# Gibbs states in statistical mechanics and dynamical systems

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# Chapter 1

# Introduction

The development of the so-called Sinai–Ruelle–Bowen (SRB) Thermodynamic Formalism emerged at the crossroads of Equilibrium Statistical Mechanics and Dynamical Systems. A pioneering step in this direction was taken by Kolmogorov, who was among the first to rigorously address how to quantify entropy – a central concept in statistical physics – within the framework of dynamical systems. In 1957, Kolmogorov, together with Sinai, introduced the concept of measure-theoretic entropy (now commonly known as Kolmogorov–Sinai entropy) to dynamical systems. This formalism provided a precise mathematical language to capture the idea of entropy in evolving systems, bridging statistical mechanics with ergodic theory. This foundational work on entropy laid the groundwork for what would later become the SRB Thermodynamic Formalism—a powerful framework that unifies thermodynamic concepts with the long-term behaviour of dynamical systems.

Efforts to construct a rigorous theory of statistical physics began long before the formal introduction of entropy as a mathematical concept. At its core, the primary aim of Equilibrium Statistical Physics is to understand the collective behaviour of systems composed of a vast number of interacting constituents. These constituents could be atoms or molecules forming a physical substance, or more abstract entities such as populations of organisms—humans, insects, or animals—interacting within a shared environment.

The earliest conceptual steps toward studying such collective phenomena can be traced back to ancient Greek philosophers such as Democritus and Aristotle, who were among the first to speculate that matter is composed of indivisible units—what we now call atoms. These foundational ideas were later extended by Islamic scholars like Ibn Sina (Avicenna) and Al-Biruni, whose correspondence explored the nature and formation of matter in a more systematic way [5, 26].

The modern understanding of how atoms and molecules are organised in gases, liquids, and crystals was significantly shaped by the work of Boltzmann

and Gibbs in the 19th century. Boltzmann, in particular, introduced a statistical definition of entropy, providing a microscopic interpretation of the Second Law of Thermodynamics—a cornerstone in the foundations of statistical mechanics.

In the early 20th century, Lenz proposed what is now known as the Lenz–Ising model (a.k.a. Ising model), originally intended to provide a mathematical explanation for the Curie temperature and the phenomenon of spontaneous magnetisation in ferromagnetic materials. While early analytical efforts failed to demonstrate phase transitions in these models rigorously, a breakthrough came in the 1930s when Peierls offered a geometric argument establishing the existence of phase transitions in the two-dimensional Ising model. Since Peierls' contribution, the Ising model has served as a prototypical framework in mathematical Statistical Mechanics, providing a fertile ground for testing hypotheses and developing rigorous methods to understand complex collective behaviours.

Systematic mathematical study of lattice spin and gas models—and their associated phase transitions—began in the late 1960s through the foundational works of Dobrushin, Lanford, and Ruelle [13,33]. They proposed a rigorous framework in which the possible state of each constituent in a large system could be described via a system of conditional probabilities. More precisely, if the interaction among constituents in the system $(\sigma_i)_{i\in\mathbb{L}}$ is described by a family $\Phi = (\Phi_\Lambda)_{\Lambda\Subset\mathbb{L}}$, then the probability that a constituent $\sigma_i$ is in a state $a \in E$, given that the other constituents $(\sigma_j)_{j\in\mathbb{L}\setminus i}$ are in the states $(x_j)_{j\in\mathbb{L}\setminus i} \in E^{\mathbb{L}\setminus i}$, is given by

$$\mu^\Phi(\sigma_i = a | \sigma_j = x_j,\ \ j \in \mathbb{L}\setminus i) = \frac{e^{-H^\Phi_{\{i\}}(a_i\, x_{\mathbb{L}\setminus i})}}{\sum_{\bar a\in E} e^{-H^\Phi_{\{i\}}(\bar a_i\, x_{\mathbb{L}\setminus i})}}, \tag{1.1}$$

where $E$ denotes the set of all possible states, assumed to be at most countable, and $H_{\{i\}} := \sum_{i\in V\Subset\mathbb{L}} \Phi_V$—known as the Hamiltonian of the system on $\{i\}$—represents the total influence of the constituents $(\sigma_j)_{j\in\mathbb{L}\setminus i}$ on $\sigma_i$. More generally, for a finite subset $\Lambda \Subset \mathbb{L}$, the joint probability that the constituents $\sigma_\Lambda := (\sigma_i)_{i\in\Lambda}$ are in the states $a_\Lambda := (a_i)_{i\in\Lambda} \in E^\Lambda$, given that the remainder of the system is in states $x_{\mathbb{L}\setminus\Lambda} := (x_j)_{j\in\mathbb{L}\setminus\Lambda}$, is described by the Boltzmann ansatz:

$$\mu^\Phi(\sigma_\Lambda = a_\Lambda | \sigma_{\mathbb{L}\setminus\Lambda} = x_{\mathbb{L}\setminus\Lambda}) = \frac{e^{-H^\Phi_\Lambda(a_\Lambda\, x_{\mathbb{L}\setminus\Lambda})}}{\sum_{\bar a_\Lambda\in E^\Lambda} e^{-H^\Phi_\Lambda(\bar a_\Lambda\, x_{\mathbb{L}\setminus\Lambda})}}. \tag{1.2}$$

A notable mathematical limitation of the Boltzmann ansatz is that different interactions $(\Phi_\Lambda)_{\Lambda\Subset\mathbb{L}}$ can lead to the same system of conditional probabilities – a phenomenon formalised under the concept of physical equivalence. To address this ambiguity, a more robust mathematical formulation of the global state of the system is given through a regular system of conditional probabilities, commonly referred to as specifications. Interestingly, the notion of specification is

closely tied to a dynamical concept known as a cocycle, which allows for a generalisation of Gibbs measures beyond simple lattice structures. By using this relationship, Bowen infused the theory of Gibbs measures with a distinctly dynamical systems perspective. Building on the foundations laid by Sinai and Ruelle, Bowen extended the notion of Gibbs measures to broader dynamical contexts, including Axiom A diffeomorphisms and general hyperbolic systems. His work provided a crucial bridge between the statistical mechanics formulation and the language of dynamical systems by redefining Gibbs measures within this new framework—a definition we revisit in Chapter 3, where we compare it with the standard DLR (Dobrushin–Lanford–Ruelle) formulation from statistical mechanics. Moreover, under fairly general conditions—such as the existence of a Markov partition—these broader dynamical systems can often be shown to be conjugate to product-type systems, effectively linking the dynamical perspective back to the more classical approaches.

In practical applications, the most commonly encountered cases for the indexing set (or set of sites) $\mathbb{L}$ include deterministic and random graphs, countable groups, and semi-groups. In the context of dynamical systems, particular attention is given to situations where $\mathbb{L}$ is a countable group or semi-group, as these naturally induce a group action on the configuration space $\Omega = E^{\mathbb{L}}$—a structure that is especially rich from the viewpoint of dynamical systems. In this thesis, we primarily focus on two specific choices for the indexing set $\mathbb{L}$: either $\mathbb{L} = \mathbb{Z}^d$ for some $d \geq 1$, or $\mathbb{L} = \mathbb{Z}_+$. Within these settings, we investigate Gibbs measures from various viewpoints, examining both their foundational definitions and a range of properties relevant to statistical mechanics and dynamical systems.

The thesis is intrinsically divided into three parts. In the first part (covered in Chapters 2 and 3), we explore the various notions of Gibbsianity as they appear in both Statistical Mechanics and Dynamical Systems, and we study the interplay between these frameworks. Among other topics, we discuss the Kozlov–Sullivan characterisation of specifications, present a form of interaction-independent variational principle which is new to the literature, and analyse the Gibbs properties of equilibrium states. The second part (encompassing Chapters 4 and 5) shifts the focus to one-dimensional systems. Here, we examine Gibbs measures defined on both the whole-line $\mathbb{Z}$ and the half-line $\mathbb{Z}_+$, and we compare the features of these two settings. A closely related subject in this section is the principal eigenfunction problem for Ruelle-Perron-Frobenius transfer operators. In fact, this part initiates a systematic study of the thermodynamic formalism for long-range potentials in dimension one. Finally, the third part (presented in Chapter 6) is devoted to an investigation of the multifractal properties of a general Gibbsian system, with special attention given to the aspects governed by large deviations.

