Gibbs states in statistical mechanics and dynamical systems

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Gibbs States in Statistical Mechanics and Dynamical Systems

Proefschrift

ter verkrijging van
de graad van doctor aan de Universiteit Leiden,
op gezag van rector magnificus prof. dr. ir. H. Bijl,
volgens besluit van het college voor promoties
te verdedigen op dinsdag 2 september 2025
klokke 10:00 uur

door

Mirmukhsin Makhmudov

geboren te Khorazm, Uzbekistan

in 1995

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Introduction

The development of the so-called Sinai–Ruelle–Bowen (SRB) Thermodynamic Formalism emerged at the crossroads of Equilibrium Statistical Mechanics and Dynamical Systems. A pioneering step in this direction was taken by Kolmogorov, who was among the first to rigorously address how to quantify entropy – a central concept in statistical physics – within the framework of dynamical systems. In 1957, Kolmogorov, together with Sinai, introduced the concept of measure-theoretic entropy (now commonly known as Kolmogorov–Sinai entropy) to dynamical systems. This formalism provided a precise mathematical language to capture the idea of entropy in evolving systems, bridging statistical mechanics with ergodic theory. This foundational work on entropy laid the groundwork for what would later become the SRB Thermodynamic Formalism—a powerful framework that unifies thermodynamic concepts with the long-term behaviour of dynamical systems.

Efforts to construct a rigorous theory of statistical physics began long before the formal introduction of entropy as a mathematical concept. At its core, the primary aim of Equilibrium Statistical Physics is to understand the collective behaviour of systems composed of a vast number of interacting constituents. These constituents could be atoms or molecules forming a physical substance, or more abstract entities such as populations of organisms—humans, insects, or animals—interacting within a shared environment.

The earliest conceptual steps toward studying such collective phenomena can be traced back to ancient Greek philosophers such as Democritus and Aristotle, who were among the first to speculate that matter is composed of indivisible units—what we now call atoms. These foundational ideas were later extended by Islamic scholars like Ibn Sina (Avicenna) and Al-Biruni, whose correspondence explored the nature and formation of matter in a more systematic way [5, 26].

The modern understanding of how atoms and molecules are organised in gases, liquids, and crystals was significantly shaped by the work of Boltzmann and Gibbs in the 19th century. Boltzmann, in particular, introduced a statistical definition of entropy, providing a microscopic interpretation of the Second Law of Thermodynamics—a cornerstone in the foundations of statistical mechanics.

In the early 20th century, Lenz proposed what is now known as the Lenz–Ising model (a.k.a. Ising model), originally intended to provide a mathematical explanation for the Curie temperature and the phenomenon of spontaneous magnetisation in ferromagnetic materials. While early analytical efforts failed to demonstrate phase transitions in these models rigorously, a breakthrough came in the 1930s when Peierls offered a geometric argument establishing the existence of phase transitions in the two-dimensional Ising model. Since Peierls' contribution, the Ising model has served as a prototypical framework in mathematical Statistical Mechanics, providing a fertile ground for testing hypotheses and developing rigorous methods to understand complex collective behaviours.

Systematic mathematical study of lattice spin and gas models—and their associated phase transitions—began in the late 1960s through the foundational works of Dobrushin, Lanford, and Ruelle [13,33]. They proposed a rigorous framework in which the possible state of each constituent in a large system could be described via a system of conditional probabilities. More precisely, if the interaction among constituents in the system $(\sigma_i)_{i\in\mathbb{L}}$ is described by a family $\Phi = (\Phi_\Lambda)_{\Lambda\Subset\mathbb{L}}$, then the probability that a constituent σ_i is in a state $a \in E$, given that the other constituents $(\sigma_j)_{j\in\mathbb{L}\setminus i}$ are in the states $(x_j)_{j\in\mathbb{L}\setminus i} \in E^{\mathbb{L}\setminus i}$, is given by

$$\mu^\Phi(\sigma_i = a|\sigma_j = x_j,\ \ j \in \mathbb{L}\setminus i) = \frac{e^{-H^\Phi_{\{i\}}(a_i x_{\mathbb{L}\setminus i})}}{\sum_{\bar{a}\in E} e^{-H^\Phi_{\{i\}}(\bar{a}_i x_{\mathbb{L}\setminus i})}}, \tag{1.1}$$

where E denotes the set of all possible states, assumed to be at most countable, and $H_{\{i\}} := \sum_{i\in V\Subset\mathbb{L}} \Phi_V$—known as the Hamiltonian of the system on $\{i\}$—represents the total influence of the constituents $(\sigma_j)_{j\in\mathbb{L}\setminus i}$ on σ_i. More generally, for a finite subset $\Lambda \Subset \mathbb{L}$, the joint probability that the constituents $\sigma_\Lambda := (\sigma_i)_{i\in\Lambda}$ are in the states $a_\Lambda := (a_i)_{i\in\Lambda} \in E^\Lambda$, given that the remainder of the system is in states $x_{\mathbb{L}\setminus\Lambda} := (x_j)_{j\in\mathbb{L}\setminus\Lambda}$, is described by the Boltzmann ansatz:

$$\mu^\Phi(\sigma_\Lambda = a_\Lambda|\sigma_{\mathbb{L}\setminus\Lambda} = x_{\mathbb{L}\setminus\Lambda}) = \frac{e^{-H^\Phi_\Lambda(a_\Lambda x_{\mathbb{L}\setminus\Lambda})}}{\sum_{\bar{a}_\Lambda\in E^\Lambda} e^{-H^\Phi_\Lambda(\bar{a}_\Lambda x_{\mathbb{L}\setminus\Lambda})}}. \tag{1.2}$$

A notable mathematical limitation of the Boltzmann ansatz is that different interactions $(\Phi_\Lambda)_{\Lambda\Subset\mathbb{L}}$ can lead to the same system of conditional probabilities – a phenomenon formalised under the concept of physical equivalence. To address this ambiguity, a more robust mathematical formulation of the global state of the system is given through a regular system of conditional probabilities, commonly referred to as specifications. Interestingly, the notion of specification is closely tied to a dynamical concept known as a cocycle, which allows for a generalisation of Gibbs measures beyond simple lattice structures. By using this relationship, Bowen infused the theory of Gibbs measures with a distinctly dynamical systems perspective. Building on the foundations laid by Sinai and Ruelle, Bowen extended the notion of Gibbs measures to broader dynamical contexts, including Axiom A diffeomorphisms and general hyperbolic systems. His work provided a crucial bridge between the statistical mechanics formulation and the language of dynamical systems by redefining Gibbs measures within this new framework—a definition we revisit in Chapter 3, where we compare it with the standard DLR (Dobrushin–Lanford–Ruelle) formulation from statistical mechanics. Moreover, under fairly general conditions—such as the existence of a Markov partition—these broader dynamical systems can often be shown to be conjugate to product-type systems, effectively linking the dynamical perspective back to the more classical approaches.

In practical applications, the most commonly encountered cases for the indexing set (or set of sites) $\mathbb{L}$ include deterministic and random graphs, countable groups, and semi-groups. In the context of dynamical systems, particular attention is given to situations where $\mathbb{L}$ is a countable group or semi-group, as these naturally induce a group action on the configuration space $\Omega = E^{\mathbb{L}}$—a structure that is especially rich from the viewpoint of dynamical systems. In this thesis, we primarily focus on two specific choices for the indexing set $\mathbb{L}$: either $\mathbb{L} = \mathbb{Z}^d$ for some $d \geq 1$, or $\mathbb{L} = \mathbb{Z}_+$. Within these settings, we investigate Gibbs measures from various viewpoints, examining both their foundational definitions and a range of properties relevant to statistical mechanics and dynamical systems.

The thesis is intrinsically divided into three parts. In the first part (covered in Chapters 2 and 3), we explore the various notions of Gibbsianity as they appear in both Statistical Mechanics and Dynamical Systems, and we study the interplay between these frameworks. Among other topics, we discuss the Kozlov–Sullivan characterisation of specifications, present a form of interaction-independent variational principle which is new to the literature, and analyse the Gibbs properties of equilibrium states. The second part (encompassing Chapters 4 and 5) shifts the focus to one-dimensional systems. Here, we examine Gibbs measures defined on both the whole-line $\mathbb{Z}$ and the half-line $\mathbb{Z}_+$, and we compare the features of these two settings. A closely related subject in this section is the principal eigenfunction problem for Ruelle-Perron-Frobenius transfer operators. In fact, this part initiates a systematic study of the thermodynamic formalism for long-range potentials in dimension one. Finally, the third part (presented in Chapter 6) is devoted to an investigation of the multifractal properties of a general Gibbsian system, with special attention given to the aspects governed by large deviations.

1.1 Gibbs Measures in Dynamical Systems

In the first part of this thesis, we explore various notions of Gibbsianity, which differ in terms of their level of generality and mathematical setup. The standard Gibbsian formalism is built in configuration spaces. Let us fix a single-site space (also known as the single-spin space in models related to magnetism), denoted by E. Throughout the thesis, we assume E to be finite. Typical examples of such spaces include the Ising spin space $\{\pm 1\}$, the lattice gas space $\{0,1\}$, or the set of colours $\{1,2,\ldots,q\}$ for $q \geq 2$, as encountered in Potts models. We always endow E with the discrete topology. Next, consider an arbitrary countable set $\mathbb{L}$, referred to as the set of sites. Examples of $\mathbb{L}$ include the vertex (or edge) set of an infinite graph – either deterministic or random – such as a Cayley tree, a countable group like $\mathbb{Z}^d$, or a semigroup such as $\mathbb{Z}_+^d$. The configuration space is then defined as $\Omega := E^{\mathbb{L}}$. Equipped with the product topology, Ω becomes a compact and metrizable space. We denote by $\mathscr{F}$ the Borel σ-algebra generated by this topology. The space of real-valued continuous functions on Ω is written as $C(\Omega)$, and $\mathscr{M}_1(\Omega)$ denotes the simplex of Borel probability measures on Ω. Given a configuration $\omega \in \Omega$ and a finite subset $\Lambda \Subset \mathbb{L}$ (which we refer to as a finite volume), we define the cylindrical set associated with ω as $[\omega_\Lambda] := \{\xi \in \Omega : \xi_i = \omega_i,\ i \in \Lambda\}$. For each site $i \in \mathbb{L}$, we define the random variable $\sigma_i : \Omega \to E$ by $\sigma_i(\omega) := \omega_i$. For any subset $V \subset \mathbb{L}$ (finite or infinite), we denote by $\mathscr{F}_V$ the smallest sub-σ-algebra of $\mathscr{F}$ with respect to which the collection $\{\sigma_i\}_{i\in V}$ is measurable. Finally, the tail σ-algebra $\mathscr{T}$ is defined as $\mathscr{T} := \cap_{\Lambda\Subset\mathbb{L}} \mathscr{F}_{\mathbb{L}\setminus\Lambda}$.

1.1.1 Classical Gibbs formalism in statistical mechanics

The standard definition of Gibbs states in statistical mechanics is related – via the Boltzmann ansatz (1.2)– to the notion of interaction. An ***interaction*** Φ on Ω is a family $(\Phi_\Lambda)_{\Lambda\Subset\mathbb{L}}$ of Λ–local functions $\Phi_\Lambda : \Omega \to \mathbb{R}$, i.e., each Φ_Λ is $\mathscr{F}_\Lambda$ measurable. An interaction $\Phi = (\Phi_\Lambda)_{\Lambda\Subset\mathbb{L}}$ is called ***uniformly absolutely convergent*** (UAC) if for every $i \in \mathbb{L}$,

$$\sum_{i\in\Lambda\Subset\mathbb{L}} \sup_{\omega\in\Omega} |\Phi_\Lambda(\omega)| < \infty. \tag{1.3}$$

For any finite volume $\Lambda \Subset \mathbb{L}$, one can associate the ***Hamiltonian***

$$H_\Lambda^\Phi := \sum_{\substack{V\Subset\mathbb{L} \\ V\cap\Lambda\neq\emptyset}} \Phi_V.$$

Note that under the UAC condition, H_Λ^Φ is a well-defined and continuous function on Ω.

Definition 1.1.1. *A probability measure $\mu \in \mathscr{M}_1(\Omega)$ is called a* ***Gibbs measure for the interaction*** Φ *(denoted by $\mu \in \mathscr{G}(\Phi)$) if for every $f \in C(\Omega)$,*

$$\mu(f|\mathscr{F}_{\Lambda^c})(\omega) = \frac{1}{Z_\Lambda^\Phi(\omega)} \sum_{\bar{\omega}_\Lambda \in E^\Lambda} f(\bar{\omega}_\Lambda \omega_{\Lambda^c}) \cdot e^{-H_\Lambda^\Phi(\bar{\omega}_\Lambda \omega_{\Lambda^c})}, \;\; \mu - a.e.\ \omega \in \Omega, \tag{1.4}$$

where $Z_\Lambda^\Phi(\omega)$, known as the partition function, normalises (1.4) to 1*:*

$$Z_\Lambda^\Phi(\omega) := \sum_{\bar{\omega}_\Lambda \in E^\Lambda} e^{-H_\Lambda^\Phi(\bar{\omega}_\Lambda \omega_{\Lambda^c})},$$

and $\bar{\omega}_\Lambda \omega_{\Lambda^c}$ denotes the concatenated configuration.

When E is compact – which is the case throughout this thesis – the set $\mathscr{G}(\Phi)$ of Gibbs measures for Φ is always non-empty, convex, and compact with respect to the weak-* topology. Since the existence of Gibbs measures is always guaranteed under these conditions, the more physically relevant question concerns their uniqueness. Specifically, one says that the interaction Φ exhibits a ***phase transition*** if there exist multiple Gibbs measures for Φ, i.e., if $\#\mathscr{G}(\Phi) > 1$.

1.1.2 DLR Gibbs formalism

The modern approach, also known as the Dobrushin-Lanford-Ruelle (DLR) approach, to Gibbs measures in mathematical Statistical Mechanics is rather different and does not involve interactions. Instead, one immediately starts with a family of regular conditional probabilities, which is known as *specification.* Since there is no initial reference measure, one needs to define the regular conditional probabilities everywhere on Ω rather than almost everywhere. A *regular conditional probability* in a volume $\Lambda \Subset \mathbb{L}$ is a proper probability kernel γ_Λ from $(\Omega, \mathscr{F})$ to $(\Omega, \mathscr{F}_{\Lambda^c})$, i.e., $\gamma_\Lambda : \mathscr{F} \times \Omega \to [0,1]$ such that

- for all $\omega \in \Omega$, $\gamma_\Lambda(\cdot|\omega) : \mathscr{F} \to [0,1]$ is a probability measure on Ω;
- for every $B \in \mathscr{F}$, $\gamma_\Lambda(B|\cdot) : \Omega \to [0,1]$ is $\mathscr{F}_{\Lambda^c}$−measurable;
- for each $B \in \mathscr{F}_{\Lambda^c}$, $\gamma_\Lambda(B|\cdot) = \mathbb{1}_B(\cdot)$.

A ***specification*** γ on Ω is a *consistent* family of regular conditional probabilities γ_Λ indexed by $\Lambda \Subset \mathbb{L}$. This means that for every pair of finite volumes $\Delta \subset \Lambda \Subset \mathbb{L}$, the consistency condition holds:

$$\gamma_\Lambda(B|\omega) = \gamma_\Lambda \gamma_\Delta(B|\omega) := \int_\Omega \gamma_\Delta(B|\xi) \gamma_\Lambda(d\xi|\omega), \;\; B \in \mathscr{F},\ \omega \in \Omega. \tag{1.5}$$

Definition 1.1.2. *A probability measure $\mu \in \mathcal{M}_1(\Omega)$ is a* ***Gibbs measure for the specification*** γ*, if for every $\Lambda \Subset \mathbb{L}$ and $f \in C(\Omega)$, one has*

$$\mu(f|\mathscr{F}_{\Lambda^c})(\omega) = \gamma_\Lambda(f|\omega) := \sum_{\bar{\omega}_\Lambda \in E^\Lambda} f(\bar{\omega}_\Lambda \omega_{\Lambda^c})\gamma_\Lambda([\bar{\omega}_\Lambda]|\omega), \; \mu - a.e. \; \omega \in \Omega. \tag{1.6}$$

Equivalently, μ is Gibbs for γ if for every $\Lambda \Subset \mathbb{L}$,

$$\mu(B) = (\mu\gamma_\Lambda)(B) := \int_\Omega \gamma_\Lambda(B|\xi)\mu(d\xi), \; B \in \mathscr{F}. \tag{1.7}$$

The set $\mathscr{G}(\gamma)$ of Gibbs measures for a specification γ is always convex and closed in the weak-* topology. However, it is not necessarily non-empty. Existence is guaranteed under an additional regularity condition: if γ is *continuous*—that is, it satisfies the *Feller property*—then $\mathscr{G}(\gamma) \neq \emptyset$. This property requires that for every $f \in C(\Omega)$ and every finite volume $\Lambda \Subset \mathbb{L}$, the function $\gamma_\Lambda(f \mid \cdot)$ belongs to $C(\Omega)$. A continuous specification γ on Ω is called a **Gibbsian specification** if it is *non-null*, i.e., for every $\Lambda \Subset \mathbb{L}$ and all $\xi, \omega \in \Omega$,

$$\gamma_\Lambda([\xi_\Lambda]|\omega) > 0. \tag{1.8}$$

In fact, one can readily check that any UAC interaction Φ gives rise to a Gibbsian specification $\gamma^\Phi = (\gamma^\Phi_\Lambda)_{\Lambda \Subset \mathbb{L}}$ defined by:

$$\gamma^\Phi_\Lambda(B|\omega) := \frac{1}{Z^\Phi_\Lambda(\omega)} \sum_{\bar{\omega}_\Lambda \in E^\Lambda} \mathbb{1}_B(\bar{\omega}_\Lambda \omega_{\Lambda^c}) \cdot e^{-H^\Phi_\Lambda(\bar{\omega}_\Lambda \omega_{\Lambda^c})}, \; B \in \mathscr{F}, \; \omega \in \Omega. \tag{1.9}$$

By the celebrated theorem of Kozlov [31], the opposite statement is also true, namely, for any Gibbsian specification γ, there exists a UAC interaction Φ such that $\gamma = \gamma^\Phi$. However, subtleties arise when the specification has certain symmetries, and if it is also required to find a UAC interaction respecting those symmetries. For example, when $\mathbb{L} = \mathbb{Z}^d$ or $\mathbb{L} = \mathbb{Z}_+$, the natural shift (left-shift) transformations S_j, $j \in \mathbb{L}$ acts on Ω by $(S_j\omega)_{i\in\mathbb{L}} = (\omega_{i+j})_{i\in\mathbb{L}}$. An interaction Φ on Ω is called translation-invariant if for every $\Lambda \Subset \mathbb{L}$ and $j \in \mathbb{L}$, $\Phi_{\Lambda+j} = \Phi_\Lambda \circ S_j$. A specification γ on the lattice $\mathbb{L} = \mathbb{Z}^d$ is translation-invariant if for every $\Lambda \Subset \mathbb{Z}^d$, $j \in \mathbb{Z}^d$, $B \in \mathscr{F}$ and $\omega \in \Omega$, $\gamma_{\Lambda+j}(B|\omega) = \gamma_\Lambda(S_j(B)|S_j\omega)$. It is worth noting that any translation-invariant Gibbsian specification admits at least one translation-invariant Gibbs measure—that is, a Gibbs measure that is invariant under the shift map. Moreover, any translation-invariant UAC interaction generates a translation-invariant Gibbsian specification (see Figure 2.1). However, a recent result by [1], in the case $\mathbb{L} = \mathbb{Z}$, demonstrates that there exist translation-invariant Gibbsian specifications that are *not* generated by any translation-invariant UAC interaction. In Chapter 2 of this thesis, we address a similar question in the setting $\mathbb{L} = \mathbb{Z}_+$.

Theorem 1.1.3. *There exists a translation-shift invariant Gibbsian specification on $\mathbb{Z}_+$ that can not be associated with a translation-invariant UAC interaction.*

Our proof of Theorem 1.1.3 is constructive, in contrast to the non-constructive approach used in [1]. In Chapter 2, we also establish a precise criterion characterising when a Gibbsian specification on $\mathbb{Z}_+$ can be generated by a translation-invariant UAC interaction. We emphasise that no analogous criterion is known for the whole line $\mathbb{Z}$.

This mismatch between the classes of translation-invariant interactions and translation-invariant specifications has important implications for the classical theory of equilibrium statistical physics. Translation-invariant Gibbs measures – those invariant under the shift maps – play a central role in statistical physics, as they are considered physically meaningful descriptions of macroscopic phases in infinite systems. It is well known that every translation-invariant Gibbs measure corresponds to a translation-invariant Gibbsian specification. However, not all such specifications arise from translation-invariant UAC interactions. This suggests that relying solely on interactions to study translation-invariant Gibbs measures may be conceptually inadequate. Nevertheless, much of classical mathematical statistical mechanics has been historically developed through the lens of interactions. Key results concerning translation-invariant Gibbs measures—such as the variational principle and large deviation principles—are typically formulated in terms of interactions. This tension naturally motivates the development of a theory of translation-invariant Gibbs measures based purely on specifications. Some progress in this direction has already been made. For instance, [32] establishes a version of the variational principle using specifications alone. Combined with results from [36], this leads to large deviation bounds applicable to all translation-invariant Gibbs measures. In Chapter 2 of this thesis, we aim to contribute further to this line of research by proving a new, more dynamical form of the variational principle—also formulated entirely in terms of specifications.

Theorem 1.1.4. *Assume γ is a translation-invariant Gibbs specification on $\Omega = E^{\mathbb{Z}^d}$. Then, for any translation-invariant measure $\rho \in \mathscr{M}_1(\Omega)$, there exists a continuous function $u_\gamma^\rho \in C(\Omega)$ associated with γ such that the translation-invariant Gibbs measures for γ are precisely the equilibrium states for the potential u_γ^ρ, i.e.,*

$$h(\mu)+\int_\Omega u_\gamma^\rho \, d\mu = \sup\left\{h(\tau)+\int_\Omega u_\gamma^\rho \, d\tau : \tau \in \mathscr{M}_{1,S}(\Omega)\right\} \quad \Longleftrightarrow \quad \mu \in \mathscr{G}_S(\gamma), \tag{1.10}$$

where $h(\tau)$ is the measure-theoretic entropy of the translation-invariant measure τ and $\mathscr{M}_{1,S}(\Omega)$ denotes the class of translation-invariant measures on Ω.

1.1.3 G-formalism and its relation to DLR formalism on $\mathbb{Z}_+$

In 1972, Keane [30] introduced the notion of a *g-measure* in ergodic theory, which is a one-sided counterpart of translation-invariant Gibbs measures. Later, in 1991, Brown and Dooley [11] extended this concept to G-measures, which generalise g-measures to the setting of non-translation-invariant measures. Let $E \simeq \mathbb{Z}/q\mathbb{Z}$ and consider the direct sum $\Gamma := \bigoplus_{\mathbb{Z}_+} E$ and its subgroup $\Gamma_\Delta := \{\omega \in \Gamma : \omega_i = 0,\ i \in \mathbb{Z}_+ \setminus \Delta\}$ for $\Delta \Subset \mathbb{Z}_+$. Then a family G of Borel measusrable functions $G_\Lambda : \Omega \to [0,1]$, $\Omega = E^{\mathbb{Z}_+}$, indexed by the finite subsets Λ of $\mathbb{Z}_+$ is called a $G-$***family*** if for all $\Lambda \Subset \mathbb{Z}_+$,

$$\sum_{\omega \in \Gamma_\Lambda} G_\Lambda(\omega + \eta) = 1, \ \ \eta \in \Omega, \tag{1.11}$$

and for all $\Delta \subset \Lambda \Subset \mathbb{Z}_+$,

$$G_\Lambda(\omega + \eta) G_\Delta(\eta) = G_\Lambda(\eta) G_\Delta(\omega + \eta), \ \ \omega \in \Gamma_\Delta, \ \eta \in \Omega, \tag{1.12}$$

here the sum $\omega + \eta$ should be understood coordinate-wise, i.e., $(\omega+\eta)_i := \omega_i + \eta_i$. For any $\Lambda \Subset \mathbb{Z}_+$, and $\nu \in \mathscr{M}_1(\Omega)$, define $\nu_\Lambda := \frac{1}{q^{|\Lambda|}} \sum_{x \in \Gamma_\Lambda} x_* \nu$, where for $x \in \Gamma_\Lambda$, $x_* : \mathscr{M}_1(\Omega) \to \mathscr{M}_1(\Omega)$ is the pushforward of the map $x : \Omega \to \Omega$, $x(\eta) = x + \eta$. It is clear that $\nu \ll \nu_\Lambda$ since $e_* \nu = \nu$, where e is the neutral element of the group Γ.

Definition 1.1.5. *A probability measure $\nu \in \mathscr{M}_1(\Omega)$ is called a $G-$measure for a $G-$family $G = (G_\Lambda)_{\Lambda \Subset \mathbb{Z}_+}$, if for all $\Lambda \Subset \mathbb{Z}_+$,*

$$\frac{d\nu}{d\nu_\Lambda} = q^{|\Lambda|} G_\Lambda. \tag{1.13}$$

We note that a $G-$measure becomes a $g-$measure if the associated $G-$family is continuous, positive and

$$\frac{G_{[0,n+1] \cap \mathbb{Z}_+}}{G_{[0,n] \cap \mathbb{Z}_+}} = G_{\{0\}} \circ S_{n+1}, \quad n \in \mathbb{Z}_+. \tag{1.14}$$

In Chapter 2, we demonstrate that the $G-$formalism introduced by Brown and Dooley coincides with the DLR Gibbs formalism on the lattice $\mathbb{Z}_+$. More precisely, we show that

Theorem 1.1.6. *The relation*

$$\gamma_\Lambda(\eta_\Lambda | \eta_{\mathbb{Z}_+ \setminus \Lambda}) := G(\eta), \ \ \Lambda \Subset \mathbb{Z}_+, \ \eta \in \Omega = E^{\mathbb{Z}_+}, \tag{1.15}$$

establishes one-to-one correspondence between the G-families and the specifications on $\mathbb{Z}_+$. Furthermore, ν is a G-measure for G if and only if ν is a Gibbs measure for γ.

1.1.4 Gibbsianity of equilibrium states

In Chapter 3, which constitutes the remaining portion of the first part of the thesis, we shift our focus to the Gibbsianity of equilibrium states on the shift space $\Omega = E^{\mathbb{Z}}$. Although the results presented in this chapter are valid for higher-dimensional lattices $\mathbb{Z}^d$, we restrict our attention to the one-dimensional case in order to avoid cumbersome formulas. Note that a translation-invariant measure μ on $\Omega = E^{\mathbb{Z}}$ is an ***equilibrium state*** for a continuous function (potential) $\phi : \Omega \to \mathbb{R}$, denoted by $\mu \in \mathscr{ES}(\phi)$, if

$$h(\mu) + \int_{\Omega} \phi \, d\mu = \sup \Big\{ h(\tau) + \int_{\Omega} \phi \, d\tau : \tau \in \mathscr{M}_{1,S}(\Omega) \Big\}. \tag{1.16}$$

We note that the class of equilibrium states rather is broad (see Figure 3.1): if $\mu_1, \ldots, \mu_k$ are some ergodic measures on Ω, then one can find a continuous potential $\phi \in C(\Omega)$ such that all these measures are equilibrium states for ϕ [17, 27, 39]. This remarkable generality suggests that equilibrium states can exhibit a wide range of behaviours, and in particular, one cannot expect them to possess any form of Gibbsianity in general. In Section 3, we investigate two types of Gibbsianity for an equilibrium state μ (for a potential ϕ):

- *DLR Gibbsianity*: we explore whether there exists a Gibbsian specification γ on $\Omega = E^{\mathbb{Z}}$ such that $\mu \in \mathscr{G}(\gamma)$;
- *(weak) Bowen-Gibbsianity*: we investigate if there exists a sequence of positive numbers $C_n > 0$, $n \in \mathbb{N}$, that is subexponential, i.e., $\lim_{n\to\infty} \frac{1}{n} \log C_n = 0$, such that for all n and $\omega \in \Omega$,

$$\frac{1}{C_n} \leq \frac{\mu(\{\tilde{\omega} \in \Omega : \tilde{\omega}_0^{n-1} = \omega_0^{n-1}\})}{e^{S_n\phi(\omega) - nP}} \leq C_n, \tag{1.17}$$

  where $S_n\phi$ is the ergodic sum, i.e., $S_n\phi := \sum_{j=0}^{n-1} \phi \circ S_j$.

The Gibbsianity properties of equilibrium states mentioned above have been extensively studied under various regularity assumptions on the potential ϕ by Sinai, Bowen, Ruelle, Haydn, Pfister, Sullivan, and more recently by Bissacott and collaborators [3, 7, 10, 23–25, 37, 40]. In Chapter 3, we establish both the DLR Gibbs and weak Bowen-Gibbs properties of equilibrium states under a barely minimal regularity assumption on the potential ϕ, which we refer to as the *extensibility condition.*

Definition 1.1.7. *A potential $\phi : \Omega \to \mathbb{R}$ satisfies the **extensibility condition** if for all $a_0, b_0 \in E$ the sequence of functions*

$$\rho_n^{a_0,b_0}(\omega) := \sum_{i=-n}^{n} \big(\phi \circ S_i(\omega_{-\infty}^{-1} b_0 \omega_1^{\infty}) - \phi \circ S_i(\omega_{-\infty}^{-1} a_0 \omega_1^{\infty}) \big)$$

converges uniformly in $\omega \in \Omega = E^{\mathbb{Z}}$ as $n \to \infty$.

The uniform limit $\lim_{n\to\infty} \rho^{a_0,b_0}(\omega)$ gives rise to a *cocycle* ρ^{ϕ} on the asymptotic equivalence relation $\mathfrak{T}(\Omega) := \{(\bar{\omega}, \omega) \in \Omega \times \Omega : \bar{\omega}_i = \omega_i$ for all but finitely many $i \in \mathbb{Z}\}$ which the same amounts to saying that for all $(\bar{\omega}, \hat{\omega}), (\hat{\omega}, \omega) \in \mathfrak{T}(\Omega)$

$$\rho^{\phi}(\bar{\omega}, \hat{\omega}) + \rho^{\phi}(\hat{\omega}, \omega) = \rho^{\phi}(\bar{\omega}, \omega). \tag{1.18}$$

One can associate a non-null specification $\overset{\rightleftharpoons}{\gamma}^{\phi} = (\overset{\rightleftharpoons}{\gamma}^{\phi}_{\Lambda})_{\Lambda \Subset \mathbb{Z}}$ on $\Omega = E^{\mathbb{Z}}$ with the cocycle ρ^{ϕ} via

$$\overset{\rightleftharpoons}{\gamma}^{\phi}_{\Lambda}(\omega_{\Lambda}|\omega_{\Lambda^c}) = \Big(\sum_{\xi_{\Lambda} \in E^{\Lambda}} e^{\rho^{\phi}(\xi_{\Lambda}\omega_{\mathbb{Z}\setminus\Lambda},\, \omega)}\Big)^{-1}, \;\; \omega \in \Omega, \; \Lambda \Subset \mathbb{Z}. \tag{1.19}$$

The uniform nature of the limit $\lim_{n\to\infty} \rho_n^{a_0,b_0}$ ensures the Gibbsianity of the specification $\overset{\rightleftharpoons}{\gamma}^{\phi}$.

The extensibility condition is not very restrictive. Notably, unlike the conditions used in the works of Sinai, Bowen, Ruelle, and Walters, it does not imply uniqueness of equilibrium states. Moreover, the extensibility condition encompasses all previously studied classes of potentials. In Chapter 3, we demonstrate that this condition is sufficient to guarantee both DLR Gibbsianity and weak Bowen-Gibbsianity of the equilibrium states.

Theorem 1.1.8. *Suppose $\phi \in C(\Omega)$ has the extensibility property. Then any equilibrium state μ for ϕ is*

(1) Gibbs in the Dobrushin-Lanford-Ruelle sense, in fact, $\mathscr{ES}(\phi) = \mathscr{G}_S(\overset{\rightleftharpoons}{\gamma}^{\phi})$;

(2) weak Bowen-Gibbs relative to the potential ϕ.

In Chapter 3, we also establish, in a certain sense, a converse to Theorem 1.1.8. Specifically, we prove that any DLR Gibbs measure, as defined in Definition 1.1.2, is in fact an equilibrium state for some extensible potential. Additionally, we remark that Theorem 1.1.8 generalises the main result of [4] by extending it to the setting of general, not necessarily normalised, potentials.

1.2 Transfer operators for long-range potentials

The second part of the thesis comprises Chapters 4 and 5, where we investigate and compare the Gibbs formalisms on the whole-line $\mathbb{Z}$ and the half-line $\mathbb{Z}_+$. A key tool in this comparison is the Perron-Frobenius-Ruelle transfer operator, which plays a central role in our analysis. These operators—viewed as infinite-dimensional analogues of positive matrices—were introduced into the thermodynamic formalism by D. Ruelle, primarily to study the mixing properties of Gibbs measures in one-dimensional systems. Throughout Chapters 4 and 5, we fix a finite set E and denote the configuration spaces by $X = E^{\mathbb{Z}}$ for the whole line and $X_+ = E^{\mathbb{Z}_+}$ for the half line.

For a potential $\phi \in C(X_+)$, the associated Perron-Frobenius-Ruelle transfer operator $\mathscr{L}_\phi$ is defined by

$$\mathscr{L}_\phi f(x) := \sum_{a\in E} e^{\phi(ax)} f(ax), \;\; f \in \mathbb{R}^{X_+}, \tag{1.20}$$

where ax denotes the configuration obtained by prepending the symbol a to x. The transfer operator maps the space of real-valued continuous functions $C(X_+)$ into itself and is bounded when restricted to $C(X_+)$. Moreover, its spectral radius equals $\lambda = e^{P(\phi)}$, where $P(\phi)$ denotes the topological pressure of the potential ϕ.

1.2.1 The role of transfer operators in the Gibbs formalism on $\mathbb{Z}_+$

The Perron-Frobenius-Ruelle transfer operators play a pivotal role in the Gibbs formalism on the half-line $\mathbb{Z}_+$. Given any potential $\phi \in C(\Omega)$, one can naturally associate a Gibbsian specification $\vec{\gamma}^{\phi} = (\vec{\gamma}^{\phi}_\Lambda)_{\Lambda \Subset \mathbb{Z}_+}$ on the configuration space X_+ via the formula:

$$\vec{\gamma}^{\phi}_\Lambda(a_\Lambda | x_{\Lambda^c}) = \frac{\exp\left((S_{n+1}\phi)(a_\Lambda x_{\Lambda^c})\right)}{\sum_{\bar{a}_\Lambda} \exp\left((S_{n+1}\phi)(\bar{a}_\Lambda x_{\Lambda^c})\right)}, \;\; a_\Lambda \in E^\Lambda, x \in X_+. \tag{1.21}$$

Then one has the following interesting relationship between the specification $\vec{\gamma}^{\phi}$ and the transfer operator $\mathscr{L}_\phi$:

$$\vec{\gamma}^{\phi}_{[0,n-1]}(f|x) = \frac{\mathscr{L}^n_\phi f(S_n x)}{\mathscr{L}^n_\phi \mathbf{1}(S_n x)}, \quad n \in \mathbb{N}, \; x \in X_+, \text{ and } f : X_+ \to \mathbb{R}. \tag{1.22}$$

Using (1.22), one can show that the eigenprobabilities of the operator $\mathscr{L}_\phi$ corresponding to the spectral radius λ are exactly the Gibbs measures for the specification $\vec{\gamma}^{\phi}$ [12, 43], i.e.,

$$\nu \in \mathscr{G}(\vec{\gamma}^{\phi}) \iff \mathscr{L}^*_\phi \nu = \lambda \nu. \tag{1.23}$$

In Chapter 4, we prove for an extensible potential ϕ that the translations of a half-line Gibbs measure $\nu \in \mathscr{G}(\vec{\gamma}^{\phi})$ converge to a two-sided Gibbs measure.

Theorem 1.2.1. *Suppose ϕ satisfies the extensibility condition and $\nu \in \mathscr{G}(\vec{\gamma}^{\phi})$. Consider $\mu_0 := \nu_- \times \nu$, where ν_- is any probability measure on $X_- := E^{-\mathbb{N}}$. Assume that a subsequence $\{\mu_{n_k} = \mu_0 \circ S^{-n_k}\}_{k\geq 0}$ converges to a probability measure*

μ in the weak topology as $k \to \infty$. Then for μ-almost all $x \in X$:*

$$\mu(x_0|x_{-\infty}^{-1}, x_1^{\infty}) = \overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\{0\}}(x_0|x_{-\infty}^{-1}, x_1^{\infty}).$$

Hence, μ is a whole-line Gibbs measure for the whole-line specification $\overset{\rightleftharpoons}{\gamma}{}^{\phi}$.

1.2.2 Spectral properties of transfer operators

If the potential ϕ related to a translation-invariant UAC interaction Φ on $\mathbb{Z}$ by

$$\phi = -\sum_{0 \in V \Subset \mathbb{Z}_+} \Phi_V, \tag{1.24}$$

then the translation-invariant Gibbs measures μ of the interaction Φ is given by $d\mu = \mathtt{h} d\nu$, where $\mathtt{h}$ is the eigenfunction and ν is an eigenprobability of $\mathscr{L}_\phi$ corresponding to λ, provided that the transfer operator $\mathscr{L}_\phi$ has an eigenfunction $\mathtt{h} \in L^1(X_+, \nu)$. It is worth noting that by (1.23), the transfer operator $\mathscr{L}_\phi$ always has an eigenprobability. However, the existence of an eigenfunction – which is a central issue in thermodynamic formalism – depends heavily on (the regularity of) the potential ϕ. A long line of research by Ruelle, Walters and others has been dedicated to studying the existence problem of eigenfunctions [18, 19, 38, 39, 41–43]. For a function $f : X_+ \to \mathbb{R}$ and $n \in \mathbb{N}$, we define the variation of f in the volume $[0, n-1] \subset \mathbb{Z}_+$ by

$$v_n(f) := \sup\{f(x) - f(y) : \ x, y \in X_+, \ x_0^{n-1} = y_0^{n-1}\}.$$

It was Ruelle who first established the following fundamental result for *Hölder continuous potentials*, which is now known as Ruelle's Theorem:

Theorem 1.2.2. *Let the potential ϕ be Hölder continuous, i.e., for some $\theta \in (0,1)$, $C > 0$ and for all $n \in \mathbb{N}$, $v_n(\phi) \le C\theta^n$. Then*

(1) there exists a unique equilibrium state μ for ϕ and the transfer operator $\mathscr{L}_\phi$ also has a unique eigenprobability ν corresponding to the spectral radius $\lambda = e^{P(\phi)}$;

(2) the transfer operator has a positive continuous eigenfunction $\mathtt{h} \in C(X_+)$ corresponding to λ;

(3) for every $f \in C(X_+)$, the sequence $\left\{\frac{1}{\lambda^n}\mathscr{L}_\phi^n f\right\}_{n \ge 1}$ converges uniformly to $\nu(f) \cdot \mathtt{h}$ as $n \to \infty$, where $\nu(f) := \int_{X_+} f \, d\nu$.

Walters later proved [41, 42] Ruelle's theorem under weaker conditions, such as the *summable variations*: $\sum_{n\in\mathbb{N}} v_n(\phi) < \infty$, and the so-called *Walters condition*, which requires that $\lim_{p\to\infty} \sup_{n\in\mathbb{N}} v_{n+p}(S_n\phi) = 0$. Walters also partly extended [43] Ruelle's theorem to the class of the so-called *Bowen potentials*, namely those ϕ satisfying $\sup_{n\in\mathbb{N}} v_n(S_n\phi) < \infty$. In the Bowen setting, Walters was only able to establish the existence of a bounded eigenfunction $\mathtt{h}$ of the transfer operator, while the existence of a continuous eigenfunction remains an open problem to this day. Independently, Fan and Jiang [18] proved Ruelle's theorem under the *Dini condition*, a refinement of Hölder continuity. These regularity conditions satisfy the strict chain

$$\text{summable variations} \implies \text{Walters condition} \implies \text{Bowen condition},$$

with no reversals. From the statistical mechanics perspective, all these conditions are considered short-range, in the sense that for every $\beta \geq 0$, the scaled potential $\beta\phi$ admits a unique equilibrium state and a unique half-line Gibbs measure. In contrast, we refer to a potential ϕ as *long-range* if there exists a finite $\beta > 0$ such that $\beta\phi$ admits multiple equilibrium states and multiple half-line Gibbs measures. In one-dimensional statistical mechanics, long-range potentials are often favoured over short-range potentials. This preference stems from the fact that long-range models exhibit non-trivial phase diagrams, including the emergence of finite critical temperatures, which are absent in short-range systems due to the lack of phase transitions at any finite temperature. At criticality, long-range models often display complex, fractal-like structures and rich critical behaviour, making them fundamentally distinct from short-range systems. Beyond their physical richness, long-range models are valued for the mathematical challenges they pose, as they often resist classical techniques and demand new analytical tools, making them both difficult and fascinating to study.

In Chapter 4, we attempt to extend Ruelle's theorem to the long-range potentials. We consider potentials ϕ which are linked to some translation-invariant UAC interaction on $X = E^{\mathbb{Z}}$ via $\phi = -\sum_{0\in V\Subset\mathbb{Z}_+} \Phi_V$. By removing all terms $V \Subset \mathbb{Z}$ with $\min V < 0 \leq \max V$ from Φ and returning them back one by one, we construct intermediate interactions $\{\Psi^{(k)} : k \in \mathbb{Z}_+\}$. Formally, we consider the family

$$\mathscr{A} = \left\{\Lambda \Subset \mathbb{Z} : \min(\Lambda) < 0,\ \max\Lambda \geq 0\right\}.$$

indexed according to some arbitrary order: $\mathscr{A} = \{\Lambda_1, \Lambda_2, \ldots\}$. Then define, for each $k \in \mathbb{Z}_+$,

$$\Psi^{(k)}_\Lambda = \begin{cases} 0, & \Lambda \in \{\Lambda_i : i \geq k+1\}, \\ \Phi_\Lambda, & \text{otherwise}. \end{cases} \tag{1.25}$$

In particular, $\Psi^{(0)}$ has no interaction between the left and right half lines. Clearly, all the constructed interactions are UAC and in addition have the following properties:

- For any finite volume $V \subset \mathbb{Z}$, $\|H_V^{\Psi^{(k)}} - H_V^{\Phi}\|_\infty \le \sum_{\substack{\Lambda_j \cap V \neq \emptyset \\ j \ge k}} \|\Phi_{\Lambda_j}\|_\infty \xrightarrow[k\to\infty]{} 0$;
- the specifications $\gamma^{\Psi^{(k)}}$ converge to γ^{Φ} as $k \to \infty$ in the sense that, for all $B \in \mathscr{F}$ and $V \Subset \mathbb{Z}$,

  $\gamma_V^{\Psi^{(k)}}(B|\omega) \xrightarrow[k\to\infty]{} \gamma_V^{\Phi}(B|\omega)$ uniformly in the boundary conditions $\omega \in X$;
- if $\nu^{(k)}$ is a Gibbs measure for $\Psi^{(k)}$, then any weak*-limit point, μ of the sequence $\{\nu^{(k)}\}_{k\ge 0}$ is a Gibbs measure for the potential Φ.

In this setup, we prove the following theorem:

Theorem 1.2.3. *Assume the following:*

1) *the interaction* Φ *satisfies:* $\sum_{0\in\Lambda\Subset\mathbb{Z}} (|\Lambda|-1)\cdot\|\Phi_\Lambda\|_\infty < 2$, *which is known as the Dobrushin uniqueness condition in literature* [22]*;*

2) $\sum_{k=1}^{\infty}\sum_{i\in\mathbb{Z}} \delta_i(\Phi_{\Lambda_k})^2 < \infty$, *where for* $F : X \to \mathbb{R}$, $\delta_i F := \sup_{\xi_{\mathbb{Z}\setminus\{i\}}=\eta_{\mathbb{Z}\setminus\{i\}}} |F(\xi)-F(\eta)|$*;*

3) $\sum_{k=1}^{\infty} \rho_k < \infty$, *where* $\rho_k := \sup_{n\in\mathbb{Z}_+} \left| \int_X \Phi_{\Lambda_k}\, d\nu^{(n)} \right|$ *and* $\nu^{(n)} \in \mathscr{G}(\Phi^{(n)})$.

Then the interactions Φ *and* $\Psi^{(0)}$ *have unique Gibbs states* μ *and* $\nu^{(0)}$ *and* $\mu \ll \nu^{(0)}$. *In particular, the restriction* μ_+ *of* μ *to* X_+ *is absolutely continuous with respect to the half-line Gibbs measure* ν *for* ϕ. *Furthermore, the transfer operator* $\mathscr{L}_\phi$ *for the potential* ϕ *has an integrable eigenfunction* $\mathrm{h} = \frac{d\mu_+}{d\nu} \in L^1(X_+, \nu)$.

1.2.3 Eigenfunctions of the transfer operator for the Dyson model

The so-called Dyson model, the long-range Ising model, is probably one of the most prominent examples in one-dimensional statistical mechanics. The Dyson model is defined on the shift space $\{\pm 1\}^{\mathbb{Z}}$ by the interaction:

$$\Phi_\Lambda^D(\omega) = \begin{cases} -\beta J_{|i-j|}\omega_i\omega_j, & \text{if } \Lambda = \{i,j\} \subset \mathbb{Z},\ i \neq j; \\ -h\omega_i, & \text{if } \Lambda = \{i\} \subset \mathbb{Z}; \\ 0, & \text{otherwise,} \end{cases} \tag{1.26}$$

where $\beta \geq 0$ represents the inverse temperature, $h \in \mathbb{R}$ is the external field and $J_{|i-j|} = |i-j|^{-\alpha}$ for the ferromagnetic model and $J_{|i-j|} = -|i-j|^{-\alpha}$ for the anti-ferromagnteic model, where $\alpha > 1$. The Dyson potential ϕ associated with the Dyson interaction is defined as

$$\phi^D(\omega) = -\sum_{0 \in V \Subset \mathbb{Z}_+} \Phi_V^D(\omega) = h\omega_0 + \sum_{k=1}^{\infty} \beta J_k \omega_0 \omega_k. \tag{1.27}$$

The following theorem about the phase diagram of the ferromagnetic Dyson model is due to Dyson [14] and Fröhlich and Spencer [21].

Theorem 1.2.4. *(i) Let $h = 0$, then for every $\alpha \in (1,2]$, there exists critical inverse temperature $\beta_c(\alpha) \in (0,\infty)$ such that for every $\beta \in [0, \beta_c(\alpha))$, the interaction Φ^D has a unique Gibbs measure and for all $\beta \geq \beta_c(\alpha)$, Φ^D has multiple Gibbs measures.*

(ii) If $h \neq 0$, then the interaction Φ^D has a unique Gibbs measure for all $\beta \geq 0$ and $\alpha > 1$.

In the phase transition region, there exist exactly two extremal Gibbs measures, often referred to as the + and − phases, denoted by $\mu^+_{\alpha,\beta,0}$ and $\mu^-_{\alpha,\beta,0}$, respectively, for the Dyson model Φ^D. In 2019, 50 years after Dyson's initial work, Johansson, Öberg and Pollicott showed [28] that there exists a similar critical value $\beta_c^+(\alpha)$ for the half-line Dyson model ϕ^D, which separates the phase transitions region from the uniqueness region.

Theorem 1.2.5. *For every $\alpha \in (1,2]$, there exists a critical value $\beta_c^+(\alpha) \in (0,\infty)$ such that for every $\beta < \beta_c^+(\alpha)$, there exists a unique half-line Gibbs measure for the specification $\overrightarrow{\gamma}^{\phi}$ and for all $\beta > \beta_c^+(\alpha)$, there are multiple Gibbs measures, i.e., $\#\mathscr{G}(\overrightarrow{\gamma}^{\phi}) > 1$.*

It has been conjectured in [28] that for all $\alpha > 1$, $\beta_c(\alpha) = \beta_c^+(\alpha)$. For $h \neq 0$, one can compare the half-line Gibbs measures with the whole-line Gibbs measures using Griffiths' inequalities [20]. This comparison implies that, for every $\beta \geq 0$ and $\alpha > 1$, there exists a unique half-line Gibbs measure for $\overrightarrow{\gamma}^{\phi}$.

In 2023, by employing the random-cluster representation of the Ising models, Johansson, Öberg and Pollicott [29] were also able to establish that for $h = 0$, $\alpha \in \left(\frac{3}{2}, 2\right]$ and sufficiently small $\beta \geq 0$, the transfer operator $\mathscr{L}_\phi$ for the ferromagnetic Dyson potential ϕ has a continuous eigenfunction. In Chapter 4, we prove a similar result with a different method, which is based on the construction of the intermediate interactions.

Theorem 1.2.6. *Let ϕ be either ferromagnetic or antiferromagnetic Dyson potential (1.27). Suppose $h = 0$, $\alpha > 1$ and $\beta > 0$ is sufficiently small. Then,*

(i) *ϕ admits a unique equilibrium state $\mu_+ \in \mathscr{ES}(\phi)$ and Gibbs state $\nu \in \mathscr{G}(\overrightarrow{\gamma}^{\phi})$;*

(ii) *for all $\alpha > 1$, μ_+ is equivalent to ν, i.e., $\mu_+ \ll \nu$ and $\nu \ll \mu_+$. In particular, the Perron-Frobenius transfer operator $\mathscr{L}_\phi$ has an eigenfunction in $L^1(X_+, \nu)$;*

(iii) *if $\alpha > \frac{3}{2}$, there exists a continuous version of the Radon-Nikodym density $\frac{d\mu_+}{d\nu}$, ensuring a continuous eigenfunction for $\mathscr{L}_\phi$.*

Our technique offers a significant advantage over the method developed in [29], as it allows us to address the antiferromagnetic Dyson potential and to handle the previously inaccessible regime $\alpha \in \left(1, \frac{3}{2}\right]$ for the ferromagnetic Dyson model in the Dobrushin uniqueness region. It is worth noting that, following the announcement of our results in [16], Johansson, Öberg, and Pollicott succeeded in extending their approach to cover the entire uniqueness region of the ferromagnetic Dyson model, namely the case $h = 0$, $\alpha \in \left(\frac{3}{2}, 2\right]$, and $\beta \in [0, \beta_c(\alpha))$, by employing a concentration inequality established in [45]. Recently, by leveraging a result of Bauerschmidt and Dagallier [2], we have also extended Theorem 1.2.6 – using the approach we developed in the second part of this thesis – to the entire uniqueness region of the ferromagnetic Dyson model (see Theorem 4.G).

Theorem 1.2.7. *Let ϕ be the ferromagnetic Dyson potential. Suppose $h = 0$. Then the following statements hold.*

(i) *For every $\alpha \in (1,2]$, we have $\beta_c(\alpha) \le \beta_c^+(\alpha)$. Hence for every $\alpha \in (1,2]$ and $\beta \in [0, \beta_c(\alpha))$, there exists a unique equilibrium state μ_+ and a (half-line) Gibbs state ν for the potential ϕ.*

(ii) *For each $\alpha \in (1,2]$ and all $\beta \in [0, \beta_c(\alpha))$, the equilibrium state μ_+ is equivalent to the half-line Gibbs state ν. In particular, $\mathscr{L}_\phi$ admits an integrable eigenfunction $\frac{d\mu_+}{d\nu} \in L^1(\nu)$ corresponding to its spectral radius.*

(iii) *If $\alpha \in \left(\frac{3}{2}, 2\right]$, then for all $\beta \in [0, \beta_c(\alpha))$, there exists a continuous version of the Radon-Nikodym derivative $\frac{d\mu_+}{d\nu}$. Hence, $\mathscr{L}_\phi$ has a continuous principal eigenfunction.*

However, none of these studies, [29] and [16], cover the case of the Dyson potential with a nonzero external field $h \neq 0$. The approach in [29] relies heavily on the random cluster representation of the Dyson model. The central obstacle to extending the method in [29] to non-zero external fields is the loss of symmetry, essential for the random cluster representation, which disrupts the cluster decay analysis. The method developed in [16] requires a certain sum of two-point functions to be uniformly bounded, a condition that fails for the Dyson model in a field. Nevertheless, one can adopt the technique in [16] to the case of non-zero external fields, as we shall do in Chapter 5.

Theorem 1.2.8. *Suppose* $\alpha \in \left(\frac{3}{2}, 2\right]$, $\beta \geq 0$ *and* $|h| > 0$ *is sufficiently large* ($|h| > 2\beta\zeta(\alpha) + \log 4\beta\zeta(\alpha)$ *is enough, here* ζ *is the Riemann zeta function*). *Then*

(i) the Dyson potential ϕ *has a unique equilibrium state* μ_+ *and there also exists a unique eigenprobability* ν *of* $\mathscr{L}_\phi^*$*;*

(ii) μ_+ *is absolutely continuous with respect to* ν*, i.e.,* $\mu_+ \ll \nu$*. In particular, the Perron-Frobenius transfer operator* $\mathscr{L}_\phi$ *admits an integrable eigenfunction corresponding to the spectral radius* $\lambda = e^{P(\phi)}$*.*

(iii) The Radon-Nikodym derivative $\frac{d\mu_+}{d\nu}$ *does not have a continuous version. In particular, the Perron-Frobenius transfer operator does not have a continuous principal eigenfunction.*

In [34], we conjectured that for $\alpha \in \left(1, \frac{3}{2}\right]$, $h \neq 0$, and all $\beta \geq 0$, the equilibrium state μ_+ and the half-line Gibbs measure ν are mutually singular.

A comparison between Theorem 1.2.6 and Theorem 1.2.8 reveals a stark contrast between the zero-field and non-zero-field regimes. In particular, the presence of an external field leads to a loss of regularity: the corresponding eigenfunction is "one degree" less regular than that of the zero-field case, for the same values of α and β.

1.3 Multifractals and Large Deviations in Dynamical Systems

In ergodic theory—and in many practical applications—it is often important to understand the size and nature of rare events. An event may be considered rare in the sense of measure theory – it does not occur almost surely, but this does not imply that it never occurs.

As a concrete example, consider the two-sided shift space $\Omega := E^{\mathbb{Z}}$, where the alphabet $E = \{\pm 1\}$ represents Ising spins. For each configuration $x \in \Omega$, define the spin observable $\sigma_n(x) := x_n \in E$. We are interested in the behaviour of the

Birkhoff averages

$$A_n(x) := \frac{1}{2n+1}\sum_{i=-n}^{n}\sigma_n(x) = \frac{1}{2n+1}\sum_{i=-n}^{n}\sigma_0 \circ S^n(x), \;\; x \in \Omega,$$

where $S : \Omega \to \Omega$ is the left shift map, defined by $S(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$. According to Birkhoff's ergodic theorem, for every ergodic probability measure μ on Ω, the sequence $A_n(x)_{n\in\mathbb{N}}$ converges μ-almost surely to the space average $\mu(\sigma_0) = \int_\Omega \sigma_0 d\mu$. This leads us to the study of *multifractal level sets*:

$$K_\alpha := \Big\{x \in \Omega : \lim_{n\to\infty} \frac{1}{2n+1}\sum_{i=-n}^{n}\sigma_n(x) = \alpha\Big\}. \tag{1.28}$$

By Birkhoff's theorem, we know that $\mu(K_{\mu(\sigma_0)}) = 1$, meaning that almost every point (with respect to μ) lies in the level set corresponding to the mean value $\mu(\sigma_0)$. For any other $\alpha \in \mathbb{R} \setminus \mu(\sigma_0)$, the set K_α is μ-null; that is, $\mu(K_\alpha) = 0$.

Now, consider another ergodic measure ν on Ω, such that $\nu(\sigma_0) \neq \mu(\sigma_0)$. Then the set $K_{\nu(\sigma_0)}$ is a rare event with respect to μ, since $\mu(K_{\nu(\sigma_0)}) = 0$, but it is of full measure with respect to ν. In this way, what is negligible for one measure may be typical for another, illustrating the subtleties of rare events in the ergodic setting. Of course, studying the size of such rare events using measures alone may not always be sufficient. In these cases, dimension theory provides a more powerful tool. Specifically, it becomes more practical to investigate the "dimension" – for example, the Hausdorff dimension – of the sets K_α.

Now, let us generalise the problem within the broader framework of dynamical systems. Assume that we are given a compact metric space Ω, which is not necessarily a shift space or a configuration space, along with a continuous transformation $T : \Omega \to \Omega$, which need not be injective. Instead of limiting ourselves to the standard Birkhoff averages $\frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i(x)$ for some observable $f : \Omega \to \mathbb{R}$, we consider a more general setup. Let $X := \{X_n\}_{n\in\mathbb{N}}$ be a sequence of Borel-measurable functions, which need not even be continuous. We now define the multifractal level sets associated with this sequence X:

$$K_\alpha := \{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} X_n(\omega) = \alpha\}, \quad \alpha \in \mathbb{R}. \tag{1.29}$$

It is worth noting that the study of the "dimension" of multifractal sets K_α – or the investigation of the so-called *multifractal spectrum* $\alpha \in \mathbb{R} \mapsto \mathtt{dim}(K_\alpha)$ – has a rich history, dating back to Besicovitch (1935) [6], Eggleston (1949) [15], and continuing to the present day. Many researchers have contributed to this field, addressing the problem at varying levels of generality. A more comprehensive list of relevant works can be found in Chapter 6. However, it is important to note that almost all previous studies, whether implicitly or explicitly, rely on ideas from large deviations theory. The objective of Chapter 6 is to explore the necessary and sufficient conditions under which one can derive the properties of the multifractal spectrum directly from the large deviation characteristics of the sequence X. Consequently, Chapter 6 aims to address the general multifractal setup as broadly as possible.

As mentioned above, our goal is to study the "dimension" of the sets K_α, for $\alpha \in \mathbb{R}$. There are several reasonable notions of "dimension" that can be used in this context, including Hausdorff dimension, box-counting dimension, packing dimension, and topological entropy for sets. Before proceeding, it is essential to specify which notion of dimension we will adopt. Among the various candidates, we choose to work with *topological entropy for non-compact sets*, a concept introduced by Bowen [8]. This choice is motivated by the fact that topological entropy is the most inherently dynamical of the aforementioned notions—it depends not only on the underlying space Ω but also on the dynamics defined by the transformation T. Furthermore, depending on the nature of the dynamical system, topological entropy can coincide with other notions of dimension. For instance, when Ω is a shift space and T is the shift map, the topological entropy of a set coincides (up to a multiplicative constant) with its Hausdorff dimension. We now proceed to define topological entropy for sets. For a set $Z \subset \Omega$, and for any $t \in \mathbb{R}$, $\epsilon > 0$, and $N \in \mathbb{N}$, define

$$m(Z,t,\epsilon,N) := \inf\Big\{\sum_{i=1}^{\infty} e^{-n_i t} : Z \subset \bigcup_{i=1}^{\infty} B_{n_i}(x_i,\epsilon),; n_i \geq N\Big\}, \tag{1.30}$$

where, for $x \in \Omega$ and $n \in \mathbb{N}$, the n-th dynamical ball is given by $B_n(x,\epsilon) := \{y \in \Omega : d(T^i y, T^i x) < \epsilon,\ 0 \leq i \leq n-1\}$. By convention, we define $m(\emptyset, t, \epsilon, N) = 0$ for all values of t, ϵ, and N. Clearly, the function $m(Z,t,\epsilon,N)$ is monotonic in N, and thus we can define

$$m(Z,t,\epsilon) := \lim_{N\to\infty} m(Z,t,\epsilon,N).$$

It can be shown (see [35]) that $m(\cdot,t,\epsilon)$ defines an *outer measure*, with properties analogous to those of the t-dimensional Hausdorff outer measure. In particular, there exists a critical value $t' \in \mathbb{R}$ such that

$$m(Z,t',\epsilon) = \begin{cases} +\infty, & \text{if } t' < t, \\ 0, & \text{if } t' > t. \end{cases}$$

We denote this critical value by $h_{\mathrm{top}}(T,Z,\epsilon)$. Thus, $h_{\mathrm{top}}(T,Z,\epsilon) = \inf\{t \in \mathbb{R} : m(Z,t,\epsilon) = 0\} = \sup\{t \in \mathbb{R} : m(Z,t,\epsilon) = +\infty\}$. Since $h_{\mathrm{top}}(T,Z,\epsilon)$ is monotonic in ϵ, we define the topological entropy of the set Z by

$$h_{\mathrm{top}}(T,Z) := \lim_{\epsilon\to 0+} h_{\mathrm{top}}(T,Z,\epsilon).$$

It is important to emphasise that the set $Z \subset \Omega$ is neither assumed to be compact nor T-invariant. Throughout our work, we also assume that the topological entropy of the full space Ω is finite, i.e., $h_{\mathsf{top}}(T,\Omega) < \infty$. The topological entropy has the following basic properties:

- *Monotonicity:* if $Z_1 \subset Z_2$, then $h_{\mathsf{top}}(T,Z_1) \leq h_{\mathsf{top}}(T,Z_2)$;
- *Countable stability:* if $Z = \bigcup_n Z_n$, then $h_{\mathsf{top}}(T,Z) = \sup_n h_{\mathsf{top}}(T,Z_n)$.

Another key concept in Chapter 6 is the *large deviation principle* (LDP). An extended function $I : \mathbb{R} \to [0,+\infty]$ is called a **large deviations rate function** (or simply a rate function) if it is lower semicontinuous, meaning that for every $\lambda \in \mathbb{R}$, the set $\{t \in \mathbb{R} : I(t) \leq \lambda\}$ is closed. A rate function I is said to be **good** if all of its sub-level sets $\{t : I(t) \leq \lambda\} \subset \mathbb{R}$ are compact for every $\lambda \in \mathbb{R}$. This additional compactness condition ensures that the function has "well-behaved" minimising properties over closed subsets of $\mathbb{R}$. For a set $E \subset \mathbb{R}$, we adopt the shorthand notation $I(E) := \inf_{t\in E} I(t)$.

Definition 1.3.1. *Let ν be a Borel probability measure on Ω and $X_n : \Omega \to \mathbb{R}$, $n \geq 1$ be random variables. The sequence $\left\{\frac{1}{n}X_n\right\}_{n\in\mathbb{N}}$ satisfies the* ***Large Deviation Principle*** *(LDP) with a rate function I if*

(1) for all closed set $F \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu\Big(\Big\{ \frac{1}{n} X_n \in F \Big\} \Big) \leq -I(F), \tag{1.31}$$

(2) for all open set $G \subset \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\Big(\Big\{ \frac{1}{n} X_n \in G \Big\} \Big) \geq -I(G). \tag{1.32}$$

We say that the sequence $\left\{\frac{1}{n}X_n\right\}_{n\in\mathbb{N}}$ satisfies the weak Large Deviation Principle (weak LDP) with a rate function I if, in place of the upper bound condition (1.31), the following weaker version holds:

$(1')$ *for all <u>compact</u> $F \subset \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu\Big(\Big\{ \frac{1}{n} X_n \in F \Big\} \Big) \leq -I(F). \tag{1.33}$$

We assume that the underlying measure ν is a Bowen-Gibbs measure with respect to a constant potential, i.e., for some $h > 0$, and for every $\epsilon > 0$ there exists $D(\epsilon) > 0$ such that for all $x \in \Omega$ and $n \in \mathbb{N}$,

$$\frac{1}{D(\varepsilon)} e^{-nh} \leq \nu\big(B_n(x,\epsilon)\big) \leq D(\varepsilon) e^{-nh}. \tag{1.34}$$

Theorem 1.3.2. *Assume the sequence $\{X_n\}_{n\in\mathbb{N}}$ satisfies the weak Bowen condition, that is, for some $\delta>0$,*

$$\lim_{n\to\infty}\frac{v_{n,\delta}(X_n)}{n}=0, \tag{A1}$$

where for $H:\Omega\to\mathbb{R}$, $v_{n,\delta}(H)$ is (n,δ)-variation of the function H, i.e., $v_{n,\delta}(H):=$ $\sup\{H(y)-H(z): d(T^i y, T^i z)\le\delta,\ i=\overline{0,n-1}\}$. If the sequence $\left\{\frac{1}{n}X_n\right\}$ satisfies the weak LDP upper bound with a rate function $I_X:\mathbb{R}\to[0,+\infty]$, then for all $\alpha\in\mathbb{R}$, one has

$$h_{top}(T,K_\alpha)\le h_{top}(T,\Omega)-I_X(\alpha). \tag{1.35}$$

To obtain the reverse inequality in (1.35), i.e., the corresponding lower bound, additional structural assumptions are required.

(A2) The sequence $\{X_n\}_n$ is *weakly almost additive*, i.e., there are non-negative constants $A_n=o(n)$ such that for all $x\in\Omega$ and $n,m\in\mathbb{N}$,

$$|X_{n+m}(x)-X_n(x)-X_m(T^n x)|\le A_n; \tag{1.36}$$

(A3) $T:\Omega\to\Omega$ is an *expansive*, i.e., there exists $\rho>0$ such that if $d(T^n(x),T^n(y))<\rho$ for all non-negative integer n, then $x=y$, and *strongly topologically exact transformation*, i.e., for any $\epsilon>0$ there is a natural number $M_1\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ and $x\in\Omega$, $T^{n+M_1}(B_n(x,\epsilon))=\Omega$.

Theorem 1.3.3. *Assume the conditions (A1), (A2) and (A3). If the sequence $\left\{\frac{1}{n}X_n\right\}_{n\in\mathbb{N}}$ satisfies LDP with an essentially strictly convex good rate function I_X, then one has the following:*

(i) there exists extended real numbers $-\infty\le\underline{\alpha}\le\overline{\alpha}\le+\infty$ such that $K_\alpha\ne\emptyset$ for every $\alpha\in(\underline{\alpha},\overline{\alpha})$ and $K_\alpha=\emptyset$ for all $\alpha\notin[\underline{\alpha},\overline{\alpha}]$.

