}$$

Then by again applying (2.68), one has

$$\frac{d(y_* \nu)}{d\nu}(\omega) = \frac{G_\Lambda(\omega - y)}{G_\Lambda(\omega)}, \quad \nu\text{-a.e. } \omega \in X_+. \tag{2.75}$$

□

2.5.4 Gibbsianness in dynamical systems

We have defined the specifications in the half-line context in Subsection 2.5.1, and established the relationship between the generalised Birkhoff sums and the specifications. In the theory of dynamical systems, a specification associated with Birkhoff sums via (2.58) is particularly interesting. Note that in this case, for a continuous function (potential) $\phi : X_+ \to \mathbb{R}$, the specification $\vec{\gamma}^{\phi} := \vec{\gamma}^{(S_n\phi)}$ is given by

$$\vec{\gamma}^{\phi}_\Lambda(a_\Lambda | x_{\mathbb{Z}_+\setminus\Lambda}) = \frac{\exp\big((S_{n+1}\phi)(a_\Lambda x_{\mathbb{Z}_+\setminus\Lambda})\big)}{\sum\limits_{\bar{a}_\Lambda \in E^\Lambda} \exp\big((S_{n+1}\phi)(\bar{a}_\Lambda x_{\mathbb{Z}_+\setminus\Lambda})\big)}, \quad a, x \in X_+, \Lambda \Subset \mathbb{Z}_+, n := \max\Lambda. \tag{2.76}$$

It should be noted that there are Gibbsian specifications on X_+ which can not be associated with any $\phi \in C(X_+)$ via (2.76). Below we give an example of such a specification.

Example 2.5.9. *Consider the following cocycle on Ω_+:*

$$\rho(\omega, \omega') = \sum_{k=0}^{\infty} (k+1)(\omega_k - \omega'_k), \ (\omega, \omega') \in \mathfrak{T}(\Omega_+). \tag{2.77}$$

Note that ρ corresponds to the generalized Birkhoff sum $\sum\limits_{k=0}^{n-1}(k+1)(\sigma_0 \circ S^k)$, $n \geq 0$. The associated specification γ^ρ can not be generated by some $\phi \in C(X_+)$ via (2.76) (c.f. Theorem C1), since

$$\sup_{(\omega,\omega') \in \mathfrak{T}(\Omega_+)} \Big| \sum_{k=1}^{\infty} \big[\sigma_0 \circ S^k(\omega) - \sigma_0 \circ S^k(\omega')\big] \Big| = \infty.$$

We call a specification γ on $\mathbb{Z}_+$ ***dynamical*** if γ can be generated by some $\phi \in C(X_+)$ via (2.76). Equivalently, γ is dynamical if and only if the associated cocycle ρ^γ (c.f., (2.12)) is ***Gibbs***, i.e., for some $\phi \in C(X_+)$,

$$\rho^\gamma(\omega,\omega') = \sum_{k=0}^{\infty}\left[\phi\circ S^k(\omega) - \phi\circ S^k(\omega')\right],\ (\omega,\omega')\in\mathfrak{T}(\Omega_+). \tag{2.78}$$

Remark 2.5.10. *In fact, by the discussion at the end of Subsection 2.5.2 (see Remark 2.5.4), the dynamical specifications on X_+ are exactly the specifications which are generated by the translation-invariant UAC interactions on X_+.*

By Example 2.5.9, it is an interesting question to know when a Gibbsian specification is dynamical. Note that this question was also asked in [9]. The following statement answers this question:

Theorem 2.C. *Assume that γ is a Gibbsian specification on X_+ and ρ^γ is the corresponding cocycle. Let $\rho_0^\gamma(\omega,\omega') = \rho^\gamma(\omega,\omega') - \rho^\gamma(S\omega, S\omega')$, $(\omega,\omega')\in\mathfrak{T}(X_+)$ be the base of the cocycle ρ^γ. Then the following statements are equivalent to each other:*

- *(i) γ is a dynamical specification, in the other words, γ is generated by a translation-invariant UAC interaction on X_+;*
- *(ii) ρ^γ is a Gibbs cocycle;*
- *(iii) the cocycle ρ_0^γ can be extended from $\mathfrak{T}(X_+)\subset X_+\times X_+$ to a continuous cocycle on the full equivalence relation $X_+\times X_+$;*
- *(iv) ρ_0^γ is uniformly continuous on $\mathfrak{T}(X_+)$ in the induced topology (which is metrizable) by $X_+\times X_+$.*

It is often difficult to check the condition of Theorem 2.C, therefore, one needs more practical conditions. The following theorems give us such necessary and sufficient conditions.