## 1.1 Gibbs Measures in Dynamical Systems

In the first part of this thesis, we explore various notions of Gibbsianity, which differ in terms of their level of generality and mathematical setup. The standard Gibbsian formalism is built in configuration spaces. Let us fix a single-site space (also known as the single-spin space in models related to magnetism), denoted by $E$. Throughout the thesis, we assume $E$ to be finite. Typical examples of such spaces include the Ising spin space $\{\pm 1\}$, the lattice gas space $\{0,1\}$, or the set of colours $\{1,2,\dots,q\}$ for $q \geq 2$, as encountered in Potts models. We always endow $E$ with the discrete topology. Next, consider an arbitrary countable set $\mathbb{L}$, referred to as the set of sites. Examples of $\mathbb{L}$ include the vertex (or edge) set of an infinite graph – either deterministic or random – such as a Cayley tree, a countable group like $\mathbb{Z}^d$, or a semigroup such as $\mathbb{Z}_+^d$. The configuration space is then defined as $\Omega := E^{\mathbb{L}}$. Equipped with the product topology, $\Omega$ becomes a compact and metrizable space. We denote by $\mathscr{F}$ the Borel $\sigma$-algebra generated by this topology. The space of real-valued continuous functions on $\Omega$ is written as $C(\Omega)$, and $\mathscr{M}_1(\Omega)$ denotes the simplex of Borel probability measures on $\Omega$. Given a configuration $\omega \in \Omega$ and a finite subset $\Lambda \Subset \mathbb{L}$ (which we refer to as a finite volume), we define the cylindrical set associated with $\omega$ as $[\omega_\Lambda] := \{\xi \in \Omega : \xi_i = \omega_i,\ i \in \Lambda\}$. For each site $i \in \mathbb{L}$, we define the random variable $\sigma_i : \Omega \to E$ by $\sigma_i(\omega) := \omega_i$. For any subset $V \subset \mathbb{L}$ (finite or infinite), we denote by $\mathscr{F}_V$ the smallest sub-$\sigma$-algebra of $\mathscr{F}$ with respect to which the collection $\{\sigma_i\}_{i\in V}$ is measurable. Finally, the tail $\sigma$-algebra $\mathscr{T}$ is defined as $\mathscr{T} := \cap_{\Lambda\Subset\mathbb{L}} \mathscr{F}_{\mathbb{L}\setminus\Lambda}$.

### 1.1.1 Classical Gibbs formalism in statistical mechanics

The standard definition of Gibbs states in statistical mechanics is related – via the Boltzmann ansatz (1.2)– to the notion of interaction. An ***interaction*** $\Phi$ on $\Omega$ is a family $(\Phi_\Lambda)_{\Lambda\Subset\mathbb{L}}$ of $\Lambda$–local functions $\Phi_\Lambda : \Omega \to \mathbb{R}$, i.e., each $\Phi_\Lambda$ is $\mathscr{F}_\Lambda$ measurable. An interaction $\Phi = (\Phi_\Lambda)_{\Lambda\Subset\mathbb{L}}$ is called ***uniformly absolutely convergent*** (UAC) if for every $i \in \mathbb{L}$,

$$\sum_{i\in\Lambda\Subset\mathbb{L}} \sup_{\omega\in\Omega} |\Phi_\Lambda(\omega)| < \infty. \tag{1.3}$$

For any finite volume $\Lambda \Subset \mathbb{L}$, one can associate the ***Hamiltonian***

$$H_\Lambda^\Phi := \sum_{\substack{V\Subset\mathbb{L}\\ V\cap\Lambda\neq\emptyset}} \Phi_V.$$

Note that under the UAC condition, $H_\Lambda^\Phi$ is a well-defined and continuous function on $\Omega$.

**Definition 1.1.1.** *A probability measure $\mu \in \mathscr{M}_1(\Omega)$ is called a **Gibbs measure for the interaction** $\Phi$ (denoted by $\mu \in \mathscr{G}(\Phi)$) if for every $f \in C(\Omega)$,*

$$\mu(f|\mathscr{F}_{\Lambda^c})(\omega) = \frac{1}{Z_\Lambda^\Phi(\omega)} \sum_{\bar{\omega}_\Lambda \in E^\Lambda} f(\bar{\omega}_\Lambda \omega_{\Lambda^c}) \cdot e^{-H_\Lambda^\Phi(\bar{\omega}_\Lambda \omega_{\Lambda^c})}, \ \mu - a.e.\ \omega \in \Omega, \tag{1.4}$$

*where $Z_\Lambda^\Phi(\omega)$, known as the partition function, normalises (1.4) to 1:*

$$Z_\Lambda^\Phi(\omega) := \sum_{\bar{\omega}_\Lambda \in E^\Lambda} e^{-H_\Lambda^\Phi(\bar{\omega}_\Lambda \omega_{\Lambda^c})},$$

*and $\bar{\omega}_\Lambda \omega_{\Lambda^c}$ denotes the concatenated configuration.*

When $E$ is compact – which is the case throughout this thesis – the set $\mathscr{G}(\Phi)$ of Gibbs measures for $\Phi$ is always non-empty, convex, and compact with respect to the weak-* topology. Since the existence of Gibbs measures is always guaranteed under these conditions, the more physically relevant question concerns their uniqueness. Specifically, one says that the interaction $\Phi$ exhibits a ***phase transition*** if there exist multiple Gibbs measures for $\Phi$, i.e., if $\#\mathscr{G}(\Phi) > 1$.

### 1.1.2 DLR Gibbs formalism

The modern approach, also known as the Dobrushin-Lanford-Ruelle (DLR) approach, to Gibbs measures in mathematical Statistical Mechanics is rather different and does not involve interactions. Instead, one immediately starts with a family of regular conditional probabilities, which is known as *specification*. Since there is no initial reference measure, one needs to define the regular conditional probabilities everywhere on $\Omega$ rather than almost everywhere. A *regular conditional probability* in a volume $\Lambda \Subset \mathbb{L}$ is a proper probability kernel $\gamma_\Lambda$ from $(\Omega, \mathscr{F})$ to $(\Omega, \mathscr{F}_{\Lambda^c})$, i.e., $\gamma_\Lambda : \mathscr{F} \times \Omega \to [0,1]$ such that

- for all $\omega \in \Omega$, $\gamma_\Lambda(\cdot|\omega) : \mathscr{F} \to [0,1]$ is a probability measure on $\Omega$;
- for every $B \in \mathscr{F}$, $\gamma_\Lambda(B|\cdot) : \Omega \to [0,1]$ is $\mathscr{F}_{\Lambda^c}$–measurable;
- for each $B \in \mathscr{F}_{\Lambda^c}$, $\gamma_\Lambda(B|\cdot) = \mathbb{1}_B(\cdot)$.

A ***specification*** $\gamma$ on $\Omega$ is a *consistent* family of regular conditional probabilities $\gamma_\Lambda$ indexed by $\Lambda \Subset \mathbb{L}$. This means that for every pair of finite volumes $\Delta \subset \Lambda \Subset \mathbb{L}$, the consistency condition holds:

$$\gamma_\Lambda(B|\omega) = \gamma_\Lambda \gamma_\Delta(B|\omega) := \int_\Omega \gamma_\Delta(B|\xi) \gamma_\Lambda(d\xi|\omega), \ B \in \mathscr{F}, \ \omega \in \Omega. \tag{1.5}$$

**Definition 1.1.2.** *A probability measure $\mu \in \mathscr{M}_1(\Omega)$ is a **Gibbs measure for the specification** $\gamma$, if for every $\Lambda \Subset \mathbb{L}$ and $f \in C(\Omega)$, one has*

$$\mu(f|\mathscr{F}_{\Lambda^c})(\omega) = \gamma_\Lambda(f|\omega) := \sum_{\bar{\omega}_\Lambda \in E^\Lambda} f(\bar{\omega}_\Lambda \omega_{\Lambda^c}) \gamma_\Lambda([\bar{\omega}_\Lambda]|\omega), \ \mu - a.e.\ \omega \in \Omega. \tag{1.6}$$

*Equivalently, $\mu$ is Gibbs for $\gamma$ if for every $\Lambda \Subset \mathbb{L}$,*

$$\mu(B) = (\mu\gamma_\Lambda)(B) := \int_\Omega \gamma_\Lambda(B|\xi)\mu(d\xi), \ B \in \mathscr{F}. \tag{1.7}$$