(ii) $h_{top}(T,K_\alpha)=h_{top}(T,\Omega)-I_X(\alpha)$ holds for all $\alpha\in(\underline{\alpha},\overline{\alpha})$.

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Part I

Gibbs Measures in Dynamical Systems

Chapter 2

On the Gibbs formalism in Dynamical Systems

Abstract: In this chapter, we review various notions of Gibbs measures, including DLR measures, G-measures, g-measures and compare them with each other. The chapter also aims to discuss the Kozlov-Sullivan characterisation of specifications in Equilibrium Statistical Mechanics within the framework of Dynamical Systems. In addition, on the whole-line shift space, we also demonstrate an interaction-independent variational principle.

Furthermore, we provide a necessary and sufficient criterion for a specification on a half-line shift space to ensure the existence of a generating translation-invariant UAC interaction, highlighting the lack of an analogous condition for whole-line shift spaces.

2.1 Introduction

The so-called Gibbs (DLR) formalism – the study of probability measures defined via their conditional probabilities, a perspective formalised in the notion of a Gibbs measure – was first initiated by Dobrushin, Lanford, and Ruelle to mathematically describe various phenomena in statistical mechanics, including phase transitions. The DLR formalism is developed based on the notion of *specifications*. Although it was formally established in the 1970s, the foundations of the theory of Gibbs measures trace back to the earlier works of Boltzmann and Gibbs. The classical Gibbs formalism starts with the well-known *Boltzmann factor*, which is a probability measure interpreted as the probability of a particular portion of a macroscopic system being in a certain energy state. This measure is connected to the interactions that define the model quantitatively. This is why the classical Gibbs formalism is developed based on the *interactions*. Subsequently, Kozlov [21] and Sullivan [28] established that the Gibbs formalism based on specifications is essentially equivalent to the one based on interactions, as every Gibbsian specification is generated by some UAC interaction. However, if one restricts oneself to the setup of translation-invariant measures, then one can see a subtle difference between those Gibbs formalisms. In fact, as shown in [2], not every translation-invariant Gibbs measure is prescribed by a translation-invariant uniformly absolutely continuous (UAC) interaction, even though it is always prescribed by a translation-invariant specification. This naturally calls for a reconstruction of classical statistical mechanics framed in terms of specifications. Moreover, the renormalisation group transformation of a Gibbs measure often results in a loss of the Gibbs property, meaning the transformed measure may lack a continuous version of its conditional probabilities. However, in more favourable cases, the transformed measure may retain an ‘almost continuous’ version of conditional probabilities, which frequently cannot be associated with any reasonable interaction. This challenge further emphasises the need to study Gibbs measures based on specifications alone. In this regard, considerable work has already been done, particularly in the context of generalised Gibbs measures, as explored in [10, 22, 24]. In particular, [22] establishes a version of the interaction-independent variational principle. Expanding on this, we present a new form of the interaction-independent variational principle, which is more suited to the framework of dynamical systems. We also discuss the Kozlov-Sullivan characterisation of the *translation-invariant* Gibbs measures. Furthermore, we also give an explicit example of a translation-invariant specification on the half-line $\mathbb{Z}_+$ which is not generated by any translation-invariant UAC interaction.

Later developments in both Statistical Mechanics and Dynamical Systems led to the definition of Gibbs measures independent of specifications or interactions, based instead on the concept of *families of multipliers* or *G-families*. The first of these alternative approaches was introduced to extend the notion of Gibbs measures to more general (semi)group actions beyond $\mathbb{Z}^d$ or $\mathbb{N}$. The idea of a G-family was inspired by Keane's work [20] and introduced into ergodic theory by Brown and Dooley [4], where it was used to study generalised Riesz products in harmonic analysis. In this chapter, we compare these alternative definitions of Gibbs measures with the classical definition based on specifications.

The chapter is structured as follows:

- In Section 2.2, we recall the different definitions of the Gibbs measures that appeared in Statistical Mechanics and Dynamical Systems literature and compare these definitions with the one based on the specifications.
- In Section 2.3, the Kozlov-Sullivan characterisation of the Gibbsian specifications on the full lattice $\mathbb{Z}^d$ will be discussed.
- In Section 2.4, we state the first set of our main results. There, we formulate an interaction-independent version of the variational principle, aligned with the spirit of dynamical systems.
- Section 2.5 focuses on the Gibbs formalism on the half-line $\mathbb{Z}_+$. In this section, we also compare the concept of G-measures, introduced by Brown and Dooley in the 1990s, with the DLR-Gibbs measures. Additionally, we provide an example of a translation-invariant Gibbsian specification on $\mathbb{Z}_+$ that is not generated by a translation-invariant UAC interaction.
- Section 2.6 is dedicated to the proofs of the main results of this chapter.

2.2 Preliminaries

2.2.1 The Gibbs measures in DLR formalism

The DLR formalism starts with the so-called *specification* – a given family of conditional probabilities. Since there is no a priori underlying probability measure, one should define the prescribing family of the conditional probabilities everywhere, not almost everywhere.

In this chapter, we consider the shift spaces in the form $\Omega := E^{\mathbb{L}}$, where E is a finite set equipped with the discrete topology and interpreted as the state-space of the system, and $\mathbb{L}$ is interpreted as the set of sites, which is at most a countable set. We denote the Borel $\sigma-$algebra in Ω by $\mathscr{F}$, and for $\Lambda \subset \mathbb{L}$, $\mathscr{F}_\Lambda$ denotes the minimal $\sigma-$algebra containing all the cylindrical $[\sigma_V]$ sets based on volume $V \Subset \Lambda$. $\mathscr{M}_1(\Omega)$ denotes the set of probability measures on Ω. A function $f : \Omega \to \mathbb{R}$ is called *local* if there exists a finite volume $\Lambda \Subset \mathbb{L}$ such that f is $\mathscr{F}_\Lambda-$measurable. Then a ***specification*** γ is a family of positive functions $\{\gamma_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ – the so-called *probability kernels* on Ω indexed by the finite volumes $\Lambda \Subset \mathbb{L}$ which has the following properties:

(P1) for each $\Lambda \Subset \mathbb{L}$ and every $B \in \mathscr{F}$, the function $(B|\omega) \in \mathscr{F} \times \Omega \xmapsto{\gamma_\Lambda} [0,1]$ is measurable in $\mathscr{F}_{\Lambda^c}$ as a function of ω;

(P2) for each $\Lambda \Subset \mathbb{L}$ and all $\omega \in \Omega$, the function $\gamma_\Lambda(\cdot|\omega) : \mathscr{F} \to [0,1]$ is a probability measure on $\mathscr{F}$;

(P3) (properness) for each $\Lambda \Subset \mathbb{L}$ and every $B \in \mathscr{F}_{\Lambda^c}$ and $\omega \in \Omega$, $\gamma_\Lambda(B|\omega) = \mathbb{1}_B(\omega)$;

(P4) (consistency) for all $\Delta \subset \Lambda \Subset \mathbb{L}$,

$$\gamma_\Lambda \gamma_\Delta = \gamma_\Lambda, \tag{2.1}$$

where $\gamma_\Lambda\gamma_\Delta(B|\omega) := \int_\Omega \gamma_\Delta(B|\eta)\gamma_\Lambda(d\eta|\omega),\ B \in \mathscr{F},\ \omega \in \Omega.$

Definition 2.2.1. *A specification γ on Ω is called **quasilocal** if for any local function f and volume $\Lambda \Subset \mathbb{L}$, one has that $\gamma_\Lambda(f) \in C(\Omega)$, where for $\omega \in \Omega$, $\gamma_\Lambda(f)(\omega) := \int_\Omega f(\xi)\gamma_\Lambda(d\xi|\omega)$.*

Describing the specifications with their *specification densities* is often convenient. The ***density*** of a specification γ in the volume $\Lambda \Subset \mathbb{L}$ is $\gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) := \gamma_\Lambda([\omega]_\Lambda|\omega)$, here $[\omega]_\Lambda$ denotes the cylinderic set based on the volume Λ and the configuration ω, i.e., $[\omega]_\Lambda := \{\xi \in \Omega : \xi_\Lambda = \omega_\Lambda\}$.

Definition 2.2.2. *A specification γ is called **non-null** (or **positive**) if for all $\Lambda \Subset \mathbb{L}$,*

$$\inf_{\omega\in\Omega} \gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) > 0.$$

We call a specification ***Gibbsian*** if it is quasilocal and non-null at the same time. The consistency property (2.1) of specifications yields the following for specification densities:

$$\gamma_\Lambda(\eta_\Lambda|\omega_{\Lambda^c}) = \gamma_\Delta(\eta_\Delta|\eta_{\Lambda/\Delta}\omega_{\Lambda^c}) \sum_{\sigma_\Delta \in E^\Delta} \gamma_\Lambda(\sigma_\Delta\eta_{\Lambda/\Delta}|\omega_{\Lambda^c}),\ \forall \Delta \subset \Lambda \Subset \mathbb{L},\ \forall \omega,\eta \in \Omega. \tag{2.2}$$

The characterising property of the non-null specification densities is the so-called *bar moving property*, which we use frequently in the chapter; therefore, we state it below:

$$\frac{\gamma_\Delta(\beta_\Delta|\omega_{\Delta^c})}{\gamma_\Delta(\alpha_\Delta|\omega_{\Delta^c})} = \frac{\gamma_\Lambda(\beta_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})}{\gamma_\Lambda(\alpha_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})},\quad \Delta \subset \Lambda \Subset \mathbb{L},\quad \alpha,\beta,\omega \in \Omega. \tag{2.3}$$

Now we turn to the *consistent probability measures* with the specifications.

Definition 2.2.3. *A probability measure $\mu \in \mathscr{M}_1(\Omega)$ is **consistent** with a specification γ if for all $\Lambda \Subset \mathbb{L}$, the conditional probability $\mu(\cdot|\mathscr{F}_{\Lambda^c})$ is prescribed by the probability kernel γ_Λ, i.e., for all $B \in \mathscr{F}$, and μ−almost every $\omega \in \Omega$,*

$$\mu(B|\mathscr{F}_{\Lambda^c})(\omega) = \gamma_\Lambda(B|\omega).$$

This is equivalent to stating that the measure μ satisfies the DLR equations, i.e., for all $\Lambda \Subset \mathbb{L}$ and $B \in \mathscr{F}$,

$$\mu(B) = (\mu\gamma_\Lambda)(B) := \int_\Omega \gamma_\Lambda(B|\omega)\mu(d\omega). \tag{2.4}$$

The set of consistent measures with a specification γ is denoted by $\mathscr{G}(\Omega,\gamma)$.

It should be stressed that any probability measure $\mu \in \mathscr{M}_1(\Omega)$ is prescribed by some specification γ [17]. Conversely, in the general case, one can't guarantee the existence of a consistent probability measure with a given specification γ [16], i.e., it can be a case that $\mathscr{G}(\Omega,\gamma) = \emptyset$. Therefore, specifications without extra properties are not very interesting. In practice, the quasilocal specifications hold more significance. One of the important properties of the quasilocal specifications is that there always exists a consistent probability measure as long as the state-space E is compact, as is the case in this chapter. In fact, for any sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathscr{M}_1(\Omega)$ of probability measures, any weak* limiting point of the set $\{\tau_n\gamma_\Lambda : n \in \mathbb{N},\ \Lambda \Subset \mathbb{L}\}$ is consistent with γ. In particular, $\mathscr{G}(\Omega,\gamma)$ is a convex closed, thus compact subset of $\mathscr{M}_1(\Omega)$ in the weak* topology. Another crucial property of the quasilocal specifications is that one probability measure can not be consistent with two distinct quasilocal specifications. Thus it would be more interesting if a measure is consistent with a quasilocal or a Gibbsian specification.

Definition 2.2.4. *A probability measure $\mu \in \mathscr{M}_1(\Omega)$ is called a **Gibbs measure** (state) if μ is prescribed by a Gibbsian specification, i.e., there exists a Gibbsian specification γ such that $\mu \in \mathscr{G}(\Omega,\gamma)$.*

Although there always exists a consistent measure with a quasilocal specification, in particular, a Gibbsian specification, it is often the case that there are multiple of such measures. In this case, one says that the underlying specification exhibits *phase transitions.*

2.2.2 The Classical Gibbs formalism

The classical Gibbs formalism starts with an ***interaction*** Φ - a family of *local functions* Φ_Λ indexed by $\Lambda \Subset \mathbb{L}$ on a configuration space $\Omega = E^{\mathbb{L}}$ such that for each Λ, Φ_Λ is $\mathscr{F}_\Lambda$−measurable. If the given interaction Φ is sufficiently regular, then one can associate a specification with it. More precisely, the interaction ***Hamiltonian*** H_Λ^Φ, a.k.a. *the interaction energy in a volume* $\Lambda \Subset \mathbb{L}$, is

$$H_\Lambda^\Phi(\omega) := \sum_{\substack{V \Subset \mathbb{L},\\ V\cap\Lambda\neq\emptyset}} \Phi_V(\omega), \quad \omega \in \Omega. \tag{2.5}$$

Without any condition on the interaction Φ, the sum in (2.5) does not need to converge. In literature, depending on the mode of the convergence of the sum in (2.5), different classes of interactions arise.

(1) An interaction Φ is called ***uniformly convergent*** if the sum (2.5) converges uniformly on $\omega \in \Omega$, i.e., for any cofinal sequence $\{V_n\}_{n\in\mathbb{N}}$ of the finite volumes, the finite sum

$$\sum_{\substack{V\subset V_n\\ V\cap\Lambda\neq\emptyset}} \Phi_V(\omega)$$

converges to $H_\Lambda^\Phi(\omega)$ uniformly in $\omega \in \Omega$ as $n \to \infty$;

(2) An interaction Φ is ***uniformly absolutely convergent*** *(UAC)* if for all $i \in \mathbb{L}$,

$$\sum_{i\in V\Subset\mathbb{L}} \sup_{\omega\in\Omega} |\Phi_V(\omega)| < \infty.$$

(3) Fix a configuration $\theta \in \Omega$. An interaction Φ is called ***relatively absolutely convergent*** if for all $j \in \mathbb{L}$, the sum

$$\sum_{j\in V\Subset\mathbb{L}} |\Phi_V(\omega) - \Phi_V(\omega_j\theta_{\{j\}^c})|$$

converges uniformly in $\omega \in \Omega$. It should be mentioned that the concept of relatively absolute convergence of an interaction is independent of the choice of the configuration θ.

(4) An interaction Φ on Ω is called ***variation-summable*** if for all $i \in \mathbb{L}$,

$$\sum_{i\in V\Subset\mathbb{L}} \delta_i\Phi_V < \infty,$$

where $\delta_i f$ denotes the *oscillation/variance* of the function $f : \Omega \to \mathbb{R}$ at the coordinate i: $\delta_i f := \sup\{f(\xi) - f(\eta) : \xi_{\mathbb{L}\setminus\{i\}} = \eta_{\mathbb{L}\setminus\{i\}}\}$.

Among the above convergence classes of interactions, the class of UAC interactions is the smallest, and the class of relatively convergent potentials is the largest one. Note that the first two summability modes guarantee, for all $\Lambda \Subset \mathbb{L}$, the existence of a continuous Hamiltonian H_Λ^Φ defined by (2.5). Then for all $\Lambda \Subset \mathbb{L}$, one can define the *Boltzmann factor* γ_Λ^Φ for Φ by

$$\gamma_\Lambda^\Phi(\eta_\Lambda|\omega_{\Lambda^c}) := \frac{e^{-H_\Lambda^\Phi(\eta_\Lambda\omega_{\Lambda^c})}}{\sum\limits_{\bar\eta_\Lambda\in E^\Lambda} e^{-H_\Lambda^\Phi(\bar\eta_\Lambda\omega_{\Lambda^c})}}, \quad \eta_\Lambda\in E^\Lambda,\ \omega\in\Omega. \tag{2.6}$$

One can readily check that the family $\{\gamma_\Lambda^\Phi\}_{\Lambda\Subset\mathbb{L}}$ comprises *specification-densities.* The associated specification γ^Φ can be recovered from the Boltzmann factor γ_Λ^Φ by

$$\gamma_\Lambda^\Phi(B|\omega) := \sum_{\eta_\Lambda\in E^\Lambda} \gamma_\Lambda^\Phi(\eta_\Lambda|\omega_{\Lambda^c})\delta_{\eta_\Lambda\omega_{\Lambda^c}}(B),\ \ B\in\mathscr{F}, \omega\in\Omega.$$

It is clear that the specification γ^Φ is non-null and quasilocal, if the interaction Φ is uniformly convergent.
One easily notices that the relative absolute convergence or variation-summability of an interaction Φ is not sufficient to guarantee the convergence of the Hamiltonian (2.5), therefore, in these classes of interactions, one may not have a defined Hamiltonian. Despite this, there is another way of associating a specification with an interaction. Note that if one writes (2.6) in the following form,

$$\begin{aligned}\gamma_\Lambda^\Phi(\xi_\Lambda|\eta_{\Lambda^c}) &= \frac{1}{\sum\limits_{\bar\xi_\Lambda\in E^\Lambda} \exp\Big(H_\Lambda^\Phi(\xi_\Lambda\eta_{\Lambda^c}) - H_\Lambda^\Phi(\bar\xi_\Lambda\eta_{\Lambda^c})\Big)} \\ &= \frac{1}{\sum\limits_{\bar\xi_\Lambda\in E^\Lambda} \exp\Big(\sum\limits_{V\cap\Lambda\neq\emptyset}\Big[\Phi_V(\xi_\Lambda\eta_{\Lambda^c}) - \Phi_V(\bar\xi_\Lambda\eta_{\Lambda^c})\Big]\Big)}\end{aligned} \tag{2.7}$$

then it becomes apparent that one does not really need the convergence of the sum $H_\Lambda = \sum\limits_{\substack{V\Subset\mathbb{L}\\ V\cap\Lambda\neq\emptyset}} \Phi_V$ to associate a specification γ^Φ with an interaction Φ. In fact, if the interaction Φ is relatively absolutely convergent, then the equation (2.7) makes sense; therefore, γ^Φ is well-defined.

Once the relationship between specifications and interactions is established by either (2.5) or (2.7), the following question arises: how generic is the class of specifications generated by interactions? In fact, this question has been studied by Kozlov, Sullivan and Grimmett and others [1, 2, 18, 21, 28]. Here we only mention a theorem by Kozlov which answers the question, and will discuss the question and Kozlov's method in greater detail in Section 2.3.

Theorem 2.2.5 (Kozlov's theorem)**.** [2, 12, 16, 21] *Every Gibbsian specification is generated by a UAC interaction. In other words, for any Gibbsian specification γ, there exists a UAC interaction Φ on Ω such that $\gamma = \gamma^\Phi$.*

Notably, the Kozlov theorem applies to any countable set $\mathbb{L}$, and this is the elegance of the theorem.

2.2.3 The parameterization of specifications with cocycles

We start this subsection by introducing the *asymptotic equivalence relation*—also known as the *Gibbs relation* or *homoclinic relation*—denoted by $\mathfrak{T}(\Omega) \subset \Omega \times \Omega$. We say that two-configurations $\omega, \omega' \in \Omega = E^{\mathbb{L}}$ are asymptotically equivalent/homoclinic, i.e., $(\omega, \omega') \in \mathfrak{T}(\Omega)$, if there exists $\Lambda \Subset \mathbb{L}$ such that $\omega_{\Lambda^c} = \omega'_{\Lambda^c}$. For each fixed $\Delta \Subset \mathbb{L}$ one can introduce a subequivalence relation $\mathfrak{T}_\Delta(\Omega)$ by $(\omega, \omega') \in \mathfrak{T}_\Delta(\Omega) \Leftrightarrow \omega_{\Delta^c} = \omega'_{\Delta^c}$. Then it is clear that $\mathfrak{T}(\Omega) = \bigcup_{\Delta \Subset \mathbb{L}} \mathfrak{T}_\Delta(\Omega)$. Note that each $\mathfrak{T}_\Delta(\Omega) \Subset \Omega \times \Omega$ is closed, thus a compact subset of the product space $\Omega \times \Omega$. One can introduce a relevant topology on $\mathfrak{T}(\Omega)$ as follows: $\mathscr{O} \subset \mathfrak{T}(\Omega)$ is open if and only if $\mathscr{O} = \bigcup_{\Delta \Subset \mathbb{L}} \mathscr{O}_\Delta$, where $\mathscr{O}_\Delta$ is an open subset of $\mathfrak{T}_\Delta(\Omega) \subset \Omega \times \Omega$ in the induced product topology on $\Omega \times \Omega$. Note that this topology in $\mathfrak{T}(\Omega)$ is strictly finer than the induced topology on $\mathfrak{T}(\Omega)$ by $\Omega \times \Omega$. Nevertheless, these two topologies in $\mathfrak{T}(\Omega)$ induce the same Borel sigma-algebra. It should also be stressed that a function $f : \mathfrak{T}(\Omega) \to \mathbb{R}$ is continuous iff for all $\Delta \Subset \mathbb{L}$, the restriction $f|_{\mathfrak{T}_\Delta(\Omega)}$ is continuous in the induced topology by $\Omega \times \Omega$.

Now we turn to cocycles on the asymptotic equivalence relation $\mathfrak{T}(\Omega)$. A map $\rho : \mathfrak{T}(\Omega) \to \mathbb{R}$ is called a *cocycle on* Ω (or $\mathfrak{T}(\Omega)$*–cocycle*) if for all $(\omega, \eta), (\eta, \xi) \in \mathfrak{T}(\Omega)$,

$$\rho(\omega, \eta) + \rho(\eta, \xi) = \rho(\omega, \xi). \tag{2.8}$$

If ρ is a cocycle, it can be readily checked that for all $(\omega, \xi) \in \mathfrak{T}(\Omega)$, $\rho(\omega, \omega) = 0$ and $\rho(\omega, \xi) = -\rho(\xi, \omega)$. Another important observation is that there is a one-to-one correspondence between the non-null specifications on Ω and measurable cocycles on Ω. In fact, let ρ be a measurable cocycle on $\mathfrak{T}(\Omega)$, then one can associate a specification (density) γ with ρ by

$$\gamma^\rho_\Lambda(\eta_\Lambda | \eta_{\Lambda^c}) := \frac{1}{\sum_{\omega_\Lambda \in E^\Lambda} e^{-\rho(\omega_\Lambda \eta_{\Lambda^c}, \eta)}}, \quad \Lambda \Subset \mathbb{L},\ \eta \in \Omega. \tag{2.9}$$

One can easily verify that (2.9) is indeed a specification density. In fact, by the cocycle condition (2.8), one has

$$\begin{aligned} \sum_{\xi_\Lambda \in E^\Lambda} \gamma^\rho_\Lambda(\xi_\Lambda | \eta_{\Lambda^c}) &= \sum_{\xi_\Lambda \in E^\Lambda} \frac{e^{-\rho(\xi_\Lambda \eta_{\Lambda^c}, \eta)}}{\sum_{\omega_\Lambda \in E^\Lambda} e^{-\rho(\omega_\Lambda \eta_{\Lambda^c}, \xi_\Lambda \eta_{\Lambda^c}) - \rho(\xi_\Lambda \eta_{\Lambda^c}, \eta)}} \\ &= \frac{\sum_{\xi_\Lambda \in E^\Lambda} e^{-\rho(\xi_\Lambda \eta_{\Lambda^c}, \eta)}}{\sum_{\omega_\Lambda \in E^\Lambda} e^{-\rho(\omega_\Lambda \eta_{\Lambda^c}, \eta)}} \\ &= 1 \end{aligned} \tag{2.10}$$

By a direct calculation, it can be confirmed that the cocycle condition implies the consistency condition (2.2) as well. In fact, for all $\Delta \subset \Lambda \Subset \mathbb{L}$ and $\eta \in \Omega$,

$$\begin{aligned}
\gamma^{\rho}_{\Delta}(\eta_\Delta|\eta_{\Delta^c})\sum_{\sigma_\Delta}\gamma^{\rho}_{\Lambda}(\sigma_\Delta\eta_{\Lambda\setminus\Delta}|\eta_{\Lambda^c}) &= \sum_{\sigma_\Delta}\Big(\sum_{\xi_\Delta,\omega_\Lambda} e^{-\rho(\xi_\Delta\eta_{\Delta^c},\eta)-\rho(\omega_\Lambda\eta_{\Lambda^c},\sigma_\Delta\eta_{\Delta^c})}\Big)^{-1}\\
&= \sum_{\sigma_\Delta}\Big(\sum_{\xi_\Delta,\omega_\Lambda} e^{-\rho(\omega_\Lambda\eta_{\Lambda^c},\eta)-\rho(\xi_\Delta\eta_{\Delta^c},\sigma_\Delta\eta_{\Delta^c})}\Big)^{-1}\\
&= \sum_{\sigma_\Delta}\Big(\sum_{\omega_\Lambda} e^{-\rho(\omega_\Lambda\eta_{\Lambda^c},\eta)}\Big)^{-1}\Big(\sum_{\xi_\Delta} e^{-\rho(\xi_\Delta\eta_{\Delta^c},\sigma_\Delta\eta_{\Delta^c})}\Big)^{-1}\\
&= \Big(\sum_{\omega_\Lambda} e^{-\rho(\omega_\Lambda\eta_{\Lambda^c},\eta)}\Big)^{-1}\sum_{\sigma_\Delta}\Big(\sum_{\xi_\Delta} e^{-\rho(\xi_\Delta\eta_{\Delta^c},\sigma_\Delta\eta_{\Delta^c})}\Big)^{-1}\\
&= \gamma^{\rho}_{\Lambda}(\eta_\Lambda|\eta_{\Lambda^c})\sum_{\sigma_\Delta}\gamma^{\rho}_{\Delta}(\sigma_\Delta|\eta_{\Delta^c})\\
&\overset{(2.10)}{=} \gamma^{\rho}_{\Lambda}(\eta_\Lambda|\eta_{\Lambda^c}).
\end{aligned} \tag{2.11}$$

The other way around, let γ be a non-null specification on the configuration space Ω, then one can associate a cocycle ρ^γ with γ by the following formula: for $(\omega,\xi)\in\mathfrak{T}_\Lambda(\Omega)$,

$$\rho^\gamma(\omega,\xi) := \log\gamma_\Lambda(\xi_\Lambda|\xi_{\Lambda^c}) - \log\gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}). \tag{2.12}$$

Note that ρ^γ is well-defined due to the bar moving property (2.3), and it is also clear that ρ^γ is measurable and satisfies (2.8).

Remark 2.2.6. *It is easy to see that (2.9) associates a Gibbsian (quasilocal) specification to a continuous cocycle, and (2.12) associates a continuous cocycle to a Gibbsian specification. Hence, there is a one-to-one correspondence between Gibbsian specifications and continuous cocycles on* Ω.

Now let the specification γ be generated by an interaction Φ, i.e., $\gamma := \gamma^\Phi$. Then the corresponding cocycle ρ^Φ has the following form:

$$\rho^\Phi(\omega,\omega') = \sum_{V\Subset\mathbb{L}}\big(\Phi_V(\omega)-\Phi_V(\omega')\big),\ \ (\omega,\omega')\in\mathfrak{T}(\Omega). \tag{2.13}$$

It is clear that if the sum in (2.5) converges for all $\omega\in\Omega$, then ρ^Φ is well-defined and for $(\omega,\omega')\in\mathfrak{T}_\Lambda(\Omega)$, one has

$$\rho^\Phi(\omega,\omega') = H^\Phi_\Lambda(\omega) - H^\Phi_\Lambda(\omega'),$$

and hence one can restore (2.7) for the specification γ^Φ. However, another important observation is that the convergence of the sum in (2.13) for all $(\omega,\omega')\in\mathfrak{T}(\Omega)$ does not imply the convergence of the sum in (2.5).

2.2.4 Capocaccia's definition of Gibbs measures

In 1976, Capocaccia extended the notion of the Gibbs measure to the general setup of the actions of the group $\mathbb{Z}^d$. This subsection aims to recall her definition of the particular setup of the full shift space $\Omega = E^{\mathbb{Z}^d}$, and we also observe that for this specific setup, Capocaccia's definition agrees with the definition of Gibbs measures in DLR formalism (see Definition 2.2.4.)

In her original paper, Capocaccia showed that any Gibbs measure in the classical DLR sense is also Gibbs in the sense of [5]. In this subsection, we prove the opposite.

Capocaccia's definition involves the notions of *conjugating homeomorphism* and a *family of multipliers*. The ***conjugating homeomorphism*** between asymptotically equivalent points $(x,y) \in \mathfrak{T}(\Omega)$ is a homeomorphism φ defined on some open neighbourhood $\mathcal{O}$ of x and taking values in Ω such that $\varphi(x) = y$ and there exists $\Lambda \Subset \mathbb{Z}^d$ satisfying, $(x', \varphi(x')) \in \mathfrak{T}_\Lambda(\Omega)$ for all $x' \in \mathcal{O}$. It should be noted that for every open subset $x \in \mathcal{O}' \subset \mathcal{O}$, $\varphi' := \varphi|_{\mathcal{O}'}$ has the same properties as $(\mathcal{O}, \varphi)$.

Remark 2.2.7. *Note that for any asymptotic pair $(x,y) \in \mathfrak{T}_\Lambda(\Omega)$, there is a canonical conjugating homeomorphism $\tilde{\varphi} : [x]_\Lambda \to [y]_\Lambda$ defined by*

$$x' \in [x]_\Lambda \overset{\tilde{\varphi}}{\mapsto} y_\Lambda x'_{\Lambda^c}. \tag{2.14}$$

Thus if $(\mathcal{O}, \varphi)$ is another conjugating homeomorphism for the pair (x,y), then by the second part of Theorem 1 in [5], there is a cylindrical set $[x]_V$ contained in $[x]_\Lambda \cap \mathcal{O}$ such that $\varphi|_{[x]_V} = \tilde{\varphi}|_{[x]_V}$. Therefore, without loss of generality, we can always assume that the conjugating homeomorphism φ has a canonical form (2.14) and the open set $\mathcal{O}$ containing x is actually a cylindrical set.

Definition 2.2.8. *A **family of multipliers** is a family $f := (f_{\mathcal{O},\varphi})$ of positive continuous functions $f_{\mathcal{O},\varphi}$ defined on $\mathcal{O}$ indexed by the conjugating homeomorphisms φ on open sets $\mathcal{O}$ such that*

(i) if $\mathcal{O}' \subset \mathcal{O}$ and $\varphi' = \varphi|_{\mathcal{O}'}$ then $f_{\mathcal{O}',\varphi'} = f_{\mathcal{O},\varphi}|_{\mathcal{O}'}$;

(ii) if $\mathcal{O} \subset \mathcal{O}' \cap (\varphi')^{-1}(\mathcal{O}'')$, then

$$f_{\mathcal{O},\varphi} = \left(f_{\mathcal{O}',\varphi'}|_{\mathcal{O}}\right) \cdot \left(f_{\mathcal{O}'',\varphi''} \circ \varphi'|_{\mathcal{O}}\right). \tag{2.15}$$

*A family $f = (f_{\mathcal{O},\varphi})$ of multipliers is called **translation-invariant** if*

(iii) for every $i \in \mathbb{Z}^d$ and all $(\mathcal{O}, \varphi)$,

$$f_{S_i(\mathcal{O}), S_i \circ \varphi \circ S_{-i}} = f_{\mathcal{O},\varphi} \circ S_{-i}, \tag{2.16}$$

where S_i is the shift by $i \in \mathbb{Z}^d$, i.e., $\omega \in \Omega \mapsto S_i \omega = (\omega_{j+i})_{j \in \mathbb{Z}^d}$

Below, we define the concept of Gibbs measure for a family of multipliers.

Definition 2.2.9. *A **Gibbs measure** for a family of multipliers $f = (f_{\mathcal{O},\varphi})$ is a probability measure $\mu \in \mathcal{M}_1(\Omega)$ such that for any pair $(\mathcal{O},\varphi)$, one has*

$$\varphi_*(f_{\mathcal{O},\varphi} \cdot \mu|_{\mathcal{O}}) = \mu|_{\varphi(\mathcal{O})}, \tag{2.17}$$

where φ_* denotes the pushforward of the map $\varphi : \mathcal{O} \to \Omega$ and for the Borel set $B \in \mathscr{F}$, $\mu|_B$ is the restriction of the measure μ to the set B, i.e., for $\tilde{B} \in \mathscr{F}$, $\mu|_B(\tilde{B}) = \mu(B \cap \tilde{B})$.

Now we show that if a probability measure is Gibbs in the sense of the above definition, then it is also Gibbs for an appropriate specification in the sense of Definition 2.2.4. In fact, if μ is a Gibbs measure for a family of multipliers $f = (f_{\mathcal{O},\varphi})$, consider a cocycle ρ^f defined as follows: let $(x,y) \in \mathfrak{T}(\Omega)$ and $(\mathcal{O},\varphi)$ be a conjugating homeomorphism corresponding to this pair (x,y), then put

$$\rho^f(x,y) := -\log f_{\mathcal{O},\varphi}(x). \tag{2.18}$$

Then, using (2.15), one can check that ρ^f satisfies the cocycle equation (2.8). Indeed, consider a triplet $(x,y),(y,z),(x,z) \in \mathfrak{T}(\Omega)$ of asymptotic pairs, assume the conjugating homeomorphisms $(\mathcal{O}',\varphi')$, $(\mathcal{O}'',\varphi'')$ and $(\mathcal{O},\varphi)$ with $\mathcal{O} \subset \mathcal{O}' \cap (\varphi')^{-1}(\mathcal{O}'')$ correspond to these pairs. Then

$$\begin{aligned}
\rho^f(x,y) + \rho^f(y,z) &= -\log\left[f_{\mathcal{O}',\varphi'}(x) \cdot f_{\mathcal{O}'',\varphi''}(y)\right] \\
&= -\log\left[f_{\mathcal{O}',\varphi'}(x) \cdot f_{\mathcal{O}'',\varphi''}(\varphi'(x))\right] \\
&\overset{(2.15)}{=} -\log f_{\mathcal{O},\varphi}(x) \\
&= \rho^f(x,z).
\end{aligned}$$

The continuity of the cocycle ρ^f is clear from its definition (2.18). Note that if the family f of multipliers is translation-invariant, then the associated cocycle ρ^f is translation-invariant in the sense

$$\rho^f(x,y) = \rho^f(S_i x, S_i y), \quad (x,y) \in \mathfrak{T}(\Omega),\ i \in \mathbb{Z}^d$$

(see also Section 2.3). In fact, let $(x,y) \in \mathfrak{T}(\Omega)$ and assume $(\mathcal{O},\varphi)$ is a corresponding conjugating homeomorphism, then for any $i \in \mathbb{Z}^d$, one has

$$\begin{aligned}
\rho^f(S_i x, S_i y) &= \rho^f(S_i x, S_i \varphi(x)) \\
&= \rho^f(S_i x, S_i \varphi S_{-i}(S_i x)) \\
&= -\log f_{S_i \mathcal{O}, S_i \varphi S_{-i}}(S_i x) \\
&\overset{(2.16)}{=} -\log f_{\mathcal{O},\varphi} \circ S_{-i}(S_i x) \\
&= \rho^f(x,y).
\end{aligned}$$

With a similar argument, one can also show the opposite; namely, if the cocycle ρ^f is translation-invariant, so is the family f of multipliers.

Now consider the Gibbsian specification γ^f associated with ρ^f via (2.9). Now we aim to show $\mu \in \mathscr{G}(\gamma^f)$. Pick any configuration $x \in \Omega$ and a finite volume $\Lambda \Subset \mathbb{Z}^d$. Then by Remark 2.2.7, one can choose a finite volume V with $\Lambda \subset V$ such that for all $y \in \Omega$ with $y_{\Lambda^c} = x_{\Lambda^c}$, the conjugating homeomorphism $(\mathcal{O}, \varphi)$ corresponding to the pair (x, y) has a canonical form (2.14) in the cylinder $[x_V]$. Now take a bounded measurable test function $g : [x]_V \to \mathbb{R}$. Then by (2.17), for all y with $y_{\Lambda^c} = x_{\Lambda^c}$, one has that

$$\int_\Omega \frac{g(y_V x'_{V^c})}{\gamma^f_V(x_V|x'_{V^c})}\Big(\gamma^f_V(y_V|x'_{V^c})\mathbb{1}_{[x]_V}(x') - \gamma^f_V(x_V|x'_{V^c})\mathbb{1}_{[y]_V}(x')\Big)\mu(dx') = 0, \quad (2.19)$$

hence

$$\int_\Omega \frac{g(y_V x'_{V^c})}{\gamma^f_V(x_V|x'_{V^c})}\Big(\gamma^f_V(y_V|x'_{V^c})\mu(x_V|x'_{V^c}) - \gamma^f_V(x_V|x'_{V^c})\mu(y_V|x'_{V^c})\Big)\mu(dx') = 0. \quad (2.20)$$

Thus, since g is chosen arbitrarily,

$$\gamma^f_V(y_V|x'_{V^c})\mu(x_V|x'_{V^c}) = \gamma^f_V(x_V|x'_{V^c})\mu(y_V|x'_{V^c}), \quad \mu-\text{a.e. } x' \in \Omega. \quad (2.21)$$

The bar moving property (2.3) and the above equation yield that

$$\gamma^f_\Lambda(y_\Lambda|x'_{\Lambda^c})\mu(x_\Lambda|x'_{\Lambda^c}) = \gamma^f_\Lambda(x_\Lambda|x'_{\Lambda^c})\mu(y_\Lambda|x'_{\Lambda^c}), \quad \mu-\text{a.e. } x' \in \Omega. \quad (2.22)$$

Then by summing up both sides of the above equation against $y_\Lambda \in E^\Lambda$, one gets that

$$\mu(x_\Lambda|x'_{\Lambda^c}) = \gamma^f_\Lambda(x_\Lambda|x'_{\Lambda^c}), \quad \mu-\text{a.e. } x' \in \Omega. \quad (2.23)$$

Thus μ is indeed consistent with the specification γ^f.

2.2.5 Equilibrium states

The notion of *equilibrium state* is developed in parallel with the notion of Gibbs measure and plays an important role both in Mathematical Statistical Mechanics and Dynamical Systems. In fact, one can see the concept of the equilibrium state as a counterpart of the concept of Gibbs states in the sense that the Gibbs states are defined in terms of "local rules" (the DLR equations), but the equilibrium states are defined in terms of a "global rule" (the variational equation). Note that a measure $\mu \in \mathscr{M}_1(\Omega)$ is translation-invariant if for all $i \in \mathbb{Z}^d$, $\mu = \mu \circ S_i^{-1}$. We denote the set of translation-invariant probability measures on Ω by $\mathscr{M}_{1,S}(\Omega)$.

Definition 2.2.10. *Let $\varphi : \Omega \to \mathbb{R}$ be a continuous function on the shift space $\Omega = E^{\mathbb{Z}^d}$. A translation-invariant probability measure $\mu \in \mathscr{M}_{1,S}(\Omega)$ is called an* ***equilibrium state*** *for φ if*

$$h(\mu) + \int_\Omega \varphi \, d\mu = \sup \Big\{ h(\tau) + \int_\Omega \varphi \, d\tau : \tau \in \mathscr{M}_{1,S}(\Omega) \Big\}, \tag{2.24}$$

where $h(\mu)$ is the measure-theoretic entropy of the measure μ. We denote the set of equilibrium states for a function φ by $\mathscr{E}\mathscr{S}(\varphi)$.

Note that any continuous function φ on Ω has at least one equilibrium state. There is an interesting relationship between the equilibrium states and Gibbs measures known as the *variational principle,* and we shall discuss it in detail in Section 2.4.

2.3 On the Kozlov-Sullivan characterisation of the Gibbsian specifications on $\mathbb{Z}^d$

In this section, we discuss Kozlov's theorem, as presented in Theorem 2.2.5, and Kozlov's regrouping method, which is key to proving the Kozlov theorem.

2.3.1 Translation-invariant generators of Gibbsian specifications

Recall that a measure $\mu \in \mathscr{M}_1(\Omega)$ is translation-invariant if for all $i \in \mathbb{Z}^d$, $\mu = \mu \circ S_i^{-1}$. An interaction $\Phi = (\Phi_V)_{V \Subset \mathbb{Z}^d}$ is called translation-invariant if for all $\Lambda \Subset \mathbb{Z}^d$ and $i \in \mathbb{Z}^d$, one has

$$\Phi_\Lambda \circ S_i = \Phi_{\Lambda+i}. \tag{2.25}$$

Similarly, a specification γ on Ω is *translation-invariant* if for all $\Lambda \Subset \mathbb{Z}$, $i \in \mathbb{Z}^d$, and every $B \in \mathscr{F}$ $\omega \in \Omega$,

$$\gamma_{\Lambda+i}(B|\omega) = \gamma_\Lambda(S_i(B)|S_i\omega). \tag{2.26}$$

Note that a translation-invariant uniform-summable (or relatively uniformly convergent) interaction gives rise to a translation-invariant Gibbsian specification γ^Φ. Surprisingly, the opposite of this statement is not true: a non-translation-invariant interaction may also give rise to a translation-invariant specification (see the interaction in (2.32)). Nonetheless, a translation-invariant non-null specification γ always generates a *translation-invariant cocycle* ρ^γ i.e., for all $(\omega, \bar{\omega}) \in \mathfrak{T}(\Omega)$ and $i \in \mathbb{Z}^d$,

$$\rho^\gamma(\omega, \bar{\omega}) = \rho^\gamma(S_i\omega, S_i\bar{\omega}) \tag{2.27}$$

and vice versa.

In the context of Gibbs measures, a translation-invariant quasilocal specification always admits a translation-invariant Gibbs measure, and vice versa. That is, a quasilocal specification prescribing a translation-invariant measure must itself be translation-invariant. However, this statement is not true for the interactions, in fact, a translation-invariant measure can be a Gibbs measure for a non-translation-invariant interaction (see the figure below)

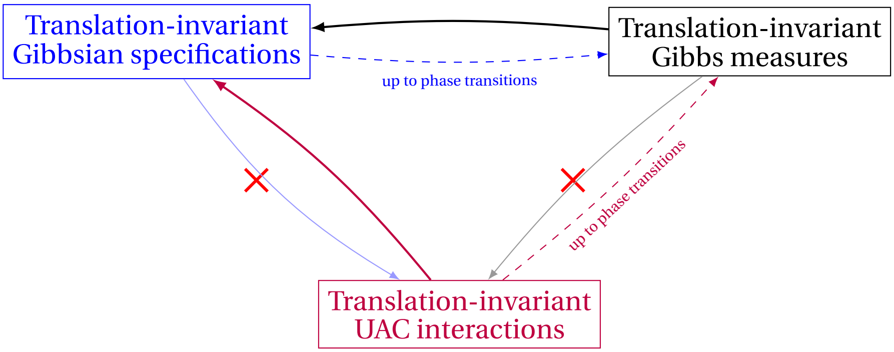

Figure 2.1

By Kozlov's theorem (Theorem 2.2.5), it is a well-known fact that any Gibbsian specification is generated by some UAC interaction on Ω. However, one may ask if every translation-invariant Gibbsian specification is also generated by a translation-invariant UAC interaction. Unfortunately, the answer to this question is not affirmative in general [2].

Theorem 2.3.1. [2] *There exists a translation-invariant Gibbsian specification on* $\Omega = \{0,1\}^{\mathbb{Z}}$ *which is not generated by any translation-invariant UAC interaction.*

The proof of Theorem 2.3.1 is not constructive but based on showing the non-surjectivity of certain bounded operators. Therefore, the specification mentioned in Theorem 2.3.1 is not explicit. It is worth noting that in Subsection 2.5.5, we consider a similar question on the one-sided full shift space $X_+ = E^{\mathbb{Z}_+}$, and prove an analogue of Theorem 2.3.1; however, the example that we provide is explicit.
The mismatch issue between translation-invariant Gibbs measures and translation-invariant UAC interactions raised by Theorem 2.3.1 can be resolved by two approaches:

(i) relaxing the UAC condition for the generating interaction;

(ii) imposing stronger regularity conditions on the specification than the quasilocality.

Below, we discuss these approaches in order.

One way of solving the issue by the first approach is done by Sullivan.

Theorem 2.3.2 (Sullivan's theorem)**.** [2,28] *Let γ be a translation-invariant Gibbsian specification on Ω. Then there exists a translation-invariant variation-summable interaction Ψ such that $\gamma = \gamma^\Psi$.*

In his original paper [21], Kozlov suggests a condition which solves the issue with the second approach. Denote the r^{th}–variation of the single-site density $\gamma_{\{0\}}$ of a Gibbsian specification γ on $E^{\mathbb{Z}^d}$ by $v(r)$, i.e.,

$$v(r) := \sup\{\gamma_{\{0\}}(\xi_0|\xi_{\{0\}^c}) - \gamma_{\{0\}}(\eta_0|\eta_{\{0\}^c}) \mid \xi_i = \eta_i,\ \forall i \in [-r,r]^d \cap \mathbb{Z}^d\}, \tag{2.28}$$

then the following is one of the results in [21].

Theorem 2.3.3. *Assume a translation-invariant Gibbsian specification γ on $\Omega = E^{\mathbb{Z}^d}$ has summable-variations in the sense that*

$$\sum_{r=1}^{\infty} r^{d-1} v(r) < \infty. \tag{2.29}$$

Then the specification γ is generated by a translation-invariant UAC interaction.

The proof of Theorem 2.3.3 is constructive and relies on the same method as the proof of Theorem 2.2.5. One can easily notice that Theorem 2.3.3 applies to the finite-range specifications since $v(r) = 0$ after some $r \in \mathbb{N}$. However, if it comes to the infinite-range specifications, (2.29) is quite a restrictive condition; for example, in dimension one, it prevents the underlying specification from the phase transitions.

Theorem 2.3.4. *Let γ be a translation-invariant Gibbsian specification on $\Omega = E^{\mathbb{Z}}$ satisfying (2.29). Then there is a unique Gibbs measure compatible with γ, i.e., $\#\mathscr{G}(\Omega,\gamma) = 1$.*

For completeness, we shall provide a proof of the above theorem in Section 2.6.

2.3.2 The challenges of extending Kozlov's method to the translation-invariant setup

In his original paper [21], Kozlov proved two theorems – Theorem 2.2.5 and Theorem 2.3.3 above – on the generating interactions of Gibbsian specifications. The idea of proofs of both theorems is based on a regrouping of the bonds in Grimmett's vacuum interaction [18]. The regrouping procedure in the proof of Kozlov's first theorem (Theorem 2.2.5 above), which we refer to as *Kozlov's first way of regrouping*, does not adhere to any order, algebraic, or graph structure in $\mathbb{Z}^d$.

Consequently, it produces an interaction that is uniformly absolutely convergent (UAC) but does not align with the structural properties of $\mathbb{Z}^d$. In contrast, the regrouping procedure in his second theorem (Theorem 2.3.3 above), referred to as *Kozlov's second way of regrouping*, respects the order and graph structure of the lattice $\mathbb{Z}^d$. As a result, while it produces an interaction with good algebraic properties, it may have very bad summability properties. Below, we demonstrate these disadvantages of Kozlov's regrouping method in a special example of the specification of the Dyson model.

Recall that the single-site density of Dyson's specification γ^D at a site $i \in \mathbb{Z}$ is defined by

$$\gamma^D_{\{i\}}(\omega_i|\omega_{\mathbb{Z}\setminus\{i\}}) := \frac{\exp\left(\beta \sum_{k=1}^{\infty} \frac{\omega_i(\omega_{i+k}+\omega_{i-k})}{k^\alpha}\right)}{\exp\left(\beta \sum_{k=1}^{\infty} \frac{\omega_{i+k}+\omega_{i-k}}{k^\alpha}\right) + \exp\left(-\beta \sum_{k=1}^{\infty} \frac{\omega_{i+k}+\omega_{i-k}}{k^\alpha}\right)}, \ \forall \omega \in \Omega, \tag{2.30}$$

where $\alpha \in (1,2]$ is the decay rate of the coupling constants and $\beta \geq 0$ is the inverse temperature. Note that the associated specification γ^D is uniquely recovered from these single-site densities [13].

The vacuum interaction for the specification γ^D corresponding to the vacuum configuration $\mathbf{-} := -1_{\mathbb{Z}}$ is given by the the following lattice-gas interaction Φ^-:

$$\Phi^-_\Lambda(\omega) := \begin{cases} -\dfrac{\beta(1+\omega_i)(1+\omega_j)}{|i-j|^\alpha}, & \text{if } \Lambda = \{i,j\} \subset \mathbb{Z},\ i \neq j; \\ 2\beta\zeta(\alpha)(1+\omega_i), & \text{if } \Lambda = \{i\} \subset \mathbb{Z}; \\ 0, & \text{otherwise,} \end{cases} \tag{2.31}$$

where ζ is the Riemann zeta function.

If we enumerate the elements in $\mathbb{Z}$ by $\{\ell_1, \ell_2, \ell_3, \ldots\}$, where for $k \in \mathbb{N}$, $\ell_{2k-1} := k-1$ and $\ell_{2k} := -k$, and define for $(k,i) \in \mathbb{N}^2$, $L^k_{2^i} := \{\ell_j : k \leq j \leq k+2^i\}$, $\mathscr{S}^k_0 := \emptyset$ and

$$\mathscr{S}^k_i := \{B \subset L^k_{2^i} : \ell_k \in B\} \setminus \mathscr{S}^k_{i-1},$$

then Kozlov's first way of regrouping produces the following interaction for γ^D:

$$\Phi^{KR1}_\Lambda := \begin{cases} 0, & \text{if } \Lambda \neq L^k_{2^i} \text{ for all } (k,i) \in \mathbb{N}^2; \\ \displaystyle\sum_{B \in \mathscr{S}^k_i} \Phi^-_B, & \text{if } \Lambda = L^k_{2^i}, \text{ for some } (k,i) \in \mathbb{N}^2. \end{cases} \tag{2.32}$$

By the construction, the interaction $\Phi^{KR1} = (\Phi^{KR1}_\Lambda)_{\Lambda \Subset \mathbb{Z}}$ generates the specification γ^D and is UAC. However, Φ^{KR1} is not translation-invariant, because

$$\Phi^{KR1}_{\{-1,0,1\}}(\omega) = \beta(1+\omega_0)(2\zeta(\alpha) - 2 - \omega_{-1} - \omega_1)$$

and for all $k \in \mathbb{Z} \setminus \{0\}$, one has $\Phi^{KR1}_{\{k-1,k,k+1\}} \equiv 0$.

Kozlov's second way of regrouping produces the following interaction for the specification γ^D:

$$\Phi_\Lambda^{KR2} := \begin{cases} \sum_{\ell \le j \le \ell+2} \Phi^-_{\{\ell,j\}}, & \text{if } \Lambda = [\ell, \ell+2] \cap \mathbb{Z},\ \ell \in \mathbb{Z}; \\ \sum_{\ell+2^{k-1} < j \le \ell+2^k} \Phi^-_{\{\ell,j\}}, & \text{if } \Lambda = [\ell, \ell+2^k] \cap \mathbb{Z},\ \ell \in \mathbb{Z},\ k \in \mathbb{N} \setminus \{1\}; \\ 0, & \text{otherwise,} \end{cases} \tag{2.33}$$

This time, by the construction, the produced interaction Φ^{KR2} is translation-invariant, but not UAC. In fact, for $k \ge 2$, one has the following for $\|\Phi^{KR2}_{[0,2^k]}\|_\infty$:

$$\|\Phi^{KR2}_{[0,2^k]}\|_\infty = \sum_{i=2^{k-1}+1}^{2^k} \frac{4\beta}{i^\alpha}, \tag{2.34}$$

thus for all $k \ge 2$,

$$\|\Phi^{KR2}_{[0,2^k]}\|_\infty \ge 2\beta \cdot 2^{(1-\alpha)k}. \tag{2.35}$$

Hence for $\alpha \in (1,2]$, one has

$$\sum_{0 \in V \Subset \mathbb{Z}} \|\Phi^{KR2}_V\|_\infty \ \ge\ \sum_{k=2}^{\infty} (2^k+1) \|\Phi^{KR2}_{[0,2^k]}\|_\infty \ \ge\ 2\beta \cdot \sum_{k=2}^{\infty} 2^{(2-\alpha)k} \ =\ \infty. \tag{2.36}$$

Thus Φ^{KM2} is not a UAC interaction.

Remark 2.3.5. *Although Kozlov's method does not yield interactions for the specification of the Dyson model having desired algebraic and summability properties, Grimmett's vacuum interaction Φ^- given by (2.31) already possesses these properties. However, Grimmett's vacuum interaction does not always have good summability properties and, in fact, it often lacks the UAC property.*

2.4 Main Results I: An Interaction-Independent Variational-Principle on $\mathbb{Z}^d$ and its Consequences

2.4.1 An Interaction-Independent Variational Principle

In probability theory, there are several notions that aim to measure the discrepancy between probability measures. One such concept is *relative entropy*, which plays a key role in both information theory and statistics and is also important in the formulation of the variational principle in Mathematical Statistical Mechanics.

Consider probability measures $\tau, \mu \in \mathscr{M}_1(\Omega)$ with full support, i.e., for any cylindrical set $[\sigma_\Lambda]$, $\sigma \in \Omega$, $\Lambda \Subset \mathbb{Z}^d$, one has $\tau([\sigma_\Lambda]) > 0$ and $\mu([\sigma_\Lambda]) > 0$. For a finite volume $\Lambda \Subset \mathbb{Z}^d$, consider the *marginals* τ_Λ and μ_Λ of probability measures τ and μ which are finite-volume measures on the sub-sigma algebra $\mathscr{F}_\Lambda$. By the full-support condition, one has $\tau_\Lambda \ll \mu_\Lambda$ and vice versa. Then the *relative entropy* $H(\tau_\Lambda|\mu_\Lambda)$ *of* τ_Λ *relative to* μ_Λ is

$$\begin{aligned} H(\tau_\Lambda|\mu_\Lambda) &:= \int_\Omega \log \frac{d\tau_\Lambda}{d\mu_\Lambda} d\tau_\Lambda \\ &= \sum_{\sigma_\Lambda \in E^\Lambda} \tau([\sigma_\Lambda]) \log \frac{\tau([\sigma_\Lambda])}{\mu([\sigma_\Lambda])}. \end{aligned} \tag{2.37}$$

By a direct application of the Jensen inequality, one can show that $H(\tau_\Lambda|\mu_\Lambda) \geq 0$. However, for the infinite volume measures τ and μ, it is often the case that $H(\tau|\mu) = +\infty$ which is not a quite useful information in comparing τ to μ. Nevertheless, in the case of infinite volume measures, the *specific relative entropy* – which can be interpreted as the relative entropy per site – gives some information. Recall that the ***specific relative entropy*** $h(\tau|\mu)$ *of* τ *relative to* μ is defined by

$$h(\tau|\mu) := \lim_{n\to\infty} \frac{1}{\Lambda_n} H(\tau_{\Lambda_n}|\mu_{\Lambda_n}) \tag{2.38}$$

provided the limit exists.

Now consider a translation-invariant UAC interaction Φ on Ω. Then one has the following, the so-called *variational principle* for translation-invariant Gibbs measures for the interaction Φ. In order to stress the difference, we state two versions of the variational principle in separate statements. Yet, these versions are equivalent, at least in the case where the specification is generated by a translation-invariant UAC interaction.

First, for the UAC interaction Φ, define a continuous function (potential) by

$$u_\Phi := -\sum_{0 \in V \Subset \mathbb{Z}^d} \frac{1}{|V|} \Phi_V$$

which is interpreted in Statistical Mechanics as the ***energy contribution from*** *(the neighbourhood of)* ***the origin***.

Theorem 2.4.1. [16, Chapter 15]

(VP1) *Let* $\mu \in \mathscr{G}_S(\Phi)$ *and* $\tau \in \mathscr{M}_{1,S}(\Omega)$, *then the* ***specific relative entropy*** $h(\tau|\mu)$ *of* τ *with respect to* μ *exists and*

$$h(\tau|\mu) = 0 \iff \tau \in \mathscr{G}_S(\Phi). \tag{2.39}$$

(VP2) *The translation-invariant Gibbs measures for the interaction* Φ *are exactly the equilibrium states for the potential* u_Φ *and vice versa, i.e.,* $\mathscr{G}_S(\Phi) = \mathscr{E}\mathscr{S}(u_\Phi)$*. In other words, for a translation-invariant measure* $\mu \in \mathscr{M}_{1,S}(\Omega)$*, one has*

$$h(\mu) + \int_\Omega u_\Phi d\mu = \sup\left\{h(\tau) + \int_\Omega u_\Phi d\tau : \tau \in \mathscr{M}_{1,S}(\Omega)\right\} \iff \mu \in \mathscr{G}_S(\Phi). \tag{2.40}$$

Remark 2.4.2. *In* [25]*, Pfister proves that for any translation-invariant probability measure* $\tau \in \mathscr{M}_{1,S}(\Omega)$ *and every translation-invariant measure* $\mu \in \mathscr{M}_{1,S}(\Omega)$ *which is* ***asymptotically decoupled from above****, the relative entropy* $h(\tau|\mu)$ *exists. Note that a measure* $\nu \in \mathscr{M}_1(\Omega)$ *is asymptotically decoupled from above if there exist* $g : \mathbb{N} \to \mathbb{N}$ *and* $c : \mathbb{N} \to [0, \infty)$ *with* $\lim_{n\to\infty} \frac{g(n)}{n} = 0$ *and* $\lim_{n\to\infty} \frac{c(n)}{|\Lambda_n|} = 0$ *such that for every* $i \in \mathbb{Z}^d$*,* $n \in \mathbb{N}$*,* $A \in \mathscr{F}_{\Lambda_n + i}$ *and all* $B \in \mathscr{F}_{i + \Lambda_{n+g(n)}}$*, one has*

$$\nu(A \cap B) \le e^{c(n)} \nu(A)\nu(B). \tag{2.41}$$

Any Gibbs measure, in the sense of Definition 2.2.4, is asymptotically decoupled from both above and below, namely together with the upper bound (2.41), the following lower bound also holds for ν*:*

$$e^{-c(n)} \nu(A)\nu(B) \le \nu(A \cap B). \tag{2.42}$$

In the first statement of Theorem 2.4.1, all participating quantities and notions are independent of the form of interaction Φ. Therefore, it is reasonable to expect a generalisation of the statement to the Gibbsian specifications. In the light of Theorem 2.3.1, such a generalisation would be strictly stronger than the first statement of Theorem 2.4.1. In [22], the authors proved such a generalisation.

Theorem 2.4.3. [22] *Let* γ *be a translation-invariant Gibbsian specification and* $\mu \in \mathscr{G}_S(\gamma)$*. Then for all* $\tau \in \mathscr{M}_{1,S}(\Omega)$*, the specific relative entropy* $h(\tau|\mu)$ *exists and*

$$h(\tau|\mu) = 0 \iff \tau \in \mathscr{G}_S(\gamma).$$

The purpose of this section is to state the VP2 in terms of specifications. To do so, first, we have to extend certain interaction-dependent thermodynamic quantities such as the notion of *the contribution to energy from the origin* to the setup of specifications.

Henceforth, $\le$ denotes the lexicographic order in the lattice $\mathbb{Z}^d$, and for $i, j \in \mathbb{Z}^d$ with $i \le j$, $[i, j] := \{k \in \mathbb{Z}^d : i \le k \le j\}$. Open and half-open intervals are also defined analogously. For $n \in \mathbb{N}$, Λ_n denotes the volume $[-n, n]^d \cap \mathbb{Z}^d$. Consider a *translation-invariant* probability measure $\rho \in \mathscr{M}_{1,S}(\Omega)$, and define a function interpreted as the ***contribution to the energy from the origin*** by

$$u_\gamma^\rho(\omega) := \int_\Omega \log \frac{\gamma_{\{\mathbf{0}\}}(\omega_\mathbf{0}|\theta_{<\mathbf{0}}\omega_{>\mathbf{0}})}{\gamma_{\{\mathbf{0}\}}(\theta_\mathbf{0}|\theta_{<\mathbf{0}}\omega_{>\mathbf{0}})}\rho(d\theta), \quad \omega \in \Omega. \tag{2.43}$$

Remark 2.4.4. *Note that the notion of the contribution to the energy from the origin that we have introduced above generalise the corresponding notion in Statistical Mechanics. This can be illustrated through the concept of physical equivalence. In fact, if a translation-invariant Gibbsian specification γ is generated by a translation-invariant UAC interaction Φ, then u_γ^ρ is physically equivalent to $u_\Phi = -\sum_{0\in V\Subset\mathbb{Z}^d} \frac{1}{|V|}\Phi_V$, i.e., there exists a constant $C \in \mathbb{R}$ such that for all $\tau \in \mathscr{M}_{1,S}(\Omega)$, one has $\int_\Omega u_\gamma^\rho \, d\tau = \int_\Omega u_\Phi \, d\tau + C$. In particular, all functions u_γ^ρ parameterised by the translation-invariant probability measures $\rho \in \mathscr{M}_{1,S}(\Omega)$ are physically equivalent to each other. It should also be noted that if the reference measure ρ is a Dirac measure δ_+ where $+$ is a constant configuration with $+_i = + \in E$ for all $i \in \mathbb{Z}^d$, then (2.43) provides the following function:*

$$u_\gamma^+(\omega) := \log \frac{\gamma_{\{\mathbf{0}\}}(\omega_\mathbf{0}|+_{<\mathbf{0}}\,\omega_{>\mathbf{0}})}{\gamma_{\{\mathbf{0}\}}(+_\mathbf{0}|+_{<\mathbf{0}}\,\omega_{>\mathbf{0}})}, \quad \omega \in \Omega. \tag{2.44}$$

The following theorem is a natural generalisation of the second part (VP2) of Theorem 2.4.1 to the setup of specifications.

Theorem 2.A. *Assume γ is a translation-invariant Gibbs specification on Ω. Then the translation-invariant Gibbs measures for γ are exactly the equilibrium states for the potential u_γ^ρ and vice versa, i.e., $\mathscr{G}_S(\gamma) = \mathscr{E}\mathscr{S}(u_\gamma^\rho)$. In other words,*

$$h(\mu) + \int_\Omega u_\gamma^\rho \, d\mu = \sup\left\{h(\tau) + \int_\Omega u_\gamma^\rho \, d\tau : \tau \in \mathscr{M}_{1,S}(\Omega)\right\} \quad \Longleftrightarrow \quad \mu \in \mathscr{G}_S(\gamma). \tag{2.45}$$

We shall present a proof of Theorem 2.A in Section 2.6. It should be noted that our proof of the theorem yields, as a byproduct, the following formula for the specific relative entropy $h(\tau|\mu)$ of a translation measure τ with respect to a Gibbs measure μ.