Corollary C1. *Let γ be a Gibbsian specification on Ω_+ and ρ^γ be the corresponding cocycle on $\mathfrak{T}(\Omega_+)$. If γ is a dynamical specification (i.e., ρ^γ is a Gibbs cocycle), then*

$$\sup_{(\omega,\omega')\in\mathfrak{T}(\Omega_+)} |\rho^\gamma(\omega,\omega') - \rho^\gamma(S\omega,S\omega')| < \infty. \tag{2.79}$$

Corollary C2. *Assume that γ is a Gibbsian specification on Ω_+ and*

$$\rho^\gamma(\omega,\omega') = \sum_{k=0}^{\infty}[\phi_k\circ S^k(\omega) - \phi_k\circ S^k(\omega')],\ (\omega,\omega')\in\mathfrak{T}(\Omega_+),$$

is the cocycle corresponding to γ. If

$$\sum_{k=1}^{\infty}\|\phi_k - \phi_{k-1}\|_\infty < \infty \tag{2.80}$$

then γ is a dynamical specification.

We shall prove Theorem 2.C and its corollaries in Section 2.6.

2.5.5 Kozlov-Sullivan characterisation on the lattice $\mathbb{Z}_+$

By motivating Theorem 2.3.1, we define the translation-invariant specifications on X_+ as follows:

Definition 2.5.11. *A translation-invariant interaction Φ on $\mathbb{Z}_+$ is called variation-summable if*

$$\sum_{0 \in V \Subset \mathbb{Z}_+} \delta_0 \Phi_V < \infty.$$

*We call a specification γ on $\mathbb{Z}_+$ a **translation-invariant Gibbsian specification** if there exists a translation-invariant variation-summable interaction Φ on $\mathbb{Z}_+$ such that $\gamma = \gamma^\Phi$.*

Remark 2.5.12. *By definition, any dynamical specification is translation-invariant. However, any translation-invariant Gibbsian specification does not need to be dynamical (c.f., Theorem 2.D)*

Then it is an interesting question to answer if any translation-invariant Gibbsian specification on $\mathbb{Z}_+$ can be associated with a translation-invariant invariant UAC interaction on $\mathbb{Z}_+$. Note that Theorem 2.2.5 implies that any one-sided Gibbsian specification, in particular, a translation-invariant one can be associated with a UAC interaction; however, the associated interaction does not always need to be translation-invariant as the following theorem states:

Theorem 2.D. *There exists a left-shift invariant Gibbsian specification on $\mathbb{Z}_+$, which can not be associated with a translation-invariant UAC interaction.*

We shall give an explicit example of an interaction satisfying conditions of Theorem 2.D in the next section.

2.6 Proofs of the Main results

2.6.1 Proof of Theorem 2.3.4

Fix a periodic configuration $\theta \in \Omega$, i.e., $S\theta = \theta$. For every $\Lambda \Subset \mathbb{Z}$ consider a function defined by

$$\Phi_\Lambda^\theta(\omega_\Lambda) := \sum_{\substack{V \subset \Lambda \\ V \neq \emptyset}} (-1)^{|\Lambda \setminus V|} \log \frac{\gamma_V(\theta_V | \theta_{V^c})}{\gamma_V(\omega_V | \theta_{V^c})}, \quad \omega_\Lambda \in E^\Lambda. \tag{2.81}$$

Then $\Phi^{\theta} := (\Phi^{\theta}_{\Lambda})_{\Lambda \Subset \mathbb{Z}}$ is a translation-invariant interaction on Ω [16, 18]. Using the interaction Φ^{θ} we define another interaction by

$$\Phi_{\Lambda} := \begin{cases} \sum_{B \in \mathscr{S}_{\ell,i}} \Phi^{\theta}_{B}, & \text{if for some } \ell \in \mathbb{Z} \text{ and } i \in \mathbb{N},\ \Lambda = [\ell, \ell + 2^{i}] \cap \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.82}$$

where

$$\mathscr{S}_{\ell,1} := \{B \Subset \mathbb{Z} : \ell \in B \subset [\ell, \ell + 2]\}.$$

and for $i \geq 2$

$$\mathscr{S}_{\ell,i} := \{B \Subset \mathbb{Z} : \ell \in B \subset [\ell, \ell + 2^{i}],\ B \not\subset [\ell, \ell + 2^{i-1}]\}.$$

It can be checked that $\Phi = (\Phi_{\Lambda})_{\Lambda \Subset \mathbb{Z}}$ is also a translation-invariant interaction on $\mathbb{Z}$, and under condition (2.29), Φ is also UAC and generates γ [21]. Furthermore, the following estimation is also valid for Φ: for all $\ell \in \mathbb{Z}$ and $i \geq 2$,

$$\left\| \Phi_{[\ell, \ell + 2^{i}]} \right\|_{\infty} \quad \leq \quad C\, v(2^{i-1}), \tag{2.83}$$

where $C > 0$ is a constant independent of both $\ell \in \mathbb{Z}$ and $i \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{\min V \leq 0 < \max V} \|\Phi_{V}\|_{\infty} &= \sum_{i=1}^{\infty} \sum_{\ell = -2^{i}+1}^{0} \|\Phi_{[\ell, \ell+2^{i}]}\|_{\infty} \\ &= \sum_{\ell=-1}^{0} \|\Phi_{[\ell, \ell+2^{i}]}\|_{\infty} + \sum_{i=2}^{\infty} \sum_{\ell=-2^{i}+1}^{0} \|\Phi_{[\ell, \ell+2^{i}]}\|_{\infty} \\ &\leq \sum_{\ell=-1}^{0} \|\Phi_{[\ell, \ell+2^{i}]}\|_{\infty} + C \sum_{i=2}^{\infty} 2^{i+2} v(2^{i-1}) \\ &< \infty. \end{aligned}$$