The set $\mathscr{G}(\gamma)$ of Gibbs measures for a specification $\gamma$ is always convex and closed in the weak-* topology. However, it is not necessarily non-empty. Existence is guaranteed under an additional regularity condition: if $\gamma$ is *continuous*—that is, it satisfies the *Feller property*—then $\mathscr{G}(\gamma) \neq \emptyset$. This property requires that for every $f \in C(\Omega)$ and every finite volume $\Lambda \Subset \mathbb{L}$, the function $\gamma_\Lambda(f \mid \cdot)$ belongs to $C(\Omega)$. A continuous specification $\gamma$ on $\Omega$ is called a **Gibbsian specification** if it is *non-null*, i.e., for every $\Lambda \Subset \mathbb{L}$ and all $\xi, \omega \in \Omega$,

$$\gamma_\Lambda([\xi_\Lambda]|\omega) > 0. \tag{1.8}$$

In fact, one can readily check that any UAC interaction $\Phi$ gives rise to a Gibbsian specification $\gamma^\Phi = (\gamma^\Phi_\Lambda)_{\Lambda \Subset \mathbb{L}}$ defined by:

$$\gamma^\Phi_\Lambda(B|\omega) := \frac{1}{Z^\Phi_\Lambda(\omega)} \sum_{\bar{\omega}_\Lambda \in E^\Lambda} \mathbb{1}_B(\bar{\omega}_\Lambda \omega_{\Lambda^c}) \cdot e^{-H^\Phi_\Lambda(\bar{\omega}_\Lambda \omega_{\Lambda^c})}, \ B \in \mathscr{F}, \ \omega \in \Omega. \tag{1.9}$$

By the celebrated theorem of Kozlov [31], the opposite statement is also true, namely, for any Gibbsian specification $\gamma$, there exists a UAC interaction $\Phi$ such that $\gamma = \gamma^\Phi$. However, subtleties arise when the specification has certain symmetries, and if it is also required to find a UAC interaction respecting those symmetries. For example, when $\mathbb{L} = \mathbb{Z}^d$ or $\mathbb{L} = \mathbb{Z}_+$, the natural shift (left-shift) transformations $S_j$, $j \in \mathbb{L}$ acts on $\Omega$ by $(S_j\omega)_{i \in \mathbb{L}} = (\omega_{i+j})_{i \in \mathbb{L}}$. An interaction $\Phi$ on $\Omega$ is called translation-invariant if for every $\Lambda \Subset \mathbb{L}$ and $j \in \mathbb{L}$, $\Phi_{\Lambda+j} = \Phi_\Lambda \circ S_j$. A specification $\gamma$ on the lattice $\mathbb{L} = \mathbb{Z}^d$ is translation-invariant if for every $\Lambda \Subset \mathbb{Z}^d$, $j \in \mathbb{Z}^d$, $B \in \mathscr{F}$ and $\omega \in \Omega$, $\gamma_{\Lambda+j}(B|\omega) = \gamma_\Lambda(S_j(B)|S_j\omega)$. It is worth noting that any translation-invariant Gibbsian specification admits at least one translation-invariant Gibbs measure—that is, a Gibbs measure that is invariant under the shift map. Moreover, any translation-invariant UAC interaction generates a translation-invariant Gibbsian specification (see Figure 2.1). However, a recent result by [1], in the case $\mathbb{L} = \mathbb{Z}$, demonstrates that there exist translation-invariant Gibbsian specifications that are *not* generated by any translation-invariant UAC interaction. In Chapter 2 of this thesis, we address a similar question in the setting $\mathbb{L} = \mathbb{Z}_+$.

**Theorem 1.1.3.** *There exists a translation-shift invariant Gibbsian specification on $\mathbb{Z}_+$ that can not be associated with a translation-invariant UAC interaction.*

Our proof of Theorem 1.1.3 is constructive, in contrast to the non-constructive approach used in [1]. In Chapter 2, we also establish a precise criterion characterising when a Gibbsian specification on $\mathbb{Z}_+$ can be generated by a translation-invariant UAC interaction. We emphasise that no analogous criterion is known for the whole line $\mathbb{Z}$.

This mismatch between the classes of translation-invariant interactions and translation-invariant specifications has important implications for the classical theory of equilibrium statistical physics. Translation-invariant Gibbs measures – those invariant under the shift maps – play a central role in statistical physics, as they are considered physically meaningful descriptions of macroscopic phases in infinite systems. It is well known that every translation-invariant Gibbs measure corresponds to a translation-invariant Gibbsian specification. However, not all such specifications arise from translation-invariant UAC interactions. This suggests that relying solely on interactions to study translation-invariant Gibbs measures may be conceptually inadequate. Nevertheless, much of classical mathematical statistical mechanics has been historically developed through the lens of interactions. Key results concerning translation-invariant Gibbs measures—such as the variational principle and large deviation principles—are typically formulated in terms of interactions. This tension naturally motivates the development of a theory of translation-invariant Gibbs measures based purely on specifications. Some progress in this direction has already been made. For instance, [32] establishes a version of the variational principle using specifications alone. Combined with results from [36], this leads to large deviation bounds applicable to all translation-invariant Gibbs measures. In Chapter 2 of this thesis, we aim to contribute further to this line of research by proving a new, more dynamical form of the variational principle—also formulated entirely in terms of specifications.

**Theorem 1.1.4.** *Assume $\gamma$ is a translation-invariant Gibbs specification on $\Omega = E^{\mathbb{Z}^d}$. Then, for any translation-invariant measure $\rho \in \mathscr{M}_1(\Omega)$, there exists a continuous function $u_\gamma^\rho \in C(\Omega)$ associated with $\gamma$ such that the translation-invariant Gibbs measures for $\gamma$ are precisely the equilibrium states for the potential $u_\gamma^\rho$, i.e.,*

$$h(\mu)+\int_\Omega u_\gamma^\rho \, d\mu = \sup\left\{h(\tau)+\int_\Omega u_\gamma^\rho \, d\tau : \tau \in \mathscr{M}_{1,S}(\Omega)\right\} \quad \Longleftrightarrow \quad \mu \in \mathscr{G}_S(\gamma), \tag{1.10}$$

*where $h(\tau)$ is the measure-theoretic entropy of the translation-invariant measure $\tau$ and $\mathscr{M}_{1,S}(\Omega)$ denotes the class of translation-invariant measures on $\Omega$.*

### 1.1.3 G-formalism and its relation to DLR formalism on $\mathbb{Z}_+$

In 1972, Keane [30] introduced the notion of a *g-measure* in ergodic theory, which is a one-sided counterpart of translation-invariant Gibbs measures. Later, in 1991, Brown and Dooley [11] extended this concept to $G$-measures, which generalise $g$-measures to the setting of non-translation-invariant measures. Let $E \simeq \mathbb{Z}/q\mathbb{Z}$ and consider the direct sum $\Gamma := \bigoplus_{\mathbb{Z}_+} E$ and its subgroup $\Gamma_\Delta := \{\omega \in \Gamma : \omega_i = 0,\ i \in \mathbb{Z}_+ \setminus \Delta\}$ for $\Delta \Subset \mathbb{Z}_+$. Then a family $G$ of Borel measusrable functions $G_\Lambda : \Omega \to [0,1]$, $\Omega = E^{\mathbb{Z}_+}$, indexed by the finite subsets $\Lambda$ of $\mathbb{Z}_+$ is called a $G-$***family*** if for all $\Lambda \Subset \mathbb{Z}_+$,

$$\sum_{\omega \in \Gamma_\Lambda} G_\Lambda(\omega + \eta) = 1,\ \ \eta \in \Omega, \tag{1.11}$$

and for all $\Delta \subset \Lambda \Subset \mathbb{Z}_+$,

$$G_\Lambda(\omega+\eta)G_\Delta(\eta) = G_\Lambda(\eta)G_\Delta(\omega+\eta),\ \ \omega \in \Gamma_\Delta,\ \eta \in \Omega, \tag{1.12}$$

here the sum $\omega+\eta$ should be understood coordinate-wise, i.e., $(\omega+\eta)_i := \omega_i + \eta_i$. For any $\Lambda \Subset \mathbb{Z}_+$, and $\nu \in \mathscr{M}_1(\Omega)$, define $\nu_\Lambda := \dfrac{1}{q^{|\Lambda|}} \sum_{x \in \Gamma_\Lambda} x_* \nu$, where for $x \in \Gamma_\Lambda$, $x_* : \mathscr{M}_1(\Omega) \to \mathscr{M}_1(\Omega)$ is the pushforward of the map $x : \Omega \to \Omega$, $x(\eta) = x + \eta$. It is clear that $\nu \ll \nu_\Lambda$ since $e_* \nu = \nu$, where $e$ is the neutral element of the group $\Gamma$.