Corollary A.1. *Let μ be a translation-invariant Gibbs measure prescribed by a Gibbsian specification γ and τ be a translation-invariant measure on Ω. Then for the specific relative entropy $h(\tau|\mu)$, one has*

$$h(\tau|\mu) \;=\; -h(\tau) - \lim_{n\to\infty} \frac{1}{|\Lambda_n|}\int_\Omega \log \gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})\tau(d\omega). \tag{2.46}$$

So far, we have not established any regularity properties of the function u^{ρ}_{γ}, which is associated with a translation-invariant Gibbsian specification γ via (2.43). We now demonstrate that u^{ρ}_{γ} possesses the *extensibility property*. Recall that a continuous potential $\phi:\Omega\to\mathbb{R}$ satisfies the **extensibility** condition if for all $a,b\in E$ the sequence

$$F_n^{b,a}(\omega):=\sum_{i\in\Lambda_n}\big(\phi\circ S_i(\omega^b)-\phi\circ S_i(\omega^a)\big)$$

converges uniformly on $\omega\in\Omega$ as $n\to\infty$, here the configuration ω^a is given by

$$\omega^a_k=\begin{cases}a, & k=0,\\ \omega_k, & k\in\mathbb{Z}^d\setminus\{0\}.\end{cases}$$

Proposition 2.4.5. *Let γ be a translation-invariant Gibbsian specification on Ω. Then u^{ρ}_{γ} satisfies the extensibility condition.*

Proof. Pick any $a,b\in E$ and $\omega\in\Omega$. One can check that for any $n\in\mathbb{N}$,

$$\begin{aligned}F_n^{b,a}(\omega)&=\sum_{i\in\Lambda_n}\Big[u^{\rho}_{\gamma}\circ S_i(\omega^b)-u^{\rho}_{\gamma}\circ S_i(\omega^a)\Big]\\ &=\sum_{\substack{i\in\Lambda_n\\ i<0}}\int_{\Omega}\log\frac{\gamma_{\{0\}}((\omega_i)_0|\theta_{<0}(\omega_{(i,0)}b_0\omega_{>0})_{>0})}{\gamma_{\{0\}}(\theta_0|\theta_{<0}(\omega_{(i,0)}b_0\omega_{>0})_{>0})}\rho(d\theta)\\ &\quad-\sum_{\substack{i\in\Lambda_n\\ i<0}}\int_{\Omega}\log\frac{\gamma_{\{0\}}((\omega_i)_0|\theta_{<0}(\omega_{(i,0)}a_0\omega_{>0})_{>0})}{\gamma_{\{0\}}(\theta_0|\theta_{<0}(\omega_{(i,0)}a_0\omega_{>0})_{>0})}\rho(d\theta)+F_0^{b,a}(\omega).\end{aligned}$$

Then, by applying the bar moving property, one gets

$$\begin{aligned}F_n^{b,a}(\omega)-F_0^{b,a}(\omega)&=\sum_{\substack{i\in\Lambda_n\\ i<0}}\int_{\Omega}\log\frac{\gamma_{\{0,-i\}}\Big((\omega_i)_0(b_0)_{-i}|\theta_{<0}(\omega_{(i,0)}\omega_{>0})_{\substack{>0\\ \neq -i}}\Big)}{\gamma_{\{0,-i\}}\Big(\theta_0(b_0)_{-i}|\theta_{<0}(\omega_{(i,0)}\omega_{>0})_{\substack{>0\\ \neq -i}}\Big)}\rho(d\theta)\\ &\quad+\sum_{\substack{i\in\Lambda_n\\ i<0}}\int_{\Omega}\log\frac{\gamma_{\{0,-i\}}\Big(\theta_0(a_0)_{-i}|\theta_{<0}(\omega_{(i,0)}\omega_{>0})_{\substack{>0\\ \neq -i}}\Big)}{\gamma_{\{0,-i\}}\Big((\omega_i)_0(a_0)_{-i}|\theta_{<0}(\omega_{(i,0)}\omega_{>0})_{\substack{>0\\ \neq -i}}\Big)}\rho(d\theta),\end{aligned}$$

and thus

$$\begin{aligned}&F_n^{b,a}(\omega)-F_0^{b,a}(\omega)=\\ &\sum_{\substack{i\in\Lambda_n\\ i<0}}\int_{\Omega}\log\frac{\gamma_{\{-i\}}((b_0)_{-i}|\theta_{<0}(\omega_{[i,0)}\omega_{>0})_{\substack{\geq 0\\ \neq -i}})}{\gamma_{\{-i\}}((a_0)_{-i}|\theta_{<0}(\omega_{[i,0)}\omega_{>0})_{\substack{\geq 0\\ \neq -i}})}\cdot\frac{\gamma_{\{-i\}}((a_0)_{-i}|\theta_{\leq 0}(\omega_{(i,0)}\omega_{>0})_{\substack{>0\\ \neq -i}})}{\gamma_{\{-i\}}((b_0)_{-i}|\theta_{\leq 0}(\omega_{(i,0)}\omega_{>0})_{\substack{>0\\ \neq -i}})}\rho(d\theta).\end{aligned}$$

Hence by the translation-invariance of γ and ρ,

$$
\begin{aligned}
&F_n^{b,a}(\omega)-F_0^{b,a}(\omega)=\\
&=\sum_{\substack{i\in\Lambda_n\\ i<0}}\int_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{<i}\omega_{[i,0)}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{<i}\omega_{[i,0)}\omega_{>0})}\rho(d\theta)-\sum_{\substack{i\in\Lambda_n\\ i<0}}\int_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{\le i}\omega_{(i,0)}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{\le i}\omega_{(i,0)}\omega_{>0})}\rho(d\theta)\\
&=\int_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{<\min\Lambda_n}\omega_{[\min\Lambda_n,0)}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{<\min\Lambda_n}\omega_{[\min\Lambda_n,0)}\omega_{>0})}\rho(d\theta)-\int_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{<0}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{<0}\omega_{>0})}\rho(d\theta).
\end{aligned}
$$

Thus

$$
F_n^{b,a}(\omega)=\int_\Omega \log\frac{\gamma_{\{0\}}(b_0|\theta_{<\min\Lambda_n}\omega_{[\min\Lambda_n,0)}\omega_{>0})}{\gamma_{\{0\}}(a_0|\theta_{<\min\Lambda_n}\omega_{[\min\Lambda_n,0)}\omega_{>0})}\rho(d\theta)\ \underset{\substack{n\to\infty\\ \omega\in\Omega}}{\rightrightarrows}\ \log\frac{\gamma_{\{0\}}(b_0|\omega_{\mathbb{Z}^d\setminus\{0\}})}{\gamma_{\{0\}}(a_0|\omega_{\mathbb{Z}^d\setminus\{0\}})}. \tag{2.47}
$$

□

2.4.2 Erasure entropies

In the rest of this section, we restrict ourselves to the one-dimensional setup, i.e., $d=1$.

For the measure-theoretic entropy of a translation-invariant measure τ on the shift space $\Omega=E^{\mathbb{Z}}$, one can prove that

$$
h(\tau)=-\int_\Omega \log\tau(\omega_0|\omega_{>0})\tau(d\omega). \tag{2.48}
$$

One can generalise the above formula for any finite set $\Lambda\Subset\mathbb{Z}_+$ by

$$
h(\tau)=-\frac{1}{|\Lambda|}\int_\Omega \log\tau(\omega_\Lambda|\omega_{\mathbb{Z}_+\setminus\Lambda})\tau(d\omega). \tag{2.49}
$$

These formulas are valid if one conditions on the *past* instead of the *future*. It is then natural to ask whether these formulas remain valid if the one-sided conditioning is replaced with two-sided conditioning. In fact, by substituting the one-sided conditioning with two-sided conditioning in (2.48), one obtains the ***erasure entropy*** $h^-(\tau)$ of the measure τ [29], namely,

$$
h^-(\tau)=-\int_\Omega \log\tau(\omega_0|\omega_{\{0\}^c})\tau(d\omega). \tag{2.50}
$$

It should be stressed that a similar formula to (2.49) does not hold for the erasure entropies; therefore, for a finite volume $\Lambda\Subset\mathbb{Z}$, the *erasure entropy of* τ *in* Λ is defined in [11] as

$$
h^-_\Lambda(\tau):=-\int_\Omega \log\tau(\omega_\Lambda|\omega_{\Lambda^c})\tau(d\omega). \tag{2.51}
$$

The erasure entropy is always less than, and generally not equal to, the measure-theoretic entropy. However, in the limit, one can expect the equality of these two notions as the following theorem states.

Theorem 2.4.6. [11] *Assume τ be a translation-invariant Gibbs measure for a translation-invariant UAC interaction Φ, i.e., $\tau \in \mathscr{G}_S(\Phi)$. Then*

$$\lim_{\Lambda\uparrow\mathbb{Z}} \frac{1}{|\Lambda|} h^-_{\Lambda}(\tau) = h(\tau). \tag{2.52}$$

The following corollary of Theorem 2.A enables us to extend the above result to any translation-invariant Gibbs measure.

Corollary 2.4.7. *Let τ be a translation-invariant Gibbs measure on $\Omega = E^{\mathbb{Z}}$. Then*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} h^-_{\Lambda_n}(\tau) = h(\tau). \tag{2.53}$$

2.5 Main Results II: Gibbs formalism on $\mathbb{Z}_+$

2.5.1 DLR-Gibbs formalism on $\mathbb{Z}_+$

In this section, we work on the half-line $\mathbb{Z}_+$, denoting the configuration space $E^{\mathbb{Z}_+}$ by X_+, with the left-shift (or translation) map S acting on X_+ as follows: for $x \in X_+$, and all $i \in \mathbb{Z}_+$, $(Sx)_i = x_{i+1}$.

Let $\overrightarrow{\gamma}$ be a non-null specification on X_+. Set

$$\tilde{g}_0(x) := \overrightarrow{\gamma}_{\{0\}}(x_0|x_1^{\infty}), \;\; x \in X_+, \tag{2.54}$$

and for all $n \geq 1$, consider a function $\tilde{g}_n : X_+ \to [0,1]$ given by

$$\tilde{g}_n(x) := \frac{\overrightarrow{\gamma}_{[0,n]}(x_0^n|x_{n+1}^{\infty})}{\overrightarrow{\gamma}_{[0,n-1]}(x_0^{n-1}|x_n^{\infty})}, \;\; x \in X_+. \tag{2.55}$$

Note that by the construction, for all $n \geq 0$, one has

$$\overrightarrow{\gamma}_{[0,n]}(x_0^n|x_{n+1}^{\infty}) = \prod_{k=0}^{n} \tilde{g}_k(x), \;\; x \in X_+. \tag{2.56}$$

The functions $\{\tilde{g}_n,\ n \in \mathbb{Z}_+\}$ has the following properties:

Proposition 2.5.1. *(1) For all $n \geq 0$, $\tilde{g}_n$ is a positive function, and $\tilde{g}_n \in \mathscr{F}_{[n,\infty)}$, i.e., $\tilde{g}_n$ is independent of the first $n-1$ coordinates.*

(2) For all $n \geq 0$, and $x \in X_+$, $\sum_{y_n \in E} \tilde{g}_n(y_n x_{n+1}^{\infty}) = 1$.

Proof. (1) Positivity follows from the non-nullness of the specification $\vec{\gamma}$. The independence from the first $n-1$ coordinates follows from the bar moving property (2.3).

(2) Firstly, since $\vec{\gamma}$ is a specification, for all $x \in X_+$, $\sum_{\bar{x}_0 \in E} \tilde{g}_0(y_0 x_1^\infty) = 1$. Fix $n \geq 1$, and assume that the statement is correct for all $k < n$. Then, since $\vec{\gamma}$ is a specification, by the first part of the proposition and from (2.56), for all $x \in X_+$, one has

$$\begin{aligned} 1 &= \sum_{y_0^n \in E^{n+1}} \prod_{k=0}^{n} \tilde{g}_k(y_k^n x_{n+1}^\infty) \\ &= \sum_{y_n \in E} \tilde{g}_n(y_n x_{n+1}^\infty) \sum_{y_{n-1} \in E} \tilde{g}_{n-1}(y_{n-1}^n x_{n+1}^\infty) \cdots \sum_{y_0 \in E} \tilde{g}_0(y_0^n x_{n+1}^\infty) \\ &= \sum_{y_n \in E} \tilde{g}_n(y_n x_{n+1}^\infty) \end{aligned}$$

□

Remark 2.5.2. *The first part of Proposition 2.5.1 yields that for all $n \in \mathbb{Z}_+$, there exists a positive measurable function g_n on X_+ such that $\tilde{g}_n = g_n \circ S^n$. Then the second part of Proposition 2.5.1 is equivalent to that for all $n \geq 0$, g_n is a g—function* [20] *on X_+, i.e., for all $x \in X_+$, $\sum_{a \in E} g_n(ax) = 1$.*

In terms of g_n's, (2.56) reads

$$\vec{\gamma}_{[0,n]}(x_0^n | x_{n+1}^\infty) = \prod_{k=0}^{n} g_k \circ S^k(x) = \exp\Big(\sum_{k=0}^{n} \log g_k \circ S^k(x)\Big), \;\; x \in X_+. \tag{2.57}$$

Thus, a non-null specification on X_+ uniquely determines a sequence of g-functions. Now consider the opposite situation: assume that a sequence $\bar{\varphi} := \{\varphi_k : k \in \mathbb{Z}_+\}$ of Borel functions on X_+ is given. For all $\Lambda \Subset \mathbb{Z}_+$, define

$$\vec{\gamma}_{\Lambda}^{\bar{\varphi}}(y_\Lambda | x_{\Lambda^c}) := \frac{\exp(\sum_{k=0}^{n} \varphi_k \circ S^k(y_\Lambda x_{\Lambda^c}))}{\sum_{\bar{y}_\Lambda \in E^\Lambda} \exp(\sum_{k=0}^{n} \varphi_k \circ S^k(\bar{y}_\Lambda x_{\Lambda^c}))}, \;\; y, x \in X_+, \tag{2.58}$$

where $n := \max \Lambda$. Then one can readily check that the family $\vec{\gamma}^{\bar{\varphi}} := (\vec{\gamma}_{\Lambda}^{\bar{\varphi}})_{\Lambda \Subset \mathbb{Z}_+}$ is indeed specification densities, therefore, one can associate a non-null specification with the sequence $\bar{\varphi}$. Then we conclude that there is an association between the non-null specifications on X_+ and the *generalized Birkhoff sums* $S_n \bar{\varphi} := \sum_{i=0}^{n-1} \phi_i \circ S^i$, $n \in \mathbb{N}$, where each φ_i is a Borel function. However, this association is not one-to-one, in fact, different generalized Birkhoff sums might give rise to the same specification which leads to the notion of *physical equivalence of potentials* [10]. In terms of cocycles on $\mathbb{Z}_+$, this means that any measurable (continuous) cocycle ρ on X_+ is given by

$$\rho(\omega,\omega')=\sum_{k=0}^{\infty}[\varphi_k\circ S^k(\omega)-\varphi_k\circ S^k(\omega')],\ (\omega,\omega')\in\mathfrak{T}(X_+), \tag{2.59}$$

where each ϕ_k is a measurable (continuous) function on X_+, and vise versa, i.e., for any sequence $\{\varphi_k\}_{k\in\mathbb{Z}_+}$ of measurable (continuous) functions $\varphi_k: X_+\to\mathbb{R},\ k\geq 0$, (2.59) defines a measurable (continuous) cocycle on $\mathfrak{T}(X_+)$.

Now consider a measurable (continuous) cocycle ρ on X_+. We define *the base of the cocycle* ρ by the following formula:

$$\rho_0(\omega,\omega')=\rho(\omega,\omega')-\rho(S\omega,S\omega'),\ (\omega,\omega')\in\mathfrak{T}(X_+). \tag{2.60}$$

Then it is clear that ρ_0 is also a measurable cocycle on $\mathfrak{T}(X_+)$, and it is continuous if ρ is continuous. Furthermore, one has the following important identity for ρ:

$$\rho(\omega,\omega')=\sum_{k=0}^{\infty}\rho_0(S^k\omega,S^k\omega'),\ (\omega,\omega')\in\mathfrak{T}(X_+). \tag{2.61}$$

2.5.2 Interactions on the lattice $\mathbb{Z}_+$

Now we turn to the interaction on $X_+=E^{\mathbb{Z}_+}$. By following the concept of translation-invariant interactions on $\mathbb{Z}$, we define such a notion for the interactions on $\mathbb{Z}_+$. Note that we also use the term one-sided interaction for the interactions on X_+. We again define the *translation-invariance of a one-sided interaction* $\vec{\Phi}$ with (2.25), but this time we only assume that Λ runs over the finite subsets of $\mathbb{Z}_+$, i.e., for all $\Lambda\Subset\mathbb{Z}_+$, the following holds true

$$\vec{\Phi}_\Lambda\circ S=\vec{\Phi}_{\Lambda+1}. \tag{2.62}$$

Note that the restriction of a translation-invariant interaction Φ on $\Omega=E^{\mathbb{Z}}$ to X_+ remains translation-invariant. Simultaneously, any left-shift invariant interaction $\vec{\Phi}$ on X_+ can be extended to Ω by translation, namely, the translated interaction Φ is defined for the volumes $\Lambda\Subset\mathbb{Z}$ with $\min\Lambda<0$ by $\Phi_\Lambda:=\vec{\Phi}_{\Lambda-\min\Lambda}\circ S^{\min\Lambda}$ (note that S is invertible on Ω). Thus there is a one-to-one correspondence between the translation-invariant interactions on Ω and X_+. It should also be mentioned that one-sided translation-invariant interaction $\vec{\Phi}$ does not give rise to a "translation-invariant" specification $\gamma^{\vec{\Phi}}$ on $\mathbb{Z}_+$ in a sense that the associated cocycle $\rho^{\vec{\Phi}}$ satisfies $\rho^{\vec{\Phi}}(\omega,\omega')=\rho^{\vec{\Phi}}(S\omega,S\omega')$ (note that we define *translation-invariant specifications* in Subsection 2.5.5 in a slightly different way).

We should note that there is a one-to-one correspondence between the translation -invariant UAC interactions on X_+ and the continuous potentials in $C(X_+)$. In fact, any translation-invariant UAC interaction Φ on X_+ can be associated with a continuous potential $\phi \in C(X_+)$ by the following formula:

$$\phi = -\sum_{0\in V \Subset \mathbb{Z}_+} \vec{\Phi}_V. \tag{2.63}$$

Then, in fact, one can check that for all $n \in \mathbb{N}$,

$$S_n\phi = \sum_{i=0}^{n-1} \phi \circ S^i = -H^{\vec{\Phi}}_{[0,n-1]} \tag{2.64}$$

where $H^{\vec{\Phi}}_{[0,n-1]}$ is the Hamiltonian in the volume V, i.e., $H^{\vec{\Phi}}_{[0,n-1]} = \sum_{\substack{V\Subset\mathbb{Z}_+ \\ V\cap[0,n-1]\neq\emptyset}} \vec{\Phi}_V$.

Hence $\vec{\gamma}^{\phi} = \gamma^{\vec{\Phi}}$, in other words, the one-sided interaction $\vec{\Phi}$ and the function ϕ are the same from the viewpoint of Thermodynamic Formalism.

The opposite of the above observation is also true. More precisely, as the following proposition states, with any continuous function ϕ, one can associate an (in fact, many) translation-invariant one-sided UAC interaction $\vec{\Phi}$ satisfying (2.63).

Proposition 2.5.3. *For any potential $\phi \in C(X_+)$, there exists a translation-invariant UAC interaction $\vec{\Phi}$ on X_+ such that $\phi = -\sum_{0\in V\Subset\mathbb{Z}_+} \vec{\Phi}_V$.*

Proof. It is an immediate application of the Stone-Weirstrass theorem to show that for the potential ϕ, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of local functions on X_+ such that $\phi = \sum_{n\in\mathbb{N}} f_n$ and $\sum_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. Let Λ_n denote the finite volume on which the local function f_n is based. We define an interaction $\vec{\Phi}$ on X_+ as follows: $\vec{\Phi}_\Lambda = -f_n$ if Λ is a translation of Λ_n, otherwise, $\vec{\Phi}_\Lambda = 0$. Then by the construction, $\vec{\Phi}$ is translation-invariant interaction on X_+ and satisfies the conditions of the proposition. □

Remark 2.5.4. *By Theorem 2.2.5 and using the arguments in Subsection 2.5.1, we summarise the correspondence between the UAC interactions and the Birkhoff sums on X_+ in the following box:*

2.5.3 G-measures and their relationship with the DLR-Gibbs measures

So far, we have not needed an algebraic structure in the state space E and the configuration space X_+. Now, we identify the state space E with the finite cyclic

(General) UAC interactions	$\Longleftrightarrow$	**Generalized Birkhoff sums**
Translation-invariant UAC interactions	$\Longleftrightarrow$	**Birkhoff sums**

group $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z} \simeq \{0, 1, \cdots, l-1\}$, where $l = |E|$. For the configurations $\omega, \omega' \in X_+$, we denote the coordinate-wise addition by $\omega + \omega'$, i.e., $\omega + \omega' = (\omega_i + \omega'_i)_{i\in\mathbb{Z}_+}$. Then, clearly, X_+ becomes a compact Abelian group. We denote the direct sum $\bigoplus_{\mathbb{Z}_+} E$ by Γ. Note that $\Gamma \subset X_+$, and $\omega \in X_+$ is in Γ iff there exists $\Delta \Subset \mathbb{Z}_+$ such that $\omega_i = 0$ for all $i \in \Delta^c$. For each $\Delta \Subset \mathbb{Z}_+$, one can define a subgroup Γ_Δ of Γ by $\Gamma_\Delta := \{\omega \in \Gamma : \omega_i = 0,\ i \in \Delta^c\}$. Then it is clear that $\Gamma = \bigcup_{\Delta \Subset \mathbb{Z}_+} \Gamma_\Delta$.

Now consider a family $G := (G_\Lambda)_{\Lambda\Subset\mathbb{Z}_+}$ of nonnegative Borel functions $G_\Lambda : X_+ \to [0,1]$. G is called *compatible* if for all $\Delta \subset \Lambda \Subset \mathbb{Z}_+$,

$$G_\Lambda(x+\omega)G_\Delta(\omega) = G_\Lambda(\omega)G_\Delta(x+\omega), \quad x \in \Gamma_\Delta, \omega \in X_+, \tag{2.65}$$

and *normalized* if for all $\Lambda \Subset \mathbb{Z}_+$,

$$\sum_{\xi_\Lambda \in E^\Lambda} G_\Lambda(\xi_\Lambda \eta_{\Lambda^c}) = 1. \tag{2.66}$$

We call a compatible normalized family $G = (G_\Lambda)_{\Lambda\Subset\mathbb{Z}_+}$ a ***G-family*** [4].

Note that each $x \in \Gamma$ defines a map $x : X_+ \to X_+$ by $\omega \mapsto x(\omega) = x + \omega$. Then let $x_* : \mathscr{M}_1(X_+) \to \mathscr{M}_1(X_+)$ denote the pushforward of x. For a probability measure $\nu \in \mathscr{M}_1(X_+)$ and a finite volume $\Lambda \Subset \mathbb{Z}_+$, define

$$\nu_\Lambda := \frac{1}{l^{|\Lambda|}} \sum_{x\in\Gamma_\Lambda} x_* \nu. \tag{2.67}$$

Then it is clear that $\nu \ll \nu_\Lambda$ since $e_*\nu = \nu$ where e is the neutral element of the group Γ.

Definition 2.5.5. [4] *A probability measure $\nu \in \mathscr{M}_1(X_+)$ is called a **G-measure**, if there exists a G-family $G = (G_\Lambda)_{\Lambda\Subset\mathbb{Z}_+}$ such that for all $\Lambda \Subset \mathbb{Z}_+$,*

$$\frac{d\nu}{d\nu_\Lambda} = l^{|\Lambda|} G_\Lambda. \tag{2.68}$$

*A $G-$measure ν is called **g-measure** if for the associated $G-$family, one has*

$$\frac{G_{[0,n+1]\cap\mathbb{Z}_+}}{G_{[0,n]\cap\mathbb{Z}_+}} = G_{\{0\}} \circ S^{n+1}, \quad n \in \mathbb{Z}_+. \tag{2.69}$$

The $g-$measures have applications in harmonic analysis, in particular, in the theory of Riesz products [20]. Inspired by these applications of g-measures, Brown and Dooley developed G-formalism by extending the concept of g-measures and utilising the formalism of Riesz products [4]. However, the following simple proposition and Theorem 2.B demonstrate that the notions of G-family and G-measure are equivalent to the notions of specification and DLR-Gibbs measures on X_+.

Proposition 2.5.6. *(i) Let $G = (G_V)_{V \Subset \mathbb{Z}_+}$ be a G-family, then $\gamma = (\gamma_V)_{V \Subset \mathbb{Z}_+}$ defined by*

$$\gamma_V(\xi_V|\eta_{V^c}) := G(\xi_V \eta_{V^c}), \ \ V \Subset \mathbb{Z}_+, \ \xi_V \in E^V, \ \eta \in X_+, \tag{2.70}$$

is a family of specification densities.

(ii) Let γ be a specification on X_+, then

$$G_V(x) := \gamma_V(x_V|x_{V^c}), \ \ x \in X_+, V \Subset \mathbb{Z}_+ \tag{2.71}$$

is a G-family.

Proof. "*(i)*" By the normalization condition (2.66), for all $\eta \in X_+$, $\sum_{\xi_V \in E^V} \gamma_V(\xi_V|\eta_{V^c}) = 1$. Then for $\Delta \subset \Lambda \Subset \mathbb{L}$, $\omega, \eta \in X_+$, by the compatibility condition (2.65),

$$\begin{aligned}
\gamma_\Delta(\eta_\Delta|\eta_{\Lambda/\Delta}\omega_{\Lambda^c}) \sum_{\sigma_\Delta \in E^\Delta} \gamma_\Lambda(\sigma_\Delta \eta_{\Lambda/\Delta}|\omega_{\Lambda^c}) &= G_\Delta(\eta_\Lambda \omega_{\Lambda^c}) \sum_{\sigma_\Delta} G_\Lambda(\sigma_\Delta \eta_{\Lambda\setminus\Delta}\omega_{\Lambda^c}) \\
&\overset{(2.65)}{=} \sum_{\sigma_\Delta} \Big(G_\Delta(\sigma_\Delta \eta_{\Lambda\setminus\Delta}\omega_{\Lambda^c}) G_\Lambda(\eta_\Lambda \omega_{\Lambda^c}) \Big) \\
&\overset{(2.66)}{=} G_\Lambda(\eta_\Lambda \omega_{\Lambda^c}) \\
&= \gamma_\Lambda(\eta_\Lambda|\omega_{\Lambda^c}).
\end{aligned}$$

"(ii)" Clearly, G is a normalized family. To prove compatibility, take any $x_\Delta \in E^\Delta$ and $\omega \in X_+$, then

$$\begin{aligned}
G_\Lambda(\omega) G_\Delta(x_\Delta \omega_{\Delta^c}) &= \gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) \gamma_\Delta(x_\Delta|\omega_{\Delta^c}) \\
&\overset{(2.2)}{=} \gamma_\Delta(x_\Delta|\omega_{\Delta^c}) \gamma_\Delta(\omega_\Delta|\omega_{\Delta^c}) \sum_{\sigma_\Delta} \gamma_\Lambda(\sigma_\Delta \omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c}) \\
&\overset{(2.2)}{=} \gamma_\Delta(\omega_\Delta|\omega_{\Delta^c}) \gamma_\Lambda(x_\Delta \omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c}) \\
&= G_\Delta(\omega) G_\Lambda(x_\Delta \omega_{\Delta^c}).
\end{aligned}$$

□

Remark 2.5.7. *The above proposition establishes a one-to-one correspondence between specifications and G−families on X_+. Furthermore, a G-family is positive (continuous) if and only if the associated specification is non-null (qusilocal).*

Theorem 2.B. *Let G be a family and γ be the associated specification. Then μ is a G-measure for G if and only if $\mu \in \mathscr{G}(\gamma)$.*

We postpone the proof of Theorem 2.B until Section 2.6.

As we have already seen in this subsection and Subsection 2.2.3, the notions of non-null specification, positive G-family and cocycle are equivalent in the half-line setup. Due to this fact, we have been able to derive equations for the associated DLR-Gibbs measures in terms of specifications and G-families. By the next theorem, we derive such an equation for the associated DLR-Gibbs measures in terms of the cocycles.

Theorem 2.5.8. *Let G be a positive G-family on X_+ and ρ be the associated measurable cocycle (c.f., (2.12) and (2.70)). Then ν is a G-measure for G, or equivalently, ν is a DLR-Gibbs measure for the corresponding specification $\gamma = \gamma^\rho$ (c.f., (2.9)), if and only if for all $y \in \Gamma$, $y_*\nu \ll \nu$, and the following equations are satisfied:*

$$\frac{d(y_*\nu)}{d\nu}(\omega) = \exp \rho(\omega, \omega - y), \quad \nu - \text{ a.e.}, \omega \in X_+. \tag{2.72}$$

Proof. ***"if":*** Assume that for all $y \in \Gamma$, $y_*\nu \ll \nu$ and (2.72) is satisfied. Then from (2.9) and (2.71), for all $\Lambda \Subset \mathbb{Z}_+$ and for $\nu-$ almost every $\omega \in X_+$,

$$\frac{d\nu_\Lambda}{d\nu}(\omega) = \frac{1}{l^{|\Lambda|}} \sum_{x \in \Gamma_\Lambda} \frac{d(x_*\nu)}{d\nu}(\omega) = \frac{1}{l^{|\Lambda|}} \sum_{x \in \Gamma_\Lambda} e^{\rho(\omega, \omega - x)} = \frac{1}{l^{|\Lambda|}} \sum_{\xi_\Lambda \in E^\Lambda} e^{-\rho(\xi_\Lambda \omega_{\Lambda^c}, \omega)}$$
$$= \frac{1}{l^{|\Lambda|} G_\Lambda(\omega)}.$$

"only if": Let ν be a G-measure for G, and take any $y \in \Gamma$, and let $y \in \Gamma_\Lambda$. Consider the measure $y_*\nu$. It can be checked that

$$(y_*\nu)_\Lambda = \frac{1}{l^{|\Lambda|}} \sum_{x \in \Gamma_\Lambda} x_*(y_*\nu) = \frac{1}{l^{|\Lambda|}} \sum_{x \in \Gamma_\Lambda} (x+y)_*\nu = \nu_\Lambda.$$

Clearly, $y_*\nu \ll \nu_\Lambda$ and let $f := \dfrac{d(y_*\nu)}{d\nu_\Lambda}$. Then for all $B \in \mathscr{F}$, one has $\nu(B - y) = \int_B f \, d\nu_\Lambda$, and hence

$$\nu(B) = \int_{X_+} f \cdot \mathbb{1}_{B+y} \, d\nu_\Lambda = \int_{X_+} f \circ y \cdot \mathbb{1}_B \, d(\nu_\Lambda \circ y). \tag{2.73}$$

Note that $\nu_\Lambda \circ y = \frac{1}{l^{|\Lambda|}} \sum_{x \in \Gamma_\Lambda} (\nu \circ x) \circ y = \frac{1}{l^{|\Lambda|}} \sum_{x \in \Gamma_\Lambda} \nu \circ (x + y) = \nu_\Lambda$. Hence (2.73) yields $\dfrac{d\nu}{d\nu_\Lambda} = f \circ y$, ν_Λ- a.e. Thus since ν is a G-measure for G, (2.68) implies that for

ν_Λ–a.e. $\omega \in X_+$, $f(\omega) = l^{|\Lambda|} G_\Lambda(\omega - y)$, therefore,

$$\frac{d(y_* \nu)}{d\nu_\Lambda} = l^{|\Lambda|} G_\Lambda \circ (-y), \quad \nu_\Lambda\text{-a.e.} . \tag{2.74}$$

Then by again applying (2.68), one has

$$\frac{d(y_* \nu)}{d\nu}(\omega) = \frac{G_\Lambda(\omega - y)}{G_\Lambda(\omega)}, \quad \nu\text{-a.e. } \omega \in X_+. \tag{2.75}$$

□

2.5.4 Gibbsianness in dynamical systems

We have defined the specifications in the half-line context in Subsection 2.5.1, and established the relationship between the generalised Birkhoff sums and the specifications. In the theory of dynamical systems, a specification associated with Birkhoff sums via (2.58) is particularly interesting. Note that in this case, for a continuous function (potential) $\phi : X_+ \to \mathbb{R}$, the specification $\vec{\gamma}^{\phi} := \vec{\gamma}^{(S_n\phi)}$ is given by

$$\vec{\gamma}^{\phi}_\Lambda(a_\Lambda | x_{\mathbb{Z}_+\setminus\Lambda}) = \frac{\exp\big((S_{n+1}\phi)(a_\Lambda x_{\mathbb{Z}_+\setminus\Lambda})\big)}{\sum\limits_{\bar{a}_\Lambda \in E^\Lambda} \exp\big((S_{n+1}\phi)(\bar{a}_\Lambda x_{\mathbb{Z}_+\setminus\Lambda})\big)}, \quad a, x \in X_+, \Lambda \Subset \mathbb{Z}_+, n := \max\Lambda. \tag{2.76}$$

It should be noted that there are Gibbsian specifications on X_+ which can not be associated with any $\phi \in C(X_+)$ via (2.76). Below we give an example of such a specification.

Example 2.5.9. *Consider the following cocycle on Ω_+:*

$$\rho(\omega, \omega') = \sum_{k=0}^{\infty} (k+1)(\omega_k - \omega'_k), \ (\omega, \omega') \in \mathfrak{T}(\Omega_+). \tag{2.77}$$

Note that ρ corresponds to the generalized Birkhoff sum $\sum\limits_{k=0}^{n-1}(k+1)(\sigma_0 \circ S^k)$, $n \geq 0$. The associated specification γ^ρ can not be generated by some $\phi \in C(X_+)$ via (2.76) (c.f. Theorem C1), since

$$\sup_{(\omega,\omega') \in \mathfrak{T}(\Omega_+)} \left| \sum_{k=1}^{\infty} \big[\sigma_0 \circ S^k(\omega) - \sigma_0 \circ S^k(\omega')\big] \right| = \infty.$$

We call a specification γ on $\mathbb{Z}_+$ ***dynamical*** if γ can be generated by some $\phi \in C(X_+)$ via (2.76). Equivalently, γ is dynamical if and only if the associated cocycle ρ^γ (c.f., (2.12)) is ***Gibbs***, i.e., for some $\phi \in C(X_+)$,

$$\rho^\gamma(\omega,\omega') = \sum_{k=0}^{\infty}\left[\phi\circ S^k(\omega) - \phi\circ S^k(\omega')\right],\ (\omega,\omega')\in\mathfrak{T}(\Omega_+). \tag{2.78}$$

Remark 2.5.10. *In fact, by the discussion at the end of Subsection 2.5.2 (see Remark 2.5.4), the dynamical specifications on X_+ are exactly the specifications which are generated by the translation-invariant UAC interactions on X_+.*

By Example 2.5.9, it is an interesting question to know when a Gibbsian specification is dynamical. Note that this question was also asked in [9]. The following statement answers this question:

Theorem 2.C. *Assume that γ is a Gibbsian specification on X_+ and ρ^γ is the corresponding cocycle. Let $\rho_0^\gamma(\omega,\omega') = \rho^\gamma(\omega,\omega') - \rho^\gamma(S\omega, S\omega')$, $(\omega,\omega')\in\mathfrak{T}(X_+)$ be the base of the cocycle ρ^γ. Then the following statements are equivalent to each other:*

- *(i) γ is a dynamical specification, in the other words, γ is generated by a translation-invariant UAC interaction on X_+;*
- *(ii) ρ^γ is a Gibbs cocycle;*
- *(iii) the cocycle ρ_0^γ can be extended from $\mathfrak{T}(X_+)\subset X_+\times X_+$ to a continuous cocycle on the full equivalence relation $X_+\times X_+$;*
- *(iv) ρ_0^γ is uniformly continuous on $\mathfrak{T}(X_+)$ in the induced topology (which is metrizable) by $X_+\times X_+$.*

It is often difficult to check the condition of Theorem 2.C, therefore, one needs more practical conditions. The following theorems give us such necessary and sufficient conditions.

Corollary C1. *Let γ be a Gibbsian specification on Ω_+ and ρ^γ be the corresponding cocycle on $\mathfrak{T}(\Omega_+)$. If γ is a dynamical specification (i.e., ρ^γ is a Gibbs cocycle), then*

$$\sup_{(\omega,\omega')\in\mathfrak{T}(\Omega_+)} |\rho^\gamma(\omega,\omega') - \rho^\gamma(S\omega, S\omega')| < \infty. \tag{2.79}$$

Corollary C2. *Assume that γ is a Gibbsian specification on Ω_+ and*

$$\rho^\gamma(\omega,\omega') = \sum_{k=0}^{\infty}[\phi_k\circ S^k(\omega) - \phi_k\circ S^k(\omega')],\ (\omega,\omega')\in\mathfrak{T}(\Omega_+),$$

is the cocycle corresponding to γ. If

$$\sum_{k=1}^{\infty}\|\phi_k - \phi_{k-1}\|_\infty < \infty \tag{2.80}$$

then γ is a dynamical specification.

We shall prove Theorem 2.C and its corollaries in Section 2.6.

2.5.5 Kozlov-Sullivan characterisation on the lattice $\mathbb{Z}_+$

By motivating Theorem 2.3.1, we define the translation-invariant specifications on X_+ as follows:

Definition 2.5.11. *A translation-invariant interaction Φ on $\mathbb{Z}_+$ is called variation-summable if*

$$\sum_{0\in V\Subset\mathbb{Z}_+} \delta_0\Phi_V < \infty.$$

*We call a specification γ on $\mathbb{Z}_+$ a **translation-invariant Gibbsian specification** if there exists a translation-invariant variation-summable interaction Φ on $\mathbb{Z}_+$ such that $\gamma = \gamma^{\Phi}$.*

Remark 2.5.12. *By definition, any dynamical specification is translation-invariant. However, any translation-invariant Gibbsian specification does not need to be dynamical (c.f., Theorem 2.D)*

Then it is an interesting question to answer if any translation-invariant Gibbsian specification on $\mathbb{Z}_+$ can be associated with a translation-invariant invariant UAC interaction on $\mathbb{Z}_+$. Note that Theorem 2.2.5 implies that any one-sided Gibbsian specification, in particular, a translation-invariant one can be associated with a UAC interaction; however, the associated interaction does not always need to be translation-invariant as the following theorem states:

Theorem 2.D. *There exists a left-shift invariant Gibbsian specification on $\mathbb{Z}_+$, which can not be associated with a translation-invariant UAC interaction.*

We shall give an explicit example of an interaction satisfying conditions of Theorem 2.D in the next section.

2.6 Proofs of the Main results

2.6.1 Proof of Theorem 2.3.4

Fix a periodic configuration $\theta \in \Omega$, i.e., $S\theta = \theta$. For every $\Lambda \Subset \mathbb{Z}$ consider a function defined by

$$\Phi^{\theta}_{\Lambda}(\omega_\Lambda) := \sum_{\substack{V\subset\Lambda \\ V\neq\emptyset}} (-1)^{|\Lambda\setminus V|} \log \frac{\gamma_V(\theta_V|\theta_{V^c})}{\gamma_V(\omega_V|\theta_{V^c})}, \quad \omega_\Lambda \in E^{\Lambda}. \tag{2.81}$$

Then $\Phi^\theta := (\Phi^\theta_\Lambda)_{\Lambda \Subset \mathbb{Z}}$ is a translation-invariant interaction on Ω [16, 18]. Using the interaction Φ^θ we define another interaction by

$$\Phi_\Lambda := \begin{cases} \sum\limits_{B \in \mathscr{S}_{\ell,i}} \Phi^\theta_B, & \text{if for some } \ell \in \mathbb{Z} \text{ and } i \in \mathbb{N},\ \Lambda = [\ell, \ell + 2^i] \cap \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.82}$$

where

$$\mathscr{S}_{\ell,1} := \{B \Subset \mathbb{Z} : \ell \in B \subset [\ell, \ell+2]\}.$$

and for $i \geq 2$

$$\mathscr{S}_{\ell,i} := \{B \Subset \mathbb{Z} : \ell \in B \subset [\ell, \ell+2^i],\ B \not\subset [\ell, \ell + 2^{i-1}]\}.$$

It can be checked that $\Phi = (\Phi_\Lambda)_{\Lambda \Subset \mathbb{Z}}$ is also a translation-invariant interaction on $\mathbb{Z}$, and under condition (2.29), Φ is also UAC and generates γ [21]. Furthermore, the following estimation is also valid for Φ: for all $\ell \in \mathbb{Z}$ and $i \geq 2$,

$$\left\| \Phi_{[\ell, \ell+2^i]} \right\|_\infty \leq C\, v(2^{i-1}), \tag{2.83}$$

where $C > 0$ is a constant independent of both $\ell \in \mathbb{Z}$ and $i \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{\min V \leq 0 < \max V} \|\Phi_V\|_\infty &= \sum_{i=1}^{\infty} \sum_{\ell=-2^i+1}^{0} \|\Phi_{[\ell,\ell+2^i]}\|_\infty \\ &= \sum_{\ell=-1}^{0} \|\Phi_{[\ell,\ell+2^i]}\|_\infty + \sum_{i=2}^{\infty} \sum_{\ell=-2^i+1}^{0} \|\Phi_{[\ell,\ell+2^i]}\|_\infty \\ &\leq \sum_{\ell=-1}^{0} \|\Phi_{[\ell,\ell+2^i]}\|_\infty + C \sum_{i=2}^{\infty} 2^{i+2} v(2^{i-1}) \\ &< \infty. \end{aligned}$$

Thus by Theorem 8.39 in [16], it can be concluded that $\#\mathscr{G}(\Omega,\Phi) = \#\mathscr{G}(\Omega,\gamma) = 1$.

2.6.2 Proof of Theorem 2.A

We start this subsection by proving the following important proposition. Fix a state $+ \in E$ and denote the constant configuration, where every component is in the state $+$ by $\mathbf{+}$.

Proposition 2.6.1. *(i) The sequence* $\left\{ -\frac{1}{|\Lambda_n|} \log \gamma_{\Lambda_n}(\mathbf{+}_{\Lambda_n} | \omega_{\Lambda_n^c}) \right\}_{n \in \mathbb{N}}$ *converges uniformly on* $\omega \in \Omega$ *to a constant* $P^+(\gamma)$ *as* $n \to \infty$.

(ii) For $\rho \in \mathscr{M}_{1,S}(\Omega)$, the sequence $\Big\{-\frac{1}{|\Lambda_n|}\int\limits_{\Omega}\log\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})\rho(d\theta)\Big\}_{n\in\mathbb{N}}$ *converges uniformly on $\omega\in\Omega$ to a constant $P^\rho(\gamma)$ which depends only on γ and ρ as $n\to\infty$.*

Proof. Part (i): The proof of the first part of Proposition 2.6.1 presented below, relies on the following lemma, whose proof can be found in [22].

Lemma 2.6.2. *For the translation-invariant Gibbsian specification γ, one has*

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sup_{\xi,\eta\in\Omega}\log\frac{\gamma_{\Lambda_n}(+_{\Lambda_n}|\xi_{\Lambda_n^c})}{\gamma_{\Lambda_n}(+_{\Lambda_n}|\eta_{\Lambda_n^c})}=0. \tag{2.84}$$

In the light of Lemma 2.6.2, the uniform convergence of the functional sequence

$$\Big\{\frac{1}{|\Lambda_n|}\log\gamma_{\Lambda_n}(+_{\Lambda_n}|\bullet_{\Lambda_n^c})\Big\}_{n\in\mathbb{N}}$$

follows if we show that the numerical sequence $\Big\{\frac{1}{|\Lambda_n|}\log\gamma_{\Lambda_n}(+_{\Lambda_n}|+_{\Lambda_n^c})\Big\}_{n\in\mathbb{N}}$ converges.

We shall consider the vacuum interaction Φ^+ corresponding to the vacuum configuration $+$. By the vacuum condition, one has that

$$\gamma_{\Lambda_n}(+_{\Lambda_n}|\omega_{\Lambda_n^c})=Z_{\Lambda_n}^{\Phi^+}(\omega_{\Lambda_n^c})^{-1}, \tag{2.85}$$

where $Z_{\Lambda_n}^{\Phi^+}(\omega_{\Lambda_n^c})$ is the partition function corresponding to the interaction Φ^+ in the volume Λ_n. Thus we have to prove that the *infinite volume pressure* is well-defined for the interaction Φ^+. Then this implies that $\lim\limits_{n\to\infty}\frac{1}{|\Lambda_n|}\log\gamma_{\Lambda_n}(+_{\Lambda_n}|\omega_{\Lambda_n^c})=-P(\Phi^+)$. Note that in general, the interaction Φ^+ is not UAC, however, it is always *uniformly convergent.* Therefore, the classical theorems in Statistical Mechanics do not apply to guarantee the existence of the pressure $P(\Phi^+)$. For every $r\in\mathbb{N}$, we consider a finite-range interaction $\Phi^{+,r}$ defined by

$$\Phi_\Lambda^{+,r}:=\begin{cases}\Phi_\Lambda^+, & \text{if } \mathtt{diam}(\Lambda)\le r;\\ 0, & \text{if } \mathtt{diam}(\Lambda)>r.\end{cases} \tag{2.86}$$

Note that for every $r\in\mathbb{N}$, the interaction $\Phi^{+,r}$ is finite-range, therefore, the sequence $\Big\{\frac{1}{|\Lambda_n|}\log Z_{\Lambda_n}^{\Phi^{+,r}}(\omega)\Big\}_{n\in\mathbb{N}}$ converges uniformly on $\omega\in\Omega$ to the pressure $P(\Phi^{+,r})$ as $n\to\infty$. For every $r\in\mathbb{N}$, one can readily check by the vacuum property of the interaction Φ^+ that

$$
\begin{aligned}
\log \frac{Z^{\Phi^+}_{\Lambda_n}(+)}{Z^{\Phi^{+,r}}_{\Lambda_n}(+)} &= \log \frac{\sum\limits_{\omega\in\Omega_{\Lambda_n}} e^{-\sum\limits_{V\subset\Lambda_n}\Phi^+_V(\omega)}}{\sum\limits_{\omega\in\Omega_{\Lambda_n}} e^{-\sum\limits_{V\subset\Lambda_n}\Phi^{+,r}_V(\omega)}} \\
&\leq \sup_{\omega\in\Omega_{\Lambda_n}} \Big| \sum_{\substack{V\subset\Lambda_n \\ \mathtt{diam}(V)>r}} \Phi^+_V(\omega)\Big| \\
&\leq \sum_{i\in\Lambda_n} \sup_{\omega\in\Omega} \Big| \sum_{\substack{i\in V\Subset\mathbb{Z} \\ \mathtt{diam}(V)>r}} \Phi^+_V(\omega)\Big| \\
&= |\Lambda_n| \sup_{\omega\in\Omega} \Big| \sum_{\substack{0\in V\Subset\mathbb{Z} \\ \mathtt{diam}(V)>r}} \Phi^+_V(\omega)\Big|. \qquad (2.87)
\end{aligned}
$$

Thus, in particular, for any $s, r \in \mathbb{N}$, one also has

$$
\begin{aligned}
\Big|\log \frac{Z^{\Phi^{+,s}}_{\Lambda_n}(+)}{Z^{\Phi^{+,r}}_{\Lambda_n}(+)}\Big| &= \Big|\log \frac{Z^{\Phi^+}_{\Lambda_n}(+)}{Z^{\Phi^{+,r}}_{\Lambda_n}(+)} - \log \frac{Z^{\Phi^+}_{\Lambda_n}(+)}{Z^{\Phi^{+,s}}_{\Lambda_n}(+)}\Big| \\
&\leq 2|\Lambda_n| \sup_{\omega\in\Omega} \Big| \sum_{\substack{0\in V\Subset\mathbb{Z} \\ \mathtt{diam}(V)>\min\{s,r\}}} \Phi^+_V(\omega)\Big|. \qquad (2.88)
\end{aligned}
$$

Thus by dividing both sides by $|\Lambda_n|$ and taking limit as $n \to \infty$, one obtains that for all $s, r \in \mathbb{N}$,

$$
|P(\Phi^{+,s}) - P(\Phi^{+,r})| \leq 2 \cdot \sup_{\omega\in\Omega} \Big| \sum_{\substack{0\in V\Subset\mathbb{Z} \\ \mathtt{diam}(V)>\min\{s,r\}}} \Phi^+_V(\omega)\Big|. \qquad (2.89)
$$

Since the interaction Φ^+ is uniformly convergent, one has that

$$
2 \cdot \sup_{\omega\in\Omega} \Big| \sum_{\substack{0\in V\Subset\mathbb{Z} \\ \mathtt{diam}(V)>n}} \Phi^+_V(\omega)\Big| \xrightarrow[n\to\infty]{} 0. \qquad (2.90)
$$

Hence (2.89) yields that the sequence $\{P(\Phi^{+,r})\}_{r\in\mathbb{N}}$ is fundamental. Thus one can immediately conclude from (2.87) that the sequence $\Big\{\frac{1}{|\Lambda_n|}\log Z^{\Phi^+}_{\Lambda_n}(+)\Big\}_{n\in\mathbb{N}}$ is convergent.

Part (ii): By the first part of Lemma 3.7 in [22], one has that

$$
\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int_\Omega \log \frac{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(+_{\Lambda_n}|\omega_{\Lambda_n^c})}\rho(d\theta) = \int_\Omega \log \frac{\gamma_{\{0\}}(\theta_{\mathbf{0}}|+_{<\mathbf{0}}\,\theta_{>\mathbf{0}})}{\gamma_{\{0\}}(+_{\mathbf{0}}|+_{<\mathbf{0}}\,\theta_{>\mathbf{0}})}\rho(d\theta), \qquad (2.91)
$$

and the limit in LHS is uniform on $\omega\in\Omega$. Thus by the first part, one can conclude the second part of the theorem, since

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\int\limits_{\Omega}\log\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})\rho(d\theta) &= \lim_{n\to\infty}\frac{1}{|\Lambda_n|}\log\gamma_{\Lambda_n}(+_{\Lambda_n}|\omega_{\Lambda_n^c})\\ &+ \int\limits_{\Omega}\log\frac{\gamma_{\{0\}}(\theta_{\mathbf{0}}|+_{<\mathbf{0}}\theta_{>\mathbf{0}})}{\gamma_{\{0\}}(+_{\mathbf{0}}|+_{<\mathbf{0}}\theta_{>\mathbf{0}})}\rho(d\theta).\end{aligned}$$

□

Below we prove an important statement for the energy contribution from the origin, which will be useful in the proof of Theorem 2.A.

Proposition 2.6.3. *Let γ be a translation-invariant Gibbsian specification on Ω and u^{ρ}_{γ} be the associated energy contribution from the origin as defined in (2.43). Then for all translation-invariant probability measure $\tau\in\mathcal{M}_{1,S}(\Omega)$, one has that*

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\int_{\Omega}\int_{\Omega}\log\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}\rho(d\theta)\tau(d\omega)=\int_{\Omega}u^{\rho}_{\gamma}(\omega)\tau(d\omega). \tag{2.92}$$

Proof of Proposition 2.6.3. For the specification densities γ_Λ, $\Lambda\Subset\mathbb{Z}^d$, we use the notations $\gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c})$ and $\gamma_\Lambda(\omega)$ interchangeably. By the bar moving the property of the specification densities, one has that

$$\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}=\prod_{i\in\Lambda_n}\frac{\gamma_{\{i\}}(\omega_i|\theta_{[\min\Lambda_n,i)}\omega_{[\min\Lambda_n,i]^c})}{\gamma_{\{i\}}(\theta_i|\theta_{[\min\Lambda_n,i)}\omega_{[\min\Lambda_n,i]^c})}. \tag{2.93}$$

For any $\ell_1,\ell_2\in\mathbb{Z}^d$ with $\ell_1\le\ell_2$ and $\theta\in\Omega$, define a transformation

$$\Theta^{\theta}_{[\ell_1,\ell_2]}(\omega):=\begin{cases}\omega_i, & \text{if } i\notin[\ell_1,\ell_2];\\ \theta_i, & \text{if } i\in[\ell_1,\ell_2].\end{cases} \tag{2.94}$$

If $\ell_1\not\le\ell_2$, then $[\ell_1,\ell_2]=\emptyset$; therefore, for all $\omega\in\Omega$, we set $\Theta_{[\ell_1\ell_2]}(\omega)=\omega$. Note that the families $\{S_\ell:\ell\in\mathbb{Z}^d\}$ and $\{\Theta_{[\ell_1,\ell_2]}:\ell_1,\ell_2\in\mathbb{Z}^d,\ell_1\le\ell_2\}$ of the transformations on Ω have the following commutativity-type property: for any $\ell_{1,2,3}\in\mathbb{Z}^d$ with $\ell_1\le\ell_2$, one has

$$S_{\ell_3}\circ\Theta^{\theta}_{[\ell_1,\ell_2]}=\Theta^{S_{\ell_3}\theta}_{[\ell_1-\ell_3,\ell_2-\ell_3]}\circ S_{\ell_3}. \tag{2.95}$$

Using the transformations $\{\Theta^{\theta}_{[\ell_1,\ell_2]}\}_{\ell_1\le\ell_2}$, (2.93) can be written as

$$\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}=\prod_{i\in\Lambda_n}\frac{\gamma_{\{i\}}\circ\Theta^{\theta}_{[\min\Lambda_n,i)}(\omega)}{\gamma_{\{i\}}\circ\Theta^{\theta}_{[\min\Lambda_n,i]}(\omega)}. \tag{2.96}$$

By the translation-invariance of the specification γ, one can write (2.96) as

$$\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}=\prod_{i\in\Lambda_n}\frac{\gamma_{\{\mathbf{0}\}}\circ S_i\circ\Theta^{\theta}_{[\min\Lambda_n,i)}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ S_i\circ\Theta^{\theta}_{[\min\Lambda_n,i]}(\omega)}. \tag{2.97}$$

Hence by the translation-invariance of the measures τ and ρ, one can obtain from (2.95) and Fubini's theorem that

$$\frac{1}{|\Lambda_n|}\int_\Omega\int_\Omega\log\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}\rho(d\theta)\tau(d\omega)=$$

$$=\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n}\int_\Omega\int_\Omega\log\frac{\gamma_{\{\mathbf{0}\}}\circ S_i\circ\Theta^{\theta}_{[\min\Lambda_n,i)}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ S_i\circ\Theta^{\theta}_{[\min\Lambda_n,i]}(\omega)}\rho(d\theta)\tau(d\omega)$$

$$=\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n}\int_\Omega\int_\Omega\log\frac{\gamma_{\{\mathbf{0}\}}\circ\Theta^{S_i\theta}_{[\min\Lambda_n-i,\mathbf{0})}\circ S_i(\omega)}{\gamma_{\{\mathbf{0}\}}\circ\Theta^{S_i\theta}_{[\min\Lambda_n-i,\mathbf{0}]}\circ S_i(\omega)}\rho(d\theta)\tau(d\omega)$$

$$=\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n+\min\Lambda_n}\int_\Omega\int_\Omega\log\frac{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0})}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0}]}(\omega)}\rho(d\theta)\tau(d\omega). \tag{2.98}$$

By quasilocality and non-nullness of γ, the net $\left\{\log\frac{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0})}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0}]}(\omega)}\right\}_{i<\mathbf{0}}$ converges uniformly in $(\theta,\omega)\in\Omega\times\Omega$ to $u_\gamma(\theta,\omega):=\log\frac{\gamma_{\{\mathbf{0}\}}(\omega_{\mathbf{0}}|\theta_{<\mathbf{0}}\omega_{>\mathbf{0}})}{\gamma_{\{\mathbf{0}\}}(\theta_{\mathbf{0}}|\theta_{<\mathbf{0}}\omega_{>\mathbf{0}})}$ as $i_1,\dots,i_d\to-\infty$. Hence, by the Stolz-Cesaro theorem, one has that

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n+\min\Lambda_n}\log\frac{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0})}(\omega)}{\gamma_{\{\mathbf{0}\}}\circ\Theta^{\theta}_{[i,\mathbf{0}]}(\omega)}=u_\gamma(\theta,\omega)\text{ uniformly on }(\theta,\omega)\in\Omega\times\Omega. \tag{2.99}$$

Thus (2.98) yields the desired claim.

□

Remark 2.6.4. *If the telescoping starts from the other end, one can end up with the following instead of (2.93):*

$$\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}=\prod_{i\in\Lambda_n}\frac{\gamma_{\{i\}}(\omega_i|\theta_{(i,\max\Lambda_n]}\omega_{[i,\max\Lambda_n]^c})}{\gamma_{\{i\}}(\theta_i|\theta_{(i,\max\Lambda_n]}\omega_{[i,\max\Lambda_n]^c})}. \tag{2.100}$$

Then, with a similar argument to the proof of Proposition 2.6.3, one can also show *that*

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\int\limits_{\Omega}\int\limits_{\Omega}\log\frac{\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})}{\gamma_{\Lambda_n}(\theta_{\Lambda_n}|\omega_{\Lambda_n^c})}\rho(d\theta)\tau(d\omega)$$
$$=\int\limits_{\Omega}\tilde{u}^{\rho}(\omega)\tau(d\omega)=\int\limits_{\Omega}\int\limits_{\Omega}\tilde{u}_{\gamma}(\theta,\omega)\rho(d\theta)\tau(d\omega),$$

where

$$\tilde{u}_{\gamma}(\theta,\omega):=\log\frac{\gamma_{\{\mathbf{0}\}}(\omega_{\mathbf{0}}|\omega_{<\mathbf{0}}\theta_{>\mathbf{0}})}{\gamma_{\{\mathbf{0}\}}(\theta_{\mathbf{0}}|\omega_{<\mathbf{0}}\theta_{>\mathbf{0}})}\ \textit{ and }\ \tilde{u}_{\gamma}^{\rho}(\omega):=\int\limits_{\Omega}\tilde{u}_{\gamma}(\theta,\omega)\rho(d\theta).$$

In particular, u_{γ}^{ρ} *and* $\tilde{u}_{\gamma}^{\rho}$ *are physically equivalent, i.e., for all* $\tau\in\mathcal{M}_{1,S}(\Omega)$,

$$\int_{\Omega}u_{\gamma}^{\rho}\,d\tau=\int_{\Omega}\tilde{u}_{\gamma}^{\rho}\,d\tau. \tag{2.101}$$

Proof of Theorem 2.A. For a translation-invariant measure $\tau\in\mathcal{M}_{1,S}(\Omega)$ and a translation -invariant Gibbs measure $\bar{\mu}\in\mathcal{G}_S(\gamma)$, it is proven in the first part of Theorem 3.3 in [22] that

$$P^{+}(\gamma)-\int_{\Omega}u_{\gamma}^{+}\,d\tau-h(\tau)=h(\tau|\bar{\mu}). \tag{2.102}$$

For any $\rho\in\mathcal{M}_{1,S}(\Omega)$, we have proven in Proposition 2.6.1 and Proposition 2.6.3 that

$$\lim_{n\to\infty}-\frac{1}{|\Lambda_n|}\int_{\Omega}\log\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})\tau(d\omega)=P^{\rho}(\gamma)-\int_{\Omega}u_{\gamma}^{\rho}(\omega)\tau(d\omega). \tag{2.103}$$

Note that LHS of (2.103) is independent of ρ, thus (2.102) yields for $\rho=\delta_{+}$,

$$\lim_{n\to\infty}-\frac{1}{|\Lambda_n|}\int_{\Omega}\log\gamma_{\Lambda_n}(\omega_{\Lambda_n}|\omega_{\Lambda_n^c})\tau(d\omega)=h(\tau)+h(\tau|\bar{\mu}). \tag{2.104}$$

Then by combining (2.104) with (2.103), we obtain that

$$\int_{\Omega}u_{\gamma}^{\rho}(\omega)\tau(d\omega)+h(\tau)=P^{\rho}(\gamma)-h(\tau|\bar{\mu}). \tag{2.105}$$

Then the rest of the proof follows from Theorem 2.4.3.

□

2.6.3 Proofs of Theorems 2.B, 2.C, and 2.D

Proof of Theorem 2.B. Consider a probability measure $\tau \in \mathscr{M}_1(X_+)$. Then for $\Lambda \Subset \mathbb{Z}_+$ and $B \in \mathscr{F}$, one has

$$\begin{aligned}(G_\Lambda \cdot \tau_\Lambda)(B) &:= \int_{X_+} G_\Lambda \mathbb{1}_B \, d\tau_\Lambda \\ &= \frac{1}{|\Gamma_\Lambda|} \sum_{x \in \Gamma_\Lambda} \int_{X_+} G_\Lambda \mathbb{1}_B \, d x_* \tau \\ &= \frac{1}{|\Gamma_\Lambda|} \sum_{x \in \Gamma_\Lambda} \int_{X_+} G_\Lambda(x+\omega) \mathbb{1}_B(x+\omega) \tau(d\omega), \end{aligned} \tag{2.106}$$

and

$$(\tau\gamma_\Lambda)(B) = \sum_{x \in \Gamma_\Lambda} \int_{X_+} G_\Lambda(\omega + x) \mathbb{1}_B(\omega + x) \tau(d\omega). \tag{2.107}$$

Then by combining (2.106) and (2.107), one concludes that for all $\Lambda \Subset \mathbb{Z}_+$,

$$\tau\gamma_\Lambda = (l^{|\Lambda|} G_\Lambda) \cdot \tau_\Lambda. \tag{2.108}$$

Note that in the light of (2.108), the DLR equation (2.4) and the consistency equation (2.68) for the G-measures are the same. Thus one immediately concludes the statement of the theorem. □

Proof of Theorem 2.C. "$(i) \Leftrightarrow (ii) \Rightarrow (iii)$" Let γ be a dynamical specification, then there exists $\varphi \in C(X_+)$ such that the corresponding cocycle ρ^γ is given by

$$\rho^\gamma(\omega, \bar{\omega}) = \sum_{k=0}^{\infty} [\varphi \circ S^k(\omega) - \varphi \circ S^k(\bar{\omega})], \; (\omega, \bar{\omega}) \in \mathfrak{T}(X_+).$$

Thus for all $(\omega, \bar{\omega}) \in \mathfrak{T}(X_+)$, $\rho_0^\gamma(\omega, \bar{\omega}) = \varphi(\omega) - \varphi(\bar{\omega})$, and then it is clear that ρ_0^γ is extended to the cocycle $\lambda : X_+ \times X_+ \to \mathbb{R}$ which is defined by $\lambda(\eta, \xi) = \varphi(\eta) - \varphi(\xi)$, $(\eta, \xi) \in X_+ \times X_+$. Clearly, λ is continuous since φ is continuous.

"$(iii) \Rightarrow (ii)$" Assume that ρ_0^γ extends to a continuous cocycle λ on the full equivalence relation $X_+ \times X_+$. Fix any $\tilde{\xi} \in X_+ \times X_+$ and for $\omega \in X_+$, set $\varphi(\omega) = \lambda(\omega, \tilde{\xi})$. Then for any $(\omega, \xi) \in X_+ \times X_+$,

$$\lambda(\omega, \xi) = \lambda(\omega, \tilde{\xi}) + \lambda(\tilde{\xi}, \xi) = \lambda(\omega, \xi) - \lambda(\xi, \tilde{\xi}) = \varphi(\omega) - \varphi(\xi). \tag{2.109}$$

The continuity of λ implies the continuity of φ. Furthermore, (2.109) immediately implies that for all $(\omega, \bar{\omega}) \in \mathfrak{T}(X_+)$, $\rho_0^\gamma(\omega, \bar{\omega}) = \lambda(\omega, \bar{\omega}) = \varphi(\omega) - \varphi(\bar{\omega})$ Then by (2.61) one concludes that ρ^γ is indeed a Gibbs cocycle.

"$(iii) \Rightarrow (iv)$" The extension of ρ_0^γ to $X_+ \times X_+$ is continuous and the product space $X_+ \times X_+$ is compact. Then as a continuous function on the compact space, the extension is uniformly continuous, thus its restriction to $\mathfrak{T}(X_+)$ which is ρ_0^γ is also uniformly continuous.

"$(iv) \Rightarrow (iii)$" First, note that $\mathfrak{T}(X_+)$ is dense in $X_+ \times X_+$ in the product topology. Now take a pair $(\xi, \eta) \in X_+ \times X_+$ of configurations, and let the sequence $\{(\xi^{(n)}, \eta^{(n)})\}_{n\in\mathbb{N}} \subset \mathfrak{T}(X_+)$ converges to (ξ, η) in the product topology. Since $\rho_0^\gamma : \mathfrak{T}(X_+) \to \mathbb{R}$ is a uniformly continuous function, the sequence $\{\rho_0^\gamma(\xi^{(n)}, \eta^{(n)})\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is convergent because a uniformly continuous function preserves fundamentality of the sequences. Hence we can define an extension $\bar{\rho}_0^\gamma : X_+ \times X_+$ by $\bar{\rho}_0^\gamma(\xi, \eta) := \lim_{n\to\infty} \rho_0^\gamma(\xi^{(n)}, \eta^{(n)})$. It is clear that the value of $\bar{\rho}_0^\gamma(\xi, \eta)$ does not depend on the choice of the sequence $\{(\xi^{(n)}, \eta^{(n)})\}_{n\in\mathbb{N}}$ and by the construction, $\bar{\rho}_0^\gamma$ is continuous on $X_+ \times X_+$ and extends ρ_0^γ. □

Below we give the proofs of corollaries of Theorem 2.C.

Proof of Corollary C1. By Theorem 2.C, ρ_0^γ can be extended to a continuous cocycle on $X_+ \times X_+$. Since $X_+ \times X_+$ is compact, the extension is bounded, thus the statement of the theorem follows. □

Proof of Corollary C2. In fact, one can check the following for ρ_0^γ: for all $(\omega, \bar{\omega}) \in \mathfrak{T}(X_+)$,

$$\rho_0^\gamma(\omega, \bar{\omega}) = \phi_0(\omega) - \phi_0(\bar{\omega}) + \sum_{k=1}^{\infty} [(\phi_k - \phi_{k-1}) \circ S^k(\omega) - (\phi_k - \phi_{k-1}) \circ S^k(\bar{\omega})].$$

Then if (2.80) is satisfied then the sum $\sum_{k=1}^{\infty} [(\phi_k - \phi_{k-1}) \circ S^k(\xi) - (\phi_k - \phi_{k-1}) \circ S^k(\eta)]$ converges uniformly on $(\xi, \eta) \in X_+ \times X_+$, therefore, the cocycle λ defined by

$$\lambda(\xi, \eta) = \phi_0(\xi) - \phi_0(\eta) + \sum_{k=1}^{\infty} [(\phi_k - \phi_{k-1}) \circ S^k(\xi) - (\phi_k - \phi_{k-1}) \circ S^k(\eta)], \ (\xi, \eta) \in \mathfrak{T}(X_+)$$

is continuous on $X_+ \times X_+$. ρ_0^γ extends to λ, thus by Theorem 2.C, γ is indeed a dynamical specification. □

Proofs of Theorem 2.D. Consider the Ising spin space $E = \{-1, 1\}$ and the following interaction Ψ on X_+: for $\omega \in X_+$, set

$$\Psi_\Lambda(\omega) := \begin{cases} -\dfrac{\omega_i \omega_j}{|i-j|^\alpha} + \kappa \omega_j, & \text{if } \Lambda = \{i, j\} \subset \mathbb{Z}_+, \ i < j; \\ 0, & \text{otherwise,} \end{cases} \tag{2.110}$$

where $\alpha > 1$ and $\kappa > 0$. Then Ψ is translation-shift invariant, but it is not UAC. For $i, j \in \mathbb{Z}_+$, with $i < j$, $||\Psi_{\{i,j\}}||_\infty = \kappa + \dfrac{1}{|i-j|^\alpha}$. Thus $\sum_{0 \in V \Subset \mathbb{Z}_+} ||\Psi_V||_\infty = \sum_{j=1}^{\infty} \Big(\kappa +$

$\left.\frac{1}{j^\alpha}\right) = \infty$. However, Ψ is a variation-summable interaction. In fact, for all $j \in \mathbb{N}$, $\delta_0 \Psi_{\{0,j\}} = \frac{2}{j^\alpha}$, therefore, $\sum_{0 \in V \Subset \mathbb{Z}_+} \delta_0 \Psi_V = \sum_{j=1}^{\infty} \frac{2}{j^\alpha} < \infty$. Thus the associated one-sided specification $\overrightarrow{\gamma}^{\Psi}$ is Gibbsian, which is the same amount as saying that the associated cocycle ρ^Ψ is continuous. Furthermore, for $(\omega, \bar{\omega}) \in \mathfrak{T}(X_+)$,

$$\begin{aligned}
\rho_0^\Psi(\omega, \bar{\omega}) &= \rho^\Psi(\omega, \bar{\omega}) - \rho^\Psi(S\omega, S\bar{\omega}) \\
&= \sum_{0 \in V \Subset \mathbb{Z}_+} [\Psi_V(\omega) - \Psi_V(\bar{\omega})] \\
&= \sum_{j=1}^{\infty} \Big[\frac{\bar{\omega}_0 \bar{\omega}_j - \omega_0 \omega_j}{j^\alpha} + \kappa(\omega_j - \bar{\omega}_j) \Big].
\end{aligned}$$

Thus, $\sup_{(\omega,\bar{\omega}) \in \mathfrak{T}(X_+)} \rho_0^\Psi(\omega, \bar{\omega}) = \infty$ which contradicts to Theorem C1. Therefore, the corresponding specification γ^Ψ is not dynamical, which is equivalent to saying that γ^Ψ is not generated by any translation-invariant UAC interaction (see Remark 2.5.10 and the discussion at the end of Subsection 2.5.1).