Thus by Theorem 8.39 in [16], it can be concluded that $\#\mathscr{G}(\Omega, \Phi) = \#\mathscr{G}(\Omega, \gamma) = 1$.

2.6.2 Proof of Theorem 2.A

We start this subsection by proving the following important proposition. Fix a state $+ \in E$ and denote the constant configuration, where every component is in the state $+$ by $\mathbf{+}$.

Proposition 2.6.1. *(i) The sequence* $\left\{ -\frac{1}{|\Lambda_n|} \log \gamma_{\Lambda_n}(+_{\Lambda_n} | \omega_{\Lambda_n^c}) \right\}_{n \in \mathbb{N}}$ *converges uniformly on* $\omega \in \Omega$ *to a constant* $P^{+}(\gamma)$ *as* $n \to \infty$.

(ii) For $\rho \in \mathcal{M}_{1,S}(\Omega)$, the sequence $\left\{ -\frac{1}{|\Lambda_n|} \int\limits_{\Omega} \log \gamma_{\Lambda_n}(\theta_{\Lambda_n} | \omega_{\Lambda_n^c}) \rho(d\theta) \right\}_{n\in\mathbb{N}}$ *converges uniformly on $\omega \in \Omega$ to a constant $P^{\rho}(\gamma)$ which depends only on γ and ρ as $n \to \infty$.*

Proof. Part (i): The proof of the first part of Proposition 2.6.1 presented below, relies on the following lemma, whose proof can be found in [22].

Lemma 2.6.2. *For the translation-invariant Gibbsian specification γ, one has*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sup_{\xi,\eta\in\Omega} \log \frac{\gamma_{\Lambda_n}(+_{\Lambda_n} | \xi_{\Lambda_n^c})}{\gamma_{\Lambda_n}(+_{\Lambda_n} | \eta_{\Lambda_n^c})} = 0. \tag{2.84}$$

In the light of Lemma 2.6.2, the uniform convergence of the functional sequence

$$\left\{ \frac{1}{|\Lambda_n|} \log \gamma_{\Lambda_n}(+_{\Lambda_n} | \bullet_{\Lambda_n^c}) \right\}_{n\in\mathbb{N}}$$

follows if we show that the numerical sequence $\left\{ \frac{1}{|\Lambda_n|} \log \gamma_{\Lambda_n}(+_{\Lambda_n} | +_{\Lambda_n^c}) \right\}_{n\in\mathbb{N}}$ converges.

We shall consider the vacuum interaction Φ^+ corresponding to the vacuum configuration $+$. By the vacuum condition, one has that

$$\gamma_{\Lambda_n}(+_{\Lambda_n} | \omega_{\Lambda_n^c}) = Z_{\Lambda_n}^{\Phi^+}(\omega_{\Lambda_n^c})^{-1}, \tag{2.85}$$

where $Z_{\Lambda_n}^{\Phi^+}(\omega_{\Lambda_n^c})$ is the partition function corresponding to the interaction Φ^+ in the volume Λ_n. Thus we have to prove that the *infinite volume pressure* is well-defined for the interaction Φ^+. Then this implies that $\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \gamma_{\Lambda_n}(+_{\Lambda_n} | \omega_{\Lambda_n^c}) = -P(\Phi^+)$. Note that in general, the interaction Φ^+ is not UAC, however, it is always *uniformly convergent*. Therefore, the classical theorems in Statistical Mechanics do not apply to guarantee the existence of the pressure $P(\Phi^+)$. For every $r \in \mathbb{N}$, we consider a finite-range interaction $\Phi^{+,r}$ defined by

$$\Phi_{\Lambda}^{+,r} := \begin{cases} \Phi_{\Lambda}^{+}, & \text{if } \mathtt{diam}(\Lambda) \le r; \\ 0, & \text{if } \mathtt{diam}(\Lambda) > r. \end{cases} \tag{2.86}$$

