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Gibbs states in statistical mechanics and dynamical systems

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Gibbs States in Statistical Mechanics and Dynamical Systems

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