**Definition 1.1.5.** *A probability measure $\nu \in \mathscr{M}_1(\Omega)$ is called a $G-$measure for a $G-$family $G = (G_\Lambda)_{\Lambda \Subset \mathbb{Z}_+}$, if for all $\Lambda \Subset \mathbb{Z}_+$,*

$$\frac{d\nu}{d\nu_\Lambda} = q^{|\Lambda|}\, G_\Lambda. \tag{1.13}$$

We note that a $G-$measure becomes a $g-$measure if the associated $G-$family is continuous, positive and

$$\frac{G_{[0,n+1]\cap\mathbb{Z}_+}}{G_{[0,n]\cap\mathbb{Z}_+}} = G_{\{0\}} \circ S_{n+1},\ \ \ n \in \mathbb{Z}_+. \tag{1.14}$$

In Chapter 2, we demonstrate that the $G-$formalism introduced by Brown and Dooley coincides with the DLR Gibbs formalism on the lattice $\mathbb{Z}_+$. More precisely, we show that

**Theorem 1.1.6.** *The relation*

$$\gamma_\Lambda(\eta_\Lambda | \eta_{\mathbb{Z}_+\setminus\Lambda}) := G(\eta),\ \ \Lambda \Subset \mathbb{Z}_+,\ \eta \in \Omega = E^{\mathbb{Z}_+}, \tag{1.15}$$

*establishes one-to-one correspondence between the G-families and the specifications on $\mathbb{Z}_+$. Furthermore, $\nu$ is a G-measure for $G$ if and only if $\nu$ is a Gibbs measure for $\gamma$.*

### 1.1.4 Gibbsianity of equilibrium states

In Chapter 3, which constitutes the remaining portion of the first part of the thesis, we shift our focus to the Gibbsianity of equilibrium states on the shift space $\Omega = E^{\mathbb{Z}}$. Although the results presented in this chapter are valid for higher-dimensional lattices $\mathbb{Z}^d$, we restrict our attention to the one-dimensional case in order to avoid cumbersome formulas. Note that a translation-invariant measure $\mu$ on $\Omega = E^{\mathbb{Z}}$ is an ***equilibrium state*** for a continuous function (potential) $\phi : \Omega \to \mathbb{R}$, denoted by $\mu \in \mathscr{ES}(\phi)$, if

$$h(\mu) + \int_{\Omega} \phi \, d\mu = \sup \Big\{ h(\tau) + \int_{\Omega} \phi \, d\tau : \ \tau \in \mathscr{M}_{1,S}(\Omega) \Big\}. \tag{1.16}$$

We note that the class of equilibrium states rather is broad (see Figure 3.1): if $\mu_1, \ldots, \mu_k$ are some ergodic measures on $\Omega$, then one can find a continuous potential $\phi \in C(\Omega)$ such that all these measures are equilibrium states for $\phi$ [17, 27, 39]. This remarkable generality suggests that equilibrium states can exhibit a wide range of behaviours, and in particular, one cannot expect them to possess any form of Gibbsianity in general. In Section 3, we investigate two types of Gibbsianity for an equilibrium state $\mu$ (for a potential $\phi$):

- *DLR Gibbsianity*: we explore whether there exists a Gibbsian specification $\gamma$ on $\Omega = E^{\mathbb{Z}}$ such that $\mu \in \mathscr{G}(\gamma)$;
- *(weak) Bowen-Gibbsianity*: we investigate if there exists a sequence of positive numbers $C_n > 0$, $n \in \mathbb{N}$, that is subexponential, i.e., $\lim_{n\to\infty} \frac{1}{n} \log C_n = 0$, such that for all $n$ and $\omega \in \Omega$,

$$\frac{1}{C_n} \leq \frac{\mu(\{\tilde{\omega} \in \Omega : \tilde{\omega}_0^{n-1} = \omega_0^{n-1}\})}{e^{S_n\phi(\omega) - nP}} \leq C_n, \tag{1.17}$$

  where $S_n\phi$ is the ergodic sum, i.e., $S_n\phi := \sum_{j=0}^{n-1} \phi \circ S_j$.

The Gibbsianity properties of equilibrium states mentioned above have been extensively studied under various regularity assumptions on the potential $\phi$ by Sinai, Bowen, Ruelle, Haydn, Pfister, Sullivan, and more recently by Bissacott and collaborators [3, 7, 10, 23–25, 37, 40]. In Chapter 3, we establish both the DLR Gibbs and weak Bowen-Gibbs properties of equilibrium states under a barely minimal regularity assumption on the potential $\phi$, which we refer to as the *extensibility condition*.

**Definition 1.1.7.** *A potential $\phi : \Omega \to \mathbb{R}$ satisfies the **extensibility condition** if for all $a_0, b_0 \in E$ the sequence of functions*

$$\rho_n^{a_0,b_0}(\omega) := \sum_{i=-n}^{n} \big( \phi \circ S_i(\omega_{-\infty}^{-1} b_0 \omega_1^{\infty}) - \phi \circ S_i(\omega_{-\infty}^{-1} a_0 \omega_1^{\infty}) \big)$$

*converges uniformly in* $\omega \in \Omega = E^{\mathbb{Z}}$ *as* $n \to \infty$.

The uniform limit $\lim_{n\to\infty} \rho^{a_0,b_0}(\omega)$ gives rise to a *cocycle* $\rho^{\phi}$ on the asymptotic equivalence relation $\mathfrak{T}(\Omega) := \{(\bar{\omega},\omega) \in \Omega \times \Omega : \bar{\omega}_i = \omega_i \text{ for all but finitely many } i \in \mathbb{Z}\}$ which the same amounts to saying that for all $(\bar{\omega},\hat{\omega}),(\hat{\omega},\omega) \in \mathfrak{T}(\Omega)$

$$\rho^{\phi}(\bar{\omega},\hat{\omega}) + \rho^{\phi}(\hat{\omega},\omega) = \rho^{\phi}(\bar{\omega},\omega). \tag{1.18}$$

One can associate a non-null specification $\overrightarrow{\overleftarrow{\gamma}}^{\phi} = (\overrightarrow{\overleftarrow{\gamma}}^{\phi}_{\Lambda})_{\Lambda \Subset \mathbb{Z}}$ on $\Omega = E^{\mathbb{Z}}$ with the cocycle $\rho^{\phi}$ via

$$\overrightarrow{\overleftarrow{\gamma}}^{\phi}_{\Lambda}(\omega_{\Lambda}|\omega_{\Lambda^c}) = \Big( \sum_{\xi_{\Lambda} \in E^{\Lambda}} e^{\rho^{\phi}(\xi_{\Lambda}\omega_{\mathbb{Z}\setminus\Lambda},\,\omega)} \Big)^{-1}, \;\; \omega \in \Omega, \; \Lambda \Subset \mathbb{Z}. \tag{1.19}$$

The uniform nature of the limit $\lim_{n\to\infty} \rho_n^{a_0,b_0}$ ensures the Gibbsianity of the specification $\overrightarrow{\overleftarrow{\gamma}}^{\phi}$.

The extensibility condition is not very restrictive. Notably, unlike the conditions used in the works of Sinai, Bowen, Ruelle, and Walters, it does not imply uniqueness of equilibrium states. Moreover, the extensibility condition encompasses all previously studied classes of potentials. In Chapter 3, we demonstrate that this condition is sufficient to guarantee both DLR Gibbsianity and weak Bowen-Gibbsianity of the equilibrium states.

**Theorem 1.1.8.** *Suppose* $\phi \in C(\Omega)$ *has the extensibility property. Then any equilibrium state* $\mu$ *for* $\phi$ *is*

*(1) Gibbs in the Dobrushin-Lanford-Ruelle sense, in fact,* $\mathscr{ES}(\phi) = \mathscr{G}_S(\overrightarrow{\overleftarrow{\gamma}}^{\phi})$;

*(2) weak Bowen-Gibbs relative to the potential* $\phi$.

In Chapter 3, we also establish, in a certain sense, a converse to Theorem 1.1.8. Specifically, we prove that any DLR Gibbs measure, as defined in Definition 1.1.2, is in fact an equilibrium state for some extensible potential. Additionally, we remark that Theorem 1.1.8 generalises the main result of [4] by extending it to the setting of general, not necessarily normalised, potentials.

## 1.2 Transfer operators for long-range potentials

The second part of the thesis comprises Chapters 4 and 5, where we investigate and compare the Gibbs formalisms on the whole-line $\mathbb{Z}$ and the half-line $\mathbb{Z}_+$. A key tool in this comparison is the Perron-Frobenius-Ruelle transfer operator, which plays a central role in our analysis. These operators—viewed as

infinite-dimensional analogues of positive matrices—were introduced into the thermodynamic formalism by D. Ruelle, primarily to study the mixing properties of Gibbs measures in one-dimensional systems. Throughout Chapters 4 and 5, we fix a finite set $E$ and denote the configuration spaces by $X = E^{\mathbb{Z}}$ for the whole line and $X_+ = E^{\mathbb{Z}_+}$ for the half line.