Note that for every $r \in \mathbb{N}$, the interaction $\Phi^{+,r}$ is finite-range, therefore, the sequence $\left\{ \frac{1}{|\Lambda_n|} \log Z_{\Lambda_n}^{\Phi^{+,r}}(\omega) \right\}_{n\in\mathbb{N}}$ converges uniformly on $\omega \in \Omega$ to the pressure $P(\Phi^{+,r})$ as $n \to \infty$. For every $r \in \mathbb{N}$, one can readily check by the vacuum property of the interaction Φ^+ that

$$\begin{aligned}
\log \frac{Z^{\Phi^+}_{\Lambda_n}(+)}{Z^{\Phi^{+,r}}_{\Lambda_n}(+)} &= \log \frac{\sum\limits_{\omega\in\Omega_{\Lambda_n}} e^{-\sum\limits_{V\subset\Lambda_n}\Phi^+_V(\omega)}}{\sum\limits_{\omega\in\Omega_{\Lambda_n}} e^{-\sum\limits_{V\subset\Lambda_n}\Phi^{+,r}_V(\omega)}} \\
&\le \sup_{\omega\in\Omega_{\Lambda_n}} \Big| \sum_{\substack{V\subset\Lambda_n\\ \mathtt{diam}(V)>r}} \Phi^+_V(\omega)\Big| \\
&\le \sum_{i\in\Lambda_n}\sup_{\omega\in\Omega}\Big| \sum_{\substack{i\in V\Subset\mathbb{Z}\\ \mathtt{diam}(V)>r}} \Phi^+_V(\omega)\Big| \\
&= |\Lambda_n|\sup_{\omega\in\Omega}\Big| \sum_{\substack{0\in V\Subset\mathbb{Z}\\ \mathtt{diam}(V)>r}} \Phi^+_V(\omega)\Big|. \qquad (2.87)
\end{aligned}$$

Thus, in particular, for any $s, r \in \mathbb{N}$, one also has

$$\begin{aligned}
\Big|\log \frac{Z^{\Phi^{+,s}}_{\Lambda_n}(+)}{Z^{\Phi^{+,r}}_{\Lambda_n}(+)}\Big| &= \Big|\log \frac{Z^{\Phi^+}_{\Lambda_n}(+)}{Z^{\Phi^{+,r}}_{\Lambda_n}(+)} - \log \frac{Z^{\Phi^+}_{\Lambda_n}(+)}{Z^{\Phi^{+,s}}_{\Lambda_n}(+)}\Big| \\
&\le 2|\Lambda_n|\sup_{\omega\in\Omega}\Big| \sum_{\substack{0\in V\Subset\mathbb{Z}\\ \mathtt{diam}(V)>\min\{s,r\}}} \Phi^+_V(\omega)\Big|. \qquad (2.88)
\end{aligned}$$

Thus by dividing both sides by $|\Lambda_n|$ and taking limit as $n \to \infty$, one obtains that for all $s, r \in \mathbb{N}$,

$$|P(\Phi^{+,s}) - P(\Phi^{+,r})| \le 2\cdot \sup_{\omega\in\Omega}\Big| \sum_{\substack{0\in V\Subset\mathbb{Z}\\ \mathtt{diam}(V)>\min\{s,r\}}} \Phi^+_V(\omega)\Big|. \qquad (2.89)$$

Since the interaction Φ^+ is uniformly convergent, one has that

$$2\cdot \sup_{\omega\in\Omega}\Big| \sum_{\substack{0\in V\Subset\mathbb{Z}\\ \mathtt{diam}(V)>n}} \Phi^+_V(\omega)\Big| \xrightarrow[n\to\infty]{} 0. \qquad (2.90)$$

Hence (2.89) yields that the sequence $\{P(\Phi^{+,r})\}_{r\in\mathbb{N}}$ is fundamental. Thus one can immediately conclude from (2.87) that the sequence $\Big\{\frac{1}{|\Lambda_n|}\log Z^{\Phi^+}_{\Lambda_n}(+)\Big\}_{n\in\mathbb{N}}$ is convergent.

Part (ii): By the first part of Lemma 3.7 in [22], one has that

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\int_\Omega \log\frac{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(+_{\Lambda_n}|\omega_{\Lambda_n^c})}\rho(d\theta) = \int_\Omega \log\frac{\gamma_{\{0\}}(\theta_{\mathbf{0}}|+_{<\mathbf{0}}\,\theta_{>\mathbf{0}})}{\gamma_{\{0\}}(+_{\mathbf{0}}|+_{<\mathbf{0}}\,\theta_{>\mathbf{0}})}\rho(d\theta), \qquad (2.91)$$

and the limit in LHS is uniform on $\omega \in \Omega$. Thus by the first part, one can conclude the second part of the theorem, since

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int_{\Omega} \log \gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})\rho(d\theta) &= \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \gamma_{\Lambda_n}(+_{\Lambda_n}|\omega_{\Lambda_n^c}) \\ &+ \int_{\Omega} \log \frac{\gamma_{\{0\}}(\theta_{\mathbf{0}}|+_{<\mathbf{0}}\theta_{>\mathbf{0}})}{\gamma_{\{0\}}(+_{\mathbf{0}}|+_{<\mathbf{0}}\theta_{>\mathbf{0}})}\rho(d\theta).\end{aligned}$$

□

Below we prove an important statement for the energy contribution from the origin, which will be useful in the proof of Theorem 2.A.