□

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Chapter 3

Gibbs Properties of Equilibrium States

Abstract: In this chapter, we consider the problem of equivalence of Gibbs states and equilibrium states for continuous potentials on full shift spaces $E^{\mathbb{Z}}$. Sinai, Bowen, Ruelle and others established equivalence under various assumptions on the potential ϕ. At the same time, it is known that every ergodic measure is an equilibrium state for some continuous potential. This means that the equivalence can occur only under some appropriate conditions on the potential function. In this chapter, we identify the necessary and sufficient conditions for the equivalence.

This chapter is based on M. Makhmudov, E. Verbitskiy, *"Gibbs Properties of Equilibrium States"*, arXiv:2503.15263.

3.1 Introduction

DLR Gibbs measures were introduced by Dobrushin (1968) and Lanford and Ruelle (1969) to describe the collective behaviour of a system composed of a large number of components, each governed by a local law. Soon after, the Gibbs measures found applications in other fields of science and various areas of mathematics. In particular, in the early 1970's, Sinai showed that natural invariant measures for hyperbolic dynamical systems are Gibbs measures. The original definition of Gibbs measures in statistical mechanics is somewhat cumbersome in the context of dynamical systems. For this reason, Bowen [3] provided a more suitable definition of Gibbs states from the dynamical systems perspective: a translation-invariant measure μ on $\Omega = E^{\mathbb{Z}}$, E is finite, is called *Gibbs in Bowen's sense*, or *Bowen-Gibbs*, for a continuous potential $\phi : \Omega \to \mathbb{R}$, if for some constants $C > 1$ and P, for all $n \in \mathbb{N}$ and every $\omega \in \Omega$

$$\frac{1}{C} \leq \frac{\mu\big(\{\tilde{\omega} \in \Omega : \tilde{\omega}_0^{n-1} = \omega_0^{n-1}\}\big)}{\exp\big(S_n\phi(\omega) - nP\big)} \leq C, \tag{3.1}$$

where $S_n\phi(\omega) = \sum_{k=0}^{n-1} \phi(S^k\omega)$ and $S : \Omega \to \Omega$ is the left shift on Ω. Subsequently, weaker versions of this notion were introduced. Namely, a translation-invariant measure μ on Ω is *weak Bowen-Gibbs* if μ satisfies

$$\frac{1}{C_n} \leq \frac{\mu(\{\tilde{\omega} \in \Omega : \tilde{\omega}_0^{n-1} = \omega_0^{n-1}\})}{e^{S_n\phi(\omega) - nP}} \leq C_n, \tag{3.2}$$

for some subexponential sequence $\{C_n\}$ of positive real numbers, i.e., $\log C_n = o(n)$.

Let us stress that Bowen's definition of Gibbs measures is actually a theorem in Statistical Mechanics. More specifically, if μ is a translation-invariant DLR-Gibbs measure (see Section 3.2 for the notion), then there exists a continuous function $\phi : \Omega \to \mathbb{R}$ and positive numbers $C_n = C_n(\phi)$ such that (3.2) holds. Therefore, we prefer to use the name of Gibbs measures for measures which are Gibbs in the DLR sense, and we refer to the measures satisfying (3.1) and (3.2) as Bowen-Gibbs and weak Bowen-Gibbs measures.

The notion of weak Gibbs states in Bowen's sense is also somewhat misleading, as it suggests some form of non-Gibbsianity and competes with a notion under the same name in Statistical Mechanics. As we will see below, weak Bowen-Gibbs measures can be bona-fide Gibbs measures in the DLR sense. We also note that there are examples of weak Gibbs measures in the DLR sense, which are weak Bowen-Gibbs as well [17].

Bowen's definition (3.1) and its weak form (3.2) are extremely convenient from the Dynamical Systems point of view. At the same time, using such definitions, one can, in principle, say very little about the conditional probabilities of the underlying measure, which is the classical approach to Gibbs measures in Statistical Mechanics. In fact, there exist Bowen-Gibbs measures that are not DLR-Gibbs measures [2, Subsection 5.2].

Another important notion is that of equilibrium states. A translation-invariant measure μ on $\Omega = E^{\mathbb{Z}}$ is called an *equilibrium state* for a (continuous) potential $\phi : \Omega \to \mathbb{R}$ if

$$h(\mu) + \int_{\Omega} \phi \, d\mu = P(\phi), \tag{3.3}$$

where $P(\phi)$ is the topological pressure of ϕ and $h(\mu)$ is the measure-theoretic entropy of μ. For expansive systems like those we consider in this chapter, equilibrium states always exist. It is easy to see that a weak Bowen-Gibbs state is also an equilibrium state for the same potential [19].

A fundamental result highlighting the breadth of the class of equilibrium states is the following: if $\mu_1, \ldots, \mu_k$ are some ergodic measures on Ω, then one can find a continuous potential $\phi \in C(\Omega)$ such that all these measures are equilibrium states for ϕ [7, 12, 22]. This remarkable generality suggests that equilibrium states can exhibit a wide range of behaviors, and in particular, one cannot expect them to possess any form of Gibbsianity in general. This leads to a natural question: under what conditions on ϕ are the equilibrium states Gibbs, either in the DLR or the Bowen sense? This question has a long history of research:

- For *Hölder continuous* potentials, Sinai proved that equilibrium states are Gibbs in the DLR sense [23, Theorem 1], and the Bowen-Gibbs property was established by Bowen [3]. Haydn extended Sinai's results to the non-symbolic setup [9, 10].

- Ruelle [21] (see also [14, Theorem 5.3.1]) studied the DLR Gibbsianity of the unique equilibrium states for potential ϕ with *summable variations*:

  $$\sum_{n\geq 1} \mathtt{var}_n \phi < \infty, \quad \mathtt{var}_n \phi := \sup\left\{\phi(\omega) - \phi(\bar{\omega}) : \omega_j = \bar{\omega}_j,\ 0 \leq j \leq n-1\right\}.$$

  The Bowen-Gibbs property was treated by Keller [14, Theorem 5.2.4, (c)]

- Walters [25] considered potentials satisfying even a weaker condition:

  $$\lim_{p\to\infty} \sup_{n\in\mathbb{N}} \mathtt{var}_{[-p,n+p]} S_{n+1}\phi = 0, \text{ here } S_{n+1}\phi = \sum_{i=0}^{n} \phi \circ S^i. \tag{3.4}$$

  Walters showed that there exists a unique equilibrium state for potentials satisfying (3.4), and they have the so-called, $g-$measure property, which amounts to saying that the equilibrium state has continuous one-sided conditional probabilities. Combining this with the result of [2], one concludes the DLR Gibbsianity of the unique equilibrium state of a potential in the Walters class. In [11], Haydn and Ruelle extended this to a more general setup than the setup of shift spaces. In the same paper, Haydn and Ruelle also established the Bowen-Gibbs property of the unique equilibrium state for a potential satisfying the *Bowen condition*:

$$\sup_{n\in\mathbb{N}} \mathtt{var}_{[-n,n]} S_{n+1}\phi < +\infty,$$

which is slightly weaker than Walters' original condition.

- More recently, Pfister and Sullivan [20] established the weak Bowen-Gibbs property of equilibrium states for potentials ϕ with *summable oscillations*:

$$\sum_{i=-\infty}^{\infty} \delta_i\phi < +\infty, \quad \delta_i\phi := \sup\{\phi(\omega)-\phi(\bar{\omega}) : \omega_j = \bar{\omega}_j,\ j\neq i\}.$$

  Unlike the preceding conditions, the summable oscillations condition does not imply the uniqueness of the corresponding equilibrium states. However, the DLR Gibbs property of equilibrium states under summable oscillations has not been addressed.

In this chapter, we continue the long line of research on the Gibbsianity of equilibrium states, as discussed above, and also extend the main result of [2], where a similar question has been answered in the case of $g-$measures. In this chapter, we show that under a similar assumption on the potential ϕ, one can establish the Gibbs properties of equilibrium states and vice versa. This assumption on the regularity of the potential $\phi : \Omega \to \mathbb{R}$ is the *extensibility condition*, which requires that for all $a_0, b_0 \in E$ the sequence of functions

$$\rho_n^{a_0,b_0}(\omega) := \sum_{i=-n}^{n} \big(\phi\circ S^i(\omega_{-\infty}^{-1} b_0 \omega_1^{\infty}) - \phi\circ S^i(\omega_{-\infty}^{-1} a_0 \omega_1^{\infty})\big)$$

converges uniformly in $\omega\in\Omega$ as $n\to\infty$. The extensibility condition is not very restrictive. For example, it does not imply the uniqueness of the equilibrium states, unlike the results by Sinai, Bowen, Ruelle and Walters. Furthermore, the extensibility condition covers the previously treated classes, including the class of Hölder continuous potentials, potentials with summable variations, Walter's class, as well as the class of potentials with summable oscillations. However, the potentials in Bowen's class do not necessarily have the extensibility property [2, Section 5.5]. An important example of extensible potentials is the Dyson potential, $\phi^D(\omega) := h\omega_0 + \sum_{n=1}^{\infty} \frac{\beta\omega_0\omega_n}{n^\alpha}$, $\omega\in\{\pm1\}^{\mathbb{Z}}$, which has been extensively studied recently [6, 13, 18], where $h, \beta\in\mathbb{R}$ and $\alpha > 1$.

The following theorem, the first of our main results in this chapter, establishes the Gibbs properties of the equilibrium states of an extensible potential.

Theorem 3.A. *Suppose $\phi \in C(\Omega)$ has the extensibility property. Then any equilibrium state $\mu \in \mathscr{ES}(\phi)$ is*

(1) Gibbs in the Dobrushin-Lanford-Ruelle sense;

(2) weak Bowen-Gibbs relative to the potential ϕ.

The proof of Theorem 3.A is given in Section 3.5, and uses the following idea: for a given potential $\phi \in C(\Omega)$ satisfying the extensibility condition, we construct a natural two-sided Gibbsian specification (a consistent family of regular probability kernels) γ^ϕ on $\Omega = E^{\mathbb{Z}}$. Then we show that any translation-invariant DLR-Gibbs state ν for the specification γ^ϕ will be an equilibrium state for ϕ. Hence, the set of Gibbs states associated with the specification γ^ϕ is a subset of the set of equilibrium states for ϕ. Finally, if we take any equilibrium state $\tau \in \mathscr{ES}(\phi)$ and a DLR-Gibbs state $\mu \in \mathscr{G}_S(\gamma^\phi)$, we will show that the relative entropy density rate $h(\tau|\mu)$ is zero. This allows us to use the classical variational principle. [15, Theorem 4.1] and conclude that $\tau \in \mathscr{G}_S(\gamma^\phi)$ as well.

Our second main result is about the translation-invariant Gibbs measures, which in some sense is a converse of Theorem 3.A.

Theorem 3.B. *Assume μ is a translation-invariant DLR Gibbs measure on $\Omega = E^{\mathbb{Z}}$. Then μ is an equilibrium state for a potential with the extensibility property.*

We should note that if μ is a Gibbs measure for a translation-invariant *uniformly absolutely convergent* (UAC) interaction, then the claim is rather standard [16, Theorem 3.2]. However, as demonstrated in [1], not all translation-invariant Gibbs measures are compatible with a translation-invariant UAC interaction. Thus, Theorem 3.B generalises the result in [16, Theorem 3.2] to a broader setting, encompassing all translation-invariant Gibbs measures, including those that are not Gibbs for any translation-invariant UAC interaction.

The proof of Theorem 3.B is constructive and is also given in Section 3.5. In fact, we construct a natural one-sided potential ϕ_γ out of the Gibbsian specification γ for μ. Then we show that ϕ_γ is extensible, and this allows us to apply Theorem 3.A to ϕ_γ.

One might observe an analogy between our results and Sullivan's theorem [24, Theorem 1] in Statistical Mechanics. In Sullivan's theorem, the role of extensible potentials is played by the so-called $\mathscr{L}$-convergent interactions, a notion that is also syntactically similar to the notion of extensibility.

We also note that the statements of Theorems 3.A and 3.B, along with their proofs presented in this chapter, naturally extend to higher-dimensional lattices $\mathbb{Z}^d$ ordered lexicographically (see Proposition 2.4.5 and Proposition 2.6.3 in Chapter 2). Since there is no substantial difference in the proofs, we only focus on the one-dimensional lattice $\mathbb{Z}$.

The diagram on the right summarises the relationship between equilibrium states and various notions of Gibbs states.

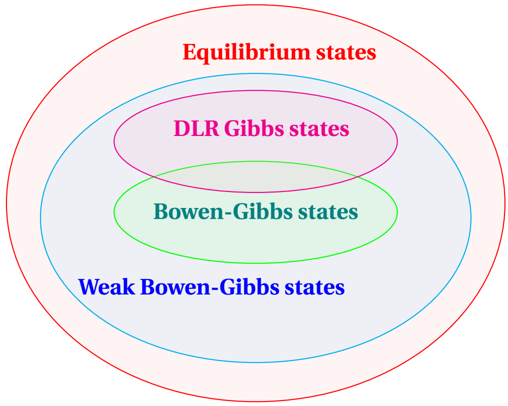

Figure 3.1

This chapter is organised as follows:

- In Section 3.2, we introduce the basic concepts in the DLR Gibbs formalism such as Gibbs measures, specifications and interactions. Here, we also recall some important results, such as the variational principle from the classical DLR Gibbs formalism.
- In Section 3.3, we discuss the motivation behind Bowen's definition of Gibbs states.
- In Section 3.4, we discuss the relationship between the extensible potentials and Gibbsian specifications.
- Section 3.5 is dedicated to the proofs of the main results in this chapter.

3.2 DLR Gibbs Formalism

The theory of Gibbs states, which is put forward by Dobrushin, Lanford, and Ruelle, is very flexible and allows one to define Gibbs states on very general lattice spaces $E^{\mathbb{L}}$, where E is a Polish space and $\mathbb{L}$ is a countable set. In the present chapter, we are primarily interested in probability measures on $\Omega = E^{\mathbb{Z}}$, E is finite, which are invariant under the left shift $S : \Omega \to \Omega$.

3.2.1 Specifications, Interactions, and Gibbs states in Statistical Mechanics

The standard Statistical Mechanics description of Gibbs states is rather different from the definitions of Bowen-Gibbs and weak Bowen-Gibbs measures. The principal point is the explicit description of the family of ***conditional expectations*** indexed by finite subsets Λ of $\mathbb{Z}$. More precisely, in Statistical Mechanics, one starts with a family of regular conditional expectations, which for $f : \Omega \to \mathbb{R}$, given by

$$\gamma_\Lambda(f|\omega) := \sum_{\xi_\Lambda \in E^\Lambda} \gamma_\Lambda(\xi_\Lambda|\omega_{\Lambda^c}) f(\xi_\Lambda \omega_{\Lambda^c}),$$

where

$$\gamma_\Lambda(\xi_\Lambda|\omega_{\Lambda^c}) = \frac{\exp\Big(-H_\Lambda(\xi_\Lambda\omega_{\Lambda^c})\Big)}{Z_\Lambda(\omega)}, \quad Z_\Lambda(\omega) = \sum_{\zeta_\Lambda\in E^\Lambda} \exp\Big(-H_\Lambda(\xi_\Lambda\omega_{\Lambda^c})\Big). \tag{3.5}$$

In order to guarantee the tower property of the conditional expectations, one needs to assume *consistency* of γ_Λ's: $\gamma_\Lambda = \gamma_\Lambda \circ \gamma_V$ for $V \subset \Lambda$, where for $f : \Omega \to \mathbb{R}$ measurable and $\omega \in \Omega$, $(\gamma_\Lambda \circ \gamma_V)(f|\omega) := \int_\Omega \gamma_V(f|\eta)\gamma_\Lambda(d\eta|\omega)$. The latter is ensured if the functions $H_\Lambda : \Omega \to \mathbb{R}$ – called a *Hamiltonian* in Λ– are of a rather special form:

$$H_\Lambda(\omega) = \sum_{\substack{V\Subset\mathbb{Z}\\ V\cap\Lambda\neq\varnothing}} \Phi_V(\omega_V), \tag{3.6}$$

here the summation is taken over all finite subsets of $\mathbb{Z}$ (denoted by $\Subset \mathbb{Z}$) which have non-empty intersection with Λ. Here $\Phi = \{\Phi_V, V \Subset \mathbb{Z}\}$ is called an ***interaction*** and each function $\Phi_V : \Omega \to \mathbb{R}$ is ***local***, meaning that the value of $\Phi_V(\omega)$ depends only on the values of ω within V, hence, we write $\Phi_V(\omega_V)$. In order for these expressions to make sense, one needs to assume a suitable form of summability in (3.6). The standard and sufficient assumption is ***uniform absolute convergence*** (UAC): for all $i \in \mathbb{Z}$,

$$\sum_{i\in V\Subset\mathbb{Z}} \|\Phi_V\|_\infty = \sum_{i\in V\Subset\mathbb{Z}} \sup_{\omega\in\Omega}|\Phi_V(\omega_V)| < \infty.$$

If Φ is an UAC interaction, the corresponding *specification* $\gamma = (\gamma_\Lambda)_{\Lambda\Subset\mathbb{Z}}$ defined by (3.5), is

- *non-null*: for all Λ, $\inf_\omega \gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) > 0$,
- *continuous*: for every Λ, $\omega \mapsto \gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c})$ is continuous.

In statistical mechanics, the second property is often referred to as quasi-locality. An important property of positive specifications, which will be used in the proofs, is the so-called *bar moving property*:

$$\frac{\gamma_\Delta(\xi_\Delta|\omega_{\Delta^c})}{\gamma_\Delta(\zeta_\Delta|\omega_{\Delta^c})} = \frac{\gamma_\Lambda(\xi_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})}{\gamma_\Lambda(\zeta_\Delta\omega_{\Lambda\setminus\Delta}|\omega_{\Lambda^c})}, \tag{3.7}$$

for all $\xi, \zeta, \omega \in \Omega$ and every $\Delta \subset \Lambda \Subset \mathbb{Z}$. The bar moving property is equivalent to the consistency condition of specifications.

A non-null continuous specification $\gamma = \{\gamma_\Lambda\}$ is called ***Gibbsian***.

Definition 3.2.1. *Suppose $\gamma = \{\gamma_\Lambda\}$ is a Gibbsian specification on Ω. The measure μ is called Gibbs for γ, denoted by $\mu \in \mathscr{G}(\gamma)$, if for every $\Lambda \Subset \mathbb{Z}$,*

$$\mu(\omega_\Lambda|\omega_{\Lambda^c}) = \gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}), \ \textit{for } \mu - \textit{a.e. } \omega \in \Omega,$$

equivalently, if the DLR equations hold: for every $f \in C(\Omega)$ and $\Lambda \Subset \mathbb{Z}$,

$$\int_\Omega \gamma_\Lambda(f|\omega)\mu(d\omega) = \int_\Omega f(\omega)\mu(d\omega).$$

For any Gibbsian specification γ, the set of corresponding Gibbs measures $\mathscr{G}(\gamma)$ is a non-empty convex set. In case, $\mathscr{G}(\gamma)$ consists of multiple measures, one says that γ exhibits phase transitions.

3.2.2 Translation-invariance and the Variational Principle

The modern approach to Gibbsian formalism is to think about Gibbsian specifications γ in an ***interaction independent*** fashion. The reason for this is that a measure μ can be consistent with at most one Gibbsian specification, while there are infinitely many interactions Φ giving rise to the same Gibbsian specification. In fact, it was proven by Kozlov that for any Gibbsian specification γ there exists a (in fact many) UAC interaction Φ such that $\gamma = \gamma^\Phi$. However, such a representation is not always possible in a way that respects translation invariance. It is shown in [1] that there exists a Gibbsian specification γ on $\{0,1\}^\mathbb{Z}$ that is *translation-invariant* – meaning $\gamma_{\Lambda+1} = \gamma_\Lambda \circ S$ for every $\Lambda \Subset \mathbb{Z}$ – but can not be associated with any *translation-invariant UAC interaction* Φ, where $\Phi_\Lambda = \Phi_\Lambda \circ S$ for all $\Lambda \Subset \mathbb{Z}$, via (3.5). Nevertheless, many important results in Statistical Mechanics, including the *variational principle*, can be formulated independently of interactions.

Theorem 3.2.2. [15] *Let γ be a translation-invariant Gibbsian specification and $\mu \in \mathscr{G}_S(\gamma)$. Then for all $\tau \in \mathscr{M}_{1,S}(\Omega)$, the specific relative entropy $h(\tau|\mu)$ exists and*

$$h(\tau|\mu) = 0 \iff \tau \in \mathscr{G}_S(\gamma).$$

In Chapter 2, we proved some technical lemmas about translation-invariant specifications, which would be useful for us. For $n \in \mathbb{N}$, we set $\Lambda_n := [-n, n] \cap \mathbb{Z}$. We fix a letter $\mathfrak{a}$ in the finite alphabet E and we will use the notation $\mathbf{a}$, to denote the constant configuration consisting of $\mathfrak{a}$'s.

Lemma 3.2.3. *Suppose γ is a translation-invariant Gibbsian specification on Ω. Then*

(i)

$$\sup_{\sigma,\omega,\eta\in\Omega} \left|\log\left(\frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\omega_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\omega_{[0,n]^c})} \cdot \frac{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})}{\gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})}\right)\right| = o(n). \tag{3.8}$$

(ii) *the sequence* $\Big\{-\frac{1}{|\Lambda_n|}\log\gamma_{\Lambda_n}(\mathbf{a}_{\Lambda_n}|\omega_{\Lambda_n^c})\Big\}_{n\in\mathbb{N}}$ *converges uniformly in* $\omega\in\Omega$ *to a constant which we denote by* $P^{\mathbf{a}}(\gamma)$.

3.3 Bowen's property of Gibbs measures

By comparing the definitions of Gibbs measures in Statistical Mechanics and that of Bowen, it is immediately clear why Bowen's definition is so attractive and popular for dynamicists: it captures the most important, from the Dynamical Systems point of view, properties of Gibbsian states – uniform estimates on measures of cylindric sets in terms of ergodic averages of the potential function. In fact, whether the measure has the Bowen property or the weak Bowen property, rarely makes any difference in Dynamical Systems: the subexponential bound is as good as the uniform bound in practically any computations.

Nevertheless, Bowen was fully aware that his definition of Gibbs states is not the same as in Statistical Mechanics: *"In statistical mechanics, Gibbs states are not defined by the above theorem. We have ignored many subtleties that come up in more complicated systems"*, [3, page 6]. To introduce the definition, Bowen was motivated by an example ([3, page 5]) of a translation-invariant pair interaction Φ, $\Phi_V\not\equiv 0$ only if $V=\{k\}$ or $V=\{k,n\}$, $k,n\in\mathbb{Z}$, satisfying a strong summability condition

$$\|\Phi_{\{0\}}\|_\infty+\frac{1}{2}\sum_{n\in\mathbb{Z}\setminus\{0\}}|n|\cdot\|\Phi_{\{0,n\}}\|_\infty<\infty. \tag{3.9}$$

The above condition is a special case of a well-known uniqueness condition in thermodynamic formalism [8, 22]:

$$\sum_{0\in V\Subset\mathbb{Z}}\frac{\mathtt{diam}(V)}{|V|}\cdot\|\Phi_V\|_\infty<\infty. \tag{3.10}$$

Let us now discuss the Bowen-Gibbs and the weak Bowen-Gibbs properties of DLR Gibbs states.

Theorem 3.3.1. *Suppose* $\Phi=\{\Phi_V\}_{V\Subset\mathbb{Z}}$ *is a translation-invariant UAC interaction and let* $\phi=-\sum_{0\in V\Subset\mathbb{Z}_+}\Phi_V$. *Then there exists a sequence* $\{C_n\}$ *with* $n^{-1}\log C_n\to 0$, *such that for every translation-invariant Gibbs measure* μ *for* Φ, *for all* n *and* $\omega\in\Omega$, *one has*

$$\frac{1}{C_n}\le\frac{\mu\big(\{\tilde\omega\in\Omega:\ \tilde\omega_0^{n-1}=\omega_0^{n-1}\}\big)}{\exp\big(S_n\phi(\omega)-nP(\phi)\big)}\le C_n.$$

If, furthermore, the interaction Φ *satisfies a stronger summability condition (3.10), then there exists a unique Gibbs measure* μ *for* Φ, *and for some* $C>1$, *every* $n\ge 1$

and all $\omega \in \Omega$,

$$\frac{1}{C} \le \frac{\mu(\{\tilde{\omega} \in \Omega : \ \tilde{\omega}_0^{n-1} = \omega_0^{n-1}\})}{\exp\big(S_n\phi(\omega) - nP(\phi)\big)} \le C. \tag{3.11}$$

Let us sketch the proof of this theorem using known results in Statistical Mechanics. The first claim is standard [8, Theorem 15.23]. Applying the DLR equations to the indicator function of the cylinder set $[\sigma_0^{n-1}]$, one concludes that

$$\mu([\sigma_0^{n-1}]) = \int \frac{\exp\big(-H_{\Lambda_n}(\sigma_{\Lambda_n}\eta_{\Lambda_n^c})\big)}{Z_{\Lambda_n}(\eta)}\mu(d\eta).$$

The sequence of functions $\frac{1}{|\Lambda_n|}\log Z_{\Lambda_n}(\eta)$ converges to the pressure $P(\Phi) = P(\phi)$ uniformly in η as $n \to \infty$ [8, Theorem 15.30, part (a)]. Thus $\log Z_{\Lambda_n}(\eta) = \exp(|\Lambda_n|P + o(n))$.

To study the numerator, we use the estimate (15.25) in [8]:

$$\sup_{\sigma,\eta\in\Omega}\Big|\sum_{i\in\Lambda_n}\phi\circ S^i(\sigma) + H_{\Lambda_n}(\sigma_{\Lambda_n}\eta_{\Lambda_n^c})\Big| \le \sum_{i\in\Lambda_n}\sum_{\substack{V\ni i\\ V\not\subset\Lambda_n}}\|\Phi_V\|_\infty, \tag{3.12}$$

and the fact that the uniformly absolute convergence of the interaction Φ ensures that

$$\sum_{i\in\Lambda_n}\sum_{\substack{V\ni i\\ V\not\subset\Lambda_n}}\|\Phi_V\|_\infty = o(|\Lambda_n|).$$

The uniqueness of the Gibbs measures under (3.10) follows from Theorem 8.39 in [8] (see also Comment 8.41 and equation (8.42) in [8]). Note that (3.10) implies that $\sum_{i\in\Lambda_n}\sum_{\substack{V\ni i\\ V\not\subset\Lambda_n}}\|\Phi_V\|_\infty$ remains bounded as $n \to \infty$, in fact, for all $n \in \mathbb{N}$, one has that

$$\sum_{i\in\Lambda_n}\sum_{\substack{V\ni i\\ V\not\subset\Lambda_n}}\|\Phi_V\|_\infty \le \sum_{0\in V\Subset\mathbb{Z}_+}\mathtt{diam}(V)\cdot\|\Phi_V\|_\infty = \sum_{0\in V\Subset\mathbb{Z}}\frac{\mathtt{diam}(V)}{|V|}\cdot\|\Phi_V\|_\infty =: D.$$

This together with (3.12) yields that

$$\sup_{\sigma,\eta\in\Omega}\Big|\sum_{i\in\Lambda_n}\phi\circ S^i(\sigma) - \sum_{i\in\Lambda_n}\phi\circ S^i(\sigma_{\Lambda_n}\eta_{\Lambda_n^c})\Big| \le 2D, \tag{3.13}$$

i.e., ϕ satisfies Bowen's condition [27]. Then (3.11) follows from Theorem 4.6 in [27] and from the fact that the translation-invariant Gibbs measures for Φ are equilibrium states for ϕ [8, Theorem 15.39].

Remark 3.3.2. *Note that it is not a coincidence that the condition (3.10) implying the Bowen property, also implies uniqueness. Indeed, suppose μ and ν are two ergodic measures on Ω with the Bowen-Gibbs property for some continuous potential ϕ. Then the Bowen-Gibbs property implies that*

$$\frac{1}{C} \leq \frac{\mu([\omega_0^n])}{\nu([\omega_0^n])} \leq C$$

for all n and every $\omega \in \Omega$, and hence the measures μ and ν are equivalent, and thus are equal.

3.4 Potentials, Cocycles, and Specifications

An alternative, more dynamical approach to DLR-Gibbs measures, also known as the Ruelle-Capocaccia approach, was introduced in [4]. However, as demonstrated in the original work [4], for lattice systems, which is the setting in this chapter, the Ruelle-Capocaccia definition of Gibbs measures coincides with the specification-based definition given in this chapter. Keller's book [14, Chapter 5] provides an excellent summary of Gibbs measures following the Ruelle-Capocaccia approach under the assumption that the underlying potential has summable variations. Below we explore how this approach extends to cases where the potential lacks summable variations but still has the extensibility property.

Recall that the extensibility condition requires that for all $\omega \in \Omega$ and every $a, \tilde{a} \in E$, the sequence of functions

$$\rho_n^{a,\tilde{a}}(\omega) = \sum_{i=-n}^{n} \left[\phi(S^i \omega^a) - \phi(S^i \omega^{\tilde{a}})\right], \quad n \geq 0, \tag{3.14}$$

converges uniformly as $n \to \infty$. Here, we use the notation $\omega^a = (\omega_k^a)_{k\in\mathbb{Z}}$ is given by

$$\omega_k^a = \begin{cases} a, & k = 0, \\ \omega_k, & k \neq 0. \end{cases}$$

Therefore, we can define a continuous function $\rho_n^{a,\tilde{a}}(\omega) : \Omega \to \mathbb{R}$,

$$\rho(\omega^a, \omega^{\tilde{a}}) = \lim_{n\to\infty} \rho_n^{a,\tilde{a}}(\omega).$$

Proposition 3.4.1. *Suppose ϕ satisfies the extensibility condition, then for any pair $\xi, \eta \in \Omega$, such that the set $\{k \in \mathbb{Z} : \xi_k \neq \eta_k\}$ is finite, the sequence of continuous functions*

$$\rho_n(\xi, \eta) = \sum_{i=-n}^{n} \left[\phi(S^i \xi) - \phi(S^i \eta)\right], \quad n \geq 0, \tag{3.15}$$

converges. Furthermore,

(1) *the limiting function* $\rho(\xi,\eta) = \lim_n \rho_n(\xi,\eta)$ *is a* ***cocycle****, i.e., for every* $\xi,\eta,\zeta \in \Omega$ *with* $\xi_i = \eta_i = \zeta_i$ *for all* i *with* $|i| \gg 1$*, one has*

$$\rho(\xi,\zeta) = \rho(\xi,\eta) + \rho(\eta,\zeta); \tag{3.16}$$

(2) ρ *is translation-invariant in the sense that for every pair* (ξ,η) *with* $\{k \in \mathbb{Z} : \xi_k \neq \eta_k\}$ *finite,*

$$\rho(\xi,\eta) = \rho(S\xi, S\eta). \tag{3.17}$$

(3) *for every* $\Lambda \Subset \mathbb{Z}$ *and* $\eta_\Lambda, \zeta_\Lambda \in E^\Lambda$, $\rho(\eta_\Lambda \xi_{\mathbb{Z}\setminus\Lambda}, \zeta_\Lambda \xi_{\mathbb{Z}\setminus\Lambda})$ *is a continuous function of* ξ*.*

Proof. We carry out the proof in two steps.
First Step: Let $\xi, \eta \in \Omega$ such that for some $k \in \mathbb{Z}$, $\xi_{\mathbb{Z}\setminus\{k\}} = \eta_{\mathbb{Z}\setminus\{k\}}$. Without loss of generality, let $k > 0$. Then

$$\begin{aligned}
\sum_{i=-n}^{n} [\phi \circ S^{i-k}(S^k\xi) - \phi \circ S^{i-k}(S^k\eta)] &= \sum_{i=-n-k}^{n-k} [\phi \circ S^{i}(S^k\xi) - \phi \circ S^{i}(S^k\eta)] \\
&= \sum_{i=-n}^{n} [\phi \circ S^{i}(S^k\xi) - \phi \circ S^{i}(S^k\eta)] \qquad (3.18)\\
&- \sum_{i=n-k+1}^{n} [\phi \circ S^{i}(S^k\xi) - \phi \circ S^{i}(S^k\eta)] \qquad (3.19)\\
&+ \sum_{i=-n-k}^{-n-1} [\phi \circ S^{i}(S^k\xi) - \phi \circ S^{i}(S^k\eta)]. \qquad (3.20)
\end{aligned}$$

Since $(S^k\xi)_{\mathbb{Z}\setminus\{0\}} = (S^k\eta)_{\mathbb{Z}\setminus\{0\}}$, the extensibility property of ϕ yields that the sum in (3.18) converges uniformly to $\rho^\phi(S^k\xi, S^k\eta)$ as $n \to \infty$. Note that for any $i \in \mathbb{Z}$,

$$|\phi \circ S^i(S^k\xi) - \phi \circ S^i(S^k\eta)| \leq \delta_i \phi \leq \mathtt{var}_{(-|i|,\,|i|)}\phi \xrightarrow[|i|\to\infty]{} 0.$$

Thus for (3.19) and (3.20), one has that

$$\Big| \sum_{i=n-k+1}^{n} [\phi \circ S^{-i}(S^k\xi) - \phi \circ S^{-i}(S^k\eta)] \Big| \leq k \cdot \mathtt{var}_{(k-n,\,n-k)}\phi$$

and

$$\Big| \sum_{i=-n-k}^{-n-1} [\phi \circ S^{-i}(S^k\xi) - \phi \circ S^{-i}(S^k\eta)] \Big| \leq k \cdot \mathtt{var}_{(k-n,\,n-k)}\phi.$$

Thus since k is fixed and $\mathtt{var}_{(k-n,\,n-k)}\phi \xrightarrow[n\to\infty]{} 0$, both sums in (3.19) and (3.20) converge uniformly to 0 as $n \to \infty$. Therefore, ρ is defined at (ξ,η) and $\rho(\xi,\eta) = \rho(S^k\xi, S^k\eta)$.

Second Step: Let $\xi,\eta\in\Omega$ such that for some $\Lambda\Subset\mathbb{Z}$, $\xi_{\mathbb{Z}\setminus\Lambda}=\eta_{\mathbb{Z}\setminus\Lambda}$. Then there exists $m\in\mathbb{N}$ such that $\Lambda\subset[-m,m]\cap\mathbb{Z}$. Then for $n>m$, one has

$$\sum_{i=-n}^{n}[\phi\circ S^i(\xi)-\phi\circ S^i(\eta)]=\sum_{j=-m}^{m}\sum_{i=-n}^{n}[\phi\circ S^i(\xi_{-\infty}^{-m-1}\eta_{-m}^{j-1}\xi_j^{\infty})-\phi\circ S^i(\xi_{-\infty}^{-m-1}\eta_{-m}^{j}\xi_{j+1}^{\infty})].$$

By the first step, the sum over i on the RHS of the last equation above converges uniformly for each j as $n\to\infty$.
We now address statements (1)-(3). Claim (1) follows directly from the definition of ρ. Equations (3.16) and (3.17) easily follow from the first and second steps discussed above. The continuity of the map $\xi\mapsto\rho(\eta_\Lambda\xi_{\mathbb{Z}\setminus\Lambda},\zeta_\Lambda\xi_{\mathbb{Z}\setminus\Lambda})$ for all $\Lambda\Subset\mathbb{Z}$ and $\eta,\zeta\in\Omega$ is a consequence of the uniform convergence of (3.15).

□

Now, we shall discuss how to associate a Gibbsian specification with an extensible potential and vice versa. We have the following theorem.

Theorem 3.4.2. *(i) Suppose $\phi:\Omega\to\mathbb{R}$ is a continuous function with extensibility property, then $\gamma^\phi=(\gamma^\phi_\Lambda)_{\Lambda\Subset\mathbb{Z}}$ given by*

$$\gamma^\phi_\Lambda(\omega_\Lambda|\omega_{\Lambda^c})=\Big(\sum_{\xi_\Lambda\in E^\Lambda}e^{\rho^\phi(\xi_\Lambda\omega_{\mathbb{Z}\setminus\Lambda},\,\omega)}\Big)^{-1},\ \ \omega\in\Omega \tag{3.21}$$

is a translation invariant Gibbsian specification.

(ii) Suppose $\gamma=(\gamma_\Lambda)_{\Lambda\Subset\mathbb{Z}}$ is a translation invariant Gibbsian specification, then

$$\phi_\gamma(\omega)=\log\frac{\gamma_{\{0\}}(\omega_0|\mathbf{a}_{-\infty}^{-1}\omega_1^{\infty})}{\gamma_{\{0\}}(\mathbf{a}_0|\mathbf{a}_{-\infty}^{-1}\omega_1^{\infty})},\ \ \omega\in\Omega \tag{3.22}$$

is a continuous function with extensibility property such that $\gamma^{\phi_\gamma}=\gamma$.

*Proof. **(i):*** This part is proven in Subsection 2.2.3.

(ii): The second part has been established in greater generality in Proposition 2.4.5; here, we consider the particular case where ρ is the Dirac measure $\delta_{\mathbf{a}}$ and $d=1$. Owing to its concreteness and reduced level of abstraction, we provide a separate proof for this case.

Pick any $a_0,b_0\in E$ and $\omega\in\Omega$. Since ϕ_γ is a one-sided function, one can check that for any $n\in\mathbb{N}$,

$$\rho_n^{a_0,b_0}(\omega)-\rho_0^{a_0,b_0}(\omega)=\sum_{i=1}^{n}\Big[\phi_\gamma\circ S^{-i}(b_0\omega_{\{0\}^c})-\phi_\gamma\circ S^{-i}(a_0\omega_{\{0\}^c})\Big] \tag{3.23}$$

and for the right-hand side of the above equation, one has

$$RHS = \sum_{i=1}^{n} \log \frac{\gamma_{\{0\}}((\omega_{-i})_0 | \mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1} b_0 \omega_1^{\infty})_1^{\infty})}{\gamma_{\{0\}}(\mathbf{a}_0 | \mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1} b_0 \omega_1^{\infty})_1^{\infty})} \cdot \frac{\gamma_{\{0\}}(\mathbf{a}_0 | \mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1} a_0 \omega_1^{\infty})_1^{\infty})}{\gamma_{\{0\}}((\omega_{-i})_0 | \mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1} a_0 \omega_1^{\infty})_1^{\infty})}.$$

Then, by applying the bar moving property, one gets

$$\begin{aligned}\rho_n^{a_0,b_0}(\omega) - \rho_0^{a_0,b_0}(\omega) = &\sum_{i=1}^{n} \log \frac{\gamma_{\{0,i\}}((\omega_{-i})_0 (b_0)_i | \mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}{\gamma_{\{0,i\}}(\mathbf{a}_0 (b_0)_i | \mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})} \\ &+ \sum_{i=1}^{n} \log \frac{\gamma_{\{0,i\}}(\mathbf{a}_0 (a_0)_i | \mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}{\gamma_{\{0,i\}}((\omega_{-i})_0 (a_0)_i | \mathbf{a}_{-\mathbb{N}}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}\end{aligned}$$

and thus, by applying the bar moving property once more,

$$\begin{aligned}\rho_n^{a_0,b_0} - \rho_0^{a_0,b_0} = &\sum_{i=1}^{n} \log \frac{\gamma_{\{i\}}((b_0)_i | \mathbf{a}_{-\mathbb{N}}(\omega_{-i}^{-1})_0^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}{\gamma_{\{i\}}((a_0)_i | \mathbf{a}_{-\mathbb{N}}(\omega_{-i}^{-1})_0^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})} \\ &+ \sum_{i=1}^{n} \log \frac{\gamma_{\{i\}}((a_0)_i | \mathbf{a}_{\mathbb{Z}_-}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})}{\gamma_{\{i\}}((b_0)_i | \mathbf{a}_{\mathbb{Z}_-}(\omega_{-i+1}^{-1})_1^{i-1}(\omega_1^{\infty})_{i+1}^{\infty})},\end{aligned}$$

here $\mathbb{Z}_-$ denotes $-\mathbb{N} \cup \{0\}$. Hence, by the translation-invariance of γ,

$$\begin{aligned}\rho_n^{a_0,b_0}(\omega) - \rho_0^{a_0,b_0}(\omega) &= \sum_{i=1}^{n} \log \frac{\gamma_{\{0\}}(b_0 | \mathbf{a}_{-\infty}^{-i-1} \omega_{-i}^{-1} \omega_1^{\infty})}{\gamma_{\{0\}}(a_0 | \mathbf{a}_{-\infty}^{-i-1} \omega_{-i}^{-1} \omega_1^{\infty})} \cdot \frac{\gamma_{\{0\}}(a_0 | \mathbf{a}_{-\infty}^{-i} \omega_{-i+1}^{-1} \omega_1^{\infty})}{\gamma_{\{0\}}(b_0 | \mathbf{a}_{-\infty}^{-i} \omega_{-i+1}^{-1} \omega_1^{\infty})} \\ &= \sum_{i=1}^{n} \log \frac{\gamma_{\{0\}}(b_0 | \mathbf{a}_{-\infty}^{-i-1} \omega_{-i}^{-1} \omega_1^{\infty})}{\gamma_{\{0\}}(a_0 | \mathbf{a}_{-\infty}^{-i-1} \omega_{-i}^{-1} \omega_1^{\infty})} - \sum_{i=0}^{n-1} \log \frac{\gamma_{\{0\}}(b_0 | \mathbf{a}_{-\infty}^{-i-1} \omega_{-i}^{-1} \omega_1^{\infty})}{\gamma_{\{0\}}(a_0 | \mathbf{a}_{-\infty}^{-i-1} \omega_{-i}^{-1} \omega_1^{\infty})} \\ &= \log \frac{\gamma_{\{0\}}(b_0 | \mathbf{a}_{-\infty}^{-n-1} \omega_{-n}^{-1} \omega_1^{\infty})}{\gamma_{\{0\}}(a_0 | \mathbf{a}_{-\infty}^{-n-1} \omega_{-n}^{-1} \omega_1^{\infty})} - \log \frac{\gamma_{\{0\}}(b_0 | \mathbf{a}_{-\infty}^{-1} \omega_1^{\infty})}{\gamma_{\{0\}}(a_0 | \mathbf{a}_{-\infty}^{-1} \omega_1^{\infty})}.\end{aligned}$$

Thus

$$\rho_n^{a_0,b_0}(\omega) = \log \frac{\gamma_{\{0\}}(b_0 | \mathbf{a}_{-\infty}^{-n-1} \omega_{-n}^{-1} \omega_1^{\infty})}{\gamma_{\{0\}}(a_0 | \mathbf{a}_{-\infty}^{-n-1} \omega_{-n}^{-1} \omega_1^{\infty})} \xrightarrow[\substack{n\to\infty \\ \omega\in\Omega}]{} \log \frac{\gamma_{\{0\}}(b_0 | \omega_{-\infty}^{-1} \omega_1^{\infty})}{\gamma_{\{0\}}(a_0 | \omega_{-\infty}^{-1} \omega_1^{\infty})}. \tag{3.24}$$

The last limit also shows that $\gamma^{\phi_\gamma} = \gamma$. □

3.5 Proofs

The proofs of Theorem 3.A and Theorem 3.B will be based on two lemmas. Our first lemma states the following.

Lemma 3.5.1. *Let $\phi \in C(\Omega)$ be an extensible function and γ^{ϕ} be the associated Gibbsian specification. Then the one-sided extensible function $\phi_{\gamma^{\phi}} \in C(\Omega_+)$ is weakly cohomologous to ϕ, i.e., there exists $C \in \mathbb{R}$ such that for all $\tau \in \mathcal{M}_{1,S}(\Omega)$,*

$$\int_{\Omega} \phi_{\gamma^{\phi}} \, d\tau = \int_{\Omega} \phi \, d\tau + C. \tag{3.25}$$

Remark 3.5.2. *The second part of Theorem 3.4.2 implies that the chain $\gamma \to \phi_{\gamma} \to \gamma^{\phi_{\gamma}}$ is closed, i.e., $\gamma = \gamma^{\phi_{\gamma}}$. For an extensible function ϕ, the diagram $\phi \to \gamma^{\phi} \to \phi_{\gamma^{\phi}}$ is, in general, not closed, i.e., it is not always true that $\phi = \phi_{\gamma^{\phi}}$.*

Proof. Note that $\phi_{\gamma^{\phi}}$ is a half-line function. For any $\omega_{\mathbb{Z}_+} \in \Omega_+$, one can easily check the following:

$$\phi_{\gamma^{\phi}}(\omega_{\mathbb{Z}_+}) = \lim_{n\to\infty} \sum_{i=-n}^{n} \left[\phi \circ S^i(\mathbf{a}_{-\infty}^{-1}\omega_0^{\infty}) - \phi \circ S^i(\mathbf{a}_{-\infty}^{0}\omega_1^{\infty})\right] \tag{3.26}$$

and the above limit is uniform on $\omega_{\mathbb{Z}_+} \in \Omega_+$. For any $\tau \in \mathcal{M}_{1,S}(\Omega_+)$ and $n \in \mathbb{N}$, denote

$$I_n(\tau) \ := \ \int_{\Omega} \sum_{i=-n}^{n} \left[\phi \circ S^i(\mathbf{a}_{-\infty}^{-1}\omega_0^{\infty}) - \phi \circ S^i(\mathbf{a}_{-\infty}^{0}\omega_1^{\infty})\right] \tau(d\omega).$$

For any $\ell_1, \ell_2 \in \mathbb{Z} \cup \{-\infty, \infty\}$ with $\ell_1 \le \ell_2$, define a transformation

$$\Theta_{[\ell_1,\ell_2]}(\omega)_i := \begin{cases} \omega_i, & \text{if } i \notin [\ell_1,\ell_2]; \\ \mathfrak{a}, & \text{if } i \in [\ell_1,\ell_2]. \end{cases} \tag{3.27}$$

If $\ell_1 \not\le \ell_2$, then $[\ell_1,\ell_2] = \emptyset$, therefore, for all $\omega \in \Omega$, we set $\Theta_{[\ell_1 \ell_2]}(\omega) = \omega$. Note that the families $\{S_\ell : \ell \in \mathbb{Z}\}$ and $\{\Theta_{[\ell_1,\ell_2]} : \ell_1, \ell_2 \in \mathbb{Z}, \ell_1 \le \ell_2\}$ of the transformations on Ω have the following commutativity-type property: for any $\ell_{1,2,3} \in \mathbb{Z}$ with $\ell_1 \le \ell_2$,

$$S_{\ell_3} \circ \Theta_{[\ell_1,\ell_2]} = \Theta_{[\ell_1-\ell_3,\ell_2-\ell_3]} \circ S_{\ell_3}. \tag{3.28}$$

Thus $I_n(\tau)$ is written in terms of the transformations Θ as follows:

$$I_n(\tau) = \sum_{i=-n}^{n} \int_{\Omega} \phi \circ \Theta_{(-\infty,-i-1]} \circ S^i(\omega)\tau(d\omega) - \sum_{i=-n}^{n} \int_{\Omega} \phi \circ \Theta_{(-\infty,-i]} \circ S^i(\omega)\tau(d\omega) \tag{3.29}$$

Then, by the translation-invariance of measure τ,

$$I_n(\tau) \ = \ \sum_{i=-n}^{n} \int_{\Omega} \phi \circ \Theta_{(-\infty,-i-1]} d\tau - \sum_{i=-n}^{n} \int_{\Omega} \phi \circ \Theta_{(-\infty,-i]} d\tau \tag{3.30}$$

$$= \ \int_{\Omega} \phi \circ \Theta_{(-\infty,-n-1]} d\tau - \int_{\Omega} \phi \circ \Theta_{(-\infty,n]} d\tau. \tag{3.31}$$

By continuity of ϕ, the sequences $\{\phi\circ\Theta_{(-\infty,-n-1]}(\omega)\}_{n\in\mathbb{Z}_+}$ and $\{\phi\circ\Theta_{(-\infty,n]}(\omega)\}_{n\in\mathbb{Z}_+}$ converge uniformly in $\omega\in\Omega$ to $\phi(\omega)$ and $\phi(\mathbf{a})$, respectively, as $n\to\infty$. Thus one concludes that

$$\lim_{n\to\infty} I_n(\tau)=\int_\Omega \phi\, d\tau-\phi(\mathbf{a}). \tag{3.32}$$

Since the limit in (3.26) is uniform, we conclude from (3.32) that

$$\int_\Omega \phi_{\gamma^\phi}(\omega)\tau(d\omega)=\int_\Omega \phi(\omega)\tau(d\omega)-\phi(\mathbf{a}). \tag{3.33}$$

□

Now, we formulate the second lemma.

Lemma 3.5.3. *Let γ be a translation-invariant Gibbsian specification and ϕ_γ be the associated extensible function. Then*

(i) Every translation-invariant Gibbs state $\mu\in\mathscr{G}_S(\gamma)$ is weak Bowen-Gibbs with respect to ϕ_γ, i.e., μ satisfies (3.2);

(ii) The set of translation-invariant Gibbs states for γ coincides with the set of the equilibrium states for ϕ_γ, i.e., $\mathscr{G}_S(\gamma)=\mathscr{E}\mathscr{S}(\phi_\gamma)$.

Proof. ***(i):*** Now we shall prove that any translation-invariant DLR Gibbs measure μ prescribed by a specification γ is weak Bowen-Gibbs relative to the potential ϕ_γ.

The first part of Lemma 3.2.3 and translation-invariance of the specification γ imply

$$\lim_{n\to\infty} -\frac{1}{n}\log\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\omega_{[0,n]^c})=P^{\mathbf{a}}(\gamma) \tag{3.34}$$

and the convergence is uniform in $\omega\in\Omega$.

Now consider a configuration $\sigma\in\Omega_+$ and a cylindric set $[\sigma_0^n]$, then by the DLR equations,

$$\begin{aligned}\mu([\sigma_0^n]) &= \int_X \gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})\mu(d\eta)\\ &= \int_X \frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})}\cdot\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})\mu(d\eta).\end{aligned} \tag{3.35}$$

By translation-invariance of the specification γ, Lemma 3.2.3 yields that

$$\mu([\sigma_0^n]) = \int_X \frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})}\cdot e^{-nP^{\mathbf{a}}(\gamma)+o(n)}\mu(d\eta),$$

here the error factor $o(n)$ is independent on η and only depends on n. Therefore,

$$\mu([\sigma_0^n]) \quad = \quad e^{-nP^{\mathbf{a}}(\gamma)+o(n)} \int_X \frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\eta_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\eta_{[0,n]^c})} \mu(d\eta), \tag{3.36}$$

Hence, by taking into account (3.8), we obtain that

$$\mu([\sigma_0^n]) = e^{-nP^{\mathbf{a}}(\gamma)} \cdot \frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\mathbf{a}_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\mathbf{a}_{[0,n]^c})} \cdot e^{o(n)}. \tag{3.37}$$

Using the bar moving property (3.7), we have that

$$\begin{aligned} \frac{\gamma_{[0,n]}(\sigma_{[0,n]}|\mathbf{a}_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,n]}|\mathbf{a}_{[0,n]^c})} &= \prod_{i=0}^{n} \frac{\gamma_{[0,n]}(\mathbf{a}_{[0,i)}\sigma_{[i,n]}|\mathbf{a}_{[0,n]^c})}{\gamma_{[0,n]}(\mathbf{a}_{[0,i]}\sigma_{(i,n]}|\mathbf{a}_{[0,n]^c})} \overset{(3.7)}{=} \prod_{i=0}^{n} \frac{\gamma_{\{i\}}(\sigma_i|\sigma_{(i,n]}\mathbf{a}_{[i,n]^c})}{\gamma_{\{i\}}(\mathbf{a}_i|\sigma_{(i,n]}\mathbf{a}_{[i,n]^c})} \\ &= \prod_{i=0}^{n} e^{\phi_\gamma \circ S^i(\sigma_{[0,n]}\mathbf{a}_{[0,n]^c})}. \end{aligned} \tag{3.38}$$

Note that in the last equation of (3.38), we used the fact that ϕ_γ is independent of the components in the negative half-line $-\mathbb{N}$. Combining (3.38) with (3.37), we get

$$\mu([\sigma_0^n]) = \frac{e^{S_{n+1}\phi_\gamma(\sigma_{[0,n]}\mathbf{a}_{[0,n]^c})}}{e^{nP^{\mathbf{a}}(\gamma)}} \cdot e^{o(n)}. \tag{3.39}$$

Note that

$$\mathtt{var}_n(S_{n+1}\phi_\gamma) \le \sum_{k=0}^{n} \mathtt{var}_k(\phi_\gamma) \tag{3.40}$$

and since ϕ_γ is continuous, $\mathtt{var}_k(\phi_\gamma) \to 0$ as $k \to \infty$. Thus $\mathtt{var}_n(S_{n+1}\phi_\gamma) = o(n)$, and hence we obtain from (3.39) that

$$\mu([\sigma_0^n]) = \frac{e^{S_{n+1}\phi_\gamma(\sigma)}}{e^{nP^{\mathbf{a}}(\gamma)}} e^{o(n)}. \tag{3.41}$$

We note that the weak Bowen-Gibbs property (3.41) implies that $P^{\mathbf{a}}(\gamma) = P(\phi_\gamma)$, where $P(\phi_\gamma)$ is the topological pressure of ϕ_γ.

(ii): Now take any $\tau \in \mathscr{M}_{1,S}(\Omega)$ and $\mu \in \mathscr{G}_S(\gamma)$. The relative entropy $H_n(\tau|\mu)$ is given by

$$\begin{aligned} H_n(\tau|\mu) &= \sum_{a_0^n \in E^{n+1}} \tau([a_0^n]) \log \frac{\tau([a_0^n])}{\mu([a_0^n])} \\ &= \sum_{a_0^n \in E^{n+1}} \tau([a_0^n]) \log \tau([a_0^n]) - \sum_{a_0^n \in E^{n+1}} \tau([a_0^n]) \log \mu([a_0^n]) \end{aligned} \tag{3.42}$$

For the first sum in (3.42), one has that

$$\frac{1}{n}\sum_{a_0^n\in E^{n+1}}\tau([a_0^n])\log\tau([a_0^n])\xrightarrow[n\to\infty]{}-h(\tau). \tag{3.43}$$

For the second sum, by inserting (3.40) and using (3.41), one has

$$\begin{aligned}\sum_{a_0^n\in E^{n+1}}\tau([a_0^n])\log\mu([a_0^n]) &= \sum_{a_0^n\in E^{n+1}}\tau([a_0^n])(S_{n+1}\phi_\gamma-(n+1)P^{\mathbf{a}}(\gamma)+o(n)\\ &= (n+1)\int_\Omega\phi_\gamma\,d\tau-(n+1)P^{\mathbf{a}}(\gamma)+o(n). \end{aligned}\tag{3.44}$$

By combining (3.43) and (3.44), and since $P^{\mathbf{a}}(\gamma)=P(\phi_\gamma)$, one concludes that the relative entropy density rate $h(\tau|\mu)$ indeed exists and

$$h(\tau|\mu)=\lim_{n\to\infty}\frac{1}{n}H_n(\tau|\mu)=-h(\tau)-\int_\Omega\phi_\gamma\,d\tau+P(\phi_\gamma). \tag{3.45}$$

Thus the Variational Principle (Theorem 3.2.2) yields that τ is a Gibbs state for γ if and only if τ is an equilibrium state for ϕ_γ, i.e., $\mathscr{G}_S(\gamma)=\mathscr{E}\mathscr{S}(\phi_\gamma)$.

□

Proof of Theorem 3.A. One can easily see that weakly cohomologous potentials have the same equilibrium states because the functionals $\tau\in\mathscr{M}_{1,S}(\Omega)\mapsto h(\tau)+\int_\Omega\phi\,d\tau$ and $\tau\in\mathscr{M}_{1,S}(\Omega)\mapsto h(\tau)+\int_\Omega\phi_{\gamma^\phi}\,d\tau$ differ by only a constant $P(\phi_{\gamma^\phi})-P(\phi)$. Furthermore, the weak cohomology between ϕ_{γ^ϕ} and ϕ also yields the following [7, Proposition 2.34]:

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{i=0}^{n-1}\Big[\phi_{\gamma^\phi}-\phi-P(\phi_{\gamma^\phi})+P(\phi)\Big]\circ S^i\right\|_\infty=0. \tag{3.46}$$

Then the first part of Theorem 3.A follows from Lemma 3.5.1 and the second part of Lemma 3.5.3 since the weak cohomologous potentials have the same set of equilibrium states, i.e., $\mathscr{E}\mathscr{S}(\phi)=\mathscr{E}\mathscr{S}(\phi_{\gamma^\phi})$. The second part of Theorem 3.A also follows from Lemma 3.5.1 and Lemma 3.5.3. In fact, by applying the first and second parts of Lemma 3.5.3 to $\gamma=\gamma^\phi$, one obtains from Lemma 3.5.1 that for any equilibrium state $\mu\in\mathscr{E}\mathscr{S}(\phi)$,

$$\mu([\sigma_0^{n-1}])=\frac{e^{(S_n\phi_{\gamma^\phi})(\sigma)}}{e^{nP(\phi_{\gamma^\phi})}}e^{o(n)},\ \sigma\in\Omega, \tag{3.47}$$

and by (3.46), one has that $S_n(\phi-P(\phi))=S_n(\phi_{\gamma^\phi}-P(\phi_{\gamma^\phi}))+o(n)$. Hence one immediately concludes the weak Bowen-Gibbs property of μ with respect to the potential ϕ.

□

Proof of Theorem 3.B. It is easy to see that the second part of Lemma 3.5.3 implies Theorem 3.B. □

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Part II

Transfer operators for long-range potentials

Chapter 4

On an extension of a theorem by Ruelle to long-range potentials

Abstract: Ruelle's transfer operator plays an important role in understanding thermodynamic and probabilistic properties of dynamical systems. In this chapter, we develop a method of finding eigenfunctions of transfer operators based on comparing Gibbs measures on the half-line $\mathbb{Z}_+$ and the whole line $\mathbb{Z}$. For a rather broad class of potentials, including both the ferromagnetic and antiferromagnetic long-range Dyson potentials, we are able to establish the existence of integrable, but not necessarily continuous, eigenfunctions. For a subset thereof we prove that the eigenfunction is actually continuous.

This chapter is based on A. van Enter, R. Fernández, M. Makhmudov, E. Verbitskiy, *"On an extension of a theorem by Ruelle to long-range potentials "*, arXiv:2404.07326.

4.1 Introduction

One of the main issues of equilibrium statistical mechanics is to derive and describe the properties of the possible (global) states of a macroscopic system starting from the knowledge of the finite-volume (local) states of the system. In order to give a mathematical framework to this problem, Dobrushin, Lanford, and Ruelle developed the so-called DLR formalism in the second half of the last century.

Shortly after their introduction of Gibbs measures, Spitzer and Averintsev [3, 61] characterised Gibbs measures for short-range potentials in terms of measures having Markov properties. These results were then simplified and generalised, in various directions by Hammersley and Clifford [44], by Sullivan, Kozlov [49, 62], and by Geoffrey Grimmett in his first paper [41].

The novel DLR formalism was immediately adapted to the theory of Dynamical Systems by Sinai [58–60]. There are, however, two important differences between the models typically studied in Statistical Mechanics and those studied in Dynamical Systems. Firstly, in Dynamical Systems one is typically interested in one-dimensional systems, that is, systems with configuration spaces $E^{\mathbb{Z}}$, where E is the set of possible spin values and the spatial dimension represents time. The conditional probabilities in Dynamical Systems are typically "one-sided" (from past to future), those in Statistical Mechanics "two-sided" (from outside to inside). For short-range potentials this does not make much of a difference, although in general regularity properties between one-sided and two-sided conditional probabilities may differ [7, 8, 27, 28, 32]. Secondly, and perhaps more important, the natural description of dynamical systems often involves *half-line* configuration spaces $E^{\mathbb{Z}_+}$, rather than *whole-line* configuration spaces of the form $E^{\mathbb{Z}}$.

Already in his original papers, Sinai addressed these questions, [58–60]. He showed that Gibbs equilibrium states for exponentially decaying interactions are, in fact, equilibrium states for half-line potentials as well. The issue of half-line versus whole-line Gibbsianness is, therefore, as old as the theory of thermodynamic formalism in Dynamical Systems. For more recent results established in this area see, e.g., [7, 33, 64–67]. One of the most important Dynamical Systems tools used in the study of Gibbs equilibrium states is the so-called Ruelle's transfer operator [57, 64]. Various probabilistic properties of chaotic dynamical systems can be characterised in terms of transfer operators. In particular, a central issue is to characterise those potentials (interactions) for which transfer operators have positive continuous eigenfunctions that is, finding those potentials for which Ruelle's theorem holds. This question has been answered by different authors [30, 54, 55, 64, 66, 67] for different regularity classes of potentials. In this chapter, we answer it for potentials beyond these earlier studied classes.

This chapter is organised as follows:

- In Section 4.2, Section 4.3, and Section 4.4 we introduce the notions of thermodynamic formalism that are important for this chapter.
- In Section 4.5, we discuss the relationship between half-line and whole-line Gibbs measures and formulate the first part of the main results (Theorem 4.A and 4.B).
- In Section 4.6, we discuss when whole-line Gibbs measures are absolutely continuous with respect to the product of two half-line ones, and we formulate the second series of our main results (Theorem 4.C, 4.D and 4.E).
- In Section 4.7, the Dyson model, the main example of the chapter, is discussed. We then discuss what is the behaviour in other regimes of the phase transitions.
- Section 4.8 and Section 4.9 are dedicated to the proofs of our main results and the final remarks.

4.2 Basic notions: I. Specifications

The Dobrushin-Lanford-Ruelle definition of Gibbs states via specifications goes well beyond the standard lattices $\mathbb{Z}^d$. Consider the lattice system $\Omega = E^{\mathbb{L}}$, where $\mathbb{L}$ is an at most countable set (lattice) and E is a set of possible spin values. In this chapter, we will focus on finite E. We denote the Borel $\sigma-$algebra of the measurable subsets of Ω by $\mathscr{F}$. For a subset $\Lambda \subset \mathbb{L}$, we define $\mathscr{F}_\Lambda$ as the minimal σ-algebra that makes the maps $\omega \in \Omega \mapsto \omega_i \in E$, $i \in \Lambda$ measurable. Additionally, we let $\mathscr{T}$ represent the tail σ-algebra, defined by $\mathscr{T} := \cap_{\Lambda \Subset \mathbb{L}} \mathscr{F}_\Lambda$. The specification is a consistent family of probability kernels (conditional probabilities) indexed by finite subsets Λ of $\mathbb{L}$ denoted by $\Lambda \Subset \mathbb{L}$. The consistency condition is the requirement that $\gamma_\Lambda \gamma_\Delta = \gamma_\Lambda$ for all $\Delta \subset \Lambda \Subset \mathbb{L}$ [39, Chapter 1].

In the sequel, for $\Delta, \Lambda \subset \mathbb{L}$, we will denote the concatenation of strings $\xi_\Delta \in E^\Delta$, $\eta_\Lambda \in E^\Lambda$ by $\xi_\Delta \eta_\Lambda$, namely, $\xi_\Delta \eta_\Lambda$ is a string such that $(\xi_\Delta \eta_\Lambda)_i = \xi_i$ if $i \in \Delta$ and $(\xi_\Delta \eta_\Lambda)_i = \eta_i$ if $i \in \Lambda$. Given the specification $\gamma = \{\gamma_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ on $\Omega = E^{\mathbb{L}}$, we say that a probability measure μ on Ω is Gibbs for the specification γ or, equivalently, that μ is consistent with γ, if

$$\mu(\sigma_\Lambda | \sigma_{\Lambda^c}) = \gamma_\Lambda(\sigma_\Lambda \sigma_{\Lambda^c}) \quad \text{for } \mu\text{-a.a. } \sigma \in \Omega,$$

or, equivalently, if the DLR equations hold:

$$\int \gamma_\Lambda f \, d\mu = \int f \, d\mu,$$

for all $f \in L^1(\Omega, \mu)$, and every $\Lambda \Subset \mathbb{L}$, where

$$\gamma_\Lambda f(\sigma) = \sum_{\xi_\Lambda \in E^\Lambda} \gamma_\Lambda(\xi_\Lambda | \sigma_{\Lambda^c}) f(\xi_\Lambda \sigma_{\Lambda^c}),$$

note that for $\omega \in \Omega$ and $\Lambda \subset \mathbb{L}$, ω_{Λ^c} denotes the (infinite) string $\omega_{\mathbb{L}\setminus\Lambda}$. The set of all Gibbs measures for γ will be denoted by $\mathscr{G}(\Omega, \gamma)$. For any probability measure μ one can find at least one specification γ such that μ is Gibbs for γ [40]. However, useful and interesting specifications have additional properties such as finite energy (non-nullness) and quasi-locality (continuity). We now turn to two particular ways of defining specifications as used in Statistical Mechanics and Dynamical Systems.