For a potential $\phi \in C(X_+)$, the associated Perron-Frobenius-Ruelle transfer operator $\mathscr{L}_\phi$ is defined by

$$\mathscr{L}_\phi f(x) := \sum_{a \in E} e^{\phi(ax)} f(ax), \ \ f \in \mathbb{R}^{X_+}, \tag{1.20}$$

where $ax$ denotes the configuration obtained by prepending the symbol $a$ to $x$. The transfer operator maps the space of real-valued continuous functions $C(X_+)$ into itself and is bounded when restricted to $C(X_+)$. Moreover, its spectral radius equals $\lambda = e^{P(\phi)}$, where $P(\phi)$ denotes the topological pressure of the potential $\phi$.

### 1.2.1 The role of transfer operators in the Gibbs formalism on $\mathbb{Z}_+$

The Perron-Frobenius-Ruelle transfer operators play a pivotal role in the Gibbs formalism on the half-line $\mathbb{Z}_+$. Given any potential $\phi \in C(\Omega)$, one can naturally associate a Gibbsian specification $\overrightarrow{\gamma}^{\phi} = (\overrightarrow{\gamma}^{\phi}_\Lambda)_{\Lambda \Subset \mathbb{Z}_+}$ on the configuration space $X_+$ via the formula:

$$\overrightarrow{\gamma}^{\phi}_\Lambda(a_\Lambda | x_{\Lambda^c}) = \frac{\exp((S_{n+1}\phi)(a_\Lambda x_{\Lambda^c}))}{\sum_{\bar{a}_\Lambda} \exp((S_{n+1}\phi)(\bar{a}_\Lambda x_{\Lambda^c}))}, \ \ a_\Lambda \in E^\Lambda, x \in X_+. \tag{1.21}$$

Then one has the following interesting relationship between the specification $\overrightarrow{\gamma}^{\phi}$ and the transfer operator $\mathscr{L}_\phi$:

$$\overrightarrow{\gamma}^{\phi}_{[0,n-1]}(f|x) = \frac{\mathscr{L}^n_\phi f(S_n x)}{\mathscr{L}^n_\phi \mathbf{1}(S_n x)}, \ \ n \in \mathbb{N}, \ x \in X_+, \text{ and } f : X_+ \to \mathbb{R}. \tag{1.22}$$

Using (1.22), one can show that the eigenprobabilities of the operator $\mathscr{L}_\phi$ corresponding to the spectral radius $\lambda$ are exactly the Gibbs measures for the specification $\overrightarrow{\gamma}^{\phi}$ [12, 43], i.e.,

$$\nu \in \mathscr{G}(\overrightarrow{\gamma}^{\phi}) \iff \mathscr{L}^*_\phi \nu = \lambda \nu. \tag{1.23}$$

In Chapter 4, we prove for an extensible potential $\phi$ that the translations of a half-line Gibbs measure $\nu \in \mathscr{G}(\overrightarrow{\gamma}^{\phi})$ converge to a two-sided Gibbs measure.

**Theorem 1.2.1.** *Suppose $\phi$ satisfies the extensibility condition and $\nu \in \mathscr{G}(\overrightarrow{\gamma}^{\phi})$. Consider $\mu_0 := \nu_- \times \nu$, where $\nu_-$ is any probability measure on $X_- := E^{-\mathbb{N}}$. Assume that a subsequence $\{\mu_{n_k} = \mu_0 \circ S^{-n_k}\}_{k \geq 0}$ converges to a probability measure*

*$\mu$ in the weak$^*$ topology as $k \to \infty$. Then for $\mu$-almost all $x \in X$:*

$$\mu(x_0|x_{-\infty}^{-1}, x_1^{\infty}) = \overrightarrow{\overleftarrow{\gamma}}^{\phi}_{\{0\}}(x_0|x_{-\infty}^{-1}, x_1^{\infty}).$$

*Hence, $\mu$ is a whole-line Gibbs measure for the whole-line specification $\overrightarrow{\overleftarrow{\gamma}}^{\phi}$.*

### 1.2.2 Spectral properties of transfer operators

If the potential $\phi$ related to a translation-invariant UAC interaction $\Phi$ on $\mathbb{Z}$ by

$$\phi = -\sum_{0 \in V \Subset \mathbb{Z}_+} \Phi_V, \tag{1.24}$$

then the translation-invariant Gibbs measures $\mu$ of the interaction $\Phi$ is given by $d\mu = \mathtt{h}\, d\nu$, where $\mathtt{h}$ is the eigenfunction and $\nu$ is an eigenprobability of $\mathscr{L}_\phi$ corresponding to $\lambda$, provided that the transfer operator $\mathscr{L}_\phi$ has an eigenfunction $\mathtt{h} \in L^1(X_+, \nu)$. It is worth noting that by (1.23), the transfer operator $\mathscr{L}_\phi$ always has an eigenprobability. However, the existence of an eigenfunction – which is a central issue in thermodynamic formalism – depends heavily on (the regularity of) the potential $\phi$. A long line of research by Ruelle, Walters and others has been dedicated to studying the existence problem of eigenfunctions [18, 19, 38, 39, 41–43]. For a function $f : X_+ \to \mathbb{R}$ and $n \in \mathbb{N}$, we define the variation of $f$ in the volume $[0, n-1] \subset \mathbb{Z}_+$ by

$$v_n(f) := \sup\{f(x) - f(y) :\ x, y \in X_+,\ x_0^{n-1} = y_0^{n-1}\}.$$

It was Ruelle who first established the following fundamental result for *Hölder continuous potentials*, which is now known as Ruelle's Theorem:

**Theorem 1.2.2.** *Let the potential $\phi$ be Hölder continuous, i.e., for some $\theta \in (0,1)$, $C > 0$ and for all $n \in \mathbb{N}$, $v_n(\phi) \le C\theta^n$. Then*

*(1) there exists a unique equilibrium state $\mu$ for $\phi$ and the transfer operator $\mathscr{L}_\phi$ also has a unique eigenprobability $\nu$ corresponding to the spectral radius $\lambda = e^{P(\phi)}$;*

*(2) the transfer operator has a positive continuous eigenfunction $\mathtt{h} \in C(X_+)$ corresponding to $\lambda$;*

*(3) for every $f \in C(X_+)$, the sequence $\left\{\frac{1}{\lambda^n}\mathscr{L}_\phi^n f\right\}_{n\ge 1}$ converges uniformly to $\nu(f)\cdot \mathtt{h}$ as $n \to \infty$, where $\nu(f) := \int_{X_+} f\, d\nu$.*

Walters later proved [41, 42] Ruelle's theorem under weaker conditions, such as the *summable variations*: $\sum_{n\in\mathbb{N}} v_n(\phi) < \infty$, and the so-called *Walters condition*, which requires that $\lim_{p\to\infty} \sup_{n\in\mathbb{N}} v_{n+p}(S_n\phi) = 0$. Walters also partly extended [43] Ruelle's theorem to the class of the so-called *Bowen potentials*, namely those $\phi$ satisfying $\sup_{n\in\mathbb{N}} v_n(S_n\phi) < \infty$. In the Bowen setting, Walters was only able to establish the existence of a bounded eigenfunction $\mathtt{h}$ of the transfer operator, while the existence of a continuous eigenfunction remains an open problem to this day. Independently, Fan and Jiang [18] proved Ruelle's theorem under the *Dini condition*, a refinement of Hölder continuity. These regularity conditions satisfy the strict chain

$$\text{summable variations} \implies \text{Walters condition} \implies \text{Bowen condition},$$

with no reversals. From the statistical mechanics perspective, all these conditions are considered short-range, in the sense that for every $\beta \geq 0$, the scaled potential $\beta\phi$ admits a unique equilibrium state and a unique half-line Gibbs measure. In contrast, we refer to a potential $\phi$ as *long-range* if there exists a finite $\beta > 0$ such that $\beta\phi$ admits multiple equilibrium states and multiple half-line Gibbs measures. In one-dimensional statistical mechanics, long-range potentials are often favoured over short-range potentials. This preference stems from the fact that long-range models exhibit non-trivial phase diagrams, including the emergence of finite critical temperatures, which are absent in short-range systems due to the lack of phase transitions at any finite temperature. At criticality, long-range models often display complex, fractal-like structures and rich critical behaviour, making them fundamentally distinct from short-range systems. Beyond their physical richness, long-range models are valued for the mathematical challenges they pose, as they often resist classical techniques and demand new analytical tools, making them both difficult and fascinating to study.