Proposition 2.6.3. *Let γ be a translation-invariant Gibbsian specification on Ω and u_γ^ρ be the associated energy contribution from the origin as defined in (2.43). Then for all translation-invariant probability measure $\tau \in \mathscr{M}_{1,S}(\Omega)$, one has that*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int_{\Omega}\int_{\Omega} \log \frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}\rho(d\theta)\tau(d\omega) = \int_{\Omega} u_\gamma^\rho(\omega)\tau(d\omega). \tag{2.92}$$

Proof of Proposition 2.6.3. For the specification densities γ_Λ, $\Lambda \Subset \mathbb{Z}^d$, we use the notations $\gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c})$ and $\gamma_\Lambda(\omega)$ interchangeably. By the bar moving the property of the specification densities, one has that

$$\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})} = \prod_{i\in\Lambda_n} \frac{\gamma_{\{i\}}(\omega_i|\theta_{[\min\Lambda_n,i)}\omega_{[\min\Lambda_n,i]^c})}{\gamma_{\{i\}}(\theta_i|\theta_{[\min\Lambda_n,i)}\omega_{[\min\Lambda_n,i]^c})}. \tag{2.93}$$

For any $\ell_1, \ell_2 \in \mathbb{Z}^d$ with $\ell_1 \le \ell_2$ and $\theta \in \Omega$, define a transformation

$$\Theta^\theta_{[\ell_1,\ell_2]}(\omega) := \begin{cases} \omega_i, & \text{if } i \notin [\ell_1,\ell_2]; \\ \theta_i, & \text{if } i \in [\ell_1,\ell_2]. \end{cases} \tag{2.94}$$

If $\ell_1 \not\le \ell_2$, then $[\ell_1,\ell_2] = \emptyset$; therefore, for all $\omega \in \Omega$, we set $\Theta_{[\ell_1\ell_2]}(\omega) = \omega$. Note that the families $\{S_\ell : \ell \in \mathbb{Z}^d\}$ and $\{\Theta_{[\ell_1,\ell_2]} : \ell_1,\ell_2 \in \mathbb{Z}^d, \ell_1 \le \ell_2\}$ of the transformations on Ω have the following commutativity-type property: for any $\ell_{1,2,3} \in \mathbb{Z}^d$ with $\ell_1 \le \ell_2$, one has

$$S_{\ell_3} \circ \Theta^\theta_{[\ell_1,\ell_2]} = \Theta^{S_{\ell_3}\theta}_{[\ell_1-\ell_3,\ell_2-\ell_3]} \circ S_{\ell_3}. \tag{2.95}$$

Using the transformations $\{\Theta^\theta_{[\ell_1,\ell_2]}\}_{\ell_1\le\ell_2}$, (2.93) can be written as

$$\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})} = \prod_{i\in\Lambda_n} \frac{\gamma_{\{i\}} \circ \Theta^\theta_{[\min\Lambda_n,i)}(\omega)}{\gamma_{\{i\}} \circ \Theta^\theta_{[\min\Lambda_n,i]}(\omega)}. \tag{2.96}$$

By the translation-invariance of the specification γ, one can write (2.96) as

$$\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}=\prod_{i\in\Lambda_n}\frac{\gamma_{\{\mathbf{0}\}}\circ S_i\circ\Theta^{\theta}_{[\min\Lambda_n,i)}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ S_i\circ\Theta^{\theta}_{[\min\Lambda_n,i]}(\omega)}. \tag{2.97}$$

Hence by the translation-invariance of the measures τ and ρ, one can obtain from (2.95) and Fubini's theorem that

$$\frac{1}{|\Lambda_n|}\int_\Omega\int_\Omega \log\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}\rho(d\theta)\tau(d\omega)=$$

$$=\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n}\int_\Omega\int_\Omega \log\frac{\gamma_{\{\mathbf{0}\}}\circ S_i\circ\Theta^{\theta}_{[\min\Lambda_n,i)}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ S_i\circ\Theta^{\theta}_{[\min\Lambda_n,i]}(\omega)}\rho(d\theta)\tau(d\omega)$$

$$=\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n}\int_\Omega\int_\Omega \log\frac{\gamma_{\{\mathbf{0}\}}\circ\Theta^{S_i\theta}_{[\min\Lambda_n-i,\mathbf{0})}\circ S_i(\omega)}{\gamma_{\{\mathbf{0}\}}\circ\Theta^{S_i\theta}_{[\min\Lambda_n-i,\mathbf{0}]}\circ S_i(\omega)}\rho(d\theta)\tau(d\omega)$$

$$=\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n+\min\Lambda_n}\int_\Omega\int_\Omega \log\frac{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0})}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0}]}(\omega)}\rho(d\theta)\tau(d\omega). \tag{2.98}$$