4.2.1 Gibbs(ian) specifications in Statistical Mechanics

An interaction Φ is a family of functions $\{\Phi_\Lambda\}$, indexed by finite subsets $\Lambda \Subset \mathbb{L}$, such that each function Φ_Λ, depends only on values of σ in Λ, that is, with a slight abuse of notation, $\Phi_\Lambda(\sigma) = \Phi_\Lambda(\sigma_\Lambda)$. One needs to impose some additional summability conditions on the interaction Φ: Φ is said to be ***uniformly absolutely convergent*** (UAC) if for all $i \in \mathbb{L}$, $\sum_{i \in V \Subset \mathbb{L}} ||\Phi_V||_\infty < \infty$. For an UAC interaction Φ, the specification (specification density) $\gamma^\Phi = \{\gamma^\Phi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ is defined as follows, for $\omega, \eta \in \Omega$,

$$\gamma^\Phi_\Lambda(\omega_\Lambda | \eta_{\Lambda^c}) := \frac{e^{-H^\Phi_\Lambda(\omega_\Lambda \eta_{\Lambda^c})}}{Z^\Phi_\Lambda(\eta)}, \tag{4.1}$$

where $H^\Phi_\Lambda(\omega) := \sum_{V \cap \Lambda \neq \emptyset} \Phi_V(\omega)$ is the Hamiltonian in the volume Λ, and Z^Φ_Λ is a normalization constant (the partition function), i.e., $Z^\Phi_\Lambda(\eta) := \sum_{\bar{\omega}_\Lambda \in E^\Lambda} e^{-H^\Phi_\Lambda(\bar{\omega}_\Lambda \eta_{\Lambda^c})}$.

It should be stressed that a Gibbsian specification γ^Φ is always *quasilocal* [5, 26, 39]. In the current setting, in which E is finite, this property is equivalent to the fact that for all $\Lambda \Subset \mathbb{L}$ and $\omega_\Lambda \in E^\Lambda$, $\gamma_\Lambda(\omega_\Lambda | \eta)$ is a continuous function of the boundary condition $\eta \in \Omega$. Another important property of the Gibbsian specifications is *non-nullness*, which means that for all volumes $\Lambda \Subset \mathbb{L}$, $\inf_{\eta, \omega \in \Omega} \gamma^\Phi_\Lambda(\omega_\Lambda | \eta_{\Lambda^c}) > 0$.

We denote the set of Gibbs states for the interaction Φ by $\mathscr{G}(\Omega, \Phi)$ (or $\mathscr{G}(\Phi)$). It is a convex set – in fact a simplex – which is always non-empty if, as is the case in this chapter, the spin space E is compact.

Depending on the symmetries of the lattice $\mathbb{L}$ and the spin space E, the interactions and specifications may also exhibit some symmetries. For example, if $\mathbb{L} = \mathbb{Z}$, then an interaction Φ on $X := E^{\mathbb{Z}}$ is called ***translation-invariant*** if for all $\Lambda \Subset \mathbb{Z}$, every $k \in \mathbb{Z}$ and $\omega \in X$, $\Phi_{\Lambda+k}(\omega) = \Phi_\Lambda(S^k(\omega))$, where $\Lambda + k := \{i + k : i \in \Lambda\}$

and $S : X \to X$ is left shift, i.e., for every $i \in \mathbb{Z}$ and $\omega \in X$, $(S\omega)_i = \omega_{i+1}$. Respectively, a specification γ on $X = E^{\mathbb{Z}}$ is called ***translation-invariant*** if for all $B \in \mathscr{F}$, $\Lambda \Subset \mathbb{Z}$, $k \in \mathbb{Z}$ and $\omega \in X$, $\gamma_{\Lambda+k}(B|\omega) = \gamma_\Lambda(S^k(B)|S^k(\omega))$. Translation-invariant interactions give rise to translation-invariant specifications. To some extent, the opposite statement is also true: Sullivan showed ([5, 62]) that for a quasilocal translation-invariant specification on $\mathbb{Z}$, one can find a translation-invariant interaction Φ such that $\gamma = \gamma^\Phi$. Recently, however, it was shown in [5] that this interaction is not necessarily uniformly absolutely convergent.

It should be noted that Gibbsian specifications can be uniquely recovered from a consistent family of single-site probability kernels (densities) $\{\gamma_{\{i\}} : i \in \mathbb{L}\}$ [33]. Therefore, it is sufficient to study only the single-site densities of a Gibbsian specification instead of studying *all* densities. Due to this fact, it is worth defining the single-site densities of a specification separately from the concept of specification as follows.

Definition 4.2.1. [31] *A collection* $\{\gamma_{\{i\}}\}_{i\in\mathbb{L}}$ *of positive functions* $\gamma_{\{i\}}(\cdot|\cdot) : E \times E^{\mathbb{L}\setminus\{i\}} \to (0,1)$ *is called the family of single-site densities of a specification if*

(i) $\sum_{a_i \in E} \gamma_{\{i\}}(a_i|\omega_{\{i\}^c}) = 1$ *for all* $\omega \in \Omega = E^{\mathbb{L}}$, $i \in \mathbb{L}$,

(ii) and for all $i, j \in \mathbb{L}$, $\alpha, \omega \in \Omega$ *the following holds*

$$\frac{\gamma_{\{i\}}(\alpha_i|\alpha_j\omega_{\{i,j\}^c})}{\sum_{\beta_{\{i,j\}}} \frac{\gamma_{\{j\}}(\beta_j|\beta_i\omega_{\{i,j\}^c})\gamma_{\{i\}}(\beta_i|\alpha_j\omega_{\{i,j\}^c})}{\gamma_{\{j\}}(\alpha_j|\beta_i\omega_{\{i,j\}^c})}} = \frac{\gamma_{\{j\}}(\alpha_j|\alpha_i\omega_{\{i,j\}^c})}{\sum_{\beta_{\{i,j\}}} \frac{\gamma_{\{i\}}(\beta_i|\beta_j\omega_{\{i,j\}^c})\gamma_{\{j\}}(\beta_j|\alpha_i\omega_{\{i,j\}^c})}{\gamma_{\{i\}}(\alpha_i|\beta_j\omega_{\{i,j\}^c})}}.$$

The following theorem signifies the importance of single-site densities of a specification, and it will be useful later.

Proposition 4.2.2. [33] *Let* $\{\gamma_{\{i\}}\}_{i\in\mathbb{L}}$ *be single-site densities of a specification. There is a unique non-null specification* γ *on* $(\Omega, \mathscr{F})$ *having* $\{\gamma_{\{i\}}\}_{i\in\mathbb{L}}$ *as its single-site densities. Furthermore,* γ *is quasilocal if and only if all functions in the collection* $\{\gamma_{\{i\}}\}_{i\in\mathbb{L}}$ *are continuous, and a probability measure* $\mu \in \mathscr{M}_1(\Omega, \mathscr{F})$ *is consistent with* γ *if and only if it is consistent with all single-site probability kernels* $\gamma_{\{i\}}$, $i \in \mathbb{L}$.

4.2.2 Gibbs(ian) specifications in Dynamical Systems and Transfer operators

As already mentioned above, in Dynamical Systems, the 'natural' lattice is the half-line $\mathbb{L} = \mathbb{Z}_+$. For an introductory treatment of some of these issues, see also [48]. Let $X_+ = E^{\mathbb{Z}_+}$ be the space of one-sided sequences $\omega = (\omega_n)_{n\ge 0}$ in alphabet E. We equip X_+ with the metric $d(\omega, \omega') = \sum_{n=0}^{\infty} \mathbb{I}[\omega_n \neq \omega'_n]2^{-n}$. We also define the left-shift S on X_+ by $y = Sx$ where $y_i = x_{i+1}$ for all $i \geq 0$. Let $\phi : X_+ \to \mathbb{R}$ be a continuous function (potential). Following [16, 17, 64], we define the corresponding specification $\overrightarrow{\gamma}^{\phi} := \{\overrightarrow{\gamma}_n = \overrightarrow{\gamma}^{\phi}_{[0,n-1]},\ n \geq 1\}$, by

$$\overrightarrow{\gamma}_n(a_0^{n-1}|x_n^\infty) = \frac{\exp\big((S_n\phi)(a_0^{n-1}x_n^\infty)\big)}{\sum_{\bar{a}_0^{n-1}}\exp\big((S_n\phi)(\bar{a}_0^{n-1}x_n^\infty)\big)}, \quad \text{where } (S_n\phi)(x) = \sum_{k=0}^{n-1}\phi(S^k x). \tag{4.2}$$

This gives a family of probability kernels on finite intervals $[0, n-1]$ in $\mathbb{Z}_+$. However, the definition extends to general volumes $\Lambda \Subset \mathbb{Z}_+$ by

$$\overrightarrow{\gamma}_\Lambda(a_\Lambda|x_{\Lambda^c}) = \frac{\exp\left((S_{n+1}\phi)(a_\Lambda x_{\Lambda^c})\right)}{\sum_{\bar{a}_\Lambda}\exp\left((S_{n+1}\phi)(\bar{a}_\Lambda x_{\Lambda^c})\right)}, \ a_\Lambda \in E^\Lambda, x \in X_+, \tag{4.3}$$

where $n = \max\Lambda$.

For $f \in C(X_+, \mathbb{R})$, one has

$$\overrightarrow{\gamma}_n(f)(x) = \frac{\sum_{\bar{a}_0^{n-1}}\exp\big((S_n\phi)(\bar{a}_0^{n-1}x_n^\infty)\big) f(\bar{a}_0^{n-1}x_n^\infty)}{\sum_{\bar{a}_0^{n-1}}\exp\big((S_n\phi)(\bar{a}_0^{n-1}x_n^\infty)\big)}. \tag{4.4}$$

It turns out that $\overrightarrow{\gamma}_n(f)$ can naturally be expressed in terms of the Ruelle-Perron-Frobenius transfer operator. This is the operator $\mathscr{L}_\phi$ acting on the space of continuous functions $C(X_+, \mathbb{R})$ as

$$\mathscr{L}_\phi f(x) = \sum_{y \in S^{-1}x} e^{\phi(y)} f(y) = \sum_{a \in E} e^{\phi(ax)} f(ax), \tag{4.5}$$

where the configuration ax is obtained by the concatenation of the letter a and the configuration x. Thus, for any $n \geq 1$,

$$\mathscr{L}_\phi^n f(x) = \sum_{a_0^{n-1} \in E^n} e^{S_n\phi(a_0^{n-1}x)} f(a_0^{n-1}x), \text{ and hence, } \overrightarrow{\gamma}_n(f)(x) = \frac{\mathscr{L}_\phi^n f(S^n x)}{\mathscr{L}_\phi^n \mathbf{1}(S^n x)}.$$

One readily checks that the family of probability kernels $\{\overrightarrow{\gamma}_n\}$ has the standard properties of specifications; most importantly, the consistency condition

$$\overrightarrow{\gamma}_m(\overrightarrow{\gamma}_n(f)) = \overrightarrow{\gamma}_n(\overrightarrow{\gamma}_m(f)) = \overrightarrow{\gamma}_m(f)$$

for all $f \in C(X_+, \mathbb{R})$ and every $m \geq n \geq 1$, and this particular property can be readily validated using properties of transfer operators [64, Theorem 2.1].

The set of all Gibbs states on X_+ for potential ϕ, i.e., the set of measures consistent with the specification $\overrightarrow{\gamma}^{\phi}$, will be denoted by $\mathscr{G}(X_+, \phi)$. The set of Gibbs measures $\mathscr{G}(X_+, \phi)$ is a closed convex set and the extremal points of $\mathscr{G}(X_+, \phi)$ are tail-trivial.

Transfer operators allow for a dual view on Gibbs measures $\mathscr{G}(X_+,\phi)$. Define the dual operator $\mathscr{L}_\phi^*$, acting on the space of measures $\mathscr{M}(X_+)$ by

$$\int f\, d(\mathscr{L}_\phi^*\nu)=\int \mathscr{L}_\phi f\, d\nu \quad \text{for all } f\in C(X_+,\mathbb{R}).$$

It is well-known that there exists at least one eigenprobability ν on X_+ for the maximal eigenvalue $\lambda=e^{P(\phi)}$, i.e.,

$$\mathscr{L}_\phi^*\nu=e^{P(\phi)}\nu,$$

where $P(\phi)=\lim\limits_{n\to\infty}\frac{1}{n}\log\sup\limits_{x\in X_+}\mathscr{L}^n\mathbf{1}(x)$ is the so-called topological pressure of ϕ. Combining the results of [64, Corollary 2.3] and [17, Theorem 4.8], we can conclude that the sets of probability eigenmeasures of $\mathscr{L}_\phi^*$ and the Gibbs states for ϕ on X_+ coincide:

$$\nu\in\mathscr{G}(X_+,\phi) \text{ if and only if } \mathscr{L}_\phi^*\nu=\lambda\nu.$$

There is an interesting phenomenon in the theory of Gibbs measures on one-sided (half-line) symbolic spaces, which has no direct analogue in the two-sided (whole-line) context. Let us say that a continuous potential $\phi: X_+\to\mathbb{R}$ is quasi-normalized if $\mathscr{L}_\phi\mathbf{1}$ is a constant function on X_+. It turns out that ϕ is quasi-normalized if and only if all Gibbs measures μ in $\mathscr{G}(X_+,\phi)$ are translation invariant, i.e., $\mu=\mu\circ S^{-1}$ [65, Theorem 1.2 (iii)].

4.2.3 Relation between Gibbsian specifications

Specifications discussed above are defined on different spaces: $X=E^{\mathbb{Z}}$ vs $X_+=E^{\mathbb{Z}_+}$, as well as, in different terms: namely, the interaction Φ vs the potential ϕ. What is the relation between these classes of specifications?

For a given interaction Φ, the potential ϕ should be interpreted as minus the contribution to the energy from (the neighborhood of) the origin [57, Section 3.2], [26, Section 2.4.5]. In fact there are multiple possibilities to define relevant ϕ, e.g.,

$$\phi(\omega):=-\sum_{0\in V\Subset\mathbb{Z}}\frac{1}{|V|}\Phi_V(\omega_V), \tag{4.6}$$

or,

$$\phi(\omega):=-\sum_{0\in V\Subset\mathbb{Z}_+}\Phi_V(\omega_V). \tag{4.7}$$

In the setup of Thermodynamic Formalism, the second choice is particularly convenient, and it will be used in this chapter.

It is worth mentioning that for every $\phi \in C(E^{\mathbb{Z}_+},\mathbb{R})$ there exists a translation-invariant interaction Φ^ϕ on $X = E^{\mathbb{Z}}$ satisfying (4.7) such that

$$\sum_{0\in V\Subset\mathbb{Z}} \frac{1}{|V|}\|\Phi_V^\phi\|_\infty < \infty,$$

nonetheless, such Φ^ϕ does not need to be unique [57]. The reciprocal property would be more interesting, namely that for each $\phi \in C(X_+)$ there would exist a translation-invariant UAC interaction Φ on X satisfying (4.7). Unfortunately, there exist counterexamples showing this to be false (c.f. Proposition 4.6.2 and Section 4.9).

4.3 Basic notions II: The Ruelle theorem and Equilibrium states

Gibbs measures are eigenmeasures of the duals of transfer operators corresponding to the maximal eigenvalue. What can we say about the eigenfunctions of transfer operators? Ruelle established the first result for smooth potentials [55, 57]: if $\phi : X_+ \to \mathbb{R}$ is Hölder continuous, then the transfer operator $\mathscr{L}_\phi$ has a positive continuous eigenfunction h; $\mathscr{L}_\phi h = \lambda h$, with $\lambda = e^{P(\phi)}$. The existence of continuous eigenfunctions has been established for larger classes of smooth potentials: for example, by Walters for potentials with summable variations ([66, 67]). For less regular potentials – the so-called Bowen class – Walters established the existence of bounded measurable eigenfunctions.

There is an interesting principal relation between the Gibbs measures $\mathscr{G}(X_+,\phi)$, eigenfunctions of transfer operators, and translation invariant equilibrium states.

Proposition 4.3.1. [51, Theorem 1] *Consider a continuous potential $\phi \in C(X_+)$, and suppose $\nu \in \mathscr{G}(X_+,\phi)$ or, equivalently, $\mathscr{L}_\phi^* \nu = e^{P(\phi)}\nu$. Then*

1) *If there exists a non-negative eigenfunction $h \in L^1(X_+,\nu)$ of the transfer operator $\mathscr{L}_\phi$ with $\mathscr{L}_\phi h = e^{P(\phi)}h$, then $\mu = h\cdot\nu$ (i.e., $d\mu = h\,d\nu$) is a translation invariant equilibrium state of ϕ: namely, μ is the measure satisfying the variational principle*

$$h(S,\mu)+\int \phi\, d\mu = \sup_{\rho\in\mathscr{M}_1(X_+,S)}\Big[h(S,\rho)+\int \phi\, d\rho\Big] =: P(\phi).$$

where $h(S,\cdot)$ is the Kolmogorov-Sinai entropy, the supremum is taken over the set of all translation invariant probability measures on X_+, and $P(\phi)$ is the (topological) pressure of ϕ.

2) *If there exists a translation invariant measure* $\mu \in \mathscr{M}_1(X_+, S)$ *such that* $\mu \ll \nu$*, then* μ *is an equilibrium state for* ϕ *and the Radon-Nikodym derivative* $h = \frac{d\mu}{d\nu}$ *is the eigenfunction of* $\mathscr{L}_\phi$ *with* $\mathscr{L}_\phi h = e^{P(\phi)} h$.

The proof of Proposition 4.3.1 follows standard arguments under the assumption that the eigenfunction (Radon-Nikodym density) is continuous [64]. While the transfer operator $\mathscr{L}_\phi$ is typically considered as an operator acting on continuous functions, it can be readily extended to $L^1(\nu)$. The standard proof is then adapted to integrable functions in a straightforward fashion.

By the above proposition, the condition that the transfer operator $\mathscr{L}_\phi$ has an eigenfunction for the maximal eigenvalue $\lambda = e^{P(\phi)}$ is equivalent to the existence of an equilibrium state for ϕ on X_+ having this eigenfunction as Radon-Nikodym derivative with respect to some Gibbs state $\nu \in \mathscr{G}(X_+, \phi)$.

We should note that the approach based on Proposition 4.3.1 has already been used in at least two particular cases: for Dyson potentials by Johansson, Öberg, Pollicott [47], and for product-type potentials by Cioletti, Denker, Lopes, Stadlbauer [15]. Let us now recall these results. The Dyson potential ϕ^D is given by

$$\phi^D(x) := \beta \sum_{n=1}^{\infty} \frac{x_0 x_n}{n^\alpha}, \quad x \in \{-1, 1\}^{\mathbb{Z}_+}, \tag{4.8}$$

where $\beta \geq 0$ is the inverse temperature and $\alpha > 1$ is the model parameter. The Dyson potential ϕ^D originates from the standard Dyson interaction Φ, by means of (4.7),

$$\Phi_\Lambda(\omega) := \begin{cases} -\dfrac{\beta \omega_i \omega_j}{|i-j|^\alpha}, & \text{if } \Lambda = \{i, j\} \subset \mathbb{Z},\ i \neq j; \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

If $\alpha > 2$, ϕ^D has summable variations, and, hence, the Ruelle-Walters theorem applies, and the transfer operator has a unique positive continuous eigenfunction [67]. For $\alpha \in (1, 2]$, in complete analogy to the classical (whole-line) Dyson model on $\{-1, 1\}^{\mathbb{Z}}$, phase transitions – the existence of multiple Gibbs measures – occur [46], and no general result in Dynamical Systems applies. The main result of [47] reads

Theorem 4.3.2. [47] *For* $\alpha \in \left(\frac{3}{2}, 2\right]$ *and all sufficiently small*[1] $\beta \in [0, +\infty)$ *there exists a positive continuous eigenfunction of the Perron-Frobenius transfer operator* $\mathscr{L}_{\phi^D}$.

[1] This result has been improved by the authors since we first submitted our paper [29]. Their result now reads that the continuous eigenfunction exists for all supercritical temperatures.

Remark 4.3.3. *Theorem 4.3.2 holds for all $\beta < \beta_c^1$, where β_c^1 is the critical value for a certain long-range Bernoulli percolation*[1]*. For further details, see Section 4.9 and* [47].

In [15], the authors introduced a product-type potential $\phi^P : \{-1,1\}^{\mathbb{Z}_+} \to \mathbb{R}$, with

$$\phi^P(x) := \beta \sum_{n=1}^{\infty} \frac{x_n}{n^\alpha}, \tag{4.10}$$

where again $\beta \geq 0$ and $\alpha > 1$. As above, ϕ^P has a summable variation for $\alpha > 2$, and thus the standard theory applies [67]. For each β and $\alpha > 1$, there is a unique Gibbs state ν which has the product form $\nu = \prod_{n=0}^{\infty} \lambda_n$, with $\lambda_n(1) = p_n = \dfrac{\exp(\beta \sum_{i=1}^n i^{-\alpha})}{2\cosh(\beta \sum_{i=1}^n i^{-\alpha})}$, while the unique equilibrium state for ϕ^P is the Bernoulli measure μ on $\{-1,1\}^{\mathbb{Z}_+}$ with $\mu([1]_0) = \lim_{n\to\infty} p_n = \dfrac{e^{\beta\zeta(\alpha)}}{2\cosh(\beta\zeta(\alpha))}$.

Theorem 4.3.4. [15]

(i) For $\alpha > 3/2$, the equilibrium state μ^P is absolutely continuous with respect to the Gibbs state ν^P, and thus the Perron-Frobenius transfer operator $\mathscr{L}_{\phi^{Pt}}$ has an eigenfunction $h^P := \dfrac{d\mu^P}{d\nu^P} \in L^1(X_+, \nu^P)$. The density h^P is continuous for $\alpha > 2$, and essentially discontinuous if $\alpha \leq 2$.

(ii) If $1 < \alpha \leq 3/2$, then μ^P and ν^P are singular measures, and therefore, the transfer operator $\mathscr{L}_{\phi^P}$ does not have an eigenfunction in $L^1(X_+, \nu^P)$.

4.4 Basic notions III: Equivalence of interactions. Dobrushin uniqueness condition

4.4.1 Equivalent specifications

We start this section with the notion of equivalent Gibbsian specifications; we follow closely the book by Georgii [39, Chapter 7].

Definition 4.4.1. *Two Gibbsian specifications $\gamma, \tilde{\gamma}$ are called equivalent, denoted by $\gamma \simeq \tilde{\gamma}$ if there exists a constant $C > 1$ such that*

$$c^{-1}\gamma_\Lambda(A\,|\,\cdot) \leq \tilde{\gamma}_\Lambda(A\,|\,\cdot) \leq c\gamma_\Lambda(A\,|\,\cdot)$$

for all $\Lambda \in \mathbb{Z}$ and $A \in \mathscr{F}$.

In particular, if Φ and Ψ are both UAC interactions, and $\gamma = \gamma^{\Phi}$, $\tilde{\gamma} = \gamma^{\Psi}$ are the corresponding Gibbsian specifications, then $\gamma^{\Phi} \simeq \gamma^{\Psi}$ if

$$\sup_{\Lambda \Subset \mathbb{Z}} \left\| H_{\Lambda}^{\Phi} - H_{\Lambda}^{\Psi} \right\| < \infty. \tag{4.11}$$

The sufficient condition (4.11) certainly holds if the collection of sets

$$\{\Lambda \Subset \mathbb{Z} : \Phi_{\Lambda}(\cdot) \neq \Psi_{\Lambda}(\cdot)\}$$

is finite. In this case, we say that Ψ is a ***finite perturbation*** of Φ, and vice versa.

The following theorem summarizes the properties of sets of Gibbs measures of two equivalent specifications.

Theorem 4.4.2. [39, Theorem 7.3] *Let γ and $\tilde{\gamma}$ be two equivalent specifications. Then $\mathscr{G}(\gamma) \neq \emptyset$ if and only if $\mathscr{G}(\tilde{\gamma}) \neq \emptyset$, and in this case there is an affine bijection $\mu \longleftrightarrow \tilde{\mu}$ between $\mathscr{G}(\gamma)$ and $\mathscr{G}(\tilde{\gamma})$ such that $\mu = \tilde{\mu}$ on $\mathscr{T}$. In particular,* $|\mathrm{ex}\mathscr{G}(\gamma)| = |\mathrm{ex}\mathscr{G}(\tilde{\gamma})|$.

4.4.2 Dobrushin uniqueness condition and its corollaries

This is one of the most general criteria for the uniqueness of Gibbs states. We discuss it in the framework of a general countable set $\mathbb{L}$ of sites and a configuration space $\Omega := E^{\mathbb{L}}$. Consider a uniformly absolutely convergent (UAC) interaction $\Phi = \{\Phi_{\Lambda}(\cdot) : \Lambda \Subset \mathbb{L}\}$ on Ω and let γ^{Φ} be the corresponding Gibbsian specification. For any sites $i, j \in \mathbb{L}$, define

$$C(\gamma^{\Phi})_{i,j} := \sup_{\eta_{\mathbb{L}\setminus\{j\}} = \overline{\eta}_{\mathbb{L}\setminus\{j\}}} \|\gamma_{\{i\}}^{\Phi}(\cdot|\eta) - \gamma_{\{i\}}^{\Phi}(\cdot|\overline{\eta})\|_{\infty},$$

where $\|\cdot\|_{\infty}$ is the supremum norm on $\mathscr{M}(\Omega)$ defined by $\|\tau\|_{\infty} := \sup_{B \in \mathscr{B}(\Omega)} |\tau(B)|$ for any finite signed Borel measure τ. The infinite matrix $C(\gamma^{\Phi}) := (C(\gamma^{\Phi})_{i,j})_{i,j \in \mathbb{L}}$ is called the Dobrushin interdependence matrix.

Definition 4.4.3. [18, 37, 39] *The specification γ^{Φ} satisfies the **Dobrushin uniqueness (contraction) condition** if*

$$c(\gamma^{\Phi}) := \sup_{i \in \mathbb{L}} \sum_{j \in \mathbb{L}} C(\gamma^{\Phi})_{i,j} < 1. \tag{4.12}$$

The Dobrushin uniqueness condition admits a slightly stronger — and easy to check— form:

Proposition 4.4.4. [39, Proposition 8.8] *Let $\mathbb{L}$ be any countable set, and suppose a UAC interaction $\Phi = \{\Phi_\Lambda(\cdot) : \Lambda \Subset \mathbb{L}\}$ is such that*

$$\bar{c}(\Phi) := \frac{1}{2} \sup_{i \in \mathbb{L}} \sum_{\Lambda \ni i} (|\Lambda| - 1)\,\delta(\Phi_\Lambda) < 1, \tag{4.13}$$

where $\delta(f) := \sup\{|f(\xi) - f(\eta)| : \xi, \eta \in E^{\mathbb{L}}\}$ is the variation of $f : E^{\mathbb{L}} \to \mathbb{R}$. Then γ^Φ satisfies the Dobrushin uniqueness condition.

The proof of Proposition 4.4.4 boils down to showing that for all i, j, $i \neq j$,

$$C(\gamma^\Phi)_{ij} \leq \frac{1}{2} \sum_{\{i,j\} \subset \Lambda} \delta(\Phi_\Lambda) =: \bar{C}(\Phi)_{ij},$$

and hence $c(\gamma^\Phi) = \sup_i \sum_j C(\gamma^\Phi)_{ij} \leq \sup_i \sum_j \bar{C}(\Phi)_{ij} = \bar{c}(\Phi)$. Notice that the non-negative matrix $\bar{C}(\Phi) := (\bar{C}(\Phi)_{i,j})_{i,j \in \mathbb{L}}$ is symmetric.

Note that by transitioning to condition (4.13), we are reinforcing the primary condition of this chapter, which is the Dobrushin uniqueness condition (4.12), but with a particular purpose. The condition (4.13) is stable under a *perturbation* of the underlying model/interaction Φ. Indeed, let $\Psi = \{\Psi_V\}_{V \Subset \mathbb{L}}$ be an interaction such that for $V \Subset \mathbb{L}$, either $\Psi_V = \Phi_V$ or $\Psi_V = 0$, then it is straightforward to check that $\bar{c}(\Psi) \leq \bar{c}(\Phi)$. Thus Ψ inherits the Dobrushin uniqueness condition from Φ as long as Φ satisfies (4.13).

The crucial property of the Dobrushin uniqueness condition is that it provides the uniqueness of the compatible probability measures with the specification γ^Φ. In fact, we have the following theorem.

Theorem 4.4.5. [18][37, Section 6.5][39, Chapter 8] *If γ^Φ satisfies the Dobrushin uniqueness condition (4.12), then $\left|\mathscr{G}(\gamma^\Phi)\right| \leq 1$.*

If E is compact, as is the case in this chapter, there is always at least one Gibbs state; hence, the inequality becomes an equality.

The validity of Dobrushin's criterion yields two important properties of the unique Gibbs state: concentration inequalities and explicit bounds on the decay of correlations.
The first property involves the coefficient $\bar{c}(\Phi)$, and it provides the tail bounds for the unique Gibbs measure. Let

$$\delta_k F := \sup\left\{F(\xi) - F(\eta) : \xi_j = \eta_j,\ j \in \mathbb{L} \setminus \{k\}\right\} \tag{4.14}$$

denote the oscillation of a local function $F : \Omega \to \mathbb{R}$ at a site $k \in \mathbb{L}$ and $\underline{\delta}(F) = (\delta_k F)_{k \in \mathbb{L}}$ is the oscillation vector, where $\Omega = E^{\mathbb{L}}$.

Theorem 4.4.6. [50] *Suppose Φ is a UAC interaction satisfying (4.13) and let μ_Φ be its unique Gibbs measure. Set*

$$D := \frac{4}{(1-\bar{c}(\Phi))^2}\,. \tag{4.15}$$

Then, for all $t>0$ and every continuous function F on Ω, one has

$$\mu_\Phi\Big(\Big\{\omega\in\Omega:\ F(\omega)-\int_\Omega F\,d\mu_\Phi \ \geq\ t\Big\}\Big) \leq e^{-\frac{2t^2}{D\|\underline{\delta}(F)\|_2^2}}, \tag{4.16}$$

where $\|\underline{\delta}(F)\|_2^2 := \sum_{k\in\mathbb{L}}(\delta_k F)^2$.

It is a well-known fact that (4.16) implies that F is *sub-Gaussian* and μ_Φ has the *moment concentration bounds* as stated in the following theorem ([63, Proposition 2.5.2]).

Theorem 4.4.7. [63] *Assume that a probability measure μ_Φ satisfies (4.16) with a constant $D = D(\mu_\Phi) > 0$. Then:*

*(i) μ_Φ satisfies a **Gaussian Concentration Bound** with the constant D, i.e., for any continuous function F on $\Omega = E^{\mathbb{L}}$, one has*

$$\int_\Omega e^{F-\int_\Omega F\,d\mu_\Phi}\,d\mu_\Phi \leq e^{D\|\underline{\delta}(F)\|_2^2}. \tag{4.17}$$

(ii) for all $m\in\mathbb{N}$ and any continuous function F on Ω, one has

$$\int_\Omega \Big|F-\int_\Omega F\,d\mu_\Phi\Big|^m d\mu_\Phi \leq \Big(\frac{D\|\underline{\delta}(F)\|_2^2}{2}\Big)^{\frac{m}{2}} m\Gamma\Big(\frac{m}{2}\Big), \tag{4.18}$$

where Γ is Euler's gamma function.

We present the second property in the particular setup $\mathbb{L}=\mathbb{Z}$, and it involves the $\mathbb{Z}\times\mathbb{Z}$ matrix

$$D(\gamma^\Phi) = \sum_{n=0}^{\infty} C(\gamma^\Phi)^n\,. \tag{4.19}$$

The sum of the $\mathbb{Z}\times\mathbb{Z}$ matrices in the right-hand side converges due to the Dobrushin condition (4.12).

Proposition 4.4.8. [36, 39] *Consider a UAC interaction Φ on $X = E^{\mathbb{Z}}$.*

(i) Assume the specification γ^Φ satisfies the Dobrushin condition (4.12) and let μ be its unique Gibbs measure. Then, for all $f,g\in C(X)$ and $i\in\mathbb{Z}$,

$$\Big|cov_\mu(f, g\circ S^i)\Big| \leq \frac{1}{4}\sum_{k,j\in\mathbb{Z}} D(\gamma^\Phi)_{jk}\cdot\delta_k f\cdot\delta_{j-i}\, g. \tag{4.20}$$

(ii) Suppose Φ *satisfies (4.13) and define the non-negative symmetric* $\mathbb{Z}\times\mathbb{Z}$*-matrix by*

$$\bar{D}(\Phi) := \sum_{n\geq 0} \bar{C}(\Phi)^n . \tag{4.21}$$

Then,

$$\sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \bar{D}(\Phi)_{ij} \leq \frac{1}{1-\bar{c}(\Phi)}. \tag{4.22}$$

4.5 Main results I: From half-line to whole-line specifications and measures

4.5.1 From half-line to whole-line specifications

The main results of this chapter are grouped into two parts. In the first part, we consider a half-line potential $\phi : X_+ \to \mathbb{R}$ satisfying minor technical assumptions and we identify the natural translation-invariant specification on $X = E^{\mathbb{Z}}$ which extends $\overrightarrow{\gamma}^{\phi}$. In the second part we provide sufficient conditions for the Gibbs/equilibrium state on X_+ to be absolutely continuous/equivalent with respect to the corresponding half-line Gibbs state ν. Then Proposition 3.1 allows us to conclude that the transfer operator admits an eigenfunction.
First, we consider the issue of whether there is a whole-line specification naturally associated with a half-line specification $\overrightarrow{\gamma}^{\phi}$. It turns out that under a very mild condition on ϕ we obtain an affirmative answer.

Definition 4.5.1. *We say that a continuous potential* $\phi : X_+ \to \mathbb{R}$ *satisfies the* ***extensibility*** *condition if for all* $a_0, b_0 \in E$ *the sequence*

$$F_n^{a_0,b_0}(x) := S_{n+1}\phi(x_{-n}^{-1} b_0 x_1^{\infty}) - S_{n+1}\phi(x_{-n}^{-1} a_0 x_1^{\infty}) = \sum_{i=0}^{n} \big(\phi(x_{-i}^{-1} b_0 x_1^{\infty}) - \phi(x_{-i}^{-1} a_0 x_1^{\infty})\big)$$

converges uniformly on $x \in X$ *as* $n \to \infty$.

This condition has first appeared in [7] in connection to a related question of whether g-measures are also Gibbs. In terms of the extensibility condition, we can reformulate the main result of [7] as follows.

Theorem 4.5.2. [7] *Let* μ *be a* $g-$*measure for a continuous* $g-$*function* $g : X_+ \to (0,1)$*, i.e.,* μ *is translation-invariant and for* $\mu-$*almost all* $x \in X_+$, $\mu(x_0|\mathscr{F}_{[1,\infty)})(x) = g(x)$. *Then* μ *(the natural extension) is a Gibbs measure on* X *if and only if* $\log g$ *satisfies the extensibility condition.*

In [7], the authors identified several sufficient conditions for ϕ to satisfy the extensibility condition, such as the Walters condition and the so-called Good Future condition. Note that a function $\phi \in C(X_+,\mathbb{R})$ satisfies the Walters condition and the Good Future condition if $\lim_{p\to\infty}\sup_{n\geq 1} v_{n+p}(S_n\phi) = 0$ and $\sum_{k=1}^{\infty}\delta_k\phi < \infty$, respectively. Here

$$v_k(\varphi) := \sup_{x_0^{k-1}=y_0^{k-1}} |\varphi(x)-\varphi(y)| \tag{4.23}$$

is the k^{th}–variation (or the oscillation in the volume $[0,k-1]$) of a function $\varphi : X_+ \to \mathbb{R}$ and

$$\delta_k\varphi \equiv \sup_{x\in X_+, a_k, b_k\in E} \left|\varphi(x_0^{k-1}a_k x_{k+1}^{\infty}) - \varphi(x_0^{k-1}b_k x_{k+1}^{\infty})\right|. \tag{4.24}$$

is the oscillation of φ at the site k. Interesting examples of potentials satisfying the extensibility condition are the Dyson potential (4.8) and the product-type potential (4.10). Note that both potentials are in the Walters class if and only if $\alpha > 2$. However, both satisfy the Good Future condition, and thus the extensibility condition as well, for all admissible values of the parameter α, since $\delta_n(\phi^D) = \mathcal{O}(n^{-\alpha})$ and $\delta_n(\phi^P) = \mathcal{O}(n^{-\alpha})$.

Gibbsianness for potentials satisfying the extensibility condition stems from the fact that they lead to a natural whole-line translation-invariant specification. Indeed, if $\phi \in C(X_+,\mathbb{R})$ satisfies the extensibility condition the limits

$$\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\{i\}}(\sigma_i|\omega_{\{i\}^c}) := \lim_{p\to\infty} \frac{e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\sigma_i\omega_{i+1}^{\infty})}}{\sum_{\bar{\omega}_i} e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\bar{\omega}_i\omega_{i+1}^{\infty})}} \tag{4.25}$$

are well defined for all $i \in \mathbb{Z}$. In turn, they lead to a full specification.

Proposition 4.5.3. *There is a unique translation-invariant quasilocal non-null specification $\overset{\rightleftharpoons}{\gamma}{}^{\phi}$ on X such that the functions (4.25) become its single-site densities.*

Proof. By Proposition 4.2.2, it is sufficient to check that $\{\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\{i\}}\}_{i\in\mathbb{Z}}$ satisfy conditions (i) and (ii) of Definition 4.2.1. Clearly, $\{\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\{i\}}\}_{i\in\mathbb{Z}}$ satisfies the first condition In order to check the second condition, consider arbitrary $i, j \in \mathbb{Z}$ with $i < j$, and $p \in \mathbb{N}$ such that $i, j \gg -p$. Then a straightforward computation shows that

$$\frac{e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\alpha_i\omega_{i+1}^{j-1}\alpha_j\omega_{j+1}^{\infty})}}{\sum_{\beta_{i,j}} e^{S_{j+p+1}\phi(\omega_{-p}^{i-1}\beta_i\omega_{i+1}^{j-1}\beta_j\omega_{j+1}^{\infty})+S_{i+p+1}\phi(\omega_{-p}^{i-1}\beta_i\omega_{i+1}^{j-1}\alpha_j\omega_{j+1}^{\infty})-S_{j+p+1}\phi(\omega_{-p}^{i-1}\beta_i\omega_{i+1}^{j-1}\alpha_j\omega_{j+1}^{\infty})}} =$$

$$= \frac{e^{S_{j+p+1}\phi(\omega_{-p}^{i-1}\alpha_i\omega_{i+1}^{j-1}\alpha_j\omega_{j+1}^{\infty})}}{\sum_{\beta_{i,j}} e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\beta_i\omega_{i+1}^{j-1}\beta_j\omega_{j+1}^{\infty})+S_{j+p+1}\phi(\omega_{-p}^{i-1}\alpha_i\omega_{i+1}^{j-1}\beta_j\omega_{j+1}^{\infty})-S_{i+p+1}\phi(\omega_{-p}^{i-1}\alpha_i\omega_{i+1}^{j-1}\beta_j\omega_{j+1}^{\infty})}}.$$

The limit $p \to \infty$ of this identity yields condition (ii).

Finally, note that for all $i \in \mathbb{Z}$,

$$\overset{\rightleftharpoons}{\gamma}^{\phi}_{\{0\}}((S^i\sigma)_0|S^i(\omega)_{\{0\}^c}) = \lim_{p\to\infty} \frac{\exp(S_{p+1}\phi(\omega_{i-p}^{i-1}\sigma_i\omega_{i+1}^{\infty}))}{\sum_{\bar{\sigma}_i}\exp(S_{p+1}\phi(\omega_{i-p}^{i-1}\bar{\sigma}_i\omega_{i+1}^{\infty}))} = \overset{\rightleftharpoons}{\gamma}^{\phi}_{\{i\}}(\sigma_i|\omega_{\{i\}^c})$$

and hence $\overset{\rightleftharpoons}{\gamma}^{\phi}$ is a translation-invariant specification. □

Example 4.5.4. *Let us illustrate the construction in the previous proof with the Dyson potential ϕ defined in (4.8). In this case a straightforward computation shows that for all $p \in \mathbb{N}$, $\beta \geq 0$, $\sigma_0 \in \{-1,1\}$ and $\omega \in \{-1,1\}^{\mathbb{Z}}$,*

$$\frac{\exp\Big(S_{p+1}\phi(\omega_{-p}^{-1}\sigma_0\omega_1^{\infty})\Big)}{\exp\big(S_{p+1}\phi(\omega_{-p}^{-1}\bar{\sigma}_0\omega_1^{\infty})\big)} = \frac{\exp\Big(\beta\sum_{k=-p}^{-1}\frac{\sigma_0\omega_k}{|k|^{\alpha}} + \beta\sum_{k=1}^{+\infty}\frac{\sigma_0\omega_k}{|k|^{\alpha}}\Big)}{\exp\Big(\beta\sum_{k=-p}^{-1}\frac{\bar{\sigma}_0\omega_k}{|k|^{\alpha}} + \beta\sum_{k=1}^{+\infty}\frac{\bar{\sigma}_0\omega_k}{|k|^{\alpha}}\Big)}.$$

Thus

$$\frac{\exp\Big(S_{p+1}\phi(\omega_{-p}^{-1}\sigma_0\omega_1^{\infty})\Big)}{\exp\big(S_{p+1}\phi(\omega_{-p}^{-1}\sigma_0\omega_1^{\infty})\big)+\exp\big(S_{p+1}\phi(\omega_{-p}^{-1}\bar{\sigma}_0\omega_1^{\infty})\big)} \to \gamma_{\{0\}}(\sigma_0|\omega_{-\infty}^{-1},\omega_1^{\infty})$$

as $p \to \infty$, where

$$\gamma_{\{0\}}(\sigma_0|\omega_{-\infty}^{-1},\omega_1^{\infty}) = \frac{\exp\Big(\beta\sum_{k=1}^{+\infty}\frac{\sigma_0(\omega_k+\omega_{-k})}{k^{\alpha}}\Big)}{\exp\Big(\beta\sum_{k=1}^{+\infty}\frac{\sigma_0(\omega_k+\omega_{-k})}{k^{\alpha}}\Big)+\exp\Big(\beta\sum_{k=1}^{+\infty}\frac{\bar{\sigma}_0(\omega_k+\omega_{-k})}{k^{\alpha}}\Big)}.$$

4.5.2 Whole-line Gibbs measures as limits of half-line ones

Once the correspondence between $\overset{\rightharpoonup}{\gamma}^{\phi}$ and $\overset{\rightleftharpoons}{\gamma}^{\phi}$ is established, it is interesting to understand the relation between $\mathscr{G}(X_+,\overset{\rightharpoonup}{\gamma}^{\phi})\big(=\mathscr{G}(X_+,\phi)\big)$ and $\mathscr{G}(X,\overset{\rightleftharpoons}{\gamma}^{\phi})$.

Suppose ν is a Gibbs probability measure on X_+ for some potential ϕ satisfying the extensibility condition, and consider an arbitrary measure ρ on $X_- = E^{-\mathbb{N}}$(e.g. a uniform Bernoulli measure), and consider the measure $\mu_0 = \rho\times\nu$ on $X = X_-\times X_+$.

For any $n \geq 0$, let

$$\mu_n = \mu_0 \circ S^{-n},$$

This is a sequence of Borel probability measures $\{\mu_n\}_{n\geq 0}$ on the compact metric space X, and hence it has weak*-converging subsequences $\{\mu_{n_k}\}$. It turns out that their limits are bona fide whole-line Gibbs measures.

Theorem 4.A. *Suppose ϕ satisfies the extensibility condition and $\nu \in \mathscr{G}(X_+, \phi)$. Consider $\mu_0 := \rho \times \nu$, where ρ is any probability measure on X_-. Assume that a subsequence $\{\mu_{n_k} = \mu_0 \circ S^{-n_k}\}_{k \geq 0}$ converges to a probability measure μ in the weak* topology as $k \to \infty$. Then for μ-almost all x:*

$$\mu(x_0|x_{-\infty}^{-1}, x_1^{\infty}) = \overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\{0\}}(x_0|x_{-\infty}^{-1}, x_1^{\infty}) = \lim_{n\to\infty} \frac{\exp\left(S_{n+1}\phi(x_{-n}^{-1}x_0x_1^{\infty})\right)}{\sum_{\bar{a}_0}\exp\left(S_{n+1}\phi(x_{-n}^{-1}\bar{a}_0x_1^{\infty})\right)}.$$

Hence μ is a whole-line Gibbs measure for the whole-line specification defined by the kernels $\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\{0\}}$.

Now we turn to translation invariant measures.

Theorem 4.B. *Assume the conditions of Theorem 4.A, let μ be a weak* limit point of the following sequence of measures*

$$\tilde{\mu}_n := \frac{1}{n}\sum_{i=0}^{n-1}\mu_0 \circ S^{-i}.$$

Then μ is a translation-invariant Gibbs measure for $\overset{\rightleftharpoons}{\gamma}{}^{\phi}$, i.e., $\mu \in \mathscr{G}_S(\overset{\rightleftharpoons}{\gamma}{}^{\phi}) := \mathscr{G}(\overset{\rightleftharpoons}{\gamma}{}^{\phi}) \cap \mathscr{M}_1(X,S)$.

We shall give the proofs of Theorem 4.A and Theorem 4.B in Section 4.8.

4.6 Main results II: Eigenfunctions of transfer operators and absolute continuity of Gibbs measures

As mentioned in the introduction, we can show the existence of an eigenfunction if we can show that the equilibrium state μ is absolutely continuous with respect to the half-line non-translation invariant Gibbs measure ν. This approach has recently been used by Johansson, Öberg and Pollicott [47] to show Theorem 4.3.2. We also use this approach to prove the following theorem.

Theorem 4.C. *Let ϕ be the Dyson potential (4.8). Suppose $\alpha > 1$ and $\beta > 0$ is sufficiently small. Then,*

(i) the half-line Dyson model ϕ on $X_+ = \{\pm 1\}^{\mathbb{Z}_+}$ admits a unique equilibrium state μ_+ and Gibbs state ν;

(ii) for all $\alpha > 1$, μ_+ is equivalent to ν, i.e., $\mu_+ \ll \nu$ and $\nu \ll \mu_+$, and thus the Perron-Frobenius transfer operator $\mathscr{L}_\phi$ has an eigenfunction in $L^1(X_+, \nu)$;

(iii) furthermore, if $\alpha > \frac{3}{2}$, there exists a continuous version of the Radon-Nikodym density $\frac{d\mu_+}{d\nu}$, and thus the Perron-Frobenius transfer operator $\mathscr{L}_\phi$ has a continuous eigenfunction.

Remark 4.6.1. *In fact, we prove Theorem 4.C under the Dobrushin uniqueness condition (c.f. (4.12)-(4.13)). Therefore, the parameter $\beta \geq 0$ should be sufficiently small so that the corresponding whole-line Dyson interaction on $X = \{\pm 1\}^{\mathbb{Z}}$ satisfies the Dobrushin uniqueness condition (4.13).*

We will show a much more general statement (Theorem 4.E below), from which one can easily obtain the first two parts of Theorem 4.C. Our method of proof of statement (iii) differs from that of [47], and can be generalised to other models. The proof of Theorem 4.C will be presented in Section 4.8.

4.6.1 From $C(X_+)$ to the space of interactions on X

It is clear from Section 4.5 that every (half-line) translation-invariant UAC interaction $\vec{\Phi}$ (i.e., $\vec{\Phi}_\Lambda \circ S = \vec{\Phi}_{\Lambda+1}$) on X_+, yields a potential $\phi \in C(X_+)$ with $\phi = -\sum_{0\in V \Subset \mathbb{Z}_+} \vec{\Phi}_V$, such that $\gamma^\phi = \gamma^{\vec{\Phi}}$.

In the opposite direction, it is known that every potential ϕ yields an equivalent interaction $\vec{\Phi}$ on X_+ [57] which in its turn can be extended by translations to an interaction Φ on X. This extension, however, is only guaranteed to belong to the so-called $\mathscr{B}_0(X)$ class —-i.e., is such that $\sum_{0\in V \Subset \mathbb{Z}_+} \frac{1}{|V|} ||\Phi_V||_\infty < \infty$— but it may fail to be UAC.

The determination of necessary and sufficient conditions for ϕ to yield a UAC potential Φ on X is an open problem that we do not address here. Rather, in the sequel we determine a class of potentials ϕ that admit a translation invariant UAC interaction Φ on $\mathbb{Z}$ satisfying

$$\phi = -\sum_{0\in V \Subset \mathbb{Z}_+} \Phi_V. \tag{4.26}$$

The following proposition shows that potentials in this class satisfy the extensibility condition.

Proposition 4.6.2. *Let $\phi \in C(X_+)$ such that there exists a translation invariant UAC interaction Φ on $\mathbb{Z}$ satisfying $\phi = -\sum_{0\in V \Subset \mathbb{Z}_+} \Phi_V$. Then ϕ satisfies the extensibility condition. Furthermore, $\overset{\rightleftharpoons}{\gamma}^\phi = \gamma^\Phi$.*

Proof. Note that

$$S_{n+1}\phi = -\sum_{V\cap[0,n]\neq\emptyset;\ V\Subset\mathbb{Z}_+} \Phi_V.$$

Thus since the interaction is translation-invariant ($\Phi\circ S^{-n} = \Phi_{V-n}$)

$$S_{n+1}\phi\circ S^{-n} = -\sum_{V\cap[-n,0]\neq\emptyset;\ V\Subset[-n,\infty]} \Phi_V.$$

(The LHS of the above equation should be considered on X, otherwise, S^{-n} is not defined on X_+.)

Pick any $\xi,\eta\in X$, such that $\xi_j=\eta_j$ if $j\neq 0$. Then

$$\begin{aligned} S_{n+1}\phi(\xi_{-n}^{\infty}) - S_{n+1}\phi(\eta_{-n}^{\infty}) &= \sum_{V\cap[-n,0]\neq\emptyset;\ V\Subset[-n,\infty]} (\Phi_V(\eta)-\Phi_V(\xi)) \\ &= \sum_{0\in V\Subset[-n,\infty]} (\Phi_V(\eta)-\Phi_V(\xi)). \end{aligned}$$

Since Φ is UAC, $\sum_{0\in V\Subset[-n,\infty]} [\Phi_V(\xi)-\Phi_V(\eta)]$ converges uniformly to $H_{\{0\}}^{\Phi}(\xi)-H_{\{0\}}^{\Phi}(\eta) = \sum_{0\in V\Subset\mathbb{Z}} (\Phi_V(\xi)-\Phi_V(\eta))$ as $n\to\infty$. Thus it is also clear that $\overset{\rightleftharpoons}{\gamma}^{\phi} = \gamma^{\Phi}$. □

Example 4.6.3. *Consider the Dyson potential ϕ (c.f. (4.8)) with the state space $E=\{-1,1\}$, and the standard (whole-line) Dyson interaction Φ on $\mathbb{Z}$ (c.f. (4.9)). Then it is easy to see that $\phi = -\sum_{0\in\Lambda\Subset\mathbb{Z}_+}\Phi_\Lambda$.*

4.6.2 Decoupling across the origin

To every UAC interaction Φ on $\mathbb{Z}$ we can associate a sequence $\Psi^{(k)}$ of interactions obtained by removing all bonds linking $-\mathbb{N}$ and $\mathbb{Z}_+$ and adding them one at a time. Formally, we consider the family

$$\mathscr{A} = \left\{\Lambda\Subset\mathbb{Z} : \min(\Lambda)<0,\ \max\Lambda\geq 0\right\}.$$

indexed according to some arbitrary order: $\mathscr{A} = \{\Lambda_1,\Lambda_2,\ldots\}$. Then define, for each $k\in\mathbb{Z}_+$,

$$\Psi_\Lambda^{(k)} = \begin{cases} 0, & \Lambda\in\{\Lambda_i : i\geq k+1\}, \\ \Phi_\Lambda, & \text{otherwise.} \end{cases} \tag{4.27}$$

In particular, $\Psi^{(0)}$ has no interaction between the left and right half lines. Clearly, all the constructed interactions are UAC and in addition have the following properties:

Remark 4.6.4. *1) Every $\Psi^{(k)}$ in (4.27) is a local (finite) perturbation of $\Psi^{(0)}$. Moreover, the sequence $\Psi^{(k)}$ tends to Φ as $k\to\infty$, in the sense that $\Psi^{(k)}_\Lambda \rightrightarrows \Phi_\Lambda$ for all $\Lambda \Subset \mathbb{Z}$.*

2) For any finite volume V,

$$||H_V^{\Psi^{(k)}} - H_V^{\Phi}||_\infty \le \sum_{\substack{\Lambda_j\cap V\neq\emptyset\\ j\ge k}} ||\Phi_{\Lambda_j}||_\infty \xrightarrow[k\to\infty]{} 0.$$

3) The specifications $\gamma^{\Psi^{(k)}}$ converge to γ^Φ as $k\to\infty$. More precisely, for all $B\in\mathscr{F}$ and $V\Subset\mathbb{Z}$,

$$\gamma_V^{\Psi^{(k)}}(B|\omega)\xrightarrow[k\to\infty]{}\gamma_V^{\Phi}(B|\omega) \text{ uniformly in the boundary conditions } \omega\in X.$$

4) In addition, if $\nu^{(k)}$ is a Gibbs measure for $\Psi^{(k)}$, then by Lemma 4.2, any weak-limit point, μ of the sequence $\{\nu^{(k)}\}_{k\ge0}$ is a Gibbs measure for the potential Φ.*

Another important observation applies to the interaction $\Psi^{(0)}$. As it is constructed from Φ by removing all interaction between $-\mathbb{N}$ and $\mathbb{Z}_+$, the corresponding specification $\gamma^{\Psi^{(0)}}$ is of product type [39, Example 7.18]: $\gamma^{\Psi^{(0)}}=\gamma^{\Phi^-}\times\gamma^{\Phi^+}$, where Φ^- and Φ^+ are the restrictions of Φ respectively to the negative and positive half-lines. Thus, the extreme Gibbs measures also factorize [39, Example 7.18]:

$$\operatorname{ex}\mathscr{G}(\gamma^{\Psi^{(0)}})=\{\nu_-\times\nu_+ : \nu_-\in\operatorname{ex}\mathscr{G}(X_-,\gamma^{\Phi^-}),\ \ \nu_+\in\operatorname{ex}\mathscr{G}(X_+,\gamma^{\Phi^+})\}. \tag{4.28}$$

4.6.3 Absolute continuity

If $\Psi^{(0)}$ does not exhibit phase transitions, i.e., has a unique Gibbs measure, then by Theorem 4.4.2, all the interactions $\Psi^{(k)}$, $k\ge1$, do not exhibit phase transitions as well. Let us denote by $\nu^{(k)}$ the unique Gibbs state for $\Psi^{(k)}$.

Theorem 4.6.5. *If the interaction Φ satisfies the Dobrushin uniqueness criterion (4.13), so do all interactions $\Psi^{(k)}$ and, furthermore, $\nu^{(k)}$ and $\nu^{(0)}$ are equivalent with*

$$\frac{d\nu^{(k)}}{d\nu^{(0)}}=\frac{e^{-\sum_{i=1}^k\Phi_{\Lambda_i}}}{\int_X e^{-\sum_{i=1}^k\Phi_{\Lambda_i}}\,d\nu^{(0)}}\quad,\quad \frac{d\nu^{(0)}}{d\nu^{(k)}}=\frac{e^{\sum_{i=1}^k\Phi_{\Lambda_i}}}{\int_X e^{\sum_{i=1}^k\Phi_{\Lambda_i}}\,d\nu^{(k)}}. \tag{4.29}$$

Proof. The hereditary character of Dobrushin's criterion follows from the obvious fact that, for all $k\in\mathbb{Z}_+$, $\bar c(\Psi^{(k)})\le\bar c(\Phi)$.

The proof of (4.29) follows, telescopically, from the partial Radon-Nikodym derivatives

$$\frac{d\,\nu^{(k)}}{d\,\nu^{(k-1)}} = \frac{e^{-\Phi_{\Lambda_k}}}{\int e^{-\Phi_{\Lambda_k}}\,d\,\nu^{(k-1)}} \quad , \quad \frac{d\,\nu^{(k-1)}}{d\,\nu^{(k)}} = \frac{e^{\Phi_{\Lambda_k}}}{\int e^{\Phi_{\Lambda_k}}\,d\,\nu^{(k)}}\,, \tag{4.30}$$

which is a particular case of the following elementary lemma. □

Lemma 4.6.6. *Let Ψ be a UAC interaction that does not exhibit phase transitions. Consider a perturbed interaction of the form $\bar{\Psi} = \Psi + \mathfrak{P}$ with $\mathfrak{P}$ a finite interaction supported on $A \Subset \mathbb{Z}$. If $\{\nu\} = \mathscr{G}(\Psi)$ and $\bar{\nu} \in \mathscr{G}(\bar{\Psi})$, then $\bar{\nu} \ll \nu$, and*

$$\frac{d\,\bar{\nu}}{d\,\nu} = \frac{e^{-H_A^{\mathfrak{P}}}}{\int_X e^{-H_A^{\mathfrak{P}}}\,d\,\nu}. \tag{4.31}$$

Proof. First, note that by Theorem 4.4.2, the model $\bar{\Psi}$ does not exhibit phase transitions since it is a finite perturbation of Ψ. Thus $\{\bar{\nu}\} = \mathscr{G}(\bar{\Psi})$. By uniqueness of the Gibbs state, we have that, for both interactions, the Gibbs measures are achieved through limits

$$\gamma_\Lambda(\cdot) \xrightarrow[\Lambda\uparrow\mathbb{Z}]{*} \nu \text{ and } \bar{\gamma}_\Lambda(\cdot) \xrightarrow[\Lambda\uparrow\mathbb{Z}]{*} \bar{\nu} \tag{4.32}$$

where $\gamma := \gamma^{\Psi}$ and $\bar{\gamma} := \gamma^{\bar{\Psi}}$ are the Gibbsian kernels corresponding to the interactions Ψ and $\bar{\Psi}$ and free boundary conditions (that is, considering only bonds within Λ). Consider any $V \subset \Lambda \Subset \mathbb{Z}$. As $H_\Lambda^{\bar{\Psi}} = H_A^{\mathfrak{P}} + H_\Lambda^{\Psi}$, for any cylindrical event $[\sigma_V]$, we have that

$$\bar{\gamma}_\Lambda(\sigma_V) = \frac{\sum_{\xi_\Lambda} \mathbf{1}_{\sigma_V}\, e^{-H_A^{\mathfrak{P}}(\xi_A)}\, e^{-H_\Lambda^{\Psi}(\xi_\Lambda)}}{\sum_{\xi_\Lambda} e^{-H_A^{\mathfrak{P}}(\xi_A)}\, e^{-H_\Lambda^{\Psi}(\xi_\Lambda)}}.$$

Dividing top and bottom by $\sum_{\xi_\Lambda} e^{-H_\Lambda^{\Psi}(\xi_\Lambda)}$ we get

$$\bar{\gamma}_\Lambda(\sigma_V) = \frac{\gamma_\Lambda\big(\mathbf{1}_{\sigma_V}\, e^{-H_A^{\mathfrak{P}}}\big)}{\gamma_\Lambda\big(e^{-H_A^{\mathfrak{P}}}\big)}.$$

Taking limits over Λ and using (4.32) we obtain

$$\bar{\nu}(\mathbf{1}_{\sigma_V}) = \nu(\mathbf{1}_{\sigma_V}\, f)$$

where f is, precisely, the right-hand side of (4.31). This concludes the proof because cylindrical events uniquely determine the measures. □

Now we turn to the restrictions $\nu_+^{(k)}$ of measures $\nu^{(k)}$ to the half-line $\mathbb{Z}_+$, i.e., for $B \in \mathscr{F}_+$, $\nu_+^{(k)}(B) := \nu^{(k)}(X_- \times B)$. Note that for the cylindrical sets $[\sigma_\Lambda]$, $\Lambda \Subset \mathbb{Z}_+$, one has $\nu_+^{(k)}([\sigma_\Lambda]) = \nu^{(k)}([\sigma_\Lambda])$. Similarly, one can define the restrictions to the left half-line $-\mathbb{N}$. If the model $\Psi^{(0)}$ does not exhibit phase transitions, then $\nu^{(0)}$ is a product measure, i.e., $\nu^{(0)} = \nu_-^{(0)} \times \nu_+^{(0)}$ (c.f. (4.28)), where $\nu_-^{(0)}$ is the unique Gibbs measure for the interaction Φ^- on X_-, and $\nu_+^{(0)}$ is the unique Gibbs measure for Φ^+ on X_+. If this is the case, then one can compute the Radon-Nikodym density $f_+^{(k)} := \dfrac{d\nu_+^{(k)}}{d\nu_+^{(0)}}$, in fact, for all $\sigma \in X_+$, $k \in \mathbb{Z}_+$,

$$f_+^{(k)}(\sigma) = \frac{\int_{X_-} e^{-\sum_{j=1}^{k} \Phi_{\Lambda_j}(\xi,\sigma)} \nu_-^{(0)}(d\xi)}{\int_{X_+}\int_{X_-} e^{-\sum_{j=1}^{k} \Phi_{\Lambda_j}(\xi,\zeta)} \nu_-^{(0)}(d\xi)\nu_+^{(0)}(d\zeta)}. \tag{4.33}$$

By Theorem 4.6.5, all the measures $\nu^{(k)}$, $k \geq 0$ are equivalent to $\nu^{(0)}$. However, it is not clear whether the weak*-limit points of the sequence $\{\nu^{(k)}\}_{k\in\mathbb{Z}_+}$ are absolutely continuous with respect to $\nu^{(0)}$ or not. The following theorem provides sufficient conditions.

Theorem 4.D. *Assume that Φ satisfies the Dobrushin uniqueness condition (4.13). Suppose the family $\{f^{(k)}\}_{k\in\mathbb{N}}$ is uniformly integrable in $L^1(\nu^{(0)})$. Then the weak* limit point of the sequence $\{\nu^{(k)}\}$ is a Gibbs measure for Φ and absolutely continuous with respect to $\nu^{(0)}$.*

The next theorem is the main theorem of this section and it provides sufficient conditions for uniform integrability of the family $\{f^{(k)}\}_{k\in\mathbb{N}}$, and thus absolute continuity of the weak* limit points with respect to $\nu^{(0)}$.

Theorem 4.E. *Assume the following*

1) *the interaction Φ satisfies the Dobrushin uniqueness condition (4.13);*

2) $\sum_{k=1}^{\infty} ||\underline{\delta}(\Phi_{\Lambda_k})||_2^2 < \infty$;

3) $\sum_{k=1}^{\infty} \rho_k < \infty$, *where* $\rho_k := \sup_{n\in\mathbb{N}} \left| \int_X \Phi_{\Lambda_k} d\nu^{(n)} \right|$.

Then $\Psi^{(0)}$ does not exhibit phase transitions, and $\{f^{(k)} : k \in \mathbb{N}\}$ is uniformly integrable in $L^1(\nu^{(0)})$.

Remark 4.6.7. *Note that the third, summability condition in Theorem 4.E is important. To illustrate this, consider the product-type potential ϕ^P defined by (4.10).*

One can readily verify that the potential ϕ^P *coincides with the half-line mean energy at* 0 *— that is,* $\phi^P = -\sum_{0\in V\Subset\mathbb{Z}_+} \Phi_V$ *— for the translation-invariant UAC interaction:*

$$\Phi_\Lambda(\omega) = \begin{cases} -\dfrac{\beta\omega_j}{|i-j|^\alpha}, & \Lambda = \{i,j\} \subset \mathbb{Z},\ j > i, \\ 0, & \textit{otherwise}\,. \end{cases}$$

This interaction Φ*, which is not spin-flip invariant, satisfies the first and second conditions of Theorem 4.E for* β *sufficiently small. In fact, for all* $\beta \geq 0$*, both Dobrushin interdependence matrices* $C(\gamma^\Phi)$ *and* $C(\gamma^{\phi^P})$ *are zero matrices, thus, both specifications (not the interactions) satisfy the condition (4.12) for all* β*'s. However,* Φ *does not satisfy the last condition of the theorem. Indeed, as the unique Gibbs measure* μ *for* Φ *is Bernoulli with*

$$\mu([1]_0) = \frac{e^{\beta\zeta(\alpha)}}{2\cosh(\beta\zeta(\alpha))},$$

one has

$$\int_X \sigma_0\, d\mu = \tanh(\beta\zeta(\alpha)) > 0.$$

Therefore, for all $i \in \mathbb{N}$*,* $j \in \mathbb{Z}_+$*,*

$$\rho_{-i,j} \geq \left| \int_X \Phi_{\{-i,j\}}\, d\mu \right| = \frac{\beta}{(i+j)^\alpha} \tanh(\beta\zeta(\alpha)),$$

and thus the sum $\sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \rho_{-i,j}$ *diverges, where* $\rho_{-i,j} := \sup_{n\in\mathbb{N}} \left| \int_X \Phi_{\{-i,j\}}\, d\nu^{(n)} \right|$*. Hence, Theorem 4.E does not apply to this particular model. Moreover, as asserted in the second part of Theorem 4.3.4, for* $1 < \alpha \leq \frac{3}{2}$*, the conclusion of Theorem 4.E fails to hold for this model.*

The combination of Theorems 4.E and 4.D implies that if Φ satisfies the conditions of the former, the unique Gibbs measure $\mu \in \mathscr{G}(\Phi)$ is absolutely continuous with respect to $\nu^{(0)}$. Note that, as a consequence, Theorem 4.E can not be true for the previous interaction if $1 < \alpha \leq 3/2$. Indeed, the second part of Theorem 4.3.4 directly implies that the measures μ and $\nu^{(0)}$ are singular for those values of α.