In Chapter 4, we attempt to extend Ruelle's theorem to the long-range potentials. We consider potentials $\phi$ which are linked to some translation-invariant UAC interaction on $X = E^{\mathbb{Z}}$ via $\phi = -\sum_{0\in V\Subset\mathbb{Z}_+} \Phi_V$. By removing all terms $V \Subset \mathbb{Z}$ with $\min V < 0 \leq \max V$ from $\Phi$ and returning them back one by one, we construct intermediate interactions $\{\Psi^{(k)} : k \in \mathbb{Z}_+\}$. Formally, we consider the family

$$\mathscr{A} = \big\{\Lambda \Subset \mathbb{Z} : \min(\Lambda) < 0,\ \max\Lambda \geq 0\big\}.$$

indexed according to some arbitrary order: $\mathscr{A} = \{\Lambda_1, \Lambda_2, \ldots\}$. Then define, for each $k \in \mathbb{Z}_+$,

$$\Psi^{(k)}_\Lambda = \begin{cases} 0, & \Lambda \in \{\Lambda_i : i \geq k+1\}, \\ \Phi_\Lambda, & \text{otherwise.} \end{cases} \tag{1.25}$$

In particular, $\Psi^{(0)}$ has no interaction between the left and right half lines. Clearly, all the constructed interactions are UAC and in addition have the following properties:

- For any finite volume $V \subset \mathbb{Z}$, $||H_V^{\Psi^{(k)}} - H_V^{\Phi}||_\infty \leq \sum_{\substack{\Lambda_j \cap V \neq \emptyset \\ j \geq k}} ||\Phi_{\Lambda_j}||_\infty \xrightarrow[k\to\infty]{} 0$;
- the specifications $\gamma^{\Psi^{(k)}}$ converge to $\gamma^{\Phi}$ as $k \to \infty$ in the sense that, for all $B \in \mathscr{F}$ and $V \Subset \mathbb{Z}$,

  $$\gamma_V^{\Psi^{(k)}}(B|\omega) \xrightarrow[k\to\infty]{} \gamma_V^{\Phi}(B|\omega) \text{ uniformly in the boundary conditions } \omega \in X;$$

- if $\nu^{(k)}$ is a Gibbs measure for $\Psi^{(k)}$, then any weak$^*$-limit point, $\mu$ of the sequence $\{\nu^{(k)}\}_{k\geq 0}$ is a Gibbs measure for the potential $\Phi$.

In this setup, we prove the following theorem:

**Theorem 1.2.3.** *Assume the following:*

1) *the interaction $\Phi$ satisfies:* $\sum_{0\in\Lambda\Subset\mathbb{Z}} (|\Lambda|-1)\cdot||\Phi_\Lambda||_\infty < 2$, *which is known as the Dobrushin uniqueness condition in literature* [22]*;*

2) $\sum_{k=1}^{\infty}\sum_{i\in\mathbb{Z}} \delta_i(\Phi_{\Lambda_k})^2 < \infty$, *where for* $F : X \to \mathbb{R}$, $\delta_i F := \sup_{\xi_{\mathbb{Z}\setminus\{i\}}=\eta_{\mathbb{Z}\setminus\{i\}}} |F(\xi)-F(\eta)|$;

3) $\sum_{k=1}^{\infty}\rho_k < \infty$, *where* $\rho_k := \sup_{n\in\mathbb{Z}_+}\left|\int_X \Phi_{\Lambda_k}\, d\,\nu^{(n)}\right|$ *and* $\nu^{(n)} \in \mathscr{G}(\Phi^{(n)})$.

*Then the interactions $\Phi$ and $\Psi^{(0)}$ have unique Gibbs states $\mu$ and $\nu^{(0)}$ and $\mu \ll \nu^{(0)}$. In particular, the restriction $\mu_+$ of $\mu$ to $X_+$ is absolutely continuous with respect to the half-line Gibbs measure $\nu$ for $\phi$. Furthermore, the transfer operator $\mathscr{L}_\phi$ for the potential $\phi$ has an integrable eigenfunction* $\mathtt{h} = \dfrac{d\mu_+}{d\nu} \in L^1(X_+, \nu)$.

### 1.2.3 Eigenfunctions of the transfer operator for the Dyson model

The so-called Dyson model, the long-range Ising model, is probably one of the most prominent examples in one-dimensional statistical mechanics. The Dyson model is defined on the shift space $\{\pm 1\}^{\mathbb{Z}}$ by the interaction:

$$\Phi_\Lambda^D(\omega) = \begin{cases} -\beta J_{|i-j|}\omega_i\omega_j, & \text{if } \Lambda = \{i,j\} \subset \mathbb{Z},\ i \neq j; \\ -h\omega_i, & \text{if } \Lambda = \{i\} \subset \mathbb{Z}; \\ 0, & \text{otherwise,} \end{cases} \tag{1.26}$$

where $\beta \geq 0$ represents the inverse temperature, $h \in \mathbb{R}$ is the external field and $J_{|i-j|} = |i-j|^{-\alpha}$ for the ferromagnetic model and $J_{|i-j|} = -|i-j|^{-\alpha}$ for the anti-ferromagnteic model, where $\alpha > 1$. The Dyson potential $\phi$ associated with the Dyson interaction is defined as

$$\phi^D(\omega) = -\sum_{0 \in V \Subset \mathbb{Z}_+} \Phi^D_V(\omega) = h\omega_0 + \sum_{k=1}^{\infty} \beta J_k \omega_0 \omega_k. \tag{1.27}$$

The following theorem about the phase diagram of the ferromagnetic Dyson model is due to Dyson [14] and Fröhlich and Spencer [21].

**Theorem 1.2.4.** *(i) Let $h = 0$, then for every $\alpha \in (1,2]$, there exists critical inverse temperature $\beta_c(\alpha) \in (0,\infty)$ such that for every $\beta \in [0, \beta_c(\alpha))$, the interaction $\Phi^D$ has a unique Gibbs measure and for all $\beta \geq \beta_c(\alpha)$, $\Phi^D$ has multiple Gibbs measures.*

*(ii) If $h \neq 0$, then the interaction $\Phi^D$ has a unique Gibbs measure for all $\beta \geq 0$ and $\alpha > 1$.*

In the phase transition region, there exist exactly two extremal Gibbs measures, often referred to as the + and − phases, denoted by $\mu^+_{\alpha,\beta,0}$ and $\mu^-_{\alpha,\beta,0}$, respectively, for the Dyson model $\Phi^D$. In 2019, 50 years after Dyson's initial work, Johansson, Öberg and Pollicott showed [28] that there exists a similar critical value $\beta^+_c(\alpha)$ for the half-line Dyson model $\phi^D$, which separates the phase transitions region from the uniqueness region.

**Theorem 1.2.5.** *For every $\alpha \in (1,2]$, there exists a critical value $\beta^+_c(\alpha) \in (0,\infty)$ such that for every $\beta < \beta^+_c(\alpha)$, there exists a unique half-line Gibbs measure for the specification $\overrightarrow{\gamma}^{\phi}$ and for all $\beta > \beta^+_c(\alpha)$, there are multiple Gibbs measures, i.e., $\#\mathcal{G}(\overrightarrow{\gamma}^{\phi}) > 1$.*

It has been conjectured in [28] that for all $\alpha > 1$, $\beta_c(\alpha) = \beta^+_c(\alpha)$. For $h \neq 0$, one can compare the half-line Gibbs measures with the whole-line Gibbs measures using Griffiths' inequalities [20]. This comparison implies that, for every $\beta \geq 0$ and $\alpha > 1$, there exists a unique half-line Gibbs measure for $\overrightarrow{\gamma}^{\phi}$.

In 2023, by employing the random-cluster representation of the Ising models, Johansson, Öberg and Pollicott [29] were also able to establish that for $h = 0$, $\alpha \in \left(\frac{3}{2}, 2\right]$ and sufficiently small $\beta \geq 0$, the transfer operator $\mathscr{L}_\phi$ for the ferromagnetic Dyson potential $\phi$ has a continuous eigenfunction. In Chapter 4, we prove a similar result with a different method, which is based on the construction of the intermediate interactions.

**Theorem 1.2.6.** *Let $\phi$ be either ferromagnetic or antiferromagnetic Dyson potential (1.27). Suppose $h = 0$, $\alpha > 1$ and $\beta > 0$ is sufficiently small. Then,*

*(i) $\phi$ admits a unique equilibrium state $\mu_+ \in \mathcal{ES}(\phi)$ and Gibbs state $\nu \in \mathcal{G}(\vec{\gamma}^{\phi})$;*

*(ii) for all $\alpha > 1$, $\mu_+$ is equivalent to $\nu$, i.e., $\mu_+ \ll \nu$ and $\nu \ll \mu_+$. In particular, the Perron-Frobenius transfer operator $\mathcal{L}_\phi$ has an eigenfunction in $L^1(X_+, \nu)$;*

*(iii) if $\alpha > \frac{3}{2}$, there exists a continuous version of the Radon-Nikodym density $\frac{d\mu_+}{d\nu}$, ensuring a continuous eigenfunction for $\mathcal{L}_\phi$.*

Our technique offers a significant advantage over the method developed in [29], as it allows us to address the antiferromagnetic Dyson potential and to handle the previously inaccessible regime $\alpha \in \left(1, \frac{3}{2}\right]$ for the ferromagnetic Dyson model in the Dobrushin uniqueness region. It is worth noting that, following the announcement of our results in [16], Johansson, Öberg, and Pollicott succeeded in extending their approach to cover the entire uniqueness region of the ferromagnetic Dyson model, namely the case $h = 0$, $\alpha \in \left(\frac{3}{2}, 2\right]$, and $\beta \in [0, \beta_c(\alpha))$, by employing a concentration inequality established in [45]. Recently, by leveraging a result of Bauerschmidt and Dagallier [2], we have also extended Theorem 1.2.6 – using the approach we developed in the second part of this thesis – to the entire uniqueness region of the ferromagnetic Dyson model (see Theorem 4.G).