By quasilocality and non-nullness of γ, the net $\left\{\log\frac{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0})}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0}]}(\omega)}\right\}_{i<\mathbf{0}}$ converges uniformly in $(\theta,\omega)\in\Omega\times\Omega$ to $u_\gamma(\theta,\omega):=\log\frac{\gamma_{\{\mathbf{0}\}}(\omega_{\mathbf{0}}|\theta_{<\mathbf{0}}\omega_{>\mathbf{0}})}{\gamma_{\{\mathbf{0}\}}(\theta_{\mathbf{0}}|\theta_{<\mathbf{0}}\omega_{>\mathbf{0}})}$ as $i_1,\dots,i_d\to-\infty$. Hence, by the Stolz-Cesaro theorem, one has that

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n+\min\Lambda_n}\log\frac{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0})}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0}]}(\omega)}=u_\gamma(\theta,\omega)\text{ uniformly on }(\theta,\omega)\in\Omega\times\Omega. \tag{2.99}$$

Thus (2.98) yields the desired claim.

□

Remark 2.6.4. *If the telescoping starts from the other end, one can end up with the following instead of (2.93):*

$$\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}=\prod_{i\in\Lambda_n}\frac{\gamma_{\{i\}}(\omega_i|\theta_{(i,\max\Lambda_n]}\omega_{[i,\max\Lambda_n]^c})}{\gamma_{\{i\}}(\theta_i|\theta_{(i,\max\Lambda_n]}\omega_{[i,\max\Lambda_n]^c})}. \tag{2.100}$$

Then, with a similar argument to the proof of Proposition 2.6.3, one can also show *that*

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\int\limits_{\Omega}\int\limits_{\Omega}\log\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}\rho(d\theta)\tau(d\omega)$$
$$=\int\limits_{\Omega}\tilde{u}^{\rho}(\omega)\tau(d\omega)=\int\limits_{\Omega}\int\limits_{\Omega}\tilde{u}_{\gamma}(\theta,\omega)\rho(d\theta)\tau(d\omega),$$

where

$$\tilde{u}_{\gamma}(\theta,\omega):=\log\frac{\gamma_{\{\mathbf{0}\}}(\omega_{\mathbf{0}}|\omega_{<\mathbf{0}}\theta_{>\mathbf{0}})}{\gamma_{\{\mathbf{0}\}}(\theta_{\mathbf{0}}|\omega_{<\mathbf{0}}\theta_{>\mathbf{0}})}\ \textit{ and }\ \tilde{u}_{\gamma}^{\rho}(\omega):=\int\limits_{\Omega}\tilde{u}_{\gamma}(\theta,\omega)\rho(d\theta).$$

In particular, u_{γ}^{ρ} *and* $\tilde{u}_{\gamma}^{\rho}$ *are physically equivalent, i.e., for all* $\tau\in\mathscr{M}_{1,S}(\Omega)$,

$$\int_{\Omega}u_{\gamma}^{\rho}\,d\tau=\int_{\Omega}\tilde{u}_{\gamma}^{\rho}\,d\tau. \tag{2.101}$$

Proof of Theorem 2.A. For a translation-invariant measure $\tau\in\mathscr{M}_{1,S}(\Omega)$ and a translation -invariant Gibbs measure $\bar{\mu}\in\mathscr{G}_S(\gamma)$, it is proven in the first part of Theorem 3.3 in [22] that

$$P^{+}(\gamma)-\int_{\Omega}u_{\gamma}^{+}\,d\tau-h(\tau)=h(\tau|\bar{\mu}). \tag{2.102}$$

For any $\rho\in\mathscr{M}_{1,S}(\Omega)$, we have proven in Proposition 2.6.1 and Proposition 2.6.3 that

$$\lim_{n\to\infty}-\frac{1}{|\Lambda_n|}\int_{\Omega}\log\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})\tau(d\omega)=P^{\rho}(\gamma)-\int_{\Omega}u_{\gamma}^{\rho}(\omega)\tau(d\omega). \tag{2.103}$$

Note that LHS of (2.103) is independent of ρ, thus (2.102) yields for $\rho=\delta_{+}$,

$$\lim_{n\to\infty}-\frac{1}{|\Lambda_n|}\int_{\Omega}\log\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})\tau(d\omega)=h(\tau)+h(\tau|\bar{\mu}). \tag{2.104}$$

Then by combining (2.104) with (2.103), we obtain that

$$\int_{\Omega}u_{\gamma}^{\rho}(\omega)\tau(d\omega)+h(\tau)=P^{\rho}(\gamma)-h(\tau|\bar{\mu}). \tag{2.105}$$

Then the rest of the proof follows from Theorem 2.4.3.