It is natural to ask whether, reciprocally, $\nu^{(0)}$ is also absolutely continuous with respect to μ. The answer is affirmative modulo conditions comparable to those of Theorem 4.E. The argument resorts also to a sequence of interactions but this time obtained by removing one by one the bonds in the volumes in $\mathscr{A} = \{\Lambda \Subset \mathbb{Z} :$

$\min\Lambda < 0, \max\Lambda \geq 0\}$ instead of adding them to $\Psi^{(0)}$. More precisely, at each step $k \in \mathbb{Z}_+$, we construct a new interaction $\Phi^{(k)}$ as follows:

$$\Phi_\Lambda^{(k)} = \begin{cases} 0, & \Lambda \in \{\Lambda_i : 1 \leq i \leq k\}, \\ \Phi_\Lambda, & \text{otherwise.} \end{cases}$$

Then as previously, for all $k \in \mathbb{Z}_+$, the matrix $\bar{C}(\Phi^{(k)})$ is dominated by $\bar{C}(\Phi)$, thus if Φ satisfies the Dobrushin uniqueness condition (4.13) so do all the interactions $\Phi^{(k)}$. Furthermore, Remark 4.6.4 still remains valid interchanging Φ and $\Psi^{(0)}$. Thus the sequence $\mu^{(k)}$ of unique Gibbs measures for each $\Phi^{(k)}$ converges in weak* sense to a Gibbs measure μ for $\Psi^{(0)}$. In addition, by Lemma 4.6.6, all the measures $\mu^{(k)}$ are equivalent to μ and their Radon-Nikodym derivatives are given by

$$f_\mu^{(k)} := \frac{d\mu^{(k)}}{d\mu} = \frac{e^{\sum_{i=1}^k \Phi_{\Lambda_i}}}{\int_X e^{\sum_{i=1}^k \Phi_{\Lambda_i}} d\mu}.$$

To conclude, we present the following analogue of Theorem 4.E which can be proven in a similar way.

Theorem 4.F. *Assume that*

1) *the interaction Φ satisfies (4.13);*

2) $\sum_{k=1}^{\infty} ||\underline{\delta}(\Phi_{\Lambda_k})||_2^2 < \infty$*;*

3) $\sum_{k=1}^{\infty} \rho_k^\mu < \infty$*, where* $\rho_k^\mu := \sup_{n\in\mathbb{N}} \left| \int_X \Phi_{\Lambda_k} \, d\mu^{(n)} \right|$.

Then $\Psi^{(0)}$ does not exhibit phase transitions, and $\{f_\mu^{(k)} : k \in \mathbb{N}\}$ is uniformly integrable in $L^1(\mu)$. In particular, $\nu^{(0)}$ is absolutely continuous with respect to μ.

We postpone the proofs of Theorem 4.D and Theorem 4.E until Section 4.8.

4.7 Application: Dyson model

This was our motivating example. Let us recall that the Dyson potential ϕ is defined on the half-line configuration space $X_+ = \{-1, +1\}^{\mathbb{Z}_+}$ as

$$\phi(x) = \beta \sum_{n=1}^{\infty} \frac{x_0 x_n}{n^\alpha}, \quad x = (x_0, x_1, \ldots) \in X_+,$$

for some $\alpha > 1$. As mentioned in Example 4.6.3, this potential is related to the whole-line Dyson model Φ, defined in (4.9), by $\phi = -\sum_{0\in V \Subset \mathbb{Z}_+} \Phi_V$, and thus $\overset{\rightleftharpoons}{\gamma}^{\phi} = \gamma^{\Phi}$. Hence by Theorem 4.A, for all $\nu \in \mathscr{G}(\phi)$, any weak* accumulation point of the sequence $\{\nu \circ S^{-n}\}$ is a restriction of some $\mu \in \mathscr{G}(\Phi)$ to $\mathbb{Z}_+$.

The Dyson potential satisfies the Good Future condition because its oscillations (4.24) $\delta_n(\phi) = \mathscr{O}(n^{-\alpha})$ are summable. As a consequence, it satisfies the extensibility condition. For $\alpha > 2$, furthermore, its variations $v_n(S_n\phi) := \sup\{S_n\phi(x) - S_n\phi(y) : x_0^{n-1} = y_0^{n-1}\}$ are also summable and, therefore, the standard theory applies [67]. The case of $\alpha \in (1,2]$ is significantly more subtle, and its theory is less developed. Its most recent advance is (Theorem 4.3.2 above), obtained by Johansson, Öberg and Pollicott [47] through the random-cluster representation for the whole-line Dyson model Φ (c.f. (4.9)).

Now, we recall some important properties of the Dyson model and its phase diagram. It is clear that the interaction Φ is translation and spin-flip invariant. One of the most interesting properties of the associated Gibbs measures is that all the measures in $\mathscr{G}(\Phi)$ are translation-invariant for all the values of the parameters α and β. However, this statement is not true as regards spin-flip invariance. In fact, there is only one spin-flip invariant Gibbs measure in $\mathscr{G}(\Phi)$ which is the cause of the phase transitions for high β's. By applying FKG inequalities, it can be shown that the (weak*) limits $\mu^+ := \lim_{\Lambda\uparrow\mathbb{Z}} \gamma_\Lambda^\Phi(\cdot|+)$ and $\mu^- := \lim_{\Lambda\uparrow\mathbb{Z}} \gamma_\Lambda^\Phi(\cdot|-)$ exist, and both are extremal. In fact, μ^- and μ^+ are the only extremal elements of $\mathscr{G}(\Phi)$ since they stochastically dominate all other Gibbs measures of the model i.e. $\int_X f\,d\mu^- \le \int_X f\,d\mu \le \int_X f\,d\mu^+$ for any Gibbs state $\mu \in \mathscr{G}(\Phi)$ and for any non-decreasing function $f \in C(X)$. The phase diagram of this model is in many ways similar to the phase diagram of the two-dimensional nearest-neighbour Ising model, in fact, we have the following result.

Theorem 4.7.1. [23, 38, 56] *For all $\alpha \in (1,2]$, there exists a critical temperature $\beta_c(\alpha) \in (0,+\infty)$ such that there is no phase transition for all $\beta \in [0,\beta_c(\alpha))$ (i.e. $\mu^+ = \mu^-$), and there is a phase transition for all $\beta \in (\beta_c(\alpha),+\infty)$ (i.e. $\mu^+ \ne \mu^-$). Furthermore, for all the values of β, $\mathscr{G}(\Phi) = [\mu^-;\mu^+]$.*

Note that if $\alpha > 2$, then for all the values of the inverse tempereature $\beta > 0$, Φ does not exhibit phase transitions, i.e., $|\mathscr{G}(\Phi)| = 1$.

In [46], the authors proved that the phase diagram of the half-line Dyson model ϕ is similar to the phase diagram of the whole-line Dyson model. In fact, they showed that for all $\alpha \in (1,2]$, there exists β_c^+, such that the for all $\beta \in (0,\beta_c^+)$, there exists a unique half-line Gibbs state for ϕ, and for $\beta > \beta_c^+$, there exist multiple Gibbs states. The authors also conjectured that the critical values β_c^+ and β_c of the half and whole-line Dyson models are, in fact, equal $\beta_c^+ = \beta_c$.

4.8 Proofs of the main results

4.8.1 Proofs of Theorems 4.A and 4.B

We shall use the following two simple lemmas in the proof of Theorem 4.A.

Lemma 4.8.1. [39, Remark 5.10] *Let γ be a specification on X and $\mu \in \mathscr{G}(\gamma)$. Then $\mu \circ S^{-1}$ is consistent with the specification $\gamma^{(1)}$, where*

$$\gamma^{(1)}_{\Lambda}(B|\omega) := \gamma_{\Lambda+1}(S^{-1}(B)|S^{-1}(\omega)),\ \forall B \in \mathscr{F},\ \omega \in X,\ \forall \Lambda \Subset \mathbb{Z}.$$

Lemma 4.8.2. [39, Theorem 4.17] *Suppose γ and $\gamma^{(n)}$, $n \geq 1$ are specifications on $X = E^{\mathbb{Z}}$. Assume that $\gamma^{(n)}$ converge uniformly to γ as $n \to \infty$, in the sense that for all $\Lambda \Subset \mathbb{Z}$ and all $\sigma \in X$,*

$$\gamma_{\Lambda}(\sigma_{\Lambda}|\omega_{\Lambda^c}) = \lim_{n\to\infty} \gamma^{(n)}_{\Lambda}(\sigma_{\Lambda}|\omega_{\Lambda^c})$$

uniformly in the boundary condition $\omega \in X$. Take $\mu^{(n)} \in \mathscr{G}(\gamma^{(n)})$, and assume that the sequence $\mu^{(n)}$ converges to some $\mu \in \mathscr{M}_1(X)$ in the weak topology. Then $\mu \in \mathscr{G}(\gamma)$.*

Proof of Theorem 4.A. We, first, prove the theorem in the case where ρ is the uniform Bernoulli measure on X_-. Let us consider a family of functions $\gamma^{(0)}_{\{n\}} : E \times E^{\mathbb{Z}\setminus\{n\}} \to (0,1)$ given by:

$$\gamma^{(0)}_{\{n\}}(a_n|x_{-\infty}^{n-1}, x_{n+1}^{\infty}) := \begin{cases} \overrightarrow{\gamma}^{\phi}_{\{n\}}(a_n|x_{-\infty}^{n-1}, x_{n+1}^{\infty}), & n \geq 0; \\ 1/|E|, & n < 0 \end{cases} \tag{4.34}$$

with

$$\overrightarrow{\gamma}^{\phi}_{\{n\}}(a_n|x_{-\infty}^{n-1}, x_{n+1}^{\infty}) = \frac{e^{S_{n+1}\phi(x_0^{n-1}a_n x_{n+1}^{\infty})}}{\sum_{\bar{a}_n} e^{S_{n+1}\phi(x_0^{n-1}\bar{a}_n x_{n+1}^{\infty})}} \tag{4.35}$$

are the single-site kernels for the half-line specification $\overrightarrow{\gamma}^{\phi}$. [To simplify we adopt the convention $x_m^n = \emptyset$ if $n < m$.] It can be easily checked that it is a family of single-site densities of a specification, therefore, by Proposition 4.2.2 there is a non-null quasilocal specification $\gamma^{(0)}$ on $(X, \mathscr{F})$ having $\{\gamma^{(0)}_{\{n\}}\}_{n\in\mathbb{Z}}$ as its single-site densities. Note that $\gamma^{(0)}$ is not translation-invariant. We claim that $\mu_0 \in \mathscr{G}(\gamma^{(0)})$. To show this, it is enough, by Proposition 4.2.2, to check that $\mu_0\big(\gamma^{(0)}_{\{n\}}(F)\big) = \mu_0(F)$ for each local cylindrical F. That is, we must show that

$$[\rho \times \nu]\big(\gamma^{(0)}_{\{n\}}(\mathbb{1}_{b_k^r})\big) = [\rho \times \nu]\big(\mathbb{1}_{b_k^r}\big) \tag{4.36}$$

for all integer $k \le r$ and n and all $b_k^r \in E^{r-k+1}$. A quick inspection shows that the only non-trivial case is $k \le n \le r$, $n \ge 0$. In this case

$$[\rho \times \nu]\big(\gamma^{(0)}_{\{n\}}(\mathbb{1}_{b_k^r})\big) = \rho\big(\mathbb{1}_{b^0_{k\wedge(-1)}}\big)\, \nu\big(\overrightarrow{\gamma}_{\{n\}}(\mathbb{1}_{b^r_{k\vee 0}})\big), \tag{4.37}$$

and, as ν is a Gibbs measure for ϕ, we conclude that

$$[\rho \times \nu]\big(\gamma^{(0)}_{\{n\}}(\mathbb{1}_{b_k^r})\big) = \rho\big(\mathbb{1}_{b^0_{k\wedge(-1)}}\big)\, \nu\big(\mathbb{1}_{b^r_{k\vee 0}}\big), \tag{4.38}$$

proving (4.36).

The proof is concluded by invoking the previous lemmas. By Lemma 4.8.1, for all $p \in \mathbb{N}$, the measure $\mu_p = \mu_0 \circ S^{-p}$ is consistent with $\gamma^{(p)}$, where

$$\gamma^{(p)}_\Lambda(B|\omega) := \gamma^{(0)}_{\Lambda+p}(S^{-p}(B)|S^{-p}(\omega)),\ \forall B \in \mathscr{F},\ \omega \in X,\ \forall \Lambda \Subset \mathbb{Z}.$$

Note that the single-site density functions of $\gamma^{(p)}$ can be calculated explicitly, namely, for all $\sigma, \omega \in X$,

$$\gamma^{(p)}_{\{i\}}(\sigma_i|\omega_{\{i\}^c}) := \begin{cases} \dfrac{e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\sigma_i\omega_{i+1}^\infty)}}{\sum_{\bar\omega_i} e^{S_{i+p+1}\phi(\omega_{-p}^{i-1}\bar\omega_i\omega_{i+1}^\infty)}}, & i \ge -p; \\ 1/|E|, & i < -p. \end{cases} \tag{4.39}$$

Thus $\overset{\rightleftharpoons}{\gamma}^{\phi}$ is the uniform limit of the sequence of specifications $\{\gamma^{(p)}\}_p$. Thus by Lemma 4.8.2, we obtain that $\mu \in \mathscr{G}(\overset{\rightleftharpoons}{\gamma}^{\phi})$. Hence, we establish the theorem in the case where ρ is the uniform Bernoulli measure.

Now let $\tilde\rho \in \mathscr{M}_1(X_-)$ be an arbitrary measure, and let ρ once again denote the uniform Bernoulli measure on X_-. Then for any local function $g : X \to \mathbb{R}$, $g \circ S^n$ becomes $F_{\mathbb{Z}_+}$ measurable, for sufficiently large $n \ge 1$. Thus by Fubini's theorem, for sufficiently large n,

$$\int_X g \circ S^n \, d(\tilde\rho \times \nu) = \int_{X_+} g \circ S^n \, d\nu = \int_X g \circ S^n \, d(\rho \times \nu).$$

Hence, one concludes that a subsequence $\{(\tilde\rho \times \nu) \circ S^{-n_k}\}_{k\ge 1}$ converges in the weak* topology if and only if $\{(\rho \times \nu) \circ S^{-n_k}\}_{k \ge 1}$ converges and the limiting points coincide. □

The following lemma will be useful in the proof of Theorem 4.B.

Lemma 4.8.3. *Under the conditions of Theorem 4.A, for all cylindrical sets $C \subset X$, and all volumes $\Lambda \Subset \mathbb{Z}$,*

$$\lim_{n\to\infty} \int_X \Big[\overset{\rightleftharpoons}{\gamma}^{\phi}_\Lambda(C|x) - \mathbb{1}_C(x)\Big] \mu_n(dx) = 0\,.$$

Proof. We will show that

$$\liminf_{n\to\infty}\int_X\Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_n(dx)=0\,; \tag{4.40}$$

the analogous result for the lim sup can be shown similarly. Take any subsequence $\{n_k\}$ such that

$$\lim_{k\to\infty}\int_X\Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_{n_k}(dx)=\liminf_{n\to\infty}\int_X\Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_n(dx).$$

By compactness of $\mathscr{M}_1(X)$ the subsequence $\{\mu_{n_k}\}_k$ converges in the weak*-topology. If μ is its limit, then $\mu\in\mathscr{G}(\overset{\rightleftharpoons}{\gamma}{}^{\phi})$ by Theorem 4.A. Thus,

$$\lim_{k\to\infty}\int_X\Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_{n_k}(dx)=\int_X\Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu(dx)=0\,.$$

□

Proof of Theorem 4.B. The proof of translation-invariance of μ is standard. Thus it is enough to check the consistency, i.e., $\mu\in\mathscr{G}(\overset{\rightleftharpoons}{\gamma}{}^{\phi})$. Let $\mu=\lim_k\tilde{\mu}_{n_k}$, and take any cylindrical event C. Then, the weak convergence implies that

$$\int_X\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)\tilde{\mu}_{n_k}(dx)\xrightarrow[k\to\infty]{}(\mu\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda})(C). \tag{4.41}$$

On the other hand, the Stolz-Cesaro theorem and Lemma 4.8.3 yield that

$$\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int_X\Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_i(dx)\xrightarrow[k\to\infty]{}0\,.$$

Thus,

$$\int_X\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)\tilde{\mu}_{n_k}(dx)=\frac{1}{n_k}\sum_{i=0}^{n_k-1}\int_X\Big[\overset{\rightleftharpoons}{\gamma}{}^{\phi}_{\Lambda}(C|x)-\mathbb{1}_C(x)\Big]\mu_i(dx)+\tilde{\mu}_{n_k}(C)\xrightarrow[k\to\infty]{}\mu(C),$$

□

4.8.2 Proofs of Theorem 4.D, 4.E and 4.C

Proof of Theorem 4.D. By Theorem 4.6.5, each measure $\nu^{(k)}$ is absolutely continuous with respect to $\nu^{(0)}$ with $f^{(k)}=\dfrac{d\,\nu^{(k)}}{d\,\nu^{(0)}}$. Furthermore, by Theorem 4.4.2, $\nu^{(k)}$

is the unique Gibbs measure for $\Psi^{(k)}$ for all $k \geq 0$. Let μ^* be a weak* limit of a subsequence $\{\nu^{(k_s)}\}_{s\in\mathbb{N}}$. By Lemma 4.8.2, $\mu^* \in \mathscr{G}(\Phi)$ (c.f. Remark 4.6.4). By the weak star convergence, we have that for all $g_0 \in C(X)$,

$$\int_X g_0 d\nu^{(k_s)} = \int_X g_0 f^{(k_s)} d\nu^{(0)} \xrightarrow[s\to\infty]{} \int_X g_0 d\mu^*. \tag{4.42}$$

Since the family $\{f^{(k)} : k \in \mathbb{Z}_+\}$ is uniformly integrable, it is relatively weakly compact in $L^1(\nu^{(0)})$ by the Dunford-Pettis theorem. Therefore, there exists a weak limit point $f \in L^1(\nu^{(0)})$ of the sequence $\{f^{(k_s)}\}_{s\in\mathbb{N}}$. Without loss of generality, assume that $f^{(k_s)} \xrightarrow[s\to\infty]{} f$. Thus for all $g \in L^\infty(X, \nu^{(0)})$,

$$\int_X g f^{(k_s)} d\nu^{(0)} \xrightarrow[s\to\infty]{} \int_X g f d\nu^{(0)}. \tag{4.43}$$

By combining (4.42) and (4.43), we conclude that for all $g_0 \in C(X)$,

$$\int_X g_0 d\mu^* = \int_X g_0 f d\nu^{(0)}.$$

□

Proof of Theorem 4.E. For all $k \in \mathbb{N}$, denote $W_k := \sum_{i=1}^{k} \Phi_{\Lambda_i}$. Our argument relies on two claims:

Claim 1:

$$\sup_{k\geq 0} \left| \int_X -W_k d\nu^{(k)} \right| < \infty .$$

Claim 2:

$$\sup_{k\geq 0} \int_X e^{-W_k} d\nu^{(0)} < \infty.$$

Then, as

$$\int_X f^{(k)} \log f^{(k)} d\nu^{(0)} = \int_X -W_k d\nu^{(k)} - \log \int_X e^{-W_k} d\nu^{(0)} ,$$

these claims imply that

$$\sup_{k\geq 0} \int_X f^{(k)} \log f^{(k)} d\nu^{(0)} < \infty. \tag{4.44}$$

Hence by applying de la Vallée Poussin's theorem to the family $\{f^{(k)} : k \in \mathbb{N}\}$ and to the function $t \in (0, +\infty) \mapsto t \log t$, one concludes that the family $\{f^{(k)} : k \in \mathbb{N}\}$ is uniformly integrable in $L^1(\nu^{(0)})$.

The proof of Claim 1 is immediate:

$$\Big|\int_X -W_k\, d\nu^{(k)}\Big| \le \sum_{i=1}^{k}\Big|\int_X \Phi_{\Lambda_i}\, d\nu^{(k)}\Big| \le \sum_{i=1}^{k}\rho_i \le \sum_{i=1}^{\infty}\rho_i < \infty\,. \tag{4.45}$$

The proof of Claim 2 relies on the Gaussian concentration bounds. Note that for all $k\in\mathbb{N}$, $\bar{c}(\Psi^{(0)}) \le \bar{c}(\Psi^{(k)}) \le \bar{c}(\Phi)$. Therefore, the Dobrushin uniqueness condition $\bar{c}(\Phi) < 1$ is inherited by all the intermediate interactions. Applying the first part of Theorem 4.4.7 we see that the (only) measure $\mu \in \mathscr{G}(\Phi)$ and all the intermediate measures $\nu^{(k)}$, $k \ge 0$, satisfy the Gaussian Concentration Bound with the same constant $D := \dfrac{4}{(1-\bar{c}(\Phi))^2}$. This implies that, for all $k\in\mathbb{N}$,

$$\int_X e^{-\Phi_{\Lambda_k}}\, d\nu^{(k-1)} \le e^{D\|\underline{\delta}(\Phi_{\Lambda_k})\|_2^2}\, e^{-\int_X \Phi_{\Lambda_k}\, d\nu^{(k-1)}}. \tag{4.46}$$

We combine this inequality with (4.30) to iterate

$$\begin{aligned}
&\int_X e^{-(\Phi_{\Lambda_k}+\Phi_{\Lambda_{k-1}})} d\nu^{(k-2)}\\
&= \int_X e^{-\Phi_{\Lambda_k}}\, d\nu^{(k-1))} \int_X e^{-\Phi_{\Lambda_{k-1}}}\, d\nu^{(k-2)}\\
&\le e^{D(\|\underline{\delta}(\Phi_{\Lambda_k})\|_2^2+\|\underline{\delta}(\Phi_{\Lambda_{k-1}})\|_2^2)}\cdot e^{-(\int_X \Phi_{\Lambda_k}\, d\nu^{(k-1)}+\int_X \Phi_{\Lambda_{k-1}}\, d\nu^{(k-2)})}.
\end{aligned} \tag{4.47}$$

By induction this yields

$$\int_X e^{-\sum_{i=1}^{k}\Phi_{\Lambda_i}}\, d\nu^{(0)} \le e^{D\sum_{i=1}^{k}\|\underline{\delta}(\Phi_{\Lambda_i})\|_2^2}\cdot e^{-\sum_{i=1}^{k}\int_X \Phi_{\Lambda_i}\, d\nu^{(i-1)}}. \tag{4.48}$$

Thus, by using (4.17) and (4.45), we have that

$$\int_X e^{-W_k}\, d\nu^{(0)} \le e^{D\sum_{i=1}^{k}\|\underline{\delta}(\Phi_{\Lambda_i})\|_2^2}\cdot e^{\sum_{i=1}^{k}\rho_i} \tag{4.49}$$

for all $k\in\mathbb{N}$, and

$$\sup_{k\in\mathbb{N}}\int_X e^{-W_k}\, d\nu^{(0)} \le e^{D\sum_{i=1}^{\infty}\|\underline{\delta}(\Phi_{\Lambda_i})\|_2^2}\cdot e^{\sum_{i=1}^{\infty}\rho_i} < \infty. \tag{4.50}$$

This proves Claim 2 and, hence, concludes the proof of the theorem. □

Proof of Theorem 4.C.

Part (i): Its proof is rather straightforward. We choose $\beta > 0$, so that the resulting Φ satisfies the Dobrushin uniqueness condition $\bar{c}(\Phi) < 1$. This condition is inherited by $\Psi^{(0)}$, hence both potentials have a unique Gibbs state. Furthermore, as direct products of Gibbs measures for the restricted interactions Φ^- and Φ^+ are Gibbs measures for $\Psi^{(0)}$, neither Φ^- nor Φ^+ may exhibit phase transitions (c.f. Equation (4.28)). In particular, $\gamma^{\phi} = \gamma^{\Phi^+}$ admits only one Gibbs state.

Part (ii): We just have to verify the hypotheses of Theorem 4.E; the absolute continuity follows then from Theorem 4.D. We chose $0 < \beta \leq \beta_{DU}$ with

$$\beta_{DU} = \big(2\sum_{i=1}^{\infty} \frac{1}{i^{\alpha}}\big)^{-1} = \frac{1}{2}\zeta(\alpha)^{-1} . \tag{4.51}$$

Hence, by Proposition 4.4.4, for all $\beta \in (0, \beta_{DU})$, Φ satisfies the Dobrushin uniqueness condition $\bar{c}(\Phi) < 1$. *Hypothesis 1):* Consequence of the Dobrushin uniqueness criterion.

Hypothesis 2): For all $i \in \mathbb{N}$, $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$,

$$\delta_k(\Phi_{\{-i,j\}}) = \begin{cases} 0, & k \notin \{-i, j\}; \\ \dfrac{2\beta}{(i+j)^{\alpha}}, & k \in \{-i, j\}. \end{cases}$$

Thus for all $i \in \mathbb{N},\ j \in \mathbb{Z}_+$, Therefore,

$$\sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} ||\underline{\delta}(\Phi_{\{-i,j\}})||_2^2 = \sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} \frac{8\beta^2}{(i+j)^{2\alpha}} < \infty .$$

Hypothesis 3): We shall use inequalities (4.20) and (4.22). We use the notation introduced in Section 4.4.2. Note that for all $k \geq 0$ we have the componentwise domination

$$C(\gamma^{\Psi^{(k)}})_{i,j} \leq \bar{C}(\Phi)_{i,j} \quad , \quad D(\gamma^{\Psi^{(k)}})_{i,j} \leq \bar{D}(\Phi)_{i,j} . \tag{4.52}$$

Thus since $\bar{c}(\Phi) < 1$ all the specifications $\gamma^{\Psi^{(k)}}$ satisfy (4.12) (c.f. Section 4.4.2). Applying (4.20) to the measures $\nu^{(k)}$, $k \geq 0$ and using (4.52), we see that

$$\sup_{k\geq 0} \Big| \int_X \sigma_{-m} \cdot \sigma_0 \circ S^n \, d\,\nu^{(k)} \Big| \leq \frac{1}{4} \sum_{r,j\in\mathbb{Z}} \bar{D}(\Phi)_{rj} \cdot \delta_r \sigma_{-m} \cdot \delta_{n-j} \sigma_0. \tag{4.53}$$

Hence for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}_+$,

$$(m+n)^{\alpha} \rho_{-m,n} \leq \frac{1}{4} \sum_{r,j\in\mathbb{Z}} \bar{D}(\Phi)_{rj} \cdot \delta_r \sigma_{-m} \cdot \delta_{n-j} \sigma_0. \tag{4.54}$$

Summing and applying inequality (4.22), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(m+n)^{\alpha}\rho_{-m,n} &\le \frac{1}{4}\sum_{r,j,n\in\mathbb{Z}}\bar{D}(\Phi)_{rj}\cdot\delta_r\sigma_{-m}\cdot\delta_{n-j}\sigma_0\\ &= \frac{1}{2}\sum_{r\in\mathbb{Z}}\Big[\delta_r\sigma_{-m}\cdot\sum_{j\in\mathbb{Z}}\bar{D}(\Phi)_{rj}\Big]\\ &\overset{(4.22)}{\le} \frac{1}{1-\bar{c}(\Phi)}.\end{aligned} \tag{4.55}$$

As a consequence, $m^{\alpha}\cdot\sum_{n=0}^{\infty}\rho_{-m,n}\le(1-\bar{c}(\Phi))^{-1}$ for all $m\in\mathbb{N}$, which implies

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\rho_{-m,n}\le\frac{1}{1-\bar{c}(\Phi)}\sum_{m=1}^{\infty}\frac{1}{m^{\alpha}}<\infty. \tag{4.56}$$

This concludes the verification of the hypotheses of Theorem 4.E, which, together with Theorem 4.D imply that the unique limit point μ of the sequence $\{\nu^{(k)}\}_{k\in\mathbb{Z}_+}$ is absolutely continuous with respect to $\nu^{(0)}$. As $\nu^{(0)}=(\nu^{(0)})^-\times(\nu^{(0)})^+$ and $(\nu^{(0)})^+$ coincides with the unique Gibbs state ν of ϕ, it follows that $\mu\ll\nu^{(0)}$ implies that $\mu^+\ll\nu$.

The proof of $\nu\ll\mu^+$ is analogous but using Theorem 4.F instead of Theorem 4.E.

Part (iii): This part follows from an application of the Arzelà–Ascoli theorem. Before proceeding with the proof, we fix an enumeration $\{\Lambda_1,\Lambda_2,\dots\}$ of the set $\mathscr{A}=\{\Lambda\Subset\mathbb{Z}:\min\Lambda<0\le\max\Lambda\}$ such that, for every $N\in\mathbb{N}$, there exists $k_N\in\mathbb{N}$ satisfying

$$\sum_{i=1}^{k_N}\Phi_{\Lambda_i}=\sum_{\substack{\min V<0\le\max V\\ V\subset[-N,N]}}\Phi_V. \tag{4.57}$$

Let us denote ξ the configurations in $X_-=\{-1,1\}^{-\mathbb{N}}$ and σ those in $X_+=\{-1,1\}^{\mathbb{Z}_+}$. For all $N\in\mathbb{N}$, and $(\xi,\sigma)\in X_-\times X_+$, define

$$W_N(\xi,\sigma):=\sum_{i=1}^{N}\sum_{j=0}^{N}-\frac{\beta\xi_{-i}\sigma_j}{(i+j)^{\alpha}}$$

and

$$f_+^{[N]}(\sigma):=\frac{d\nu_+^{[N]}}{d\nu_+^{(0)}}(\sigma)=\frac{\int_{X_-}e^{-W_N(\xi,\sigma)}\,\nu_-^{(0)}(d\xi)}{\int_{X_+}\int_{X_-}e^{-W_N(\xi,\zeta)}\,\nu_-^{(0)}(d\xi)\,\nu_+^{(0)}(d\zeta)}. \tag{4.58}$$

The sequence $\{f_+^{[N]}\}_{N\in\mathbb{N}}$ is a subsequence of the sequence $\{f_+^{(k)}\}_{k\in\mathbb{Z}_+}$ defined in (4.33), since $f_+^{[N]}=f_+^{k_N}$. All $f_+^{[N]}$ are local functions on X_+, thus continuous. We claim that it is enough to prove that the family $\{f_+^{[N]} : N \in \mathbb{N}\}$ is relatively compact in $C(X_+)$. Indeed, if this is true, there exists a function $f_+ \in C(X_+)$ and a subsequence $\{f_+^{[N_k]}\}_{k\in\mathbb{N}}$ such that $f_+^{[N_k]} \rightrightarrows f_+$ as $k \to \infty$. Thus, by the argument presented in the proof of Theorem 4.D, f_+ is the Radon-Nikodym density of μ_+ with respect to ν, and

$$\mathscr{L}_\phi f_+(x) = e^{P(S,\phi)} f_+(x)$$

for all for $x \in X_+$ [in principle, the identity holds for ν–almost all $x \in X_+$, but ν is fully supported]. Hence f_+ is the continuous eigenfunction of the transfer operator $\mathscr{L}_\phi$ corresponding to the largest eigenvalue. Note that we can also conclude from this argument that the entire sequence $\{f_+^{[N]}\}_{N\in\mathbb{N}}$ converges in the uniform topology.

To conclude, we turn to the proof of the relative compactness of the family $\{f_+^{[N]} : N \in \mathbb{N}\}$. By the Arzela-Ascoli theorem, it is enough to show that this family is uniformly bounded and equicontinuous. These properties are proven separately.

Uniform boundedness: As

$$\bar{c}(\Phi^-) = \frac{1}{2}\sup_{i\in -\mathbb{N}} \sum_{i\in V\Subset -\mathbb{N}} (|V|-1)\delta(\Phi_V) \le \frac{1}{2}\sup_{i\in\mathbb{Z}} \sum_{i\in V\Subset\mathbb{Z}} (|V|-1)\delta(\Phi_V) = \bar{c}(\Phi) < 1\,,$$

the interaction Φ^- on $-\mathbb{N}$ satisfies the Dobrushin uniqueness condition, therefore, the unique Gibbs measure $\nu_- \in \mathscr{G}(\Phi^-)$ satisfies the Gaussian Concentration Bound (Theorem 4.4.7) with the constant $D = 4(1-\bar{c}(\Phi))^{-2}$. Fix any $\sigma \in X_+$ and consider $W_N(\xi,\sigma)$ as a function of $\xi \in X_-$. Clearly, it is a local function, thus by the first part of Theorem 4.4.7, for all $\kappa \in \mathbb{R}$,

$$\int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \le e^{D\kappa^2 \|\underline{\delta}(W_N(\cdot,\sigma))\|_2^2} \cdot e^{\kappa \int_{X_-} W_N(\xi,\sigma)\nu_-(d\xi)}. \tag{4.59}$$

First, note that the interaction Φ^- is invariant under the global spin-flip transformation, therefore, so is the unique Gibbs measure ν_-. Thus for all $N \in \mathbb{N}$ and $\sigma \in X_+$, $\int_{X_-} W_N(\xi,\sigma)\nu_-(d\xi) = 0$. Second, for all $k \in \mathbb{N}$,

$$\delta_{-k}(W_N(\cdot,\sigma)) = 2\beta \left| \sum_{j=0}^{N} \frac{\sigma_j}{(k+j)^\alpha} \right| \le 2\beta \sum_{j=0}^{N} \frac{1}{(k+j)^\alpha}.$$

Hence if $\alpha > \frac{3}{2}$, for all $N \in \mathbb{N}$, $\sigma \in X_+$,

$$\|\underline{\delta}(W_N(\cdot,\sigma))\|_2^2 \le 4\beta^2 \sum_{k=1}^{\infty} \Big(\sum_{j=k}^{\infty} \frac{1}{j^\alpha}\Big)^2 =: 4\beta^2 C_1(\alpha) < \infty. \tag{4.60}$$

Then (4.59) implies that for all $N \in \mathbb{N}$, $\sigma \in X_+$, one has

$$\int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \leq e^{4D\kappa^2\beta^2 C_1(\alpha)}. \tag{4.61}$$

Changing $\kappa \to -\kappa$ in (4.61) we also obtain

$$\int_{X_-} e^{-\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \leq e^{4D\kappa^2\beta^2 C_1(\alpha)}. \tag{4.62}$$

Thus by applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} 1 &= \left[\int_{X_-} e^{\kappa W_N(\xi,\sigma)/2}\, e^{-\kappa W_N(\xi,\sigma)/2}\, \nu_-(d\xi)\right]^2 \\ &\leq \int_{X_-} e^{-\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \\ &\leq e^{4D\kappa^2\beta^2 C_1(\alpha)} \cdot \int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \end{aligned}$$

which yields the lower bound

$$e^{-4D\kappa^2\beta^2 C_1(\alpha)} \leq \int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi) \tag{4.63}$$

for all $\sigma \in X_+$, $N \in \mathbb{N}$. Putting (4.61) and (4.63) together yields the bounds

$$e^{-4D\kappa^2\beta^2 C_1(\alpha)} \leq \int_{X_+}\int_{X_-} e^{\kappa W_N(\xi,\sigma)} \nu_-(d\xi)\nu_+(d\sigma) \leq e^{4D\kappa^2\beta^2 C_1(\alpha)} \tag{4.64}$$

which implies that the family $\{f_+^{[N]} : N \in \mathbb{N}\}$ is uniformly bounded from above and below:

$$e^{-8D\beta^2 C_1(\alpha)} \leq f_+^{[N]}(\sigma) \leq e^{8D\beta^2 C_1(\alpha)}. \tag{4.65}$$

Equicontinuity: As the denominator $\int_X e^{-W_N} d\nu^{(0)}$ is uniformly bounded from above and below as shown in (4.64), it is enough to show that the family

$$\left\{\int_{X_-} e^{-W_N(\xi,\cdot)} \nu_-(d\xi) : N \in \mathbb{N}\right\}$$

is equicontinuous. Consider $n \in \mathbb{N}$ and configurations $\sigma, \tilde{\sigma} \in X_+$ such that $\sigma_0^{n-1} = \tilde{\sigma}_0^{n-1}$. Then

$$\left|\int_{X_-} \left[e^{-W_N(\xi,\sigma)} - e^{-W_N(\xi,\tilde{\sigma})}\right] \nu_-(d\xi)\right| \leq \int_{X_-} e^{-W_N(\xi,\sigma)} \cdot \left|e^{W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})} - 1\right| \nu_-(d\xi). \tag{4.66}$$

Thus, by the Cauchy-Schwarz inequality,

$$RHS \le \Big(\int_{X_-} e^{-2W_N(\xi,\sigma)} \nu_-(d\xi)\Big)^{\frac{1}{2}} \Big(\int_{X_-} \Big[e^{W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})} - 1\Big]^2 \nu_-(d\xi)\Big)^{\frac{1}{2}} \tag{4.67}$$

and (4.61) yields that

$$\Big| \int_{X_-} \Big[e^{-W_N(\xi,\sigma)} - e^{-W_N(\xi,\tilde{\sigma})}\Big] \nu_-(d\xi)\Big| \le C_2 \Big(\int_{X_-} \Big[e^{W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})} - 1\Big]^2 \nu_-(d\xi)\Big)^{\frac{1}{2}} \tag{4.68}$$

with $C_2 := e^{8D\beta^2 C_1(\alpha)}$. We will bound the last integral by bounding the exponent.

Note that

$$W_N(\xi,\sigma) - W_N(\xi,\tilde{\sigma}) = -\beta \sum_{j=n}^{N} (\sigma_j - \tilde{\sigma}_j) \sum_{i=1}^{N} \frac{\xi_{-i}}{(i+j)^\alpha} \tag{4.69}$$

and, thus, for all $k \in \mathbb{N}$,

$$\delta_{-k}(W_N(\cdot,\sigma) - W_N(\cdot,\tilde{\sigma})) = 2\beta \Big| \sum_{j=n}^{N} \frac{\sigma_j - \tilde{\sigma}_j}{(k+j)^\alpha} \Big| \le 4\beta \sum_{j=n}^{N} \frac{1}{(k+j)^\alpha}\,. \tag{4.70}$$

Hence, for sufficiently large n and $N > n$,

$$||\underline{\delta}(W_N(\cdot,\sigma) - W_N(\cdot,\tilde{\sigma}))||_2^2 \le 16\beta^2 \sum_{k=1}^{\infty} \Big(\sum_{j=n}^{N} \frac{1}{(k+j)^\alpha}\Big)^2 \le 32\beta^2 \sum_{k=n+1}^{\infty} \frac{1}{k^{2(\alpha-1)}} =: u_n. \tag{4.71}$$

with

$$\lim_{n\to\infty} u_n = 0 \ \text{ if and only if } \ \alpha > \frac{3}{2}\,.$$

The spin-flip invariance of ν_- implies that for all $N \in \mathbb{N}$,

$$\int_{X_-} [W_N(\xi,\sigma) - W_N(\xi,\tilde{\sigma})]\, \nu_-(d\xi) = 0\,.$$

Then since Φ^- satisfies (4.13), we have, from the second part of Theorem 4.4.7, that for all $m \in \mathbb{N}$,

$$\begin{aligned} \int_{X_-} \Big| W_N(\xi,\sigma) - W_N(\xi,\tilde{\sigma})\Big|^m \nu_-(d\xi) \ &\le\ \Big(\frac{D||\underline{\delta}(W_N(\cdot,\sigma) - W_N(\cdot,\tilde{\sigma}))||_2^2}{2}\Big)^{\frac{m}{2}} m\Gamma\Big(\frac{m}{2}\Big) \\ &=\ m\, v_n^m\, \Gamma\Big(\frac{m}{2}\Big) \end{aligned} \tag{4.72}$$

with

$$v_n := \Big(\frac{D\, u_n}{2}\Big)^{\frac{1}{2}}\,.$$

We conclude by expanding the square in the right-hand side in (4.68):

Expanding the square in the right-hand side of (4.68),

$$\begin{aligned}
&\int_{X_-} \left[e^{W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})}-1\right]^2 \nu_-(d\xi) \\
&= \int_{X_-} \left[e^{2[W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})]}-2\,e^{W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})}+1\right]\nu_-(d\xi) \\
&\leq 1+\sum_{m=1}^{\infty}\frac{2^m+2}{m!}\int_{X_-}\left|W_N(\xi,\sigma)-W_N(\xi,\tilde{\sigma})\right|^m \nu_-(d\xi).
\end{aligned} \tag{4.73}$$

Through elementary analysis one can show that

$$\sum_{m=1}^{\infty}\frac{2^m+2}{m!}m\,v_n^m\,\Gamma\Big(\frac{m}{2}\Big)\leq 4v_n+3e^{v_n^2}+2e^{4v_n^2}-5. \tag{4.74}$$

Therefore, we obtain from (4.68) and (4.72),

$$\left|\int_{X_-}\left[e^{-W_N(\xi,\sigma)}-e^{-W_N(\xi,\tilde{\sigma})}\right]\nu_-(d\xi)\right|\leq C_2(4v_n+3e^{v_n^2}+2e^{4v_n^2}-5)^{\frac{1}{2}}. \tag{4.75}$$

Since $\lim_{n\to\infty} v_n=0$, we conclude that the family $\left\{\int_{X_-} e^{-W_N(\xi,\cdot)}\nu_-(d\xi) : N\in\mathbb{N}\right\}$ is indeed equicontinuous. □

4.9 Final Remarks and Future Directions

The main results of the present work rely on two major assumptions: uniqueness of the Gibbs measure and validity of the Gaussian concentration inequalities for that measure. We informally refer to the combination of these two conditions as strong uniqueness. Strong uniqueness holds for a wide class of Gibbs interactions (potentials).

However, we would like to end with a discussion of one particular model – the Dyson model, which served as the primary motivation for the present work. The picture below (Figure 4.1) summarises the current state of the art in the eigenfunction problem for the Dyson potential with the parameters $\alpha>1$, $\beta>0$.

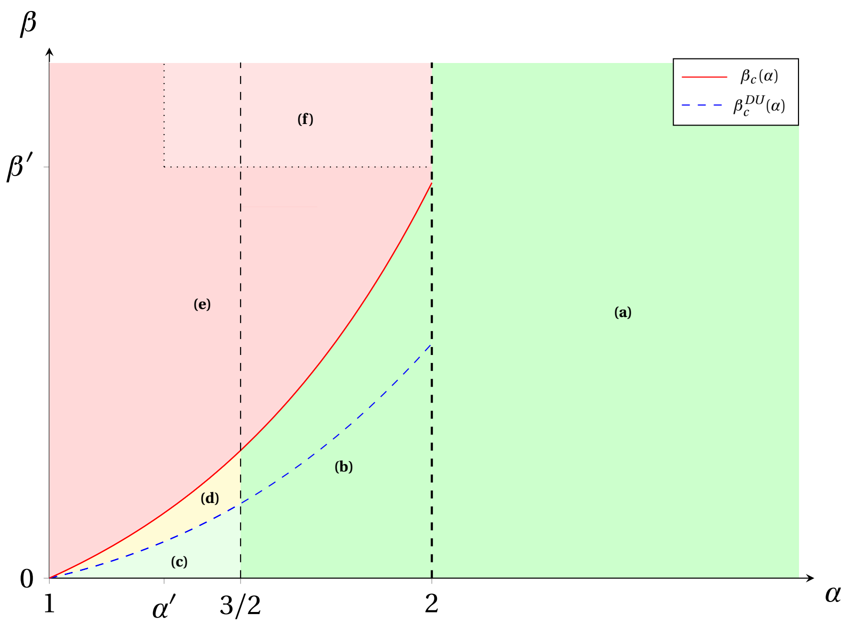

Figure 4.1: Eigenfunctions for the Dyson model across the phase diagram [2]

(a) For $\alpha > 2$, at all temperatures, the Dyson potential has summable variation, and hence the classical results of Walters [66, 67] allow one to conclude that the transfer operator admits a continuous eigenfunction with summable variation. Note also that for $\alpha > 2$ we also have strong uniqueness: uniqueness is due to Bowen [9], and the Gaussian Concentration Bounds have been established in [12, 13]. Thus our results are also applicable, although summable variation of the unique continuous positive eigenfunction requires a separate argument.

(b) For $\alpha \in (3/2, 2]$ existence of a continuous eigenfunction was first proved by Johansson, Öberg, and Pollicott for sufficiently small β and then extended to the whole subcritical region [47]. The proof in [47] relies on rather specific properties of the Dyson model, e.g., representation of the model via the random cluster model, which typically restricts one to ferromagnetic models [22, 45] We strongly believe that the Gaussian concentration inequality, and hence, the method used in this chapter can be extended to the whole uniqueness region as well (See the Appendix).

[2]The monotonicity of the function $\alpha \mapsto \beta_c(\alpha)$ follows from the GKS inequality [37], while the monotonicity of the function $\alpha \mapsto \beta_c^{DU}(\alpha)$ follows from (4.51)

(c) In this region, we have established the existence of an integrable eigenfunction. (c.f. the second part of Theorem 4.C). We conjecture that the result is sharp: for $\alpha \leq 3/2$, the transfer operator does not have a continuous eigenfunction.

(d) We conjecture that the result obtained for the region (c) must hold for all β's below the critical value $\beta_c(\alpha)$.

(e) In the supercritical regime: $\alpha > 1$ and $\beta > \beta_c(\alpha)$, we believe that transfer operators do not admit integrable eigenfunctions.

(f) The only result which applies to the supercritical phase is [8], where it has been shown that in a particular region, both pure phases of the Dyson model Φ are not $g-$measures. That immediately implies that the transfer operator does not have a continuous eigenfunction; otherwise, the normalized function $g = \dfrac{h \cdot e^{\phi}}{\lambda \cdot h \circ S}$ would become a g*–function* for all the phases, in particular, the pure phases of the model.

To summarize the picture, we conjecture that for the Dyson potential, transfer operator does not have an eigenfunction in the supercritical regime, and does have an eigenfunction in the subcritical regime. The smoothness of the eigenfunction is varying with α: from summable variation for $\alpha > 2$, to continuous for $\alpha \in (3/2, 2]$, to L^1 but not continuous for $\alpha \leq 3/2$.

The key to establishing properties of transfer operators for the Dyson potential is, in our opinion, a proper understanding of the probabilistic properties of the left-right interaction energy function:

$$W(\xi,\sigma) = \sum_{i<0\leq j} -\frac{\beta \xi_i \sigma_j}{(j-i)^\alpha}, \quad \xi \in X_-, \sigma \in X_+.$$

For example, in the case $\alpha > 2$, the results follow almost immediately from the simple observation that the interaction energy function is uniformly bounded

$$\sup_{\xi \in X_-, \sigma \in X_+} |W(\xi,\sigma)| = \beta \sum_{i<0\leq j} \frac{1}{(j-i)^\alpha} < \infty,$$

and all expressions for densities above automatically lead to continuous functions. The next interesting 'critical value' $\alpha = 3/2$ also appears quite naturally: The left-right interaction energies $W(\xi,\sigma)$ for a fixed $\sigma \in X_+$, but ν_--random configurations $\xi \in X_-$, can be represented as

$$W(\xi,\sigma) = \sum_{i<0} Z_i, \quad Z_i = \xi_i \Big(-\sum_{j\geq 0} \frac{\sigma_j}{(j-i)^\alpha} \Big).$$

The variances $\mathsf{var}(Z_i) = \mathcal{O}(|i|^{-2(\alpha-1)})$ become summable for $\alpha > 3/2$ [c.f. (4.60)]. Hence, assuming weak correlations, the condition $\alpha > 3/2$ corresponds to the almost sure existence of the left-right interaction energy for random conditions on the left half-line interacting with any (in particular, the all plus or all minus) configuration on the right half-line. The concentration inequality can be interpreted as the rigorous transcription of this observation, and the Dobrushin condition as the guarantor of weak correlations.

We strongly believe that the analysis of the left-right interaction energy function can be extended to the whole subcritical regime $\beta < \beta_c(\alpha)$.

We finish the discussion with two interesting questions. As customary in dynamical systems, we study continuous potentials $\phi \in C(X_+)$. In Section 4.6.1, however, we switch to the language of Statistical Mechanics and assume that the potential can be represented as $\phi = -\sum_{0 \in V \Subset \mathbb{Z}_+} \Phi_V$ for some translation invariant UAC interaction Φ on Z, c.f. [57]. This is clearly the case for the Dyson potential. In general, however, we do not know any reasonable description of the class of such potentials ϕ.

Finally, in the opposite direction, under which conditions can a Gibbsian specification γ, on the half-lattice $\mathbb{Z}_+$, be represented as $\gamma = \gamma^{\phi}$ for some $\phi \in C(X_+)$? One possible approach to finding such representations would be extending the Kozlov-Sullivan characterisation on $\mathbb{Z}$ to the half-line $\mathbb{Z}_+$ [5].

4.10 Appendix: Eigenfunctions for the near-critical ferromagnetic Dyson model

The appendix aims to present an extension of the results of Chapter 4 to the entire uniqueness region of the ferromagnetic Dyson model and sketch its proof by building on recent advances by Bauerschmidt and Dagallier [6], as well as a fundamental result by Duminil-Copin and Tassion [21].

In [6], the authors established *log-Sobolev inequalities* for the finite volume Gibbs measures of ferromagnetic Ising models, building on newly obtained correlation inequality by Ding, Song, and Sun [19].

Definition 4.10.1. *A measure τ on $X_{\mathscr{S}} := \{\pm 1\}^{\mathscr{S}}$, where $\mathscr{S} \subseteq \mathbb{Z}$ is a finite or infinite subset, satisfies the **log-Sobolev inequality** (LSI) if there exists a constant $D = D(\tau) > 0$ such that, for every local function $f : X_{\mathscr{S}} \to \mathbb{R}$,*

$$\mathtt{Ent}_{\tau}(f^2) \leq 2D \int_{X_{\mathscr{S}}} \sum_{i \in \mathscr{S}} (f(\omega) - f(\omega^{(i)}))^2 \tau(d\omega), \tag{4.76}$$

This appendix is based on M. Makhmudov, *"Concentration inequalities and Transfer operators for supercritical Dyson models"*, arXiv:2508.01703.

where $\omega^{(i)}$ denotes the configuration obtained from ω by flipping the spin at site i and keeping all other spins unchanged, i.e.,

$$\omega_j^{(i)} := \begin{cases} \omega_j, & i \neq j; \\ -\omega_i, & i = j, \end{cases}$$

and for a non-negative local function $\tilde{f} : X_{\mathcal{S}} \to \mathbb{R}$, the entropy of $\tilde{f}$ with respect to τ is defined as $\mathtt{Ent}_\tau(\tilde{f}) := \int_{X_{\mathcal{S}}} \tilde{f} \log \tilde{f}\, d\tau - \int_{X_{\mathcal{S}}} \tilde{f}\, d\tau \cdot \log \int_{X_{\mathcal{S}}} \tilde{f}\, d\tau$. *The smallest such constant D is called the log-Sobolev constant for τ.*

Bauerschmidt and Dagallier studied ferromagnetic Ising models in a general setting on finite lattices, under suitable assumptions on the model's coupling matrix. Below, we recall their result in the absence of external fields. Specifically, consider a probability measure τ on $X_V := \{\pm 1\}^V$, where V is finite, defined by

$$\tau^\beta(\{\omega_V\}) = \frac{e^{-\frac{\beta}{2}(\omega_V, A\omega_V)}}{\sum_{\omega_V \in X_V} e^{-\frac{\beta}{2}(\omega_V, A\omega_V)}}, \ \ \omega_V \in X_V,$$

where A is a symmetric $V \times V$ matrix and $\beta \geq 0$. Then, under suitable conditions on the coupling matrix A, the measure τ^β satisfies the log-Sobolev inequality (4.76) with a constant

$$D \leq \frac{1}{4} + \frac{\beta}{2} e^{2\beta\chi_{\beta,V}}, \tag{4.77}$$

where $\chi_{\beta,V}$ denotes the *susceptibility* of τ^β, defined as

$$\chi_{\beta,V} := \sup_{j\in V} \sum_{i\in V} \int_{X_V} \omega_i \omega_j \, \tau^\beta(d\omega). \tag{4.78}$$

The following are the conditions that the coupling matrix A must satisfy for (4.77) to hold:

(A1) Off-diagonal entries of A are non-positive, i.e., $A_{ij} \leq 0$ for $i \neq j$;

(A2) A is positive definite;

(A3) The spectral radius $\rho(A)$ of A is bounded by 1.

The coupling matrices of the Dyson model (4.9) satisfy condition (A1), but generally fail to satisfy (A2) and (A3). By carefully modifying the coupling matrices of the Dyson model to fulfill conditions (A2) and (A3), one can ensure that the log-Sobolev inequalities hold for the finite-volume Gibbs measures of the Dyson model with free boundary conditions throughout the entire uniqueness region.

However, to obtain meaningful results for infinite-volume measures, it is necessary to control the log-Sobolev constants across different volumes V, since the right-hand side of (4.77) depends on V. Thanks to the second GKS inequality [37], for every supercritical $\beta < \beta_c(\alpha) := \beta_c(\Phi)$, the following bound holds:

$$\chi_{\beta,V} \le \sum_{i\in\mathbb{Z}} \int_X \omega_0\omega_i\,\mu(d\omega) =: \chi_\beta, \tag{4.79}$$

where μ denotes the unique Gibbs measure of Φ. In 2018, Duminil-Copin and Tassion [21] proved the finiteness of the infinite-volume susceptibility for all $\beta < \beta_c(\alpha)$. As a consequence, one obtains a uniform log-Sobolev constant $D_0 := \frac{1}{4} + \frac{\beta}{2}e^{2\beta\chi_\beta}$ valid for all finite-volume Gibbs measures $\mu_V^{\emptyset}$ of Φ. This, in turn, implies that the infinite-volume Gibbs measure μ satisfies the log-Sobolev inequality with constant $D(\mu) \le D_0$. Analogously, one can establish the log-Sobolev inequalities with the same constant $D_0 = \frac{1}{4} + \frac{\beta}{2}e^{2\beta\chi_\beta}$ for the Gibbs measures corresponding to the intermediate interactions $\{\Psi^{(k)}\}_{k\in\mathbb{Z}_+}$. This can be achieved through the following two steps:

Step (1): First, we establish the uniqueness of the Gibbs measures for the interactions $\Psi^{(k)}$, $k \in \mathbb{Z}_+$, if $\beta < \beta_c(\alpha)$. In fact, by using the GKS inequalities, one can show that $\mu_{(+)} = \mu_{(-)}$ implies $\nu_{(-)}^{(k)} = \nu_{(+)}^{(k)}$ for every $k \in \mathbb{Z}_+$, where $\mu_{(\pm)}$ and $\nu_{(\pm)}^{(k)}$ denote the plus and minus phases of Φ and $\Psi^{(k)}$, respectively. In light of the FKG inequalities, this yields that $\beta_c(\alpha) \le \beta_c^{(k)}(\alpha) := \beta_c(\Psi^{(k)})$.

Step (2): For $\beta < \beta_c(\alpha)$, one can apply the same technique used for the finite-volume Gibbs measures of Φ to show that the finite-volume Gibbs measure $\nu_V^{(k),\emptyset}$ of $\Psi^{(k)}$ in a volume $V \Subset \mathbb{Z}$ satisfies the LSI with a constant not exceeding $\frac{1}{4} + \frac{\beta}{2}e^{2\beta\chi_{\beta,V}^{(k)}}$, where $\chi_{\beta,V}^{(k)} := \sup_{j\in V}\sum_{i\in V}\nu_V^{(k),\emptyset}(\sigma_i\sigma_j)$. By applying the GKS inequalities again, it follows for every $k \in \mathbb{Z}_+$ and $V \Subset \mathbb{Z}$ that $\chi_{\beta,V}^{(k)} \le \chi_\beta$. Therefore, by passing to the limit as $V \uparrow \mathbb{Z}$, one obtains that the unique infinite-volume Gibbs measure $\nu^{(k)}$ of $\Psi^{(k)}$ satisfies the LSI with the constant $D_0 = \frac{1}{4} + \frac{\beta}{2}e^{2\beta\chi_\beta}$.

In the next stage, we obtain the Gaussian Concentration Bound (4.17) from the LSI using the Herbst argument [4]. In fact, the following statement is valid.

Proposition 4.10.2. *Assume $\tau \in \mathcal{M}_1(X)$ is a unique Gibbs measure for an interaction $\Psi = (\Psi_V)_{V\Subset\mathbb{Z}}$ with*

$$\sup_{i\in\mathbb{Z}} \sum_{\substack{V\ni i\\ V\Subset\mathbb{Z}}} \|\Psi_V\|_\infty < \infty. \tag{4.80}$$

If τ satisfies the log Sobolev inequality with a constant $D > 0$, then τ has the Gaussian concentration bounds (4.17) with the constant $\frac{D(M+1)}{2}$, where $M := \exp\Big(2\sup_{i\in\mathbb{Z}}\sum_{V\ni i}\|\Psi_V\|_\infty\Big)$

By applying Proposition 4.10.2, along with the techniques used in the proofs of Theorems 4.D, 4.E, and 4.C, we have established the theorem below.

Theorem 4.G. *Let Φ be the ferromagnetic Dyson interaction (4.9), and let ϕ be the corresponding Dyson potential (4.8). Then the following statements hold.*

(i) For every $\alpha \in (1,2]$, we have $\beta_c(\alpha) \le \beta_c^+(\alpha)$, where $\beta_c^+(\alpha)$ is the critical temperature for ϕ. (See Section 4.7 for details.)

(ii) For all $\alpha \in (1,2]$ and for every $\beta \in [0,\beta_c(\alpha))$, the GCB (4.17) holds for the unique Gibbs measure μ of Φ and the unique half-line Gibbs measure ν_+ of ϕ with the constant $D = \frac{(1+2\beta e^{2\beta\chi_\beta})(1+e^{4\beta\zeta(\alpha)})}{8}$, where χ_β is the susceptibility of the Dyson model at β (c.f., (4.79)) and $\zeta(\alpha) := \sum_{n=1}^{\infty} n^{-\alpha}$.

(iii) For each $\alpha \in (1,2]$ and all $\beta \in [0,\beta_c(\alpha))$, the restriction $\mu|_{X_+}$ of μ to X_+ is equivalent to ν_+, i.e., $\mu|_{X_+} \ll \nu_+$ and $\nu_+ \ll \mu|_{X_+}$. In particular, $\mathscr{L}_\phi$ admits an integrable eigenfunction $\frac{d\mu|_{X_+}}{d\nu_+} \in L^1(\nu_+)$ corresponding to its spectral radius.

(iv) If $\alpha \in \left(\frac{3}{2},2\right]$, then for all $\beta \in [0,\beta_c(\alpha))$, there exists a continuous version of the Radon-Nikodym derivative $\frac{d\mu|_{X_+}}{d\nu_+}$. Hence, $\mathscr{L}_\phi$ has a continuous principal eigenfunction.

Remark 4.10.3. *The fourth part of Theorem 4.G has also been established by Johansson, Öberg, and Pollicott by using the random cluster representation of the ferromagnetic Ising models in combination with the concentration inequality established specifically for the random cluster models in* [45].

A detailed proof of Theorem 4.G, along with further developments based on the log-Sobolev approach, can be found in [52].

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Chapter 5

The Eigenfunctions of the transfer operator for the Dyson model in a field

Abstract: In this chapter, we prove that, for high temperatures or strong magnetic fields, there exists a non-negative, integrable (with respect to the unique half-line Gibbs measure) eigenfunction of the transfer operator for the Dyson model if $\alpha \in \left(\frac{3}{2}, 2\right]$. However, unlike in the zero-magnetic-field case, this eigenfunction is not continuous.

This chapter is based on M. Makhmudov, *"The Eigenfunctions of the Transfer Operator for the Dyson model in a field"*, arXiv:2502.09588.

5.1 Introduction

The study of the so-called *equilibrium states*, specific types of invariant measures, has been a central topic in the ergodic theory of dynamical systems since the 1970s. The foundations of this theory were laid in the seminal works of Bowen and Ruelle at the end of the decade. Equilibrium states also play a crucial role in equilibrium statistical mechanics. A pivotal result in this context is the classical variational principle, proved by Lanford and Ruelle in 1969, which establishes that, under certain conditions, translation-invariant Gibbs measures coincide with equilibrium states. Recall that an equilibrium state μ for a continuous potential ϕ on the one-sided shift space (we also refer to it as the half-line shift space) $X_+ := E^{\mathbb{Z}_+}$, $|E| < \infty$ are probability measures on X_+ which are invariant under the shift (translation) map $S : X_+ \to X_+$, $(Sx)_j = x_{j+1}$, $j \in \mathbb{Z}_+$ and solves the variational equation:

$$\int_{X_+} \phi \, d\mu + \mathfrak{h}(\mu) = \sup\Big\{ \int_{X_+} \phi \, d\tau + \mathfrak{h}(\tau) : \ \tau \in \mathscr{M}_{1,S}(X_+) \Big\}, \tag{5.1}$$

where $\mathscr{M}_{1,S}(X_+)$ is the simplex of translation-invariant probability measures on X_+ and $\mathfrak{h}(\tau)$ denotes the measure-theoretic entropy of the translation-invariant measure τ.

A natural and effective approach to studying equilibrium states for a potential ϕ is through the transfer operator $\mathscr{L}_\phi$, which acts on the space of functions on X_+, particularly on continuous functions, via:

$$\mathscr{L}_\phi f(x) := \sum_{a \in E} e^{\phi(ax)} f(ax), \ \ f \in \mathbb{R}^{X_+}, \ \ x \in X_+. \tag{5.2}$$

In fact, if $\mathfrak{h}$ is an eigenfunction of $\mathscr{L}_\phi$, i.e., $\mathscr{L}_\phi \mathfrak{h} = \lambda \mathfrak{h}$, and ν is an eigenprobability for its adjoint $\mathscr{L}_\phi^* : C(X_+)^* \to C(X_+)^*$, i.e., $\mathscr{L}_\phi^* \nu = \lambda \nu$, where λ is the spectral radius of $\mathscr{L}_\phi$, then the measure $d\mu = \mathfrak{h} \cdot d\nu$ is an equilibrium state for ϕ and vice versa (see Proposition 3.1 in [5]). Classical fixed-point theorems ensure the existence of an eigenprobability for the adjoint $\mathscr{L}_\phi^*$, however, the existence of an eigenfunction for $\mathscr{L}_\phi$ is not always guaranteed. This issue has been extensively studied by Ruelle, Walters and others under various regularity conditions on ϕ [18, 20–23]. Notably, every potential ϕ satisfying these regularity conditions shares two key properties:

(P1) for all $\beta > 0$, the potential $\beta\phi$ admits a unique equilibrium state;

(P2) the transfer operator $\mathscr{L}_\phi$ is stable under "smooth" perturbations of ϕ in the sense that if $\mathscr{L}_\phi$ has a (principal) eigenfunction, then $\mathscr{L}_{\phi+\upsilon}$ also has a (principal) eigenfunction with the same regularity properties for every local function $\upsilon : X_+ \to \mathbb{R}$.

Note that the first property rules out the possibility of phase transitions, i.e., the existence of multiple equilibrium states for some β, a phenomenon of significant interest in statistical mechanics. Consequently, these classical conditions on ϕ exclude important one-dimensional models. Efforts to extend the theory to more general one-dimensional models have emerged in recent studies. In 2019, Cioletti et al. analyzed transfer operators for product-type potentials, which fall outside existing classes but still satisfy (P1) and (P2) [2]. A relevant study conducted in [1], though in a slightly different setup, provides a result that has implications for the transfer operator of the so-called Dyson potential. This potential corresponds to the 1D long-range Ising model [15] and is defined by:

$$\phi(x) = hx_0 + \beta \sum_{n=1}^{\infty} \frac{x_0 x_n}{n^\alpha}, \quad x \in X_+ := \pm 1^{\mathbb{Z}_+}, \tag{5.3}$$

where $\alpha > 1$ is the decay rate of the coupling, $\beta \geq 0$ is the inverse temperature, and $h \in \mathbb{R}$ represents the strength of the external field. The study in [1] shows that the transfer operator does not have a continuous eigenfunction when $h = 0$ and β is sufficiently large. A more notable development came in 2023 when Johansson, Öberg, and Pollicott analysed the transfer operator for the Dyson potential in the supercritical regime. Shortly thereafter, I, in collaboration with van Enter, Fernández, and Verbitskiy, introduced a new approach to addressing the eigenfunction problem for long-range models [5]. Following the developments in [5, 15], the following result on the Dyson model is known:

Theorem 5.1.1. [5, 15] *Let ϕ be the Dyson potential (5.3). Suppose $h = 0$, then:*

(i) for all $\alpha > 1$ and for $\beta \geq 0$ sufficiently small, the unique equilibrium state μ_+ for ϕ is equivalent to the unique half-line Gibbs state ν for ϕ, i.e., $\mu_+ \ll \nu$ and $\nu \ll \mu_+$. In particular, the Perron-Frobenius transfer operator $\mathscr{L}_\phi$ has an eigenfunction in $L^1(X_+, \nu)$;

(ii) furthermore, if $\alpha > \frac{3}{2}$, then for all $0 < \beta < \beta_c$ there exists a continuous version of the Radon-Nikodym density $\frac{d\mu_+}{d\nu}$, and thus the Perron-Frobenius transfer operator $\mathscr{L}_\phi$ has a positive continuous eigenfunction, where β_c is the critical temperature of the phase transitions for the standard Dyson model (see Theorem 5.2.1 in the next section).