**Theorem 1.2.7.** *Let $\phi$ be the ferromagnetic Dyson potential. Suppose $h = 0$. Then the following statements hold.*

*(i) For every $\alpha \in (1, 2]$, we have $\beta_c(\alpha) \le \beta_c^+(\alpha)$. Hence for every $\alpha \in (1, 2]$ and $\beta \in [0, \beta_c(\alpha)$, there exists a unique equilibrium state $\mu_+$ and a (half-line) Gibbs state $\nu$ for the potential $\phi$.*

*(ii) For each $\alpha \in (1, 2]$ and all $\beta \in [0, \beta_c(\alpha))$, the equilibrium state $\mu_+$ is equivalent to the half-line Gibbs state $\nu$. In particular, $\mathcal{L}_\phi$ admits an integrable eigenfunction $\frac{d\mu_+}{d\nu} \in L^1(\nu)$ corresponding to its spectral radius.*

*(iii) If $\alpha \in \left(\frac{3}{2}, 2\right]$, then for all $\beta \in [0, \beta_c(\alpha))$, there exists a continuous version of the Radon-Nikodym derivative $\frac{d\mu_+}{d\nu}$. Hence, $\mathcal{L}_\phi$ has a continuous principal eigenfunction.*

However, none of these studies, [29] and [16], cover the case of the Dyson potential with a nonzero external field $h \neq 0$. The approach in [29] relies heavily on

the random cluster representation of the Dyson model. The central obstacle to extending the method in [29] to non-zero external fields is the loss of symmetry, essential for the random cluster representation, which disrupts the cluster decay analysis. The method developed in [16] requires a certain sum of two-point functions to be uniformly bounded, a condition that fails for the Dyson model in a field. Nevertheless, one can adopt the technique in [16] to the case of non-zero external fields, as we shall do in Chapter 5.

**Theorem 1.2.8.** *Suppose $\alpha \in \left(\frac{3}{2}, 2\right]$, $\beta \geq 0$ and $|h| > 0$ is sufficiently large ($|h| > 2\beta\zeta(\alpha) + \log 4\beta\zeta(\alpha)$ is enough, here $\zeta$ is the Riemann zeta function). Then*

*(i) the Dyson potential $\phi$ has a unique equilibrium state $\mu_+$ and there also exists a unique eigenprobability $\nu$ of $\mathscr{L}_\phi^*$;*

*(ii) $\mu_+$ is absolutely continuous with respect to $\nu$, i.e., $\mu_+ \ll \nu$. In particular, the Perron-Frobenius transfer operator $\mathscr{L}_\phi$ admits an integrable eigenfunction corresponding to the spectral radius $\lambda = e^{P(\phi)}$.*

*(iii) The Radon-Nikodym derivative $\frac{d\mu_+}{d\nu}$ does not have a continuous version. In particular, the Perron-Frobenius transfer operator does not have a continuous principal eigenfunction.*

In [34], we conjectured that for $\alpha \in \left(1, \frac{3}{2}\right]$, $h \neq 0$, and all $\beta \geq 0$, the equilibrium state $\mu_+$ and the half-line Gibbs measure $\nu$ are mutually singular.

A comparison between Theorem 1.2.6 and Theorem 1.2.8 reveals a stark contrast between the zero-field and non-zero-field regimes. In particular, the presence of an external field leads to a loss of regularity: the corresponding eigenfunction is "one degree" less regular than that of the zero-field case, for the same values of $\alpha$ and $\beta$.

## 1.3 Multifractals and Large Deviations in Dynamical Systems

In ergodic theory—and in many practical applications—it is often important to understand the size and nature of rare events. An event may be considered rare in the sense of measure theory – it does not occur almost surely, but this does not imply that it never occurs.

As a concrete example, consider the two-sided shift space $\Omega := E^{\mathbb{Z}}$, where the alphabet $E = \{\pm 1\}$ represents Ising spins. For each configuration $x \in \Omega$, define the spin observable $\sigma_n(x) := x_n \in E$. We are interested in the behaviour of the

Birkhoff averages

$$A_n(x) := \frac{1}{2n+1}\sum_{i=-n}^{n}\sigma_n(x) = \frac{1}{2n+1}\sum_{i=-n}^{n}\sigma_0\circ S^n(x),\ \ x\in\Omega,$$

where $S:\Omega\to\Omega$ is the left shift map, defined by $S(x)_i = x_{i+1}$ for all $i\in\mathbb{Z}$. According to Birkhoff's ergodic theorem, for every ergodic probability measure $\mu$ on $\Omega$, the sequence $A_n(x)_{n\in\mathbb{N}}$ converges $\mu$-almost surely to the space average $\mu(\sigma_0) = \int_\Omega \sigma_0 d\mu$. This leads us to the study of *multifractal level sets*:

$$K_\alpha := \Big\{x\in\Omega:\lim_{n\to\infty}\frac{1}{2n+1}\sum_{i=-n}^{n}\sigma_n(x) = \alpha\Big\}. \tag{1.28}$$

By Birkhoff's theorem, we know that $\mu(K_{\mu(\sigma_0)}) = 1$, meaning that almost every point (with respect to $\mu$) lies in the level set corresponding to the mean value $\mu(\sigma_0)$. For any other $\alpha\in\mathbb{R}\setminus\mu(\sigma_0)$, the set $K_\alpha$ is $\mu$-null; that is, $\mu(K_\alpha) = 0$.

Now, consider another ergodic measure $\nu$ on $\Omega$, such that $\nu(\sigma_0)\neq\mu(\sigma_0)$. Then the set $K_{\nu(\sigma_0)}$ is a rare event with respect to $\mu$, since $\mu(K_{\nu(\sigma_0)}) = 0$, but it is of full measure with respect to $\nu$. In this way, what is negligible for one measure may be typical for another, illustrating the subtleties of rare events in the ergodic setting. Of course, studying the size of such rare events using measures alone may not always be sufficient. In these cases, dimension theory provides a more powerful tool. Specifically, it becomes more practical to investigate the "dimension" – for example, the Hausdorff dimension – of the sets $K_\alpha$.

Now, let us generalise the problem within the broader framework of dynamical systems. Assume that we are given a compact metric space $\Omega$, which is not necessarily a shift space or a configuration space, along with a continuous transformation $T:\Omega\to\Omega$, which need not be injective. Instead of limiting ourselves to the standard Birkhoff averages $\frac{1}{n}\sum_{i=0}^{n-1} f\circ T^i(x)$ for some observable $f:\Omega\to\mathbb{R}$, we consider a more general setup. Let $X := \{X_n\}_{n\in\mathbb{N}}$ be a sequence of Borel-measurable functions, which need not even be continuous. We now define the multifractal level sets associated with this sequence $X$:

$$K_\alpha := \{\omega\in\Omega:\lim_{n\to\infty}\frac{1}{n}X_n(\omega) = \alpha\},\quad \alpha\in\mathbb{R}. \tag{1.29}$$

It is worth noting that the study of the "dimension" of multifractal sets $K_\alpha$ – or the investigation of the so-called *multifractal spectrum* $\alpha\in\mathbb{R}\mapsto \dim(K_\alpha)$ – has a rich history, dating back to Besicovitch (1935) [6], Eggleston (1949) [15], and continuing to the present day. Many researchers have contributed to this field, addressing the problem at varying levels of generality. A more comprehensive list of

relevant works can be found in Chapter 6. However, it is important to note that almost all previous studies, whether implicitly or explicitly, rely on ideas from large deviations theory. The objective of Chapter 6 is to explore the necessary and sufficient conditions under which one can derive the properties of the multifractal spectrum directly from the large deviation characteristics of the sequence $X$. Consequently, Chapter 6 aims to address the general multifractal setup as broadly as possible.