□

2.6.3 Proofs of Theorems 2.B, 2.C, and 2.D

Proof of Theorem 2.B. Consider a probability measure $\tau \in \mathcal{M}_1(X_+)$. Then for $\Lambda \Subset \mathbb{Z}_+$ and $B \in \mathscr{F}$, one has

$$\begin{aligned}(G_\Lambda \cdot \tau_\Lambda)(B) &:= \int_{X_+} G_\Lambda \mathbb{1}_B d\tau_\Lambda \\ &= \frac{1}{|\Gamma_\Lambda|}\sum_{x\in\Gamma_\Lambda}\int_{X_+} G_\Lambda \mathbb{1}_B d x_* \tau \\ &= \frac{1}{|\Gamma_\Lambda|}\sum_{x\in\Gamma_\Lambda}\int_{X_+} G_\Lambda(x+\omega)\mathbb{1}_B(x+\omega)\tau(d\omega), \end{aligned} \tag{2.106}$$

and

$$(\tau\gamma_\Lambda)(B) = \sum_{x\in\Gamma_\Lambda}\int_{X_+} G_\Lambda(\omega+x)\mathbb{1}_B(\omega+x)\tau(d\omega). \tag{2.107}$$

Then by combining (2.106) and (2.107), one concludes that for all $\Lambda \Subset \mathbb{Z}_+$,

$$\tau\gamma_\Lambda = (l^{|\Lambda|}G_\Lambda)\cdot\tau_\Lambda. \tag{2.108}$$

Note that in the light of (2.108), the DLR equation (2.4) and the consistency equation (2.68) for the G-measures are the same. Thus one immediately concludes the statement of the theorem. □

Proof of Theorem 2.C. "$(i) \Leftrightarrow (ii) \Rightarrow (iii)$" Let γ be a dynamical specification, then there exists $\varphi \in C(X_+)$ such that the corresponding cocycle ρ^γ is given by

$$\rho^\gamma(\omega,\bar{\omega}) = \sum_{k=0}^{\infty}[\varphi\circ S^k(\omega) - \varphi\circ S^k(\bar{\omega})],\ (\omega,\bar{\omega})\in\mathfrak{T}(X_+).$$

Thus for all $(\omega,\bar{\omega}) \in \mathfrak{T}(X_+)$, $\rho_0^\gamma(\omega,\bar{\omega}) = \varphi(\omega)-\varphi(\bar{\omega})$, and then it is clear that ρ_0^γ is extended to the cocycle $\lambda : X_+\times X_+ \to \mathbb{R}$ which is defined by $\lambda(\eta,\xi) = \varphi(\eta)-\varphi(\xi)$, $(\eta,\xi)\in X_+\times X_+$. Clearly, λ is continuous since φ is continuous.

"$(iii) \Rightarrow (ii)$" Assume that ρ_0^γ extends to a continuous cocycle λ on the full equivalence relation $X_+\times X_+$. Fix any $\tilde{\xi} \in X_+\times X_+$ and for $\omega \in X_+$, set $\varphi(\omega) = \lambda(\omega,\tilde{\xi})$. Then for any $(\omega,\xi)\in X_+\times X_+$,

$$\lambda(\omega,\xi) = \lambda(\omega,\tilde{\xi})+\lambda(\tilde{\xi},\xi) = \lambda(\omega,\xi)-\lambda(\xi,\tilde{\xi}) = \varphi(\omega)-\varphi(\xi). \tag{2.109}$$

The continuity of λ implies the continuity of φ. Furthermore, (2.109) immediately implies that for all $(\omega,\bar{\omega})\in\mathfrak{T}(X_+)$, $\rho_0^\gamma(\omega,\bar{\omega}) = \lambda(\omega,\bar{\omega}) = \varphi(\omega)-\varphi(\bar{\omega})$ Then by (2.61) one concludes that ρ^γ is indeed a Gibbs cocycle.

"$(iii) \Rightarrow (iv)$" The extension of ρ_0^γ to $X_+\times X_+$ is continuous and the product space $X_+\times X_+$ is compact. Then as a continuous function on the compact space, the extension is uniformly continuous, thus its restriction to $\mathfrak{T}(X_+)$ which is ρ_0^γ is also uniformly continuous.

"$(iv) \Rightarrow (iii)$" First, note that $\mathfrak{T}(X_+)$ is dense in $X_+ \times X_+$ in the product topology. Now take a pair $(\xi, \eta) \in X_+ \times X_+$ of configurations, and let the sequence $\{(\xi^{(n)}, \eta^{(n)})\}_{n\in\mathbb{N}} \subset \mathfrak{T}(X_+)$ converges to (ξ, η) in the product topology. Since $\rho_0^\gamma : \mathfrak{T}(X_+) \to \mathbb{R}$ is a uniformly continuous function, the sequence $\{\rho_0^\gamma(\xi^{(n)}, \eta^{(n)})\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is convergent because a uniformly continuous function preserves fundamentality of the sequences. Hence we can define an extension $\bar{\rho}_0^\gamma : X_+ \times X_+$ by $\bar{\rho}_0^\gamma(\xi, \eta) := \lim_{n\to\infty} \rho_0^\gamma(\xi^{(n)}, \eta^{(n)})$. It is clear that the value of $\bar{\rho}_0^\gamma(\xi, \eta)$ does not depend on the choice of the sequence $\{(\xi^{(n)}, \eta^{(n)})\}_{n\in\mathbb{N}}$ and by the construction, $\bar{\rho}_0^\gamma$ is continuous on $X_+ \times X_+$ and extends ρ_0^γ. □

Below we give the proofs of corollaries of Theorem 2.C.