However, both studies, [15] and [5], do not cover the case of the Dyson potential with a nonzero external field $h \neq 0$. The approach in [15] relies heavily on the random cluster representation of the Dyson model. The central obstacle to extending the method in [15] to non-zero external fields is the loss of symmetry, essential for the random cluster representation, which disrupts the cluster decay analysis. The method developed in [5] requires a certain sum of two-point functions to be uniformly bounded, a condition that fails for the Dyson model in a field. In this chapter, we adopt the technique from the previous chapter to treat the Dyson model in a nonzero field. The result of this chapter is as follows:

Theorem 5.A. *Suppose* $\alpha \in (\frac{3}{2}, 2]$, $\beta \geq 0$ *and* $|h| > 0$ *is sufficiently large* ($|h| > 2\beta\zeta(\alpha) + \log 4\beta\zeta(\alpha)$ *is enough, here* ζ *is the Riemann zeta function*). *Then*

(i) the Dyson potential ϕ *has a unique equilibrium state* μ_+ *and there also exists a unique eigenprobability* ν *of* $\mathscr{L}_\phi^*$;

(ii) μ_+ *is absolutely continuous with respect to* ν, *i.e.,* $\mu_+ \ll \nu$. *In particular, the Perron-Frobenius transfer operator* $\mathscr{L}_\phi$ *admits an integrable eigenfunction corresponding to the spectral radius* $\lambda = e^{P(\phi)}$.

(iii) The Radon-Nikdoym derivative $\frac{d\mu_+}{d\nu}$ *does not have a continuous version. In particular, the Perron-Frobenius transfer operator does not have a continuous principal eigenfunction.*

Remark 5.1.2. *(i) Theorem 5.A is proven under a strong uniqueness condition for the Dyson interaction* Φ *(see (5.6) and Section 5.2), a concept introduced in* [5]. *This condition is implied by the Dobrushin uniqueness condition (DUC) on* Φ *(Section 5.3). The interaction* Φ *satisfies DUC if either 1)* β *is sufficiently small so that* $\beta \leq \frac{1}{2\zeta(\alpha)}$ *or 2)* $|h|$ *is sufficiently large so that* $|h| > 2\beta\zeta(\alpha) + \log 4\beta\zeta(\alpha)$. *Note that in the first version of the Dobrushin uniqueness condition, there is no constraint on the external field* h. *We prove Theorem 5.A for large* $|h|$*'s, however, the statement of Theorem 5.A remains valid and the same proof works for the first version of the DUC which covers the domain* $0 \leq \beta < \frac{1}{2\zeta(\alpha)}$ *and* $h \neq 0$.

We, in fact, believe that Theorem 5.A is true for all $h \in \mathbb{R}$ *but* $h = 0$, *i.e., the transfer operator* $\mathscr{L}_\phi$ *has an integrable but not continuous eigenfunction for all* $\beta \geq 0$, $h \neq 0$ *(See Figure 5.1).*

(ii) It is worth noting that Theorem 5.A suggests a certain similarity between the Dyson potential in a field and product-type potentials [2]. *Note that for a product-type potential, there exists a unique equilibrium state* μ_+ *and eigenprobability* ν *which both are Bernoulli measures such that* $\nu = \prod_{n=0}^{\infty} \lambda_n$ *with* $\lambda_n(1) = p_n = \frac{\exp(\beta \sum_{i=1}^{n} i^{-\alpha})}{2\cosh(\beta \sum_{i=1}^{n} i^{-\alpha})}$ *and* $\mu_+ = \prod_{n=0}^{\infty} \lambda_\infty$ [2,5]. *Thus one can check*

that

$$\int_E \sqrt{\frac{d\lambda_\infty}{d\lambda_n}}d\lambda_n = \sqrt{p_\infty p_n} + \sqrt{(1-p_\infty)(1-p_n)} = 1 + O^*((p_n - p_\infty)^2),$$

therefore,

$$\log \int_E \sqrt{\frac{d\lambda_\infty}{d\lambda_n}}d\lambda_n = O^*((p_n - p_\infty)^2) = O^*(n^{-2\alpha+2}), \tag{5.4}$$

here $u(t) = O^*(v(t))$ *means the existence of* $C_1, C_2 > 0$ *such that* $C_1|u(t)| \leq |v(t)| \leq C_2|u(t)|$ *for all sufficiently small* t. *Then by (5.4), Kakutani's theorem implies that* $\mu_+ \ll \nu$ *if* $\alpha > \frac{3}{2}$ *and* $\mu_+ \perp \nu$ *if* $\alpha \leq \frac{3}{2}$.

In the case of the Dyson potential, in the presence of a strong external field, spin-spin correlations become negligible, as they decay rapidly. Consequently, individual spins behave like independent random variables. Thus, we expect a phenomenon similar to that observed with product-type potentials, as described above, to occur in the Dyson model under an external field: for $\alpha \in \left(\frac{3}{2}, 2\right]$, *we have* $\mu_+ \ll \nu$, *while for* $\alpha \in \left(1, \frac{3}{2}\right]$, $\mu_+ \perp \nu$ *(See Figure 5.1).*

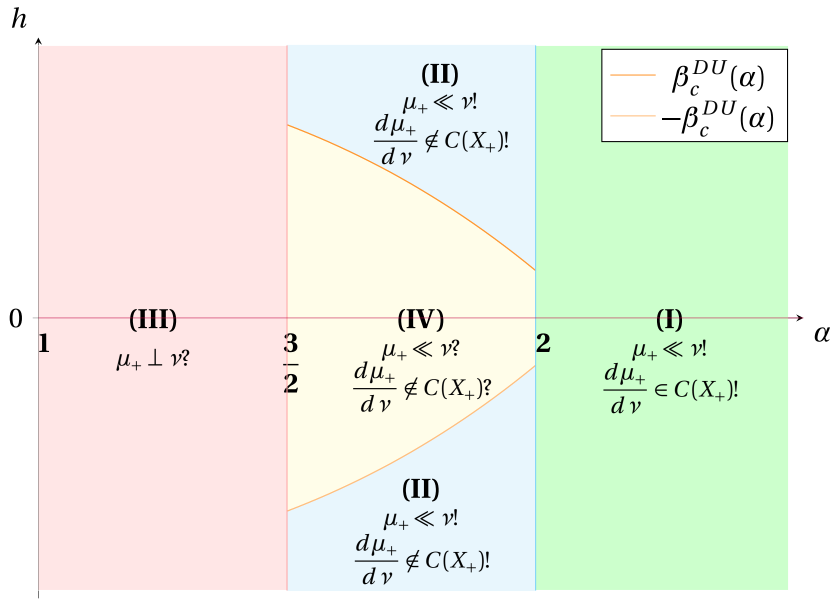

Figure 5.1: Eigenfunctions diagram for Dyson's model in a field for a fixed $\beta > 0$.

Note that the potential $\phi_0(x) = \beta \sum_{n=1}^{\infty} \frac{x_0 x_n}{n^\alpha}$, $x \in X_+ = \{\pm 1\}^{\mathbb{Z}_+}$ violates both (P1) and (P2) properties mentioned above. In fact, for all $\alpha \in (1, 2]$, there exists $\beta_c(\alpha) > 0$ such that for all $\beta > \beta_c(\alpha)$, $\phi_{\mathbf{0}}$ has multiple equilibrium states [4, 8]. As regards the second property, the second parts of Theorem 5.1.1 and Theorem 5.A yields that the transfer operator $\mathscr{L}_{\phi_0}$ is unstable under "smooth" perturbations of ϕ_0 since for $\alpha \in (\frac{3}{2}, 2]$ and $\beta < \beta_c$, the principal eigenfunction of $\mathscr{L}_{\phi_0}$ is continuous whereas for $h \neq 0$, $\mathscr{L}_{\phi_0 + h\sigma_0}$ admits an integrable but not continuous eigenfunction, where $\sigma_0(x) = x_0$ for $x \in X_+$.

The chapter is organised as follows. In the Section 5.2 we define the notions of specifications and interactions. Here, we also introduce the interaction of the Dyson model. Section 5.3 is dedicated to the Dobrushin uniqueness condition and its consequences, which are important for this chapter. In Section 5.4, we discuss the construction of the intermediate interactions which are instrumental in the proof of Theorem 5.A. Section 5.5 is dedicated to the proof of Theorem 5.A.

5.2 Specifications and Interactions

In this chapter, we again consider the ferromagnetic spin space $E = \{\pm 1\}$. For a countable set $\mathbb{L}$ of sites, we consider a configuration space $\Omega = E^{\mathbb{L}}$. For $\mathbb{L}$, we either consider the set of integers $\mathbb{Z}$, or the set of negative integers $-\mathbb{N}$, or the set of non-negative integers $\mathbb{Z}_+$. As a product of compact sets, Ω is a compact and also a metrizable topological space. In fact, for any $\theta \in (0, 1)$, $d(x, y) = \theta^{\mathbf{n}(x,y)}$ metrizes X_+, here for $x, y \in X_+$, $\mathbf{n}(x, y) = \min\{j \in \mathbb{Z}_+ : x_j \neq y_j\}$. $\mathscr{F}$ denotes the Borel sigma-algebra of X. For any $\Lambda \subset \mathbb{L}$, a sub sigma-algebra $\mathscr{F}_\Lambda \subset \mathscr{F}$ is defined as the minimal sigma-algebra so that for all $j \in \Lambda$, the functions $\sigma_j : \Omega \to E$ are measurable where $\xi \in \Omega \mapsto \sigma_j(\xi) = \xi_j \in \{\pm 1\}$. For configurations $\xi, \eta \in \Omega$ and a volume $\Lambda \subset \mathbb{L}$, $\xi_\Lambda \eta_{\Lambda^c}$ denotes the concatenated configuration, i.e., $(\xi_\Lambda \eta_{\Lambda^c})_i = \xi_i$ if $i \in \Lambda$ and $(\xi_\Lambda \eta_{\Lambda^c})_i = \eta_i$ if $i \in \Lambda^c = \mathbb{L} \setminus \Lambda$.

A *specification* γ, a regular family of conditional probabilities, is defined as a *consistent* family of proper probability kernels $\gamma_\Lambda : \mathscr{F} \times \Omega \to [0, 1]$ indexed by finite subsets Λ of $\mathbb{L}$, denoted by $\Lambda \Subset \mathbb{L}$ [9, Chapter 1]. A probability measure $\mu \in \mathscr{M}_1(\Omega)$ is *Gibbs* for a specification γ if it is consistent with γ, i.e., for all $\Lambda \Subset \mathbb{L}$, $\mu = \mu\gamma_\Lambda$, here $B \in \mathscr{F} \mapsto \mu\gamma_\Lambda(B) := \int_\Omega \gamma_\Lambda(B|\eta)\mu(d\eta)$. The set of Gibbs measures for the specification γ is denoted by $\mathscr{G}(\gamma)$.

The classical approach to specifications is via interactions. An interaction Φ is a family $(\Phi_\Lambda)_{\Lambda \Subset \mathbb{L}}$ of functions $\Phi_\Lambda : \Omega \to \mathbb{R}$ such that $\Phi_\Lambda \in \mathscr{F}_\Lambda$, i.e., each Φ_Λ is independent of coordinates outside Λ. An interaction is called *Uniformly Absolutely*

Convergent (UAC) if for all $j \in \mathbb{L}$,

$$\sum_{j \in V \Subset \mathbb{L}} \sup_{\omega \in \Omega} |\Phi_V(\omega)| < \infty.$$

For a UAC interaction Φ, the Hamiltonian in a volume $\Lambda \Subset \mathbb{L}$ is $H_\Lambda := \sum_{V \cap \Lambda \neq \emptyset} \Phi_V$ and the associated specification (density) γ^Φ is defined by

$$\gamma^\Phi_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) := \frac{e^{-H_\Lambda(\omega)}}{Z_\Lambda(\omega_{\Lambda^c})}, \ \ \omega \in \Omega, \tag{5.5}$$

where Z_Λ is the normalization constant, also known as the partition function, which is $Z_\Lambda(\omega_{\Lambda^c}) := \sum_{\xi_\Lambda \in E^\Lambda} e^{-H_\Lambda(\xi_\Lambda \omega_{\Lambda^c})}$. Note that the specification $\gamma^\Phi = (\gamma^\Phi_\Lambda)_{\Lambda \Subset \mathbb{L}}$ is restored from (5.5) by

$$\gamma^\Phi_\Lambda(B|\omega) := \sum_{\xi_\Lambda \in E^\Lambda} \mathbb{1}_B(\xi_\Lambda \omega_{\Lambda^c}) \gamma^\Phi_\Lambda(\xi_\Lambda|\omega_{\Lambda^c}), \ \ B \in \mathscr{F}, \ \omega \in \Omega.$$

This chapter focuses on the Dyson model, a cornerstone of one-dimensional statistical mechanics renowned for its long-range interactions and critical phenomena. It is defined on the lattice $\mathbb{L} = \mathbb{Z}$ by

$$\Phi_\Lambda(\omega) := \begin{cases} -\dfrac{\beta \omega_i \omega_j}{|i-j|^\alpha}, & \text{if } \Lambda = \{i, j\} \subset \mathbb{Z}, \ i \neq j; \\ -h\omega_i, & \Lambda = \{i\}; \\ 0, & \text{otherwise,} \end{cases} \tag{5.6}$$

where $\alpha \in (1, 2]$ is a parameter describing the decay of the interaction strength, $\beta \geq 0$ is the inverse temperature and $h \in \mathbb{R}$ represents the external field. In fact, the Dyson potential defined in (5.3) is related to the Dyson interaction Φ by

$$\phi = -\sum_{0 \in \Lambda \Subset \mathbb{Z}_+} \Phi_\Lambda.$$

The following theorem about the phase diagram of the Dyson model is well-known.

Theorem 5.2.1. [4, 7, 8] *Suppose* Φ *be the Dyson interaction given in (5.6).*

(i) If $h \neq 0$*, then* Φ *has a unique Gibbs measure, i.e.,* $|\mathscr{G}(\gamma^\Phi)| = 1$*;*

(ii) If $h = 0$*, there exists a finite critical temperature* $\beta_c(\alpha) > 0$ *such that for all* $\beta \in [0, \beta_c(\alpha))$*,* $|\mathscr{G}(\gamma^\Phi)| = 1$ *and for* $\beta > \beta_c(\alpha)$*,* Φ *exhibits phase transitions, i.e.,* $|\mathscr{G}(\gamma^\Phi)| > 1$*.*

5.3 The Dobrushin uniqueness condition for strong external fields

Dobrushin uniqueness condition is one of the most general criteria for the uniqueness of Gibbs states. We discuss it in the framework of a general countable set $\mathbb{L}$ of sites and a configuration space $\Omega := E^{\mathbb{L}}$. In fact, we use the Dobrushin uniqueness condition in the two cases of $\mathbb{L}$ which are $\mathbb{L} \in \{-\mathbb{N}, \mathbb{Z}\}$.

Consider a uniformly absolutely convergent (UAC) interaction $\Phi = \{\Phi_\Lambda(\cdot) : \Lambda \Subset \mathbb{L}\}$ on Ω and let γ^Φ be the corresponding Gibbsian specification. For any sites $i, j \in \mathbb{L}$, define

$$C(\gamma^\Phi)_{i,j} := \sup_{\eta_{\mathbb{L}\setminus\{j\}} = \overline{\eta}_{\mathbb{L}\setminus\{j\}}} \|\gamma^\Phi_{\{i\}}(\cdot|\eta) - \gamma^\Phi_{\{i\}}(\cdot|\overline{\eta})\|_\infty,$$

where $\|\cdot\|_\infty$ is the supremum norm on $\mathscr{M}(\Omega)$ defined by $\|\tau\|_\infty := \sup_{B \in \mathscr{F}} |\tau(B)|$ for any finite signed Borel measure τ. The infinite matrix $C(\gamma^\Phi) := (C(\gamma^\Phi)_{i,j})_{i,j \in \mathbb{L}}$ is called the Dobrushin interdependence matrix.

Definition 5.3.1. [9] *The specification γ^Φ satisfies the* ***Dobrushin uniqueness condition*** *if*

$$c(\gamma^\Phi) := \sup_{i \in \mathbb{L}} \sum_{j \in \mathbb{L}} C(\gamma^\Phi)_{i,j} < 1. \tag{5.7}$$

The Dobrushin uniqueness condition admits slightly stronger — and easy to check— forms. One of them is a high-temperature condition, which is considered in [5]. Another version is known as the strong magnetic field condition, which we formulate here:

Proposition 5.3.2. [9, Example 8.13] *Let $\mathbb{L}$ be any countable set and $E = \{\pm 1\}$. Assume Φ be a UAC interaction on $\Omega = \{\pm 1\}^{\mathbb{L}}$ such that $\Phi_{\{i\}} = -h\sigma_i$ for all $i \in \mathbb{L}$ and for some $h \in \mathbb{R}$. Suppose*

$$|h| > \log \sup_{i \in \mathbb{L}} \Big\{ \exp\Big(\frac{1}{2} \sum_{A \ni i, |A| \geq 2} \delta(\Phi_A)\Big) \cdot \sum_{A \ni i} (|A| - 1)\delta(\Phi_A) \Big\}, \tag{5.8}$$

where $\delta(f) := \sup\{|f(\xi) - f(\eta)| : \xi, \eta \in E^{\mathbb{L}}\}$ is the variation of $f : E^{\mathbb{L}} \to \mathbb{R}$. Then γ^Φ satisfies the Dobrushin uniqueness condition.

The proof of Proposition 5.8 boils down to showing that for all i, j, $i \neq j$,

$$C(\gamma^\Phi)_{ij} \leq \exp\Big(-|h| + \frac{1}{2} \sum_{A \ni i, |A| \geq 2} \delta(\Phi_A)\Big) \cdot \sum_{A \supset \{i,j\}} \delta(\Phi_A) =: \bar{C}(\Phi)_{ij},$$

and hence $c(\gamma^{\Phi}) = \sup_i \sum_j C(\gamma^{\Phi})_{ij} \le \sup_i \sum_j \bar{C}(\Phi)_{ij} = \bar{c}(\Phi)$. Note that, here without loss of generality, we set $\bar{C}(\Phi)_{ii} = 0$ for all $i \in \mathbb{L}$. Notice that the non-negative matrix $\bar{C}(\Phi) := (\bar{C}(\Phi)_{i,j})_{i,j\in\mathbb{L}}$ is symmetric.

The condition (5.8) is stable under a *perturbation* of the underlying model/interaction Φ. Indeed, let $\Psi = \{\Psi_V\}_{V\Subset\mathbb{L}}$ be an interaction such that for $V \Subset \mathbb{L}$, either $\Psi_V = \Phi_V$ or $\Psi_V = 0$, then it is straightforward to check that the matrix $\bar{C}(\Psi)$ is dominated by $\bar{C}(\Phi)$, i.e., for all $i, j \in \mathbb{L}$, $\bar{C}(\Psi)_{ij} \le \bar{C}(\Phi)_{ij}$, thus, in particular, $\bar{c}(\Psi) \le \bar{c}(\Phi)$. Hence Ψ inherits the Dobrushin uniqueness condition from Φ as long as Φ satisfies (5.8).

The crucial property of the Dobrushin uniqueness condition is that it provides the uniqueness of the compatible probability measures with the specification γ^{Φ}. In fact, we have the following theorem.

Theorem 5.3.3. [9, Chapter 8] *If γ^{Φ} satisfies the Dobrushin uniqueness condition (5.7), then $\left|\mathscr{G}(\gamma^{\Phi})\right| = 1$.*

The validity of Dobrushin's criterion yields several important properties of the unique Gibbs state such as concentration inequalities and explicit bounds on the decay of correlations.
The first property involves the coefficient $\bar{c}(\Phi)$, which governs the deviation behaviour of the unique Gibbs measure. Let

$$\delta_k F := \sup\left\{F(\xi) - F(\eta) : \xi_j = \eta_j,\ j \in \mathbb{L}\setminus\{k\}\right\} \tag{5.9}$$

denote the oscillation of a local function $F : \Omega \to \mathbb{R}$ at a site $k \in \mathbb{L}$ and $\underline{\delta}(F) = (\delta_k F)_{k\in\mathbb{L}}$ is the oscillation vector, where $\Omega = E^{\mathbb{L}}$.

Theorem 5.3.4. [17][19] *Suppose Φ is a UAC interaction satisfying (5.8) and let μ_Φ be its unique Gibbs measure. Set*

$$D := \frac{4}{(1-\bar{c}(\Phi))^2}\ . \tag{5.10}$$

Then μ_Φ satisfies a ***Gaussian Concentration Bound*** *with the constant D, i.e., for any continuous function F on $\Omega = E^{\mathbb{L}}$, one has*

$$\int_\Omega e^{F - \int_\Omega F\, d\mu_\Phi}\, d\mu_\Phi \le e^{D\|\underline{\delta}(F)\|_2^2}.$$

The second important consequence of the Dobrushin's uniqueness condition is the following:

Theorem 5.3.5. [9, Theorem 8.20] *Let γ and $\tilde{\gamma}$ be two specifications on $\Omega = E^{\mathbb{L}}$. Suppose γ satisfies Dobrushin's condition. For each $i \in \mathbb{L}$, we let b_i be a measurable function on Ω such that*

$$\|\gamma_{\{i\}}(\cdot|\omega_{\{i\}^c}) - \tilde{\gamma}_{\{i\}}(\cdot|\omega_{\{i\}^c})\|_\infty \le b_i(\omega) \tag{5.11}$$

for all $\omega \in \Omega$. If $\mu \in \mathscr{G}(\gamma)$ and $\tilde{\mu} \in \mathscr{G}(\tilde{\gamma})$ then for all $f \in C(\Omega)$,

$$|\mu(f) - \tilde{\mu}(f)| \leq \sum_{i,j \in \mathbb{L}} \delta_i(f) D(\gamma)_{i,j} \tilde{\mu}(b_j). \tag{5.12}$$

We present the third property in the particular setup $\mathbb{L} = \mathbb{Z}$, and it involves the $\mathbb{Z} \times \mathbb{Z}$ matrix

$$D(\gamma^\Phi) = \sum_{n=0}^{\infty} C(\gamma^\Phi)^n . \tag{5.13}$$

The sum of the $\mathbb{Z} \times \mathbb{Z}$ matrices in the right-hand side converges due to the Dobrushin condition (5.7).

Proposition 5.3.6. [6, 9] *Consider a UAC interaction Φ on $X = E^{\mathbb{Z}}$. Assume the specification γ^Φ satisfies the Dobrushin condition (5.7) and let μ be its unique Gibbs measure. Then, for all $f, g \in C(X)$ and $i \in \mathbb{Z}$,*

$$\left| cov_\mu(f, g \circ S^i) \right| \leq \frac{1}{4} \sum_{k,j \in \mathbb{Z}} D(\gamma^\Phi)_{jk} \cdot \delta_k f \cdot \delta_{j-i} g. \tag{5.14}$$

Suppose Φ satisfies the Dobrushin condition (5.8) and define the non-negative symmetric $\mathbb{Z} \times \mathbb{Z}$–matrix by

$$\bar{D}(\Phi) := \sum_{n \geq 0} \bar{C}(\Phi)^n . \tag{5.15}$$

The sum in the right-hand side of (5.15) converges due to (5.8). Furthermore, $\bar{D}(\Phi)$ is invertible and $\bar{D}(\Phi)^{-1} = I_{\mathbb{Z}\times\mathbb{Z}} - \bar{C}(\Phi)$, where $I_{\mathbb{Z}\times\mathbb{Z}}$ denotes the identity matrix (operator) on $l^2(\mathbb{Z})$.

In general, one can show that the off-diagonal elements $\bar{D}(\Phi)$ decay as they get far away from the diagonal, i.e., $\bar{D}(\Phi)_{ij} = o(1)$ as $|i - j| \to \infty$. However, the decay rate largely depends on the interaction Φ. Now we discuss the decay rate in the case of Dyson interaction Φ defined by (5.6). For $i \neq j$, one can readily check that

$$\bar{C}(\Phi)_{ij} = \frac{2\beta \cdot \exp(-|h| + 2\beta\zeta(\alpha))}{|i-j|^\alpha} = O(|i-j|^{-\alpha}). \tag{5.16}$$

Thus by Jaffard's theorem ([14, Proposition 3] and [13, Theorem 1.1]), one has the same asymptotics for the off-diagonal elements of the inverse matrix $(I_{\mathbb{Z}\times\mathbb{Z}} - \bar{C}(\Phi))^{-1} = \bar{D}(\Phi)$, i.e., $\bar{D}(\Phi)_{ij} = O(|i-j|^{-\alpha})$. Thus there exists a constant $C_J > 0$ such that for all $i, j \in \mathbb{Z}$,

$$\bar{D}(\Phi)_{i,j} \leq \frac{C_J}{(1+|i-j|)^\alpha}. \tag{5.17}$$

5.4 Construction of intermediate interactions

In this section, we recall the construction of the intermediate interaction from [5]. For the Dyson interaction Φ given by (5.6) and each $k \in \mathbb{Z}_+$, we construct intermediate interactions $\Psi^{(k)}$ in the following way. We will represent $\mathbb{Z} = -\mathbb{N} \cup \mathbb{Z}_+$. Consider the countable collection of finite subset of $\mathbb{Z}$ with endpoints in $-\mathbb{N}$ and $\mathbb{Z}_+$:

$$\mathscr{A} = \{\Lambda \Subset \mathbb{Z} : \min(\Lambda) < 0,\ \max\Lambda \geq 0\}.$$

Index elements of $\mathscr{A}$ in an order $\mathscr{A} = \{\Lambda_1, \Lambda_2, \ldots\}$ in a way that for every $N \in \mathbb{N}$, there exists $k_N \in \mathbb{N}$ such that

$$\sum_{i=1}^{k_N} \Phi_{\Lambda_i} = \sum_{\substack{\min V < 0 \leq \max V \\ V \subset [-N,N]}} \Phi_V. \tag{5.18}$$

Then define

$$\Psi_\Lambda^{(k)} = \begin{cases} \Phi_\Lambda, & \Lambda \notin \{\Lambda_i : i \geq k+1\}, \\ 0, & \Lambda \in \{\Lambda_i : i \geq k+1\}, \end{cases}$$

In other words, we first remove all Φ_Λ's with $\Lambda \in \mathscr{A}$ from Φ, and then add them one by one. Clearly, all the constructed interactions are UAC.

Remark 5.4.1. *1) Every $\Psi^{(k)}$ is a local (finite) perturbation of $\Psi^{(0)}$, and $\Psi^{(k)}$ tend to Φ as $k \to \infty$, in the sense that $\Psi_\Lambda^{(k)} \rightrightarrows \Phi_\Lambda$ for all $\Lambda \Subset \mathbb{Z}$.*

2) For any finite volume V, one can readily check that

$$\|H_V^{\Psi^{(k)}} - H_V^{\Phi}\|_\infty \leq \sum_{\substack{\Lambda_j \cap V \neq \emptyset \\ j \geq k}} \|\Phi_\Lambda\|_\infty \xrightarrow[k\to\infty]{} 0.$$

3) For specifications, it can also be concluded that $\gamma^{\Psi^{(k)}}$ converges to γ^Φ as $k \to \infty$, i.e., for all $B \in \mathscr{F}$ and $V \Subset \mathbb{Z}$,

$$\gamma_V^{\Psi^{(k)}}(B|\omega) \xrightarrow[k\to\infty]{} \gamma_V^{\Phi}(B|\omega) \text{ uniformly on the b.c. } \omega \in \Omega.$$

4) In addition, if $\nu^{(k)}$ is a Gibbs measure for $\Psi^{(k)}$, then by Theorem 4.17 in [9], any weak-limit point of the sequence $\{\nu^{(k)}\}_{k\geq 0}$ becomes a Gibbs measure for the potential Φ.*

Another important observation is the following: since we have constructed $\Psi^{(0)}$ from Φ by removing all the interactions between the left $-\mathbb{N}$ and the right $\mathbb{Z}_+$

half-lines, the corresponding specification $\gamma^{\Psi^{(0)}}$ becomes product type [9, Example 7.18]. More precisely, $\gamma^{\Psi^{(0)}} = \gamma^{\Phi^-} \times \gamma^{\Phi^+}$, where Φ^- and Φ^+ are the restrictions of Φ to the half-lines $-\mathbb{N}$ and $\mathbb{Z}_+$, respectively. Thus we have the following for the extreme Gibbs measures [9, Example 7.18]:

$$\operatorname{ex}\mathscr{G}(\gamma^{\Psi^{(0)}}) = \{\nu^- \times \nu^+ : \nu^- \in \operatorname{ex}\mathscr{G}(\gamma^{\Phi^-}),\ \ \nu^+ \in \operatorname{ex}\mathscr{G}(\gamma^{\Phi^+})\} \tag{5.19}$$

For $k \geq 0$, let $\nu^{(k)} \in \mathscr{G}(\Psi^{(k)})$. For any $k \geq 1$, we also consider the following function:

$$f^{(k)} = \frac{e^{-\sum_{i=1}^{k} \Phi_{\Lambda_i}}}{\int_\Omega e^{-\sum_{i=1}^{k} \Phi_{\Lambda_i}} \, d\nu^{(0)}}. \tag{5.20}$$

Now we state two important lemmas which will be used in the proof of Theorem 5.A.

Lemma 5.4.2. [5, Theorem 6.5] *If the interaction Φ satisfies the Dobrushin uniqueness condition (5.8), so do all interactions $\Psi^{(k)}$. Furthermore, $\nu^{(k)}$ and $\nu^{(0)}$ are equivalent with*

$$\frac{d\nu^{(k)}}{d\nu^{(0)}} = \frac{e^{-\sum_{i=1}^{k} \Phi_{\Lambda_i}}}{\int_X e^{-\sum_{i=1}^{k} \Phi_{\Lambda_i}} \, d\nu^{(0)}} \quad , \quad \frac{d\nu^{(0)}}{d\nu^{(k)}} = \frac{e^{\sum_{i=1}^{k} \Phi_{\Lambda_i}}}{\int_X e^{\sum_{i=1}^{k} \Phi_{\Lambda_i}} \, d\nu^{(k)}} \,.$$

Lemma 5.4.2 further implies that for any $k \geq 1$,

$$\frac{d\nu^{(k)}}{d\nu^{(k-1)}} = \frac{e^{-\Phi_{\Lambda_k}}}{\int e^{-\Phi_{\Lambda_k}} \, d\nu^{(k-1)}}. \tag{5.21}$$

Lemma 5.4.3. [5, Theorem D] *Assume that Φ satisfies the Dobrushin uniqueness condition (5.8). Suppose the family $\{f^{(k)}\}_{k\in\mathbb{N}}$ is uniformly integrable in $L^1(\nu^{(0)})$. Then the weak* limit point of the sequence $\{\nu^{(k)}\}$ is a Gibbs measure for Φ and absolutely continuous with respect to $\nu^{(0)}$.*

Proofs of both Lemma 5.4.2 and Lemma 5.4.3 can be found in [5].

We end this section by noting that if $h \geq 0$, then the Griffiths–Kelly–Sherman (GKS) inequality applies to the Gibbs measures $\nu^{(k)}$ and μ [10–12, 16]. Namely, for every $\tau \in \{\mu, \nu^{(k)}, \ k \geq 0\}$ and all $A, B \Subset \mathbb{Z}$, one has

$$\int_\Omega \sigma_A \, d\tau \geq 0 \tag{5.22}$$

and

$$\int_\Omega \sigma_A \sigma_B \, d\tau \geq \int_\Omega \sigma_A \, d\tau \cdot \int_\Omega \sigma_B \, d\tau, \tag{5.23}$$

where $\sigma_A := \prod_{i\in A} \sigma_i$ and for $i \in \mathbb{Z}$, $\omega \in X \mapsto \sigma_i(\omega) = \omega_i$. The GKS inequalities also imply the following inequality for $\tau \in \{\mu, \nu^{(k)}, k \geq 0\}$: for every $A, B \Subset \mathbb{Z}$ and $t \geq 0$,

$$\int_\Omega \sigma_B e^{t\sigma_A} d\tau \geq \int_\Omega \sigma_B d\tau \cdot \int_\Omega e^{t\sigma_A} d\tau. \tag{5.24}$$

5.5 Proof of Theorem 5.A

Part (i): Firstly, note that the uniqueness of Gibbs measures for Φ and $\Psi^{(0)}$ easily follows from Theorem 5.3.3. Thus by (5.19), the restriction Φ^+ of Φ to the half-line $\mathbb{Z}_+$ also has a unique Gibbs measure ν^+. Furthermore, Theorem 4.8 in [3] implies that the transfer operator $\mathscr{L}_\phi$ has a unique eigenprobability ν, and $\nu = \nu^+$ (see also Subsection 2.2 in [5]).

Part (ii): We start the proof of the second part by noting that the positive and negative fields are related by the global-spin flip transformation $\mathscr{S} : X_+ \to X_+$ defined by $\mathscr{S}(x)_n = -x_n$ for all $n \in \mathbb{Z}_+$. In fact, for the Dyson potential ϕ given by (5.3), one has that

$$\phi \circ \mathscr{S} = -h\sigma_0 + \beta \sum_{n=1}^{\infty} \frac{\sigma_0 \sigma_n}{n^\alpha}. \tag{5.25}$$

For any continuous potential $\psi \in C(X_+)$ one can easily check that

$$\left(\mathscr{L}_{\psi\circ\mathscr{S}}\, \mathfrak{f}\right) \circ \mathscr{S} = \mathscr{L}_\psi(\mathfrak{f} \circ \mathscr{S}), \;\; \mathfrak{f} \in \mathbb{R}^{X_+}. \tag{5.26}$$

Thus, the topological pressures of ψ and $\psi \circ \mathscr{S}$ coincide, i.e., $p(\psi) = p(\psi \circ \mathscr{S})$ and $\nu \in \mathscr{M}_1(X_+)$ is an eigenprobability of $\mathscr{L}^*_\psi$ if and only if $\nu \circ \mathscr{S}$ is an eigenprobability of $\mathscr{L}^*_{\psi\circ\mathscr{S}}$. Furthermore, $\mathfrak{h}$ is an eigenfunction of $\mathscr{L}_\psi$ corresponding to $\lambda = e^{p(\psi)}$ if and only if $\mathfrak{h} \circ \mathscr{S}$ is an eigenfunction of $\mathscr{L}_{\psi\circ\mathscr{S}}$ corresponding to λ.

By the above argument, we may assume without loss of generality that the external field h is positive for the remainder of the proof.

We below show that the unique Gibbs measure μ for Φ is absolutely continuous with respect to $\nu^{(0)}$. Then this yields that the restriction μ_+ of μ to $\mathbb{Z}_+$ is absolutely continuous with respect to ν and the Radon-Nikdoym density $f_+ := \frac{d\mu_+}{d\nu}$ is given by

$$f_+(\omega_0^\infty) = \int_{X_-} f(\xi_{-\infty}^{-1}\omega_0^\infty)\, \nu_-(d\xi), \tag{5.27}$$

where $f = \frac{d\mu}{d\nu^{(0)}}$. Then Proposition 3.1 in [5] also yields that the Radon-Nikodym density $\frac{d\mu_+}{d\nu}$ is an eigenfunction of the transfer operator $\mathscr{L}_\phi$ corresponding to $\lambda = e^{P(\phi)}$.

Note that for the Dyson interaction Φ, one has, for all $i \in \mathbb{N}$, $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$,

$$\delta_k(\Phi_{\{-i,j\}}) = \begin{cases} 0, & k \notin \{-i, j\}; \\ \dfrac{2\beta}{(i+j)^\alpha}, & k \in \{-i, j\}. \end{cases}$$

Thus for all $i \in \mathbb{N}$, $j \in \mathbb{Z}_+$, $\|\underline{\delta}(\Phi_{\{-i,j\}})\|_2^2 = \dfrac{8\beta^2}{(i+j)^{2\alpha}}$. Therefore, $\sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} \|\underline{\delta}(\Phi_{\{-i,j\}})\|_2^2 = \sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} \dfrac{8\beta^2}{(i+j)^{2\alpha}} < \infty.$

Consider any $\Lambda_k \in \mathscr{A}$ and $n \in \mathbb{N}$. It follows from (5.21) that

$$\int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n)} = \frac{-J_{\Lambda_k} \int_\Omega \sigma_{\Lambda_k} e^{J_{\Lambda_n}\sigma_{\Lambda_n}} d\,\nu^{(n-1)}}{\int_\Omega e^{J_{\Lambda_n}\sigma_{\Lambda_n}} d\,\nu^{(n-1)}},$$

here for $\Lambda \Subset \mathbb{Z}_+$, $J_\Lambda := h$ if $\Lambda = \{i\}$ and $J_\Lambda := \dfrac{\beta}{|i-j|^\alpha}$ if $\Lambda = \{i, j\}$. Then by (5.24),

$$\int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n)} \le \frac{\int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n-1)} \cdot \int_\Omega e^{J_{\Lambda_n}\sigma_{\Lambda_n}} d\,\nu^{(n-1)}}{\int_\Omega e^{J_{\Lambda_n}\sigma_{\Lambda_n}} d\,\nu^{(n-1)}} = \int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n-1)}. \tag{5.28}$$

Note that $\nu^{(n)}$ converges to μ as $n \to \infty$ in the weak star topology. Thus one can obtain from (5.28) that for all $\Lambda_k \in \mathscr{A}$, $n \in \mathbb{N}$,

$$\int_\Omega \Phi_{\Lambda_k} d\,\nu^{(0)} \ge \int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n)} \ge \lim_{n\to\infty} \int_\Omega \Phi_{\Lambda_k} d\,\nu^{(n)} = \int_\Omega \Phi_{\Lambda_k} d\mu. \tag{5.29}$$

For all $k \in \mathbb{N}$, denote $W_k := \sum_{i=1}^{k} \Phi_{\Lambda_i}$. Then we aim to prove

$$\sup_{k\ge 0} \int_X f^{(k)} \log f^{(k)} d\,\nu^{(0)} < \infty. \tag{5.30}$$

Hence by applying de la Vallée Poussin's theorem to the family $\{f^{(k)} : k \in \mathbb{N}\}$ and to the function $t \in (0, +\infty) \mapsto t \log t$, one concludes that the family $\{f^{(k)} : k \in \mathbb{N}\}$ is uniformly integrable in $L^1(\nu^{(0)})$. Then Theorem 5.4.3 imply that the unique limit point μ is absolutely continuous with respect to $\nu^{(0)}$.

Note that for every $k \in \mathbb{N}$,

$$\int_X f^{(k)} \log f^{(k)} d\,\nu^{(0)} = \int_X -W_k\, d\,\nu^{(k)} - \log \int_X e^{-W_k} d\,\nu^{(0)}. \tag{5.31}$$

Here one can easily check that

$$\sup_{k\in\mathbb{N}}\int_X e^{-W_k}\,d\,\nu^{(0)} = +\infty.$$

Under conditions of our main theorem, we argue that

$$\sup_{k\in\mathbb{N}}\int_X -W_k\,d\,\nu^{(k)} = +\infty.$$

Therefore, the method of the proof of Theorem E in [5] does not work here. We instead prove the following two claims:

Claim 1:

$$\sup_{k\in\mathbb{N}}\left|\int_X W_k\,d\,\nu^{(k)} - \int_X W_k\,d\mu\right| < \infty;$$

Claim 2:

$$\sup_{k\in\mathbb{N}}\left|-\log\int_X e^{-W_k}\,d\,\nu^{(0)} - \int_X W_k\,d\mu\right| < \infty.$$

One can easily see from (5.31) that Claim 1 and Claim 2 indeed imply (5.30).

Note that for all $k\in\mathbb{N}$, $\bar{c}(\Psi^{(0)}) \le \bar{c}(\Psi^{(k)}) \le \bar{c}(\Phi)$. Therefore, the Dobrushin uniqueness condition $\bar{c}(\Phi) < 1$ is inherited by all the intermediate interactions. Applying the first part of Theorem 4.4.7 we see that the (only) measure $\mu\in\mathscr{G}(\Phi)$ and all the intermediate measures $\nu^{(k)}$, $k\ge 0$, satisfy the Gaussian Concentration Bound with the same constant $D := \dfrac{4}{(1-\bar{c}(\Phi))^2}$. This implies that, for all $k\in\mathbb{N}$,

$$\int_X e^{-\Phi_{\Lambda_k}}\,d\,\nu^{(k-1)} \le e^{D\|\underline{\delta}(\Phi_{\Lambda_k})\|_2^2}\,e^{-\int_X \Phi_{\Lambda_k}\,d\,\nu^{(k-1)}}. \tag{5.32}$$

We combine this inequality with (5.21) to iterate

$$\begin{aligned}
&\int_X e^{-(\Phi_{\Lambda_k}+\Phi_{\Lambda_{k-1}})}\,d\,\nu^{(k-2))}\\
&= \int_X e^{-\Phi_{\Lambda_k}}\,d\,\nu^{(k-1))}\int_X e^{-\Phi_{\Lambda_{k-1}}}\,d\,\nu^{(k-2))}\\
&\le e^{D(\|\underline{\delta}(\Phi_{\Lambda_k})\|_2^2+\|\underline{\delta}(\Phi_{\Lambda_{k-1}})\|_2^2)}\cdot e^{-(\int_X \Phi_{\Lambda_k}\,d\,\nu^{(k-1)}+\int_X \Phi_{\Lambda_{k-1}}\,d\,\nu^{(k-2)})}.
\end{aligned} \tag{5.33}$$

By induction this yields

$$\int_X e^{-\sum_{i=1}^k \Phi_{\Lambda_i}}\,d\,\nu^{(0)} \le e^{D\sum_{i=1}^k \|\underline{\delta}(\Phi_{\Lambda_i})\|_2^2}\cdot e^{-\sum_{i=1}^k\int_X \Phi_{\Lambda_i}\,d\,\nu^{(i-1)}}. \tag{5.34}$$

Similarly, one can also obtain the lower bound:

$$\int_X e^{-\sum_{i=1}^k \Phi_{\Lambda_i}} d\nu^{(0)} \geq e^{-D\sum_{i=1}^k ||\underline{\delta}(\Phi_{\Lambda_i})||_2^2} \cdot e^{-\sum_{i=1}^k \int_X \Phi_{\Lambda_i} d\nu^{(i-1)}}. \tag{5.35}$$

Hence for all $k \in \mathbb{N}$,

$$-C_1 - \sum_{i=1}^k \int_X \Phi_{\Lambda_i} d\nu^{(i-1)} \leq \log \int_X e^{-\sum_{i=1}^k \Phi_{\Lambda_i}} d\nu^{(0)} \leq C_1 - \sum_{i=1}^k \int_X \Phi_{\Lambda_i} d\nu^{(i-1)}, \tag{5.36}$$

where $C_1 := D \cdot \sum_{i=1}^{\infty} ||\underline{\delta}(\Phi_{\Lambda_i})||_2^2 = \sum_{\substack{i\in\mathbb{N},\\ j\in\mathbb{Z}_+}} \frac{8\beta^2}{(i+j)^{2\alpha}} < \infty$. Thus instead of Claim 2, it suffices to show the following:

$$\sup_{k\in\mathbb{N}} \left| \sum_{i=1}^k \int_X \Phi_{\Lambda_i} d\nu^{(i-1)} - \int_X W_k d\mu \right| < \infty. \tag{5.37}$$

Thus, in the light of (5.29), Claim 1 and Claim 2 are implied by the following:

Claim 3:

$$\sup_{k\in\mathbb{N}} \left| \int_X W_k d\nu^{(0)} - \int_X W_k d\mu \right| < \infty. \tag{5.38}$$

By the GKS inequality (c.f., 5.29), we, in fact, have that $\int_X \Phi_{\Lambda_i} d\nu^{(0)} - \int_X \Phi_{\Lambda_i} d\mu \geq 0$, hence the sequence $\{\int_X W_k d\nu^{(0)} - \int_X W_k d\mu\}_k$ increases with k. Thus instead of considering all W_k's, it is enough to consider the subsequence $\{W_{[N]}\}_{N\in\mathbb{N}}$s, where

$$W_{[N]}(\omega) := \sum_{i=1}^N \sum_{j=0}^N \Phi_{\{-i,j\}}(\omega) = \sum_{i=1}^N \sum_{j=0}^N -\frac{\beta\omega_{-i}\omega_j}{(i+j)^\alpha}.$$

Then

$$\begin{aligned}
\int_X W_{[N]} d\nu^{(0)} - \int_X W_{[N]} d\mu &= \sum_{i=1}^{N}\sum_{j=0}^{N}\int_X -\frac{\beta\omega_{-i}\omega_j}{(i+j)^\alpha}\nu^{(0)}(d\omega) \\
&- \sum_{i=1}^{N}\sum_{j=0}^{N}\int_X -\frac{\beta\omega_{-i}\omega_j}{(i+j)^\alpha}\mu(d\omega) \\
&= \sum_{i=1}^{N}\sum_{j=0}^{N}\frac{-\beta\nu_-(\sigma_{-i})\cdot\nu_+(\sigma_j)}{(i+j)^\alpha} \\
&- \sum_{i=1}^{N}\sum_{j=0}^{N}\frac{-\beta\,\mathrm{cov}_\mu(\sigma_{-i},\sigma_j)-\beta\mu(\sigma_0)^2}{(i+j)^\alpha} \\
&= \sum_{i=1}^{N}\sum_{j=0}^{N}\frac{\beta\,\mathrm{cov}_\mu(\sigma_{-i},\sigma_j)}{(i+j)^\alpha} \qquad (5.39)\\
&+ \beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{\mu(\sigma_0)^2-\nu_-(\sigma_{-i})\cdot\nu_+(\sigma_j)}{(i+j)^\alpha}, \qquad (5.40)
\end{aligned}$$

here we used the fact that $\nu^0 = \nu_- \times \nu_+$. By (5.14) and (5.17), we know that in the Dobrushin uniqueness region (5.39) is bounded, i.e.,

$$\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{\beta|\mathrm{cov}_\mu(\sigma_{-i},\sigma_j)|}{(i+j)^\alpha} < \infty.$$

Therefore, to show

$$\sup_{N\in\mathbb{N}}\Big(\int_X W_{[N]} d\nu^{(0)} - \int_X W_{[N]} d\mu\Big) < \infty \qquad (5.41)$$

it is enough to show that

$$\beta\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{|\mu(\sigma_0)^2-\nu_-(\sigma_{-i})\cdot\nu_+(\sigma_j)|}{(i+j)^\alpha} < \infty. \qquad (5.42)$$

By Theorem 4.A in Chapter 4, we have that $\nu_+(\sigma_j)\xrightarrow[j\to\infty]{}\mu(\sigma_0)$, similarly, one can also argue that $\nu_-(\sigma_{-i})\xrightarrow[i\to\infty]{}\mu(\sigma_0)$. For $n\in\mathbb{Z}$, denote

$$t_n := |\nu^{(0)}(\sigma_n) - \mu(\sigma_0|. \qquad (5.43)$$

Then (5.40) can be rewritten in the following form

$$\beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{|\mu(\sigma_0)^2-\nu_-(\sigma_{-i})\cdot\nu_+(\sigma_j)|}{(i+j)^\alpha} = \beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{|\mu(\sigma_0)t_j+\mu(\sigma_0)t_{-i}-t_{-i}t_j|}{(i+j)^\alpha}. \qquad (5.44)$$

Now we show that there exists a constant $C_2 = C_2(\alpha,\beta,h) > 0$ such that for every $n \in \mathbb{Z}$,

$$t_n \le \frac{C_2}{(|n|+1)^{\alpha-1}}. \tag{5.45}$$

In fact, we prove the above inequality for positive n's and the case of negative n's can be treated similarly. Fix $j \in \mathbb{N}$. Then by Theorem 5.3.5, one has that

$$t_j = \left| \int_X \sigma_j \, d\mu - \int_X \sigma_j \, d\nu^{(0)} \right| \le 2(\bar{D}b)_j, \tag{5.46}$$

where $\bar{D} = \bar{D}(\Phi)$ which is given by (5.15) and $b = (b_s)_{s\in\mathbb{Z}}$ is given by the following:

$$b_s := \sup_{\omega\in X} \|\gamma^{\Phi}_{\{s\}}(\cdot|\omega_{\{s\}^c}) - \gamma^{\Psi^{(0)}}_{\{s\}}(\cdot|\omega_{\{s\}^c})\|_\infty. \tag{5.47}$$

For any $\xi, \omega \in X$, one has that

$$\gamma^{\Phi}_{\{s\}}(\xi_s|\omega_{\{s\}^c}) - \gamma^{\Psi^{(0)}}_{\{s\}}(\xi_s|\omega_{\{s\}^c})$$
$$= \frac{e^{-H^{\Phi}_s(\xi_s\omega_{\{s\}^c}) - H^{\Psi^{(0)}}_s(\xi_s\omega_{\{s\}^c})} \sum_{\eta_s} \left[e^{H^{\Psi^{(0)}}_s(\xi_s\omega_{\{s\}^c}) - H^{\Psi^{(0)}}_s(\eta_s\omega_{\{s\}^c})} - e^{H^{\Phi}_s(\xi_s\omega_{\{s\}^c}) - H^{\Phi}_s(\eta_s\omega_{\{s\}^c})} \right]}{\left(\sum_{\eta_s} e^{-H^{\Phi}_s(\eta_s\omega_{\{s\}^c})} \right) \cdot \left(\sum_{\eta_s} e^{-H^{\Phi^{(0)}}_s(\eta_s\omega_{\{s\}^c})} \right)}.$$

Thus since

$$\sup_{\xi,\eta,\omega} \left| H^{\Phi}_s(\xi_s\omega_{\{s\}^c}) - H^{\Phi}_s(\eta_s\omega_{\{s\}^c}) - \left(H^{\Psi^{(0)}}_s(\xi_s\omega_{\{s\}^c}) - H^{\Psi^{(0)}}_s(\eta_s\omega_{\{s\}^c}) \right) \right| = O\Big(\frac{1}{(|s|+1)^{\alpha-1}} \Big)$$

$$\sup_{\omega\in X} \|\gamma^{\Phi}_{\{s\}}(\cdot|\omega_{\{s\}^c}) - \gamma^{\Psi^{(0)}}_{\{s\}}(\cdot|\omega_{\{s\}^c})\|_\infty = O\Big(\frac{1}{(|s|+1)^{\alpha-1}} \Big). \tag{5.48}$$

Hence there exists $C_3 > 0$ such that for all $s \in \mathbb{Z}$,

$$b_s \le \frac{C_3}{(|s|+1)^{\alpha-1}}. \tag{5.49}$$

Then (5.46) yields that

$$t_j \le 2C_3 \sum_{s\in\mathbb{Z}} \frac{\bar{D}_{j,s}}{(|s|+1)^{\alpha-1}}. \tag{5.50}$$

Hence by using (5.17), we obtain that

$$t_j \le 2C_3 C_J \sum_{s\in\mathbb{Z}} \frac{1}{(1+|s-j|)^{\alpha}(|s|+1)^{\alpha-1}}. \tag{5.51}$$

The right-hand side of the above inequality can be written as

$$
\begin{aligned}
\sum_{s\in\mathbb{Z}} \frac{1}{(1+|s-j|)^{\alpha}(|s|+1)^{\alpha-1}} &= \sum_{s=1}^{\infty} \frac{1}{(1+s+j)^{\alpha}(1+s)^{\alpha-1}} + \frac{1}{(1+j)^{\alpha}} \\
&+ \sum_{s=1}^{j-1} \frac{1}{(1+j-s)^{\alpha}(s+1)^{\alpha-1}} + \frac{1}{(1+j)^{\alpha-1}} \\
&+ \sum_{s=j+1}^{\infty} \frac{1}{(1+s-j)^{\alpha}(s+1)^{\alpha-1}} \\
&\leq \sum_{s=1}^{\infty} \frac{1}{(s+j)^{\alpha}} + \frac{1}{j^{\alpha}} + \sum_{s=1}^{\lfloor j/2 \rfloor} \frac{1}{(j-s)^{\alpha-1} s^{\alpha}} \qquad (5.52) \\
&+ \sum_{s=\lceil j/2 \rceil}^{j-1} \frac{1}{(j-s)^{\alpha-1} s^{\alpha}} + \frac{1}{j^{\alpha-1}} + \frac{\zeta(\alpha)}{j^{\alpha-1}},
\end{aligned}
$$

here $\lfloor t \rfloor$ and $\lceil t \rceil$ denote respectively the floor and ceiling of $t \in \mathbb{R}$. Note that

$$
\sum_{s=1}^{\lfloor j/2 \rfloor} \frac{1}{(j-s)^{\alpha-1} s^{\alpha}} + \sum_{s=\lceil j/2 \rceil}^{j-1} \frac{1}{(j-s)^{\alpha-1} s^{\alpha}} \leq \frac{2^{\alpha-1}}{j^{\alpha-1}} \sum_{s=1}^{\infty} \frac{1}{s^{\alpha}} + \frac{2^{\alpha}}{j^{\alpha}} \sum_{s=1}^{j} \frac{1}{s^{\alpha-1}}
$$

and by the Stolz-Cesaro theorem

$$
\sum_{s=1}^{j} \frac{1}{s^{\alpha-1}} = o(j^{-1}),
$$

consequently,

$$
\sum_{s=1}^{j-1} \frac{1}{(1+j-s)^{\alpha}(s+1)^{\alpha-1}} = O(j^{1-\alpha}). \tag{5.53}
$$

By combining (5.52) and (5.53), one obtains that

$$
\sum_{s\in\mathbb{Z}} \frac{1}{(1+|s-j|)^{\alpha}(|s|+1)^{\alpha-1}} = O(j^{1-\alpha}). \tag{5.54}
$$

Hence, in light of (5.51), we conclude that (5.45) holds in the case of $n > 0$.

Using (5.45), we can now estimate (5.44) from the above. In fact, one has

$$
\begin{aligned}
&\sum_{i=1}^{N} \sum_{j=0}^{N} \frac{|\mu(\sigma_0) t_j + \mu(\sigma_0) t_{-i} - t_{-i} t_j|}{(i+j)^{\alpha}} \\
&\qquad \leq \sum_{i=1}^{N} \sum_{j=0}^{N} \frac{|\mu(\sigma_0)| t_j}{(i+j)^{\alpha}} + \sum_{i=1}^{N} \sum_{j=0}^{N} \frac{|\mu(\sigma_0)| t_{-i}}{(i+j)^{\alpha}} + \sum_{i=1}^{N} \sum_{j=0}^{N} \frac{t_{-i} t_j}{(i+j)^{\alpha}}.
\end{aligned}
$$

Now we only show that the first term remains bounded as $N \to \infty$, and the boundedness of the other terms can be shown similarly. In fact, there exists $C_4 > 0$ such that for all N,

$$\sum_{j=0}^{N}\sum_{i=1}^{N}\frac{t_j}{(i+j)^\alpha} \le \sum_{j=0}^{N}\sum_{i=1}^{\infty}\frac{t_j}{(i+j)^\alpha} \le C_4\sum_{j=1}^{N}\frac{t_j}{j^{\alpha-1}}.$$

Thus by (5.45) and by taking into account the fact that $\alpha > \frac{3}{2}$, one immediately concludes that for all $N \in \mathbb{N}$,

$$\sum_{j=0}^{N}\sum_{i=1}^{N}\frac{t_j}{(i+j)^\alpha} \le C_2C_4\sum_{j=1}^{N}\frac{1}{j^{2\alpha-2}} \le C_2C_4\sum_{j=1}^{\infty}\frac{1}{j^{2\alpha-2}} < \infty. \tag{5.55}$$

Part (iii): By the first part, we can also conclude that the entire sequence $\{f^{(k)}\}_{k\in\mathbb{N}}$ converges to $f = \frac{d\mu}{d\nu^{(0)}}$ as $k \to \infty$ in the weak topology in $L^1(\nu^{(0)})$. This follows from the fact that a limit point of the sequence $\{f^{(k)}\}_{k\in\mathbb{N}}$ should be a Radon-Nikodym density of some $\mu^* \in \mathscr{G}(\Phi)$, but $\mathscr{G}(\Phi) = \{\mu\}$, therefore, $\frac{d\mu}{d\nu^{(0)}}$ is the only limit point. Thus

$$f = \lim_{k\to\infty}\frac{e^{-W_k}}{\int_X e^{-W_k}d\nu^{(0)}}, \tag{5.56}$$

here the limit is understood in the weak topology in $L^1(\nu^{(0)})$.

Note that the weak convergence of $\{f^{(k)}\}$ to f in the weak topology in $L^1(\nu^{(0)})$ also implies the weak convergence of $f_+^{(k)} := \int_{X_-} f^{(k)}d\nu_-$ to f_+ in the weak topology in $L^1(\nu)$. In fact, for any bounded $g \in \mathscr{F}_{\mathbb{Z}_+}$, by taking into account the fact that $\nu^{(0)} = \nu_- \times \nu$, one has

$$\begin{aligned}\int_{X_+}(f_+^{(k)} - f_+)g\,d\nu &= \int_{X_+}\Big[\int_{X_-}(f^{(k)} - f)d\nu_-\Big]g\,d\nu \\ &= \int_{X_+}\int_{X_-} g(f^{(k)} - f)d\nu_-d\nu \\ &= \int_X g(f^{(k)} - f)d\nu^{(0)} \xrightarrow[k\to\infty]{} 0.\end{aligned}$$

Hence we obtain an analog of (5.56) for f_+, i.e.,

$$f_+ = \lim_{k\to\infty}\frac{\int_{X_-} e^{-W_k}d\nu_-}{\int_X e^{-W_k}d\nu^{(0)}}. \tag{5.57}$$

Here the above limit should be again understood in an appropriate topology.

Now assume the transfer operator $\mathscr{L}_\phi$ has a continuous eigenfunction $\tilde{f}$, or in other words, the Radon-Nikodym derivative $\frac{d\mu_+}{d\nu}$ has a continuous version $\tilde{f}$. Then for ν–a.e. $x \in X_+$, $\tilde{f}(x) = f_+(x)$, and $\tilde{f}$, as a function on compact space X_+, is bounded. Hence $f_+ \in L^\infty(\nu)$, therefore,

$$\sup\left\{\frac{1}{\nu([y_\Lambda])}\int_{[y_\Lambda]} f_+\, d\nu : y \in X_+,\ \Lambda \Subset \mathbb{Z}_+\right\} \le \|f_+\|_{L^\infty(\nu)} < \infty. \tag{5.58}$$

Hence to prove the second part of Theorem 5.A, it is enough to show that the supremum in (5.58) is infinite. In fact, below we show that

$$\sup_{n\in\mathbb{N}} \frac{1}{\nu([1_0^n])}\int_{[1_0^n]} f_+ d\nu = +\infty. \tag{5.59}$$

In other words, we show that $1_{\mathbb{Z}_+} = (+1_i)_{i\in\mathbb{Z}_+} \in X_+$ is an *essential discontinuity point* of the Radon-Nikdoym density $f_+ = \frac{d\mu_+}{dd\nu}$. To do so, fix $n \in \mathbb{N}$ and consider $g = \mathbb{1}_{[1_0^n]}$. By (5.57), one has that

$$\int_{X_+} f_+ \mathbb{1}_{[1_0^n]} d\nu = \lim_{k\to\infty} \int_{[1_0^n]} \frac{\int_{X_-} e^{-W_k(\xi,\eta)}\nu_-(d\xi)}{\int_{X_+}\int_{X_-} e^{-W_k(\xi,\bar{\eta})}\nu_-(d\xi)\nu(d\bar{\eta})}\nu(d\eta). \tag{5.60}$$

Thus, since $\{W_{[N]}\}_{N\in\mathbb{N}}$ (for the definition, see the proof of the first part) is a subsequence of $\{W_k\}_{k\in\mathbb{N}}$,

$$\int_{X_+} f_+ \mathbb{1}_{[1_0^n]} d\nu = \lim_{N\to\infty} \int_{[1_0^n]} \frac{\int_{X_-} e^{-W_{[N]}(\xi,\eta)}\nu_-(d\xi)}{\int_{X_+}\int_{X_-} e^{-W_{[N]}(\xi,\bar{\eta})}\nu_-(d\xi)\nu(d\bar{\eta})}\nu(d\eta). \tag{5.61}$$

Now fix $N \in \mathbb{N}$ and $\eta \in X_+$, consider $W_{[N]}$ as a function of ξ. Clearly, it is a local function, thus by the first part of Theorem 5.3.4, for all $\kappa \in \mathbb{R}$,

$$\int_{X_-} e^{\kappa[W_{[N]}(\xi,\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)]}\nu_-(d\xi) \le e^{D\kappa^2 \|\underline{\delta}(W_{[N]}(\cdot,\eta))\|_2^2}, \tag{5.62}$$

where $D = 4(1-\bar{c}(\Phi))^{-2}$. By the Cauchy-Schwarz inequality, one can also obtain a lower bound for the integral, in fact,

$$e^{-D\kappa^2\|\underline{\delta}(W_{[N]}(\cdot,\eta))\|_2^2} \le \int_{X_-} e^{\kappa[W_{[N]}(\xi,\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)]}\nu_-(d\xi). \tag{5.63}$$

For all $s \in \mathbb{N}$,

$$\delta_{-s}(W_{[N]}(\cdot,\eta)) = 2\beta \left| \sum_{j=0}^{N} \frac{\eta_j}{(s+j)^\alpha} \right| \le 2\beta \sum_{j=0}^{N} \frac{1}{(s+j)^\alpha}.$$

Hence for $\alpha > \frac{3}{2}$,

$$||\underline{\delta}(W_{[N]}(\cdot,\eta))||_2^2 \le 4\beta^2 \sum_{s=1}^{\infty} \Big(\sum_{j=s}^{\infty} \frac{1}{j^\alpha} \Big)^2 =: 4\beta^2 C_1(\alpha) < \infty.$$

Hence one obtains from (5.62) and (5.63) that

$$C_5^{-1} \cdot e^{-\int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)} \le \int_{X_-} e^{-W_{[N]}(\xi,\eta)} \nu_-(d\xi) \le C_5 \cdot e^{-\int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)}, \quad (5.64)$$

where $C_5 := e^{4D\beta^2 C_1(\alpha)}$.

