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Gibbs states in statistical mechanics and dynamical systems

Makhmudov, M.

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Gibbs States in Statistical Mechanics and Dynamical Systems

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Mirmukhsin Makhmudov

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Promotores:

Prof. dr. E.A. Verbitskiy (Universiteit Leiden, Universiteit van Amsterdam)

Prof. dr. W.T.F. den Hollander

Promotiecommissie:

Prof. dr. ir. G.L.A. Derks

Prof. dr. R.M. van Luijk

Dr. L. Coquille (Université Grenoble Alpes)

Prof. dr. Ch. Külske (Ruhr-Universität Bochum)

Prof. dr. F.H.J. Redig (Technische Universiteit Delft)

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