Proof of Corollary C1. By Theorem 2.C, ρ_0^γ can be extended to a continuous cocycle on $X_+ \times X_+$. Since $X_+ \times X_+$ is compact, the extension is bounded, thus the statement of the theorem follows. □

Proof of Corollary C2. In fact, one can check the following for ρ_0^γ: for all $(\omega, \bar{\omega}) \in \mathfrak{T}(X_+)$,

$$\rho_0^\gamma(\omega, \bar{\omega}) = \phi_0(\omega) - \phi_0(\bar{\omega}) + \sum_{k=1}^{\infty} [(\phi_k - \phi_{k-1}) \circ S^k(\omega) - (\phi_k - \phi_{k-1}) \circ S^k(\bar{\omega})].$$

Then if (2.80) is satisfied then the sum $\sum_{k=1}^{\infty}[(\phi_k - \phi_{k-1}) \circ S^k(\xi) - (\phi_k - \phi_{k-1}) \circ S^k(\eta)]$ converges uniformly on $(\xi, \eta) \in X_+ \times X_+$, therefore, the cocycle λ defined by

$$\lambda(\xi, \eta) = \phi_0(\xi) - \phi_0(\eta) + \sum_{k=1}^{\infty}[(\phi_k - \phi_{k-1}) \circ S^k(\xi) - (\phi_k - \phi_{k-1}) \circ S^k(\eta)], \ (\xi, \eta) \in \mathfrak{T}(X_+)$$

is continuous on $X_+ \times X_+$. ρ_0^γ extends to λ, thus by Theorem 2.C, γ is indeed a dynamical specification. □

Proofs of Theorem 2.D. Consider the Ising spin space $E = \{-1, 1\}$ and the following interaction Ψ on X_+: for $\omega \in X_+$, set

$$\Psi_\Lambda(\omega) := \begin{cases} -\dfrac{\omega_i \omega_j}{|i-j|^\alpha} + \kappa\omega_j, & \text{if } \Lambda = \{i, j\} \subset \mathbb{Z}_+,\ i < j; \\ 0, & \text{otherwise,} \end{cases} \tag{2.110}$$

where $\alpha > 1$ and $\kappa > 0$. Then Ψ is translation-shift invariant, but it is not UAC. For $i, j \in \mathbb{Z}_+$, with $i < j$, $\|\Psi_{\{i,j\}}\|_\infty = \kappa + \dfrac{1}{|i-j|^\alpha}$. Thus $\sum_{0 \in V \Subset \mathbb{Z}_+} \|\Psi_V\|_\infty = \sum_{j=1}^{\infty}\Big(\kappa +$

$\left.\frac{1}{j^{\alpha}}\right)=\infty$. However, Ψ is a variation-summable interaction. In fact, for all $j\in\mathbb{N}$, $\delta_0\Psi_{\{0,j\}}=\frac{2}{j^{\alpha}}$, therefore, $\sum_{0\in V\Subset\mathbb{Z}_+}\delta_0\Psi_V=\sum_{j=1}^{\infty}\frac{2}{j^{\alpha}}<\infty$. Thus the associated one-sided specification $\overrightarrow{\gamma}^{\Psi}$ is Gibbsian, which is the same amount as saying that the associated cocycle ρ^{Ψ} is continuous. Furthermore, for $(\omega,\bar{\omega})\in\mathfrak{T}(X_+)$,

$$\begin{aligned}\rho_0^{\Psi}(\omega,\bar{\omega})&=\rho^{\Psi}(\omega,\bar{\omega})-\rho^{\Psi}(S\omega,S\bar{\omega})\\&=\sum_{0\in V\Subset\mathbb{Z}_+}[\Psi_V(\omega)-\Psi_V(\bar{\omega})]\\&=\sum_{j=1}^{\infty}\Big[\frac{\bar{\omega}_0\bar{\omega}_j-\omega_0\omega_j}{j^{\alpha}}+\kappa(\omega_j-\bar{\omega}_j)\Big].\end{aligned}$$

Thus, $\sup_{(\omega,\bar{\omega})\in\mathfrak{T}(X_+)}\rho_0^{\Psi}(\omega,\bar{\omega})=\infty$ which contradicts to Theorem C1. Therefore, the corresponding specification γ^{Ψ} is not dynamical, which is equivalent to saying that γ^{Ψ} is not generated by any translation-invariant UAC interaction (see Remark 2.5.10 and the discussion at the end of Subsection 2.5.1).

□

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