By applying the Gaussian Concentration Bounds to $W_{[N]}$ as a function of both ξ and η, one can obtain an analogue of (5.64) for $\nu^{(0)}$. In fact, by using the Cauchy-Schwarz inequality, one can easily check that

$$e^{-D||\underline{\delta}(W_{[N]})||_2^2} \cdot e^{\int_X -W_{[N]} d\nu^{(0)}} \le \int_X e^{-W_{[N]}} d\nu^{(0)} \le e^{D||\underline{\delta}(W_{[N]})||_2^2} \cdot e^{\int_X -W_{[N]} d\nu^{(0)}}. \quad (5.65)$$

For any $s \in \mathbb{Z}$,

$$\delta_s(W_{[N]}) \le \frac{8\beta}{(|s|+1)^\alpha} + 2\beta \sum_{i=|s|+1}^{\infty} \frac{1}{i^\alpha} \le \frac{\beta C_6}{(|s|+1)^{\alpha-1}}, \quad (5.66)$$

where $C_6 > 0$ is only dependent on α. Hence, since $\alpha > 3/2$,

$$||\underline{\delta}(W_{[N]})||_2^2 = \sum_{s\in\mathbb{Z}} (\delta_s(W_{[N]}))^2 \le (\beta C_6)^2 \sum_{s\in\mathbb{Z}} \frac{1}{(|s|+1)^{2\alpha-2}} =: C_7(\alpha,\beta) < \infty. \quad (5.67)$$

Then one obtains from (5.65) that

$$C_8^{-1} \cdot e^{\int_X -W_{[N]} d\nu^{(0)}} \le \int_X e^{-W_{[N]}} d\nu^{(0)} \le C_8 \cdot e^{\int_X -W_{[N]} d\nu^{(0)}}, \quad (5.68)$$

here $C_8 = C_8(\alpha,\beta,h) := D \cdot C_7(\alpha,\beta)$. By combining (5.61), (5.64) and (5.68), one concludes that for any $N \in \mathbb{N}$ and $\eta \in X_+$,

$$C_9^{-1} \cdot e^{\int_X W_{[N]} d\nu^{(0)} - \int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)} \le \frac{\int_{X_-} e^{-W_{[N]}(\xi,\eta)} \nu_-(d\xi)}{\int_X e^{-W_{[N]}} d\nu^{(0)}} \quad (5.69)$$

$$\le C_9 \cdot e^{\int_X W_{[N]} d\nu^{(0)} - \int_{X_-} W_{[N]}(\xi,\eta)\nu_-(d\xi)}, \quad (5.70)$$

here $C_9 = C_9(\alpha,\beta,h) := \max\{C_5, C_8\}$. Hence, by Jensen's inequality,

$$\frac{1}{\nu([1_0^n])}\int\limits_{[1_0^n]} \frac{\int\limits_{X_-} e^{-W_{[N]}(\xi,\eta)}\,\nu_-(d\xi)}{\int\limits_X e^{-W_{[N]}}\,d\,\nu^{(0)}}\,\nu(d\eta) \geq \frac{C_9^{-1}}{\nu([1_0^n])}\cdot\int\limits_{[1_0^n]} e^{\int\limits_X W_{[N]}\,d\,\nu^{(0)} - \int\limits_{X_-} W_{[N]}(\xi,\eta)\,\nu_-(d\xi)}\,\nu(d\eta)$$

$$\geq C_9^{-1}\cdot\exp\left(\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}\left[\int_X W_{[N]}(\xi,\bar\eta)\,\nu^{(0)}(d\xi,d\bar\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\,\nu_-(d\xi)\right]\nu(d\eta)\right) \tag{5.71}$$

Note that for any $\eta\in X_+$, one has

$$\int_X W_{[N]}(\xi,\bar\eta)\,\nu^{(0)}(d\xi,d\bar\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\,\nu_-(d\xi) = \beta\sum_{i=1}^N\sum_{j=0}^N \frac{\nu_-(\sigma_{-i})[\eta_j - \nu(\sigma_j)]}{(i+j)^\alpha} \tag{5.72}$$

Hence

$$\begin{aligned}\int_{[1_0^n]}\Big[\int_X W_{[N]}(\xi,\bar\eta)\,\nu^{(0)}(d\xi,d\bar\eta) - \int_{X_-} W_{[N]}(\xi,\eta)\,\nu_-(d\xi)\Big]\nu(d\eta)&\\ = \beta\int_{[1_0^n]}\sum_{i=1}^N\sum_{j=0}^N \frac{\nu_-(\sigma_{-i})[\eta_j-\nu(\sigma_j)]}{(i+j)^\alpha}\,\nu(d\eta)&\\ = \beta\sum_{i=1}^N\sum_{j=0}^N\frac{\nu_-(\sigma_{-i})}{(i+j)^\alpha}\int_{[1_0^n]}[\sigma_j-\nu(\sigma_j)]\,d\,\nu.&\end{aligned} \tag{5.73}$$

Then since both σ_j and $\mathbb{1}_{[1_0^n]}$ are non-decreasing functions and ν is a positively correlated measure, by the FKG inequality [7], for any $j\in\mathbb{Z}_+$,

$$\int_{[1_0^n]}[\sigma_j - \nu(\sigma_j)]\,d\,\nu = \int_{X_+}\sigma_j\mathbb{1}_{[1_0^n]} - \int_{X_+}\sigma_j\,d\,\nu\int_{X_+}\mathbb{1}_{[1_0^n]}\,d\,\nu \geq 0. \tag{5.74}$$

Furthermore, for $1\leq j\leq n$, one has

$$\int_{[1_0^n]}[\sigma_j-\nu(\sigma_j)]\,d\,\nu = \nu([1_0^n])(1-\nu(\sigma_j)). \tag{5.75}$$

We also note that by (5.22), for all $i\in\mathbb{N}$,

$$\nu_-(\sigma_{-i})\geq 0. \tag{5.76}$$

By combining these arguments,

$$\beta\sum_{i=1}^{N}\sum_{j=0}^{N}\frac{\nu_-(\sigma_{-i})}{(i+j)^\alpha}\int_{[1_0^n]}[\sigma_j-\nu(\sigma_j)]d\nu \geq \beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})}{(i+j)^\alpha}\int_{[1_0^n]}[\sigma_j-\nu(\sigma_j)]d\nu$$
$$=\beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})}{(i+j)^\alpha}\nu([1_0^n])(1-\nu(\sigma_j)). \quad (5.77)$$

Hence by (5.73),

$$\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}\Big[\int_X W_{[N]}(\xi,\bar\eta)\,\nu^{(0)}(d\xi,d\bar\eta)-\int_{X_-}W_{[N]}(\xi,\eta)\,\nu_-(d\xi)\Big]\nu(d\eta)$$
$$\geq \frac{\beta}{\nu([1_0^n])}\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))\,\nu([1_0^n])}{(i+j)^\alpha}$$
$$=\beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))}{(i+j)^\alpha} \quad (5.78)$$

Then (5.71) and (5.78) yields that for any $N\in\mathbb{N}$,

$$\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}\frac{\int_{X_-}e^{-W_{[N]}(\xi,\eta)}\,\nu_-(d\xi)}{\int_X e^{-W_{[N]}}\,d\nu^{(0)}}\,\nu(d\eta)\geq C_9^{-1}\cdot\exp\Big(\beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))}{(i+j)^\alpha}\Big) \quad (5.79)$$

Hence (5.61) implies

$$\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}f_+\,d\nu \geq \lim_{N\to\infty}C_9^{-1}\cdot\exp\Big(\beta\sum_{j=0}^{n}\sum_{i=1}^{N}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))}{(i+j)^\alpha}\Big)$$
$$= C_9^{-1}\cdot\exp\Big(\beta\sum_{j=0}^{n}\sum_{i=1}^{\infty}\frac{\nu_-(\sigma_{-i})(1-\nu(\sigma_j))}{(i+j)^\alpha}\Big). \quad (5.80)$$

It follows from (5.45) that $\lim_{i\to\infty}\nu_-(\sigma_{-i})=\lim_{j\to\infty}\nu(\sigma_j)=\mu(\sigma_0)>0$. Thus

$$\tilde\kappa:=\sup_{j\in\mathbb{Z}_+}\nu(\sigma_j)<1$$

and there exists $R\in\mathbb{N}$ which depends only on $\nu^{(0)}=\nu_-\times\nu$ and μ (thus R only depends on the model parameters α,β,h) such that $\nu_-(\sigma_{-i})\geq\frac{\mu(\sigma_0)}{2}$ for all $i\geq R$. Hence we obtain from (5.80) that

$$\frac{1}{\nu([1_0^n])}\int_{[1_0^n]}f_+\,d\nu\geq C_9^{-1}\cdot\exp\Big(\frac{\beta(1-\tilde\kappa)\mu(\sigma_0)}{2}\sum_{j=0}^{n}\sum_{i=R}^{\infty}\frac{1}{(i+j)^\alpha}\Big). \quad (5.81)$$

One can readily check that the sum on the right-hand side of (5.81) diverges as $n \to \infty$. In fact, there exists $C_{10} = C_{10}(\alpha) \in (0,1)$ such that $\sum_{i=R}^{\infty} \frac{1}{(i+j)^\alpha} \geq C_{10}(R+j)^{1-\alpha}$, therefore,

$$\sum_{j=0}^{\infty}\sum_{i=R}^{\infty} \frac{1}{(i+j)^\alpha} \geq C_{10} \sum_{j=0}^{\infty} \frac{1}{(R+j)^{\alpha-1}} = \infty.$$

Hence, one indeed concludes that

$$\sup_{n\in\mathbb{N}} \frac{1}{\nu([1_0^n])} \int_{[1_0^n]} f_+ d\nu = +\infty.$$

□

Remark 5.5.1. *The statements of Theorem 5.A remain valid for the Dyson interaction* $\hat{\Phi} = (\hat{\Phi}_\Lambda)_{\Lambda \Subset \mathbb{Z}}$ *with inhomogeneous external fields:*

$$\hat{\Phi}_\Lambda(\omega) := \begin{cases} \dfrac{-\beta\,\omega_i\omega_j}{|i-j|^\alpha}, & \textit{if } \Lambda = \{i,j\},\ i \neq j; \\ h_i\omega_i, & \textit{if } \Lambda = \{i\}; \\ 0, & \textit{otherwise}, \end{cases} \tag{5.82}$$

as long as the fields $h_i \in \mathbb{R}$ *are sufficiently strong. Specifically, if* $\alpha \in \left(\frac{3}{2}, 2\right]$, $\beta \geq 0$, *and*

$$\inf_{i\in\mathbb{Z}} |h_i| \geq 2\beta\zeta(\alpha) + \log(4\beta\zeta(\alpha)),$$

then the interactions $\hat{\Phi}$ *and* $\hat{\Psi}^{(0)}$ *admit unique Gibbs measures* $\mu \in \mathscr{G}(\hat{\Phi})$ *and* $\nu^{(0)} \in \mathscr{G}(\hat{\Psi}^{(0)})$*, and we have* $\mu \ll \nu^{(0)}$*. Moreover, the Radon-Nikodym density of the restriction* $\mu|_{X_+}$ *with respect to* $\nu^{(0)}|_{X_+}$ *does not have a continuous version. However, in this general (inhomogeneous) case, we cannot always associate a potential to* $\hat{\Phi}$ *via the formula* $\sum_{0\in V\Subset\mathbb{Z}_+} \hat{\Phi}_V$*, due to the lack of translation invariance of* $\hat{\Phi}$*. We note that* $\hat{\Phi}$ *is translation-invariant if and only if* $h_i \in \mathbb{R}$ *is constant over* $i \in \mathbb{Z}$*.*

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Part III

Multifractals and Large Deviations in Dynamical Systems

Chapter 6

Multifractal Formalism from Large Deviations

Abstract: It has often been observed that the multifractal formalism and the large deviation principles are intimately related. In fact, multifractal formalism was heuristically derived using the ideas of large deviations. In many cases where multifractal properties have been rigorously established, the corresponding large deviation results hold as well. This naturally raises the question of under what conditions the multifractal formalism can be directly deduced from the corresponding large deviation results.

In this chapter, we take the first steps in establishing a systematic program for deriving multifractal formalism directly from large deviations. We consider a sequence $\{X_n\}_{n\in\mathbb{N}}$ of measurable functions on a metric space Ω, satisfying a large deviation principle, and consider the level sets $K_\alpha = \{\omega : \lim_{n\to\infty} \frac{X_n(\omega)}{n} = \alpha\}$. Under additional technical conditions, typical in multifractal analysis, we estimate the entropy spectrum of K_α directly from the large deviation rate function associated with the sequence $\{X_n\}$, and demonstrate that many known results in multifractal formalism can be immediately recovered within this framework.

This chapter is based on M. Makhmudov, E. Verbitskiy, Q. Xiao, *"Multifractal Formalism from Large Deviations"*, arXiv:2402.15642.

6.1 Introduction

The concept of multifractal formalism (MF) was proposed in the 1980's by Parisi, Frisch, Hentchel [4, 28] in the context of the study of turbulence. It has been suggested that natural local quantities have multiple fluctuation scales. These possible fluctuations can be described using a singluarity spectrum. The multifractal formalism states that the singularity spectrum is dual, in the sense of Legendre transform, to some integral 'free energy'-type function of the system. The core idea of multifractal formalism is based on the assumption that a specific probabilistic Large Deviation Principle (LDP) holds for the system.

The first rigorous mathematical results [3, 11, 35, 36] have been obtained in the late 1980s and early 1990s (Collet, Lebowitz, Riedli, Falconer, Olsen, Pesin), and since the late 1990's there was an explosion of research in this area [9, 18, 20–22, 31–34, 37, 43]. Multifractal formalism has been rigorously established for very large classes of dynamical systems and a plethora of local observables. However, a somewhat curious phenomenon has occurred. Though the original motivation and the first rigorous results relied rather heavily on probabilistic methods of Large Deviations, in subsequent works the link to Large Deviations have become somewhat less pronounced. In fact, we are not aware of a single rigorous multifractal result without an accompanying Large Deviations result. However, it is also clear that more assumptions are required for the validity of multifractal formalism than for the validity of Large Deviations Principles. For example, for multifractal analysis, the phase space must be a metric space, while this is not really a requirement for Large Deviations.

This brings us to the natural question: assuming that the local quantity of interest is a pointwise limit of quantities, whose probabilistic behaviour is governed by Large Deviations, what could be said about the corresponding singularity spectrum and multifractal formalism?

Let us compare two old results, first LDP for Bernoulli random variables, and second, what we would now call a multifractal result for frequencies of digits in binary expansions. However, this result precedes the multifractal idea by nearly 50 years.

Theorem 6.1.1 (LDP for coin-tossing)**.** *Suppose (X_k) are i.i.d. Bernoulli random variables with $\mathbb{P}(X_1=0)=\mathbb{P}(X_1=1)=\frac{1}{2}$. Set $S_n=\sum_{i=1}^{n} X_i$, then $\{S_n/n\}$ satisfies LDP with the rate function $I(p)=\log 2-H(p)$, where $H(p)=-p\log p-(1-p)\log(1-p)$ if $p\in[0,1]$, otherwise $H(p)=-\infty$. Namely,*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\frac{S_n}{n}\in(a,b)\Big)=-\inf_{p\in(a,b)} I(p),\quad \textit{for all } -\infty\le a\le b\le\infty.$$

The above theorem is one of the first Large Deviation results, and it is useful to compare it with possibly the first multifractal result by Besicovitch [5] and Eggleston [15] in the 1930s and the 1940s. This multifractal result was, first, proved by Besicovitch [5] for $N=2$ (binary shift), and later, extended by Eggleston [15] for general N (N is the cardinality of the alphabet).

Theorem 6.1.2 (Besicovitch-Eggleston)**.** *Let* $\Omega=\{0,1\}^{\mathbb{Z}_+}$ *then*

$$\dim_H\left\{\omega\in\Omega: \lim_n \frac{1}{n}\sum_{j=0}^{n-1}\omega_j = p\right\} = \frac{H(p)}{\log 2},\ \forall p\in[0,1],$$

where $\dim_H$ *denotes the Hausdorff dimension.*

It is important to stress that not only rate functions in the above theorems are related, but in fact, the proofs are also very similar.

Next to the Hausdorff dimension, there is another, but in some sense more dynamical set characteristic, known as *topological entropy*, denoted by $h_{\text{top}}(\sigma,\cdot)$, which is used frequently to analyse non-compact non-invariant sets. However, in the setting of symbolic dynamics $(\Omega:=(0,\dots,l-1)^{\mathbb{Z}_+},\sigma)$, one has $h_{\text{top}}(\sigma,Z)=\log l\cdot\dim_H(Z)$ for all $Z\subset\Omega$, in particular, $h_{\text{top}}(\sigma,\Omega)=\log l$, where $\sigma:\Omega\to\Omega$ is the left shift map. Using the topological entropy, the previous results can be summarized as

$$\boxed{h_{\text{top}}(\sigma,K_p)=h_{\text{top}}(\sigma,\Omega)-I(p),\ \forall p\in\mathbb{R}}, \tag{6.1}$$

where $K_p:=\{\omega\in\Omega:\lim_n S_n(\omega)/n=p\}$ and $S_n(\omega):=\sum_{i=0}^{n-1}\omega_i,\ \omega\in\Omega.$

6.1.1 Multifractal analysis of Birkhoff averages of Hölder continuous functions

Most of the first results in the multifractal analysis of dynamical systems can be reduced to the results in the multifractal analysis of Birkhoff averages of certain potentials. For simplicity, suppose $\Omega=\mathscr{A}^{\mathbb{Z}_+}$ is the set of all infinite sequences $\omega=(\omega_0,\omega_1,\dots)$ in a finite alphabet $\mathscr{A}$, equipped with a metric ρ, generating the product topology. Denote by $\sigma:\Omega\to\Omega$ the left shift on Ω. The ***topological pressure*** of a continuous function $\phi:\Omega\to\mathbb{R}$ is defined as

$$P(\phi)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{(a_0,\dots,a_{n-1})\in A^n}\Big(\sup_{\omega\in[a_0^{n-1}]}\exp(S_n\phi(\omega))\Big),\quad S_n\varphi(\omega)=\sum_{k=0}^{n-1}\phi(\sigma^k\omega),$$

where $[a_0^{n-1}]:=\{\omega\in\Omega:\omega_i=a_i,\ i=0,\dots,n-1\}$ is the cylinder of the length n. It is well-known that the pressure function $q\mapsto P_q(\phi):=P(q\phi)$, is convex. Moreover, by the celebrated result of Ruelle, the pressure function for a ***Hölder continuous*** potential ϕ is ***real-analytic***.

It turns out that the real-analyticity of the pressure function immediately gives us the Large Deviations Principle for ergodic averages $X_n(\omega) = S_n\phi(\omega)$. More specifically, one has the following result[1]

Theorem 6.1.3 (Gärtner-Ellis)**.** *Suppose $\{X_n\}$ is a sequence of real-valued random variables defined on a common probability space $(\Omega, \mathscr{F}, m)$. Assume that for every $q \in \mathbb{R}$, the logarithmic moment generating function, defined as the limit*

$$\Lambda(q) \triangleq \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_m e^{qX_n} = \lim_{n\to\infty} \frac{1}{n} \log \int e^{qX_n(\omega)} m(d\omega)$$

exists and is finite. Denote by Λ^* the Legendre transform (convex dual) of Λ, defined by

$$\Lambda^*(\alpha) = \sup_{q\in\mathbb{R}} \Big(\alpha q - \Lambda(q)\Big).$$

Then

(a) for any closed set $F \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log m\left(\left\{\omega \in \Omega : \frac{1}{n} X_n(\omega) \in F\right\}\right) \leq -\inf_{\alpha\in F} \Lambda^*(\alpha);$$

(b) if, furthermore $\Lambda(q)$ is differentiable on $\mathbb{R}$, for any open set $G \subset \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log m\left(\left\{\omega \in \Omega : \frac{1}{n} X_n(\omega) \in G\right\}\right) \geq -\inf_{\alpha\in G} \Lambda^*(\alpha).$$

In order to apply the Gärtner-Ellis theorem to ergodic averages $X_n(\omega) = S_n\phi(\omega)$ it is necessary to introduce the reference measure m. A good choice would be the *uniform* or the *measure of maximal entropy* for Ω. Namely, let $l = |\mathscr{A}|$ the number of different letters in alphabet $\mathscr{A}$, and let $m = \rho^{\mathbb{Z}+}$ be the product of uniform measures ρ on $\mathscr{A}$. Then one immediately concludes that

$$\Lambda_\phi(q) = P_\phi(q) - \log l = P(q\phi) - \log l.$$

Therefore, for Hölder continuous functions ϕ's, the logarithmic moment generating function $\Lambda_\phi(q)$ is also real-analytic, and hence the Gärtner-Ellis theorem is applicable. The so-called large deviations rate function is then

$$\mathscr{I}_\phi(\alpha) = \Lambda^*_\phi(\alpha) = \sup_{q\in\mathbb{R}} \Big[q\alpha - \Lambda_\phi(q)\Big] = \sup_{q\in\mathbb{R}} \Big[q\alpha - P_\phi(q) + \log l\Big] = \log l + P^*_\phi(\alpha).$$

Note also that $h_{\text{top}}(\sigma, \Omega) = \log l$. Hence,

$$\mathscr{I}_\phi(\alpha) = h_{\text{top}}(\sigma, \Omega) + P^*_\phi(\alpha)$$

[1] This is not the most general form of the Gärtner-Ellis theorem.

Theorem 6.1.4 ([3]). *Suppose $\Omega = A^{\mathbb{Z}_+}$ and $\phi : \Omega \to \mathbb{R}$ is Hölder continuous. For $\alpha \in \mathbb{R}$, consider the set*

$$K_\alpha = \left\{ \omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(\sigma^k \omega) = \alpha \right\}.$$

Then there exist $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}$ such that

- *for every $\alpha \in (\underline{\alpha}, \overline{\alpha})$, $K_\alpha \neq \emptyset$ and*

$$\dim_H(K_\alpha) = \frac{1}{\log l} \inf_q \Big(P(q\phi) - q\alpha \Big) = -\frac{1}{\log l} \sup_q \Big(q\alpha - P(q\phi) \Big) = -\frac{P^*_\phi(\alpha)}{\log l}.$$

- *the multifractal spectrum*

$$\alpha \mapsto \mathscr{D}_\phi(\alpha) := \dim_H(K_\alpha)$$

  is a real-analytic function of α on $(\underline{\alpha}, \overline{\alpha})$.

Again, if we consider the topological entropy $h_{\mathsf{top}}(\sigma, \cdot)$ instead of the Hausdorff dimension $\dim_H(\cdot)$, taking into account, that $h_{\mathsf{top}}(\sigma, \Omega) = \log l$, we obtain that the large deviations rate function and the topological entropy of level sets of ergodic averages are related

$$h_{\mathsf{top}}(\sigma, K_\alpha) = h_{\mathsf{top}}(\sigma, \Omega) - \mathscr{I}_\phi(\alpha).$$

It was observed in [46], that the result of Theorem 6.1.4 can be extended to a much larger class of dynamical systems and observables ϕ with the key property being that for every $q \in \mathbb{R}$, there is a unique equilibrium state μ_q for $q\phi$. To ensure uniqueness, one has to make two types of assumptions: on the map $f : \Omega \to \Omega$ and the potential (observable) $\phi : \Omega \to \mathbb{R}$. For example, in [46] it was assumed that

- the map $f : \Omega \to \Omega$ is an expansive homeomorphism with specification property

- the continuous observable $\phi : \Omega \to \mathbb{R}$ is in the Bowen class, i.e., for all $\epsilon > 0$,

$$\sup_{d_n(\omega,\omega')<\epsilon} \Big| (S_n\phi)(\omega) - (S_n\phi)(\omega') \Big| \leq K(\epsilon) < \infty$$

  where the supremum is taken over all ω, ω' such that

$$d_n(\omega, \omega') = \max_{k=0,\ldots,n-1} d(f^k(\omega), f^k(\omega')) < \epsilon.$$

Under these conditions, the Gärtner-Ellis theorem holds for ergodic sums $X_n(\omega) = S_n\phi(\omega)$ with a differentiable logarithmic moment generating function $\Lambda(q)$, and the multifractal spectrum for ergodic averages is given by the Legendre transform of the pressure function, i.e., for all $\alpha \in (\underline{\alpha}, \overline{\alpha})$,

$$h_{\text{top}}(f, K_\alpha) = \inf_q \Big(P(q\phi) - q\alpha \Big) = -P^*_\phi(\alpha).$$

Furthermore, the logarithmic moment-generating function Λ is related to the pressure function P_ϕ by $\Lambda(q) = P_\phi(q) - h_{\text{top}}(f, \Omega)$, $q \in \mathbb{R}$.

6.1.2 Multifractal analysis of Birkhoff averages of continuous functions

If one considers simply continuous observables $\phi : \Omega \to \mathbb{R}$, it is no longer true that there is a unique equilibrium state μ_q for the potential $q\phi$ for all $q \in \mathbb{R}$. Therefore, a different approach is required.

In the case of symbolic systems ($\Omega = A^{\mathbb{Z}_+}$, $\sigma : \Omega \to \Omega$ is the left-shift), the first results were obtained independently by Fan and Feng [18, 20] and Olivier [31, 32]. The methods are quite different: Olivier relied on the density of Hölder continuous function in the space of all continuous functions. Fan and Feng constructed the so-called Moran fractal. The approach of Fan and Feng turned out to be very suitable for generalization to abstract compact spaces. In [48] the following variational principle for multifractal spectra has been obtained:

Theorem 6.1.5 ([48])**.** *Suppose $T : \Omega \to \Omega$ is a continuous transformation of a compact metric space (Ω, d) with the specification property and $\phi : \Omega \to \mathbb{R}$ is a continuous observable. Then there exist $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}$ such that for every $\alpha \in (\underline{\alpha}, \overline{\alpha})$, $K_\alpha \neq \varnothing$, and*

$$h_{top}(T, K_\alpha) = \sup_{\mu \in \mathcal{M}(\Omega, T)} \Big\{ h_\mu(T) : \quad \int \phi \, d\mu = \alpha \Big\},$$

where supremum is taken over $\mathcal{M}(\Omega, T)$ of all T-invariant measures on Ω, and $h_\mu(T)$ is the Kolmogorov–Sinai entropy of μ.

For such systems, the Large Deviation Principle has been established in 1990 by Young [51]. We will not recall the most general form of the result, but reformulate it in a such form that the relation to the multifractal formalism becomes apparent immediately.

Theorem 6.1.6 ([51], Theorem 1)**.** *Suppose m is a not necessarily invariant Ahlfors-Bowen reference measure on Ω. Suppose $\varphi : \Omega \to \mathbb{R}$ is continuous. Then for all $c \in \mathbb{R}$, one has*

$$\limsup_{n\to\infty} \frac{1}{n} \log m \left\{ \frac{1}{n} S_n \varphi \geq c \right\} \leq \sup_{\mu \in \mathcal{M}(\Omega, T)} \left\{ h_\mu(T) : \int \varphi \, d\mu \geq c \right\} - h_{top}(T, \Omega).$$

If, furthermore, T has a specification property, then for all $c \in \mathbb{R}$, one has

$$\liminf_{n\to\infty} \frac{1}{n} \log m\left\{\frac{1}{n} S_n\varphi > c\right\} \geq \sup_{\mu\in\mathcal{M}(\Omega,T)} \left\{h_\mu(T) : \int \varphi\, d\mu > c\right\} - h_{top}(T,\Omega).$$

This result allows us to say, that under the above conditions, the sequence of random variables $\{X_n(\omega) = (S_n\phi)(\omega)\}$ satisfies the Large Deviation Principle with the rate function

$$I_\phi(\alpha) = h_{\mathrm{top}}(T,\Omega) - \sup_{\mu\in\mathcal{M}(\Omega,T)} \left\{h_\mu(T) : \quad \int \phi\, d\mu = \alpha\right\}.$$

As in the cases considered above, combining the results of Theorems 6.1.5 and 6.1.6, one concludes that for topological dynamical systems with specification, the large deviations rate function I_ϕ and the multifractal spectrum $\mathcal{E}_\phi(\alpha) := h_{\mathrm{top}}(T, K_\alpha)$ are related by

$$\mathcal{E}_\phi(\alpha) = h_{\mathrm{top}}(T,\Omega) - I_\phi(\alpha).$$

In fact, the methods developed by Young are instrumental in the proof of Theorem 6.1.5 in [48].

6.1.3 Multifractal analysis beyond Birkhoff sums

A number of multifractal results have been obtained for the level sets of the form

$$K_\alpha = \left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} X_n(\omega) = \alpha\right\},$$

where X_n's are not necessarily Birkhoff sums of the form $S_n\phi$, for some ϕ. For example, such non-additive observable arise naturally in the multifractal analysis of pointwise Lyapunov exponents for the products of matrices. Namely, suppose a map $M : \Omega \to M_d(\mathbb{R})$ is given where $M_d(\mathbb{R})$ is the algebra of $d \times d$ real matrices. We can associate to every $x \in \Omega$ and $n \in \mathbb{N}$, a random variable $X_n(x) := \log\|M(T^{n-1}x)M(T^{n-2}x)\cdots M(x)\|$ provided that $\|M(T^{n-1}x)M(T^{n-2}x)\cdots M(x)\| \neq 0$, where $\|\cdot\|$ is a norm on $M_d(\mathbb{R})$. Clearly, $\{X_n\}_{n\in\mathbb{N}}$ is not an additive sequence if $d \geq 2$.

A non-additive sequence of potentials can also be encountered if one considers weighted ergodic sums $\{S_n^w f\}$ of a continuous potential f on the space Ω, i.e., $S_n^w f := \sum_{i=0}^{n-1} w_i f \circ T^i$, $n \in \mathbb{Z}_+$ and $w_n \in \mathbb{R}$ are weights [19]. It is easy to see that $\{S_n^w f\}$ is not additive unless all the weights w_n, $n \in \mathbb{Z}_+$ are the same.

Another example of non-additive sequences arises in studying pointwise entropies of probability measures. Suppose μ is a fully supported probability measure on Ω. For a fixed $\delta > 0$, define a sequence of random variables as $X_n(x) :=$

$-\log\mu(B_n(x,\delta))$, $n\in\mathbb{N}$, $x\in\Omega$, where $B_n(x,\delta):=\{y\in\Omega : d(T^i y, T^i x)<\delta,\ \ i=0,\dots,n-1\}$, $n\geq 1$.

We shall discuss these and other examples in greater detail in Section 6.7 and show how our results can be applied in these cases.

6.2 Preliminaries and Notations

In this section, we assume that we are given a compact metric space (Ω, d) equipped with a continuous transformation $T:\Omega\to\Omega$. We denote by $M(\Omega), M(\Omega,T)$ the sets of all Borel probability measures and T-invariant Borel probability measures on Ω, respectively. For $x, y\in\Omega$ and $n\in\mathbb{N}$, the Bowen metric d_n is defined as $d_n(x,y):=\max\limits_{0\leq i\leq n-1} d(T^i x, T^i y)$. By $B_n(x,\varepsilon)$ we denote an open ball of radius $\varepsilon>0$ in the metric d_n centred at $x\in\Omega$, i.e., $B_n(x,\varepsilon):=\{y\in\Omega : d_n(x,y)<\varepsilon\}$.

6.2.1 Topological Entropy

The notion of topological entropy of non-compact and non-invariant sets has been introduced by Bowen [7] in 1973. This chapter uses an equivalent definition of Bowen's topological entropy given in [41]. For $Z\subset\Omega$, and all $t\in\mathbb{R}$, $\epsilon>0$, and $N\in\mathbb{N}$, let

$$m(Z,t,\epsilon,N):=\inf\left\{\sum_{i=1}^{\infty} e^{-n_i t} : Z\subset\bigcup_{i=1}^{\infty} B_{n_i}(x_i,\epsilon),\ n_i\geq N\right\}. \tag{6.2}$$

By the standard convention, $m(\emptyset,t,\epsilon,N)=0$, for all t,ϵ,N. Obviously, $m(Z,t,\epsilon,N)$ does not decrease on N, thus we can define

$$m(Z,t,\epsilon):=\lim_{N\to\infty} m(Z,t,\epsilon,N).$$

One can show [41] that $m(\cdot,t,\epsilon)$ is an outer measure with properties similar to those of t-dimensional Hausdorff outer measure. In particular, there exists a critical value $t'\in\mathbb{R}$ such that

$$m(Z,t',\epsilon)=\begin{cases}+\infty, & \text{if } t'<t,\\ 0, & \text{if } t'>t.\end{cases}$$

We denote this critical value by $h_{\mathsf{top}}(T,Z,\epsilon)$. Thus

$$h_{\mathsf{top}}(T,Z,\epsilon)=\inf\{t\in\mathbb{R} : m(Z,t,\epsilon)=0\}=\sup\{t\in\mathbb{R} : m(Z,t,\epsilon)=+\infty\}.$$

Furthermore, since $h_{\mathsf{top}}(T,Z,\epsilon)$ is monotonic in ϵ, we can define the ***topological entropy*** of Z by

$$h_{\mathsf{top}}(T,Z):=\lim_{\epsilon\to 0+} h_{\mathsf{top}}(T,Z,\epsilon).$$

It should be stressed that the set Z is not assumed to be compact nor T-invariant. Finally, by our convention, $h_{\mathrm{top}}(T,\emptyset) = -\infty$, and $h_{\mathrm{top}}(T,Z) \geq 0$ for all non-empty subsets $Z \subset \Omega$. We summarize the basic properties of the topological entropy in the following remark.

Remark 6.2.1 ([41]). *(i) Monotonicity: if $Z_1 \subset Z_2$, then $h_{top}(T,Z_1) \leq h_{top}(T,Z_2)$;*

(ii) Countable stability: if $Z = \bigcup_n Z_n$, then $h_{top}(T,Z) = \sup_n h_{top}(T,Z_n)$.

Remark 6.2.2. *The notions of topological entropy and Hausdorff dimension coincide if the underlying space Ω is a symbolic space. Suppose $\Omega = \{1,2,\ldots,l\}^{\mathbb{Z}_+}$ with integer $l \geq 2$, and the metric on Ω is defined as*

$$d(x,y) = \frac{1}{l^k}, \quad where \;\; k = \min\{i \geq 0 : x_i \neq y_i\},$$

then $Z \subseteq \Omega$, $\dim_H(Z) = \dfrac{h_{top}(T,Z)}{\log l}$.

6.2.2 Local (pointwise) entropies

Consider a Borel probability measure μ on the metric space (Ω, d), and define the lower and upper pointwise entropies of μ at $x \in \Omega$ as

$$\underline{h}_\mu(T,x) := \lim_{\varepsilon\to 0}\liminf_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\varepsilon)), \;\; \overline{h}_\mu(T,x) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log\mu(B_n(x,\varepsilon)).$$

Note that the limit in ε exists due to the monotonicity. Feng and Huang [26] extended the notion of measure-theoretic entropy to non-invariant measures by motivating the result of Brin and Katok. In fact, for a probability measure $\mu \in \mathcal{M}(\Omega)$, Feng and Huang [26] defined the *upper* and *lower measure-theoretic entropies* of μ relative to the transformation T by

$$\underline{h}_\mu(T) := \int_\Omega \underline{h}_\mu(T,x)\mu(dx) \text{ and } \overline{h}_\mu(T) := \int_\Omega \overline{h}_\mu(T,x)\mu(dx).$$

Note that the above formulas are consistent with the classical notion of measure-theoretic entropy due to the Brin-Katok theorem [8]. In fact, if $\mu \in \mathcal{M}(\Omega, T)$, then $\underline{h}_\mu(T) = \overline{h}_\mu(T) = h_\mu(T)$, where $h_\mu(T)$ is the measure-theoretic entropy of μ with respect to T. In [26], the authors also extended the classical variational principle to the non-invariant setting. More precisely, they showed for a compact, but not necessarily invariant set $K \subseteq \Omega$ that

$$h_{\mathrm{top}}(T,K) = \sup\{\underline{h}_\mu(T) : \mu \in \mathcal{M}(\Omega),\ \mu(K) = 1\}. \tag{6.3}$$

The next theorem is a dynamical analogue of the mass distribution principle and is very useful for the estimation of topological entropies of various sets.

Theorem 6.2.3 ([30]). *Let μ be a Borel probability measure on Ω, and E be a Borel subset of Ω.*

(1) If $\underline{h}_\mu(T,x) \le t$ for some $t < \infty$ and all $x \in E$, then $h_{top}(T,E) \le t$.

(2) If $\mu(E) > 0$, and $\underline{h}_\mu(T,x) \ge t$ for some $t > 0$ and all $x \in E$, then $h_{top}(T,E) \ge t$.

6.2.3 Ahlfors-Bowen measures

Definition 6.2.4. *We say that a not necessarily invariant Borel probability measure m on Ω is **Ahlfors-Bowen** if there exists $h > 0$ such that for every $\epsilon > 0$ there exists a finite positive constant, $D(\varepsilon)$ such that for every $n \ge 1$ and all $\omega \in \Omega$, one has*

$$\frac{1}{D(\varepsilon)} e^{-nh} \le m\big(B_n(\omega,\epsilon)\big) \le D(\varepsilon) e^{-nh}. \tag{6.4}$$

It is easy to see from the definition that an Ahlfors-Bowen measure has a very simple spectrum of local entropies. More specifically, $h_m(T,\omega) := h$ for all $\omega \in \Omega$. Furthermore, one has the following simple lemma for the constant h.

Lemma 6.2.5. *Suppose m is Ahlfors-Bowen measure for T on Ω, then $h = h_{top}(T,\Omega)$.*

Proof. Since m is an Ahlfors-Bowen measure, for any $\omega \in \Omega$, one has $h_m(T,\omega) = h$. Therefore, by Theorem 6.2.3, we immediately conclude that $h_{\text{top}}(T,\Omega) = h$. □

6.2.4 Conditions on the underlying transformation

In applications, topological dynamical systems (TDS) usually have some sort of expansivity and mixing properties. Therefore, there is a vast amount of literature devoted to studying dynamical systems with these properties. We also impose expansivity and mixing properties on our underlying transformation to prove some of our results.

Definition 6.2.6. *A continuous transformation $T : \Omega \to \Omega$ is called **expansive** if there exists a constant $\rho > 0$ such that if*

$$d(T^n(x), T^n(y)) < \rho \text{ for all non-negative integer } n, \text{ then } x = y.$$

The maximal ρ with such a property is called the expansive constant.

A slightly stronger version of topological mixing is topological exactness. Topological exactness of the transformation T is that for every $\epsilon > 0$ and $x \in \Omega$ there is $N := N(x,\epsilon) \in \mathbb{N}$ such that $T^N(B(x,\epsilon)) = \Omega$. Basically, by using the compactness of the underlying space Ω, one can uniformise the constant N with $x \in \Omega$. A relatively stronger version of topological exactness which we call strong topological exactness also gives uniform control over the Bowen balls of the transformation.

Definition 6.2.7. *We call a continuous transformation $T:\Omega\to\Omega$* ***strongly topologically exact*** *if for any $\epsilon>0$ there is a natural number $M_1\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ and $x\in\Omega$,*

$$T^{n+M_1}(B_n(x,\epsilon))=\Omega. \tag{6.5}$$

Note that topologically mixing subshifts of finite type are strongly topologically exact. Walters ([50]) showed that an expansive, strongly topologically exact transformation has a weak form of the specification property, namely, one has the following theorem:

Theorem 6.2.8 ([50])**.** *Let $T:\Omega\to\Omega$ be an expansive strongly topologically exact transformation, then T satisfies the weak specification condition, i.e., for any $\varepsilon>0$ there exists a natural number $M_2\in\mathbb{N}$ such that for all $x,x'\in\Omega$ and every $n_1,n_2\in\mathbb{N}$, there exists $w\in\Omega$ with $d(T^iw,T^ix)<\varepsilon, 0\le i\le n_1-1$ and $d(T^{n_1+M_2+j}w,T^jx')<\varepsilon$, $0\le j\le n_2-1$. Note that the latter condition is equivalent to the following: for every $\epsilon>0$ there exists an integer $M_2>0$ such that for all $x,x'\in\Omega$ and $n_1,n_2\in\mathbb{N}$, one has*

$$B_{n_1}(x,\epsilon)\cap T^{-(n_1+M_2)}(B_{n_2}(x',\epsilon))\neq\emptyset. \tag{6.6}$$

Remark 6.2.9. *If $T:\Omega\to\Omega$ is an expansive homeomorphism with weak specification property, then the measure of maximal entropy for T is Ahlfors-Bowen* [50, Theorem 4.6].

6.2.5 Class of random variables

We say that the sequence $\{X_n\}_{n\ge1}$ of real-valued random variables (observables) on Ω is ***subadditive*** if for every $m,n\ge1$,

$$X_{n+m}\le X_n+X_m\circ T^n.$$

We call a sequence $\{X_n\}_{n\ge1}$ of random variables on Ω ***weakly almost additive*** (or simply, almost additive), if there are non-negative constants $A_n=o(n)$ such that for all $x\in\Omega$ and $n,m\in\mathbb{N}$,

$$|X_{n+m}(x)-X_n(x)-X_m(T^nx)|\le A_n. \tag{6.7}$$

Recently, Cuneo [12, Theorem 1.2] showed that weakly almost additive sequences of continuous functions can be approximated uniformly by Birkhoff's sums of continuous functions.

For a function $H:\Omega\to\mathbb{R}$ and $n\in\mathbb{N}$, $\delta>0$, we denote the (n,δ)-variation of the function H by $v_{n,\delta}(H):=\sup\limits_{d_n(y,z)\le\delta}|H(y)-H(z)|$.

Definition 6.2.10. *A sequence* $\{X_n\}$ *of random variables on* Ω *satisfies the* ***Bowen condition****, if there exists* $\delta > 0$ *such that* $\sup_{n\in\mathbb{N}} v_{n,\delta}(X_n) < +\infty$. *We say that the sequence* $\{X_n\}$ *satisfies the* ***weak Bowen condition*** *if there exists* $\delta > 0$ *such that*

$$\lim_{n\to\infty} \frac{v_{n,\delta}(X_n)}{n} = 0. \tag{6.8}$$

It should be mentioned that the continuity of the random variables X_n, $n \geq 1$, is not assumed in the definition of the weak Bowen condition, and neither the Bowen condition nor the weak Bowen condition implies the continuity of each individual random variable X_n. However, if a sequence of functions $\{X_n\}$ satisfies the weak Bowen condition, then each function X_n is bounded. Indeed, since Ω is compact, for every n, there exists a finite cover $\{B_n(\omega_i, \delta/2)\}_i$, and the observable X_n is a bounded function on each ball $B_n(\omega_i, \delta/2)$ in the cover.

Remark 6.2.11. *One can observe that if* T *is expansive, then the sequence of ergodic sums of any continuous potential* $\psi : \Omega \to \mathbb{R}$ *satisfies weak Bowen's condition. Namely, if* $X_n = S_n\psi = \sum_{i=0}^{n-1} \psi \circ T^i$, *then* $v_{n,\delta}(X_n) \leq \sum_{i=1}^{n} v_{i,\delta}(\psi)$, *and thus*

$$\lim_{n\to\infty} \frac{v_{n,\delta}(X_n)}{n} \leq \lim_{n\to\infty} \frac{\sum_{i=1}^{n} v_{i,\delta}(\psi)}{n} = 0$$

by Stolz-Cesaro's theorem, since $\psi : \Omega \to \mathbb{R}$ *is continuous and hence* $\lim_{n\to\infty} v_{n,\delta}(\psi) = 0$, *where positive number* δ *is smaller than the expansive constant of* T.

6.3 Large Deviations

6.3.1 Some concepts from convex analysis

In this section, we consider functions with values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. A convex function $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ is called a **proper convex function** if $\phi(t) > -\infty$ for all $t \in \mathbb{R}$ and $\phi \not\equiv +\infty$. The ***essential domain*** of the convex function ϕ is $\mathrm{dom}(\phi) := \{t \in \mathbb{R} : \phi(t) < +\infty\}$. Note that $\mathrm{dom}(\phi) \subset \mathbb{R}$ is a convex set.

Suppose $\phi : \mathbb{R} \to (-\infty, +\infty]$ is a proper convex function and assume that ϕ is finite at some $t \in \mathbb{R}$. A real number t^* is a **subgradient** of ϕ at t if for all $s \in \mathbb{R}$, $\phi(t) + t^*(s-t) \leq \phi(s)$ holds. Denote the set of all subgradients of ϕ at t by $\partial\phi(t)$. It is easy to check that $\partial\phi(t) = \mathbb{R} \bigcap \Big[\sup_{s>0} \frac{\phi(t)-\phi(t-s)}{s},\ \inf_{s>0} \frac{\phi(t+s)-\phi(t)}{s}\Big] = \mathbb{R} \bigcap [\phi'_-(t), \phi'_+(t)]$, where $\phi'_-(t),\ \phi'_+(t) \in \overline{\mathbb{R}}$ are left and right derivatives of ϕ at t. We define the domain of the multivalued mapping $\partial\phi : t \in \mathrm{dom}(\phi) \mapsto \partial\phi(t) \subset \mathbb{R}$ by $\mathrm{dom}(\partial\phi) := \{t \in \mathrm{dom}(\phi) : \partial\varphi(t) \neq \emptyset\}$. Note that in general, $\mathrm{dom}(\partial\phi)$ is not a convex set.

Definition 6.3.1. *A proper convex function $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ is called **essentially strictly convex** if ϕ is strictly convex on every convex subset of dom$(\partial\phi)$. Note that a convex function ϕ is called strictly convex in a convex set $D \subset \mathbb{R}$, if for every $t_1, t_2 \in D$ with $t_1 \neq t_2$ and $\lambda \in (0,1)$, $\phi(\lambda t_1 + (1-\lambda)t_2) < \lambda\phi(t_1) + (1-\lambda)\phi(t_2)$.*

It is a well-known fact that finite convex functions are very close to differentiable functions. However, if it comes to extended convex functions, then one needs to be more careful about the essential domain of the function.

Definition 6.3.2. *A proper convex function $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ is **essentially smooth** if it satisfies the following conditions:*

(a) dom$(\phi)^\circ \neq \emptyset$, where A° denotes the interior of a set A;

(b) ϕ is differentiable on dom$(\phi)^\circ$;

(c) ϕ is steep, namely, $\lim_{n\to\infty} \left|\phi(t_n)\right| = \infty$ whenever $\{t_n\}$ is a sequence in dom$(\phi)^\circ$ converging to a boundary point of dom$(\phi)^\circ$.

An extended, not necessarily convex, function $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ is ***lower semicontinuous*** if for all $\lambda \in \mathbb{R}$, $\{t \in \mathbb{R} : \phi(t) \leq \lambda\}$ is a closed subset of $\mathbb{R}$. Equivalently, a function ϕ is lower semicontinuous if and only if $\liminf_{x\to x_0} \phi(x) \geq \phi(x_0)$ for all $x_0 \in \mathbb{R}$. Note that a lower semicontinuous function achieves its minimum in any compact set.

The following theorem shows that essentially strictly convex and essentially smooth convex functions are intimately related in terms of the Legendre transform.

Theorem 6.3.3 ([45])**.** *A lower semicontinuous proper convex function ϕ is essentially strictly convex if and only if its Legendre transform, $\phi^*(t) := \sup_{q\in\mathbb{R}}\{tq - \phi(q)\}$, $t \in \mathbb{R}$, is essentially smooth.*

6.3.2 Large deviations

Definition 6.3.4. *An extended function $I : \mathbb{R} \to [0, +\infty]$ is called a **large deviations rate function** (or simply a rate function) if it is lower semicontinuous. A rate function I is called **good** if the sub-level sets, $\{t : I(t) \leq \lambda\} \subset \mathbb{R}$ are compact for all $\lambda \in \mathbb{R}$. For $E \subset \mathbb{R}$, $I(E)$ abbreviates $\inf_{t\in E} I(t)$ i.e., $I(E) := \inf_{t\in E} I(t)$.*

Note that a good rate function achieves its minimum in any closed set.

Suppose $(\Omega, \mathscr{F}, \mu)$ is a probability space and $X_n : \Omega \to \mathbb{R}$, $n \geq 1$, are random variables.

Definition 6.3.5. *The sequence of random variables* $\{\frac{1}{n}X_n\}_{n=1}^{+\infty}$ *satisfies the* ***large deviations principle*** *(LDP) with a* ***rate function*** I *if for all Borel sets* $E \subset \mathbb{R}$,

$$-I(E^\circ) \le \liminf_{n\to\infty} \frac{1}{n} \log\mu\Big(\Big\{\omega : \frac{X_n(\omega)}{n} \in E\Big\}\Big) \le$$
$$\le \limsup_{n\to\infty} \frac{1}{n} \log\mu\Big(\Big\{\omega : \frac{X_n(\omega)}{n} \in E\Big\}\Big) \le -I(\overline{E}),$$

where $\overline{E}$, E° *denote the closure and the interior of* E, *respectively.*

One of the most convenient ways to establish the validity of the LDP is provided by the Gärtner-Ellis theorem which we have stated in a partial case in Theorem 6.1.3. Denote the normalised log-moment generating function of X_n by ϕ_n, i.e., for $t \in \mathbb{R}$

$$\phi_n(t) := \frac{1}{n} \log \int_\Omega e^{tX_n} d\mu.$$

Note that ϕ_n may take value $+\infty$. Define the log-moment generating function of the sequence $X := \{X_n\}$ by $\phi_X(t) := \limsup_{n\to\infty} \phi_n(t)$, $t \in \mathbb{R}$. Then the Gärtner-Ellis theorem states that if for all $t \in \mathbb{R}$, the limit, $\lim_{n\to\infty} \phi_n(t)$ exists and is finite, and the log-moment generating function ϕ_X is essentially smooth, then $\{\frac{1}{n}X_n\}_{n\in\mathbb{N}}$ satisfies the LDP with a good rate function $I_X := \phi_X^*$. An inverse statement to this in some sense is the celebrated Varadhan's lemma, which we reformulate slightly for our purposes below. One can find a general version of the theorem in [14, Theorem 4.3.1].

Theorem 6.3.6 ([14])**.** *Suppose that* $\{\frac{1}{n}X_n\}_{n=1}^{+\infty}$ *satisfies the LDP with a good rate function* $I : \mathbb{R} \to [0, \infty]$, *and*

$$\limsup_{n\to 0} \frac{1}{n} \log \int_\Omega e^{tX_n} d\mu = \phi_X(t) < \infty$$

hold for all $t \in \mathbb{R}$. *Then the limit* $\lim_{n\to 0} \frac{1}{n} \log \mathbb{E}_\mu\left[e^{tX_n}\right]$ *exists and*

$$\phi_X(t) = \lim_{n\to 0} \frac{1}{n} \log \int_\Omega e^{tX_n} d\mu = \sup_{q\in\mathbb{R}}\{tq - I(q)\} = I^*(t), \quad \textit{for all } t \in \mathbb{R}.$$

Sometimes, a weak version of LDP is more convenient for purposes, since it is easier to establish than the full LDP.

Definition 6.3.7. *Let* $(\Omega, \mathscr{F}, \mu)$ *be a probability space and* $X_n : \Omega \to \mathbb{R}$, $n \ge 1$ *be random variables. The sequence* $\{\frac{1}{n}X_n\}_{n\in\mathbb{N}}$ *satisfies the* ***weak Large Deviation Principle*** *(weak LDP) with a rate function* I *if*

1) for all compact set $F \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu\Big(\Big\{ \frac{1}{n} X_n \in F \Big\} \Big) \le -I(F),$$

2) for all open set $G \subset \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu\Big(\Big\{ \frac{1}{n} X_n \in G \Big\} \Big) \ge -I(G).$$

Note that the only difference between LDP and weak LDP is in the upper large deviation bound.

As one can see in the Gärtner-Ellis theorem, the LDP upper bound requires fewer conditions than the lower bound. However, one still needs to make rather mild assumptions (for example, the existence of the limit log-moment generating function as in Theorem 6.1.3) to obtain the upper bound. Nevertheless, one can establish the upper *weak large deviation* bound without any additional assumption about the sequence as the following folklore theorem states.

Theorem 6.3.8. *Let $\{X_n\}$ be a sequence of random variables on a probability space $(\Omega, \mathscr{F}, \mu)$. Then the sequence $\{\frac{1}{n} X_n\}_{n\in\mathbb{N}}$ satisfies the weak large deviations upper bound with the rate function $I_X := \phi_X^*$, i.e., for any compact set $F \Subset \mathbb{R}$, the following holds*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu\Big(\Big\{ \frac{1}{n} X_n \in F \Big\} \Big) \le -I_X(F). \tag{6.9}$$

6.4 Main results

The main goal of this chapter is to obtain the Multifractal spectra for the sequence $\Big\{\frac{1}{n} X_n\Big\}$ under the assumption that the sequence satisfies LDP. As a byproduct, we also obtain a formula that relates the entropy spectra with the LDP rate function. Although we assume in our theorems that the reference measure satisfies the Ahlfors-Bowen condition, one should be able to generalise the results to a multifractal measure τ with the smooth multifractal spectrum, i.e. $\alpha \mapsto h_{\mathsf{top}}(T, E_\alpha)$ is a smooth function, where $E_\alpha := \{\omega : h_\tau(\omega) = \alpha\}$.

Assume that the ambient space (Ω, d) is compact, and we have a continuous transformation $T : \Omega \to \Omega$ with finite topological entropy, i.e., $h_{\mathsf{top}}(T, \Omega) < \infty$. It is also assumed that a sequence $X := \{X_n\}$ of random variables on Ω and a reference measure $\nu \in \mathscr{M}(\Omega)$ satisfying Ahlfors-Bowen condition with the parameter $h = h_{\mathsf{top}}(T, \Omega)$ are given. It should be stressed that we are not assuming $\{X_n : n \in \mathbb{N}\}$

are continuous functions, and ν is T-invariant. We denote, for $\alpha \in \mathbb{R}$, the α-level set of the sequence $\{\frac{1}{n} X_n\}$ by

$$K_\alpha := \{x \in \Omega : \lim_{n\to\infty} \frac{1}{n} X_n(x) = \alpha\},$$

and the domain of the multifractal spectrum by $\mathscr{L}_X := \{\alpha \in \mathbb{R} : K_\alpha \neq \emptyset\}$. We shall study the entropy spectrum $\mathscr{E}_X(\alpha) := h_{\text{top}}(T, K_\alpha)$. In the statements of our theorems, we shall assume some of the following main conditions:

(A1) The sequence $\{X_n\}_n$ satisfies the weak Bowen condition;

(A2) The sequence $\{X_n\}_n$ is weakly almost additive;

(A3) $T : \Omega \to \Omega$ is an expansive strongly topologically exact transformation.

We now present our first main result that estimates the entropy spectrum $\mathscr{E}_X$ from above in terms of the LDP rate function.

Theorem 6.A. *Assume (A1), and the sequence* $\left\{\frac{1}{n} X_n\right\}$ *satisfies* the weak large deviations upper bound *with a rate function* $I_X : \mathbb{R} \to [0, +\infty]$*, i.e. for all compact* $F \subset \mathbb{R}$ *one has*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu\left\{\frac{1}{n} X_n \in F\right\} \leq -I_X(F). \tag{6.10}$$

Then for all $\alpha \in \mathscr{L}_X$*, one has*

$$\mathscr{E}_X(\alpha) \leq h_{top}(T, \Omega) - I_X(\alpha). \tag{6.11}$$

Let $\phi_n(q) := \frac{1}{n} \log \mathbb{E}_\nu e^{qX_n}$ and $\phi_X := \limsup_{n\to\infty} \phi_n$ be the log and log-limit moment-generating functions of the random variable X_n, respectively. Notice that the moment-generating functions ϕ_n exist and are finite for all $n \in \mathbb{N}$ since the observables X_n are bounded functions by the weak Bowen condition. The next result estimates the spectrum $\mathscr{E}_X$ from above in terms of the log-moment generating function ϕ_X of X.

Theorem 5.B. *Assume (A1), then for all* $\alpha \in \mathscr{L}_X$

$$\mathscr{E}_X(\alpha) \leq h_{top}(T, \Omega) - \phi_X^*(\alpha), \tag{6.12}$$

where ϕ_X^* *is the Legendre-Fenchel transform of* ϕ_X*.*

Remark 6.4.1. *If one drops the condition (A1) in the above theorems, then the statements of the theorems will be false. Assume* $\Omega := \{0,1\}^{\mathbb{Z}_+}$*,* $T : \Omega \to \Omega$ *is the left shift transformation,* $x \in \Omega$ *and* $\kappa \in \mathbb{R} \setminus \{1\}$*. Set* $X_n(\omega) := n$ *if* $\omega_i = x_i$*,* $i = 0, 1, \ldots, n^2 - 1$*, and* $X_n(\omega) := \kappa n$ *otherwise. Then* $\phi_X(q) = \kappa q$ *for every* $q \in \mathbb{R}$*, hence by Theorem 6.1.3, the sequence* $\{X_n\}_{n\in\mathbb{N}}$ *satisfies LDP with a good rate function* ϕ_X^**. However, one has* $h_{top}(T, K_1) = 0 > h_{top}(T, \Omega) - \varphi_X^*(1) = \log 2 - \infty = -\infty$*.*

Remark 6.4.2. *We should note that one can prove the same statements as in Theorem 6.A and Theorem 5.B with the following condition which is slightly weaker than the weak Bowen condition: there exists $\delta > 0$ such that*

$$\liminf_{n\to\infty} \frac{v_{n,\delta}(X_n)}{n} = 0.$$

One might expect equality in Theorem 6.A and Theorem 5.B, but it is easy to show by examples that the conditions of these theorems are "too general" to prove such equality. The subsequent theorem provides sufficient conditions for the equality to hold in Theorem 6.A and Theorem 5.B.

Theorem 6.C. *Assume the conditions (A1), (A2) and (A3). If the sequence $\{\frac{1}{n}X_n\}_{n\in\mathbb{N}}$ satisfies LDP with an essentially strictly convex good rate function I_X, then one has the following:*

(i) There exists extended real numbers $-\infty \le \underline{\alpha} \le \overline{\alpha} \le +\infty$ such that $(\underline{\alpha}, \overline{\alpha}) \subset \mathscr{L}_X \subset [\underline{\alpha}, \overline{\alpha}]$.

(ii) $\mathscr{E}_X(\alpha) = h_{top}(T, \Omega) - I_X(\alpha)$ holds for all $\alpha \in (\underline{\alpha}, \overline{\alpha})$.

Finally, it is well-known that a result of multifractal formalism for random variables can be generalized straightforwardly to random vectors (see [48]). In particular, one can directly extend the above theorems to sequences $\{\overrightarrow{X}_n\}_{n\in N}$ of random vectors $\overrightarrow{X}_n : \Omega \to \mathbb{R}^d$ with straightforward modification of the conditions (A1)-(A3).

6.5 Proofs of Theorem 6.A and Theorem 5.B: Upper bounds

In the proofs of our theorems, we frequently use the Vitali covering lemma in the following form.

Lemma 6.5.1 (Vitali covering lemma)**.** *Let (Ξ, d) be a metric space, and suppose $\mathfrak{F}$ is a non-empty finite collection of balls. Then there exists a disjoint subcollection $\mathfrak{F}' \subseteq \mathfrak{F}$, such that $\bigcup_{B\in\mathfrak{F}} B \subset \bigcup_{B'\in\mathfrak{F}'} 3B'$, where $3B'$ denotes a ball with the same centre as the ball B', but with three times the radius of B'.*

Proof of Theorem 6.A. First, we will show for any $\alpha \in \mathscr{L}_X := \{\alpha \in \mathbb{R} : K_\alpha \neq \emptyset\}$ that

$$I_X(\alpha) \le h, \tag{6.13}$$

where $h = h_{\mathsf{top}}(T, \Omega)$ (c.f. Lemma 6.2.5). This, in particular, implies that $\mathscr{L}_X \subset \mathrm{dom}(I_X)$.

Fix any decreasing sequence $\{\delta_m\}$ of positive numbers such that $\delta_m \to 0$ as $m \to \infty$. For $\alpha \in \mathscr{L}_X$ set

$$K^{(n)}_{\alpha,m} := \{w \in \Omega : -\delta_m < \frac{1}{n} X_n(w) - \alpha < \delta_m\}, \text{ and } Z_{m,j} := \bigcap_{n \ge j} K^{(n)}_{\alpha,m} \bigcap K_\alpha. \quad (6.14)$$

Then for every $m \in \mathbb{N}$, one has $K_\alpha = \bigcup_j Z_{m,j}$.

We prove (6.13) by assuming the contradiction. Suppose that $I_X(\alpha) > h$ for some $\alpha \in \mathscr{L}_X$. Then there is $\Delta > 0$ such that $h < \Delta < I_X(\alpha)$. Take $\delta > 0$, and consider the closed interval $[\alpha - \delta, \alpha + \delta]$. Since I_X is a lower semi-continuous function, I_X attains its minimum on $[\alpha - \delta, \alpha + \delta]$ at some point $\alpha_\delta \in [\alpha - \delta, \alpha + \delta]$, i.e., $I_X(\alpha_\delta) = I_X([\alpha - \delta, \alpha + \delta])$. Thus for all $\delta > 0$, $I_X(\alpha_\delta) \le I_X(\alpha)$, therefore, $\limsup_{\delta \to 0} I_X(\alpha_\delta) \le I_X(\alpha)$. But on the other hand, $\liminf_{\delta \to 0} I_X(\alpha_\delta) \ge I_X(\alpha)$ since I_X is lower semi-continuous and $\alpha_\delta \to \alpha$ as $\delta \to 0$. Thus one can conclude that $\lim_{\delta \to 0} I_X([\alpha - \delta, \alpha + \delta]) = I_X(\alpha)$, therefore, there exists $\delta > 0$ such that $I_X([\alpha - \delta, \alpha + \delta]) > \Delta$. Take any $\epsilon > 0$, and fix it. Then since $\{X_n\}$ satisfies the weak Bowen condition, there exists $N_0 \in \mathbb{N}$ such that $\frac{v_{n,\epsilon}(X_n)}{n} < \frac{\delta}{2}$ for all $n \ge N_0$. Furthermore, since $\delta_m \to 0$, there is $m^* \in \mathbb{N}$ with $\delta_{m^*} < \delta/2$. Finally, since $\bigcup_j Z_{m^*,j} = K_\alpha \neq \emptyset$ and $\{Z_{m^*,j}\}_j$ is an ascending chain of the sets, there exists $z \in K_\alpha$ and $j^* \in \mathbb{N}$ such that for all $j \ge j^*$, $z \in Z_{m^*,j}$. Thus, for all $N \ge j^*$, one can readily check that

$$B_N(z,\epsilon) \subset \Big\{ w \in \Omega : \frac{1}{N} X_N(w) \in \Big(\alpha - \delta_{m^*} - \frac{v_{N,\epsilon}(X_N)}{N}, \alpha + \delta_{m^*} + \frac{v_{N,\epsilon}(X_N)}{N} \Big) \Big\}. \quad (6.15)$$

Therefore, for every $N \ge \max\{j^*, N_0\}$ one has

$$B_N(z,\epsilon) \subset \Big\{ \frac{1}{N} X_N \in \Big[\alpha - \delta, \alpha + \delta \Big] \Big\}. \quad (6.16)$$

Since ν is an Ahlfors-Bowen measure, for every $N \ge \max\{j^*, N_0\}$ one obtains

$$\frac{1}{N} \log \nu \Big\{ \frac{1}{N} X_N \in [\alpha - \delta, \alpha + \delta] \Big\} \ge \frac{1}{N} \log \nu(B_N(z,\epsilon)) \ge -h - \frac{\log D(\epsilon)}{N}. \quad (6.17)$$

Then (6.10) leads

$$-h \le \limsup_{N \to \infty} \frac{1}{N} \log \nu \Big\{ \frac{1}{N} X_N \in [\alpha - \delta, \alpha + \delta] \Big\} \le -I_X([\alpha - \delta, \alpha + \delta]) < -\Delta.$$

But this is a contradiction to $h < \Delta$, therefore, $I_X(\alpha) \le h$ for every $\alpha \in \mathscr{L}_X$.

Now we prove the inequality (6.11). Let $\{\delta_m\}$ and $Z_{m,j}$ be as above. Take any $\alpha \in \mathscr{L}_X$ and $\epsilon > 0$. Since I_X is a lower semi-continuous function, $\lim_{\delta \to 0} I_X([a - \delta, a +$

$\delta]) = I_X(a)$ for all $a \in \mathbb{R}$. Since $I_X(\alpha) < +\infty$ for $\alpha \in \mathscr{L}_X$, for arbitrary $\sigma > 0$, there exists $m' \in \mathbb{N}$, and $\delta' > 0$ such that, for all $m \geq m'$ and $\delta \leq \delta'$

$$\left| I_X([\alpha - \delta_m - \delta; \alpha + \delta_m + \delta]) - I_X(\alpha) \right| < \sigma.$$

Again $\lim_{n\to\infty} \frac{v_{n,\epsilon}(X_n)}{n} = 0$, choose sufficiently large $N_1 \in \mathbb{N}$ such that $\frac{v_{n,\epsilon}(X_n)}{n} < \delta'$ for all $n \geq N_1$. From (6.10) one can choose $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$

$$\nu\left\{\frac{1}{n} X_n \in [\alpha - \delta_{m'} - \delta', \alpha + \delta_{m'} + \delta']\right\} \leq e^{n(-I_X([\alpha - \delta_{m'} - \delta', \alpha + \delta_{m'} + \delta']) + \sigma)} \leq e^{n(-I_X(\alpha) + 2\sigma)}. \tag{6.18}$$

Now take any $j \in \mathbb{N}$ with $Z_{m',j} \neq \emptyset$, and for all $N \geq \max\{j, N_1, N_2\}$ consider the cover $\{B_N(z, \epsilon/3)\}_{z \in Z_{m',j}}$ of $Z_{m',j}$. Clearly, it also covers $\overline{Z}_{m',j}$, therefore, by compactness, there is a finite subcover $\{B_N(z_i, \epsilon/3)\}_i$ of $Z_{m',j}$. Then by Vitali covering lemma (Lemma 6.5.1) there exists a disjoint subcollection $\{B_N(z_{i_k}, \epsilon/3)\}_k$ such that $Z_{m',j} \subset \cup_i B_N(z_i, \epsilon/3) \subset \cup_k B_N(z_{i_k}, \epsilon)$. Therefore, without loss of generality, one can assume that there is a finite disjoint collection $\mathscr{B}_N := \{B_N(z_i, \epsilon/3)\}_{z_i \in Z_{m',j}}$ of the balls such that $Z_{m',j} \subset \bigcup_i B_N(z_i, \epsilon)$. Since $z_i \in Z_{m',j}$, one can easily check for every i that

$$B_N(z_i, \epsilon) \subset \left\{ w \in \Omega : \frac{1}{N} X_N(w) \in \left(\alpha - \delta_{m'} - \frac{v_{N,\epsilon}(X_N)}{N}, \alpha + \delta_{m'} + \frac{v_{N,\epsilon}(X_N)}{N} \right) \right\}.$$

Thus by the choice of N, for all i,

$$B_N(z_i, \epsilon) \subset \left\{ \frac{1}{N} X_N \in (\alpha - \delta_{m'} - \delta', \alpha + \delta_{m'} + \delta') \right\},$$

and hence

$$\bigcup_i B_N(z_i, \epsilon) \subset \left\{ \frac{1}{N} X_N \in (\alpha - \delta_{m'} - \delta', \alpha + \delta_{m'} + \delta') \right\}.$$

Since $\mathscr{B}_N := \{B_N(z_i, \epsilon/3)\}$ is a disjoint collection of the balls, one can easily get the following from the above

$$\sum_i \nu(B_N(z_i, \epsilon/3)) \leq \nu\left(\left\{ \frac{1}{N} X_N \in (\alpha - \delta_{m'} - \delta', \alpha + \delta_{m'} + \delta') \right\} \right).$$

Thus it follows from (6.18) that

$$\sum_i \nu(B_N(z_i, \epsilon/3)) \leq e^{N(-I_X(\alpha) + 2\sigma)}.$$

Since ν is an Ahlfors-Bowen measure, we can thus conclude that

$$\#\mathscr{B}_N \leq D(\epsilon/3) e^{N(h - I_X(\alpha) + 2\sigma)}.$$

Since $\{B_N(z_i,\epsilon)\}_i$ is a cover of $Z_{m',j}$, one can immediately obtain from (6.2) that

$$m(Z_{m',j},t,\epsilon,N) \le \#\mathscr{B}_N \cdot e^{-Nt}.$$

Then by combining the last two inequalities, one gets

$$m(Z_{m',j},t,\epsilon,N) \le D(\epsilon/3)e^{-N(t-h+I_X(\alpha)-2\sigma)}. \tag{6.19}$$

Notice that σ is not chosen yet, therefore, if $t > h - I_X(\alpha)$ then one can choose σ in the interval $\Big(0, \dfrac{t-h+I_X(\alpha)}{2}\Big)$. Let $t > h - I_X(\alpha)$ be any number. Then one can obtain from (6.19) that $h_{\mathrm{top}}(T, Z_{m',j},\epsilon) \le t$, and notice that this inequality holds for any $\epsilon > 0$ and for all large $j \in \mathbb{N}$. Therefore, $h_{\mathrm{top}}(T, Z_{m',j}) \le t$ for all sufficiently large j, thus by the countable stability of the topological entropy, we can get $h_{\mathrm{top}}(T, K_\alpha) \le t$ since $\{Z_{m',j}\}_j \uparrow K_\alpha$. Thus we can obtain the desired result since t is chosen arbitrarily in $(h - I_X(\alpha), \infty)$. □

Proof of Theorem 5.B. One can readily check that ϕ_X^* is an extended convex lower-semi-continuous function on $\mathbb{R}$. Indeed, the lower semi-continuity of ϕ_X