As mentioned above, our goal is to study the "dimension" of the sets $K_\alpha$, for $\alpha \in \mathbb{R}$. There are several reasonable notions of "dimension" that can be used in this context, including Hausdorff dimension, box-counting dimension, packing dimension, and topological entropy for sets. Before proceeding, it is essential to specify which notion of dimension we will adopt. Among the various candidates, we choose to work with *topological entropy for non-compact sets*, a concept introduced by Bowen [8]. This choice is motivated by the fact that topological entropy is the most inherently dynamical of the aforementioned notions—it depends not only on the underlying space $\Omega$ but also on the dynamics defined by the transformation $T$. Furthermore, depending on the nature of the dynamical system, topological entropy can coincide with other notions of dimension. For instance, when $\Omega$ is a shift space and $T$ is the shift map, the topological entropy of a set coincides (up to a multiplicative constant) with its Hausdorff dimension. We now proceed to define topological entropy for sets. For a set $Z \subset \Omega$, and for any $t \in \mathbb{R}$, $\epsilon > 0$, and $N \in \mathbb{N}$, define

$$m(Z,t,\epsilon,N) := \inf\Big\{ \sum_{i=1}^{\infty} e^{-n_i t} : Z \subset \bigcup_{i=1}^{\infty} B_{n_i}(x_i,\epsilon),; n_i \geq N \Big\}, \tag{1.30}$$

where, for $x \in \Omega$ and $n \in \mathbb{N}$, the $n$-th dynamical ball is given by $B_n(x,\epsilon) := \{y \in \Omega : d(T^i y, T^i x) < \epsilon,\ 0 \leq i \leq n-1\}$. By convention, we define $m(\emptyset,t,\epsilon,N) = 0$ for all values of $t$, $\epsilon$, and $N$. Clearly, the function $m(Z,t,\epsilon,N)$ is monotonic in $N$, and thus we can define

$$m(Z,t,\epsilon) := \lim_{N\to\infty} m(Z,t,\epsilon,N).$$

It can be shown (see [35]) that $m(\cdot,t,\epsilon)$ defines an *outer measure*, with properties analogous to those of the $t$-dimensional Hausdorff outer measure. In particular, there exists a critical value $t' \in \mathbb{R}$ such that

$$m(Z,t',\epsilon) = \begin{cases} +\infty, & \text{if } t' < t, \\ 0, & \text{if } t' > t. \end{cases}$$

We denote this critical value by $h_{\text{top}}(T,Z,\epsilon)$. Thus, $h_{\text{top}}(T,Z,\epsilon) = \inf\{t \in \mathbb{R} : m(Z,t,\epsilon) = 0\} = \sup\{t \in \mathbb{R} : m(Z,t,\epsilon) = +\infty\}$. Since $h_{\text{top}}(T,Z,\epsilon)$ is monotonic in $\epsilon$, we define the topological entropy of the set $Z$ by

$$h_{\text{top}}(T,Z) := \lim_{\epsilon\to 0+} h_{\text{top}}(T,Z,\epsilon).$$

It is important to emphasise that the set $Z \subset \Omega$ is neither assumed to be compact nor $T$-invariant. Throughout our work, we also assume that the topological entropy of the full space $\Omega$ is finite, i.e., $h_{\mathsf{top}}(T,\Omega) < \infty$. The topological entropy has the following basic properties:

- *Monotonicity:* if $Z_1 \subset Z_2$, then $h_{\mathsf{top}}(T,Z_1) \leq h_{\mathsf{top}}(T,Z_2)$;
- *Countable stability:* if $Z = \bigcup_n Z_n$, then $h_{\mathsf{top}}(T,Z) = \sup_n h_{\mathsf{top}}(T,Z_n)$.

Another key concept in Chapter 6 is the *large deviation principle* (LDP). An extended function $I : \mathbb{R} \to [0,+\infty]$ is called a **large deviations rate function** (or simply a rate function) if it is lower semicontinuous, meaning that for every $\lambda \in \mathbb{R}$, the set $\{t \in \mathbb{R} : I(t) \leq \lambda\}$ is closed. A rate function $I$ is said to be **good** if all of its sub-level sets $\{t : I(t) \leq \lambda\} \subset \mathbb{R}$ are compact for every $\lambda \in \mathbb{R}$. This additional compactness condition ensures that the function has "well-behaved" minimising properties over closed subsets of $\mathbb{R}$. For a set $E \subset \mathbb{R}$, we adopt the shorthand notation $I(E) := \inf_{t \in E} I(t)$.

**Definition 1.3.1.** *Let $\nu$ be a Borel probability measure on $\Omega$ and $X_n : \Omega \to \mathbb{R}$, $n \geq 1$ be random variables. The sequence $\left\{\frac{1}{n}X_n\right\}_{n\in\mathbb{N}}$ satisfies the **Large Deviation Principle** (LDP) with a rate function $I$ if*

*(1) for all closed set $F \subset \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu\Big(\left\{\frac{1}{n}X_n \in F\right\}\Big) \leq -I(F), \tag{1.31}$$

*(2) for all open set $G \subset \mathbb{R}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\Big(\left\{\frac{1}{n}X_n \in G\right\}\Big) \geq -I(G). \tag{1.32}$$

We say that the sequence $\left\{\frac{1}{n}X_n\right\}_{n\in\mathbb{N}}$ satisfies the weak Large Deviation Principle (weak LDP) with a rate function $I$ if, in place of the upper bound condition (1.31), the following weaker version holds:

(1′) *for all <u>compact</u> $F \subset \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu\Big(\left\{\frac{1}{n}X_n \in F\right\}\Big) \leq -I(F). \tag{1.33}$$

We assume that the underlying measure $\nu$ is a Bowen-Gibbs measure with respect to a constant potential, i.e., for some $h > 0$, and for every $\epsilon > 0$ there exists $D(\epsilon) > 0$ such that for all $x \in \Omega$ and $n \in \mathbb{N}$,

$$\frac{1}{D(\varepsilon)} e^{-nh} \leq \nu\big(B_n(x,\epsilon)\big) \leq D(\varepsilon) e^{-nh}. \tag{1.34}$$

**Theorem 1.3.2.** *Assume the sequence* $\{X_n\}_{n\in\mathbb{N}}$ *satisfies the weak Bowen condition, that is, for some* $\delta > 0$,

$$\lim_{n\to\infty} \frac{v_{n,\delta}(X_n)}{n} = 0, \tag{A1}$$

*where for* $H : \Omega \to \mathbb{R}$, $v_{n,\delta}(H)$ *is* $(n,\delta)$*-variation of the function* $H$, *i.e.,* $v_{n,\delta}(H) := \sup\{H(y) - H(z) : d(T^i y, T^i z) \le \delta,\ i = \overline{0,n-1}\}$. *If the sequence* $\left\{\frac{1}{n}X_n\right\}$ *satisfies the weak LDP upper bound with a rate function* $I_X : \mathbb{R} \to [0,+\infty]$, *then for all* $\alpha \in \mathbb{R}$, *one has*

$$h_{top}(T, K_\alpha) \le h_{top}(T,\Omega) - I_X(\alpha). \tag{1.35}$$

To obtain the reverse inequality in (1.35), i.e., the corresponding lower bound, additional structural assumptions are required.

(A2) The sequence $\{X_n\}_n$ is *weakly almost additive*, i.e., there are non-negative constants $A_n = o(n)$ such that for all $x \in \Omega$ and $n, m \in \mathbb{N}$,

$$|X_{n+m}(x) - X_n(x) - X_m(T^n x)| \le A_n; \tag{1.36}$$

(A3) $T : \Omega \to \Omega$ is an *expansive*, i.e., there exists $\rho > 0$ such that if $d(T^n(x), T^n(y)) < \rho$ for all non-negative integer $n$, then $x = y$, and *strongly topologically exact transformation*, i.e., for any $\epsilon > 0$ there is a natural number $M_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $x \in \Omega$, $T^{n+M_1}(B_n(x,\epsilon)) = \Omega$.

**Theorem 1.3.3.** *Assume the conditions (A1), (A2) and (A3). If the sequence* $\left\{\frac{1}{n}X_n\right\}_{n\in\mathbb{N}}$ *satisfies LDP with an essentially strictly convex good rate function* $I_X$, *then one has the following:*

*(i) there exists extended real numbers* $-\infty \le \underline{\alpha} \le \overline{\alpha} \le +\infty$ *such that* $K_\alpha \neq \emptyset$ *for every* $\alpha \in (\underline{\alpha}, \overline{\alpha})$ *and* $K_\alpha = \emptyset$ *for all* $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$.

*(ii)* $h_{top}(T, K_\alpha) = h_{top}(T,\Omega) - I_X(\alpha)$ *holds for all* $\alpha \in (\underline{\alpha}, \overline{\alpha})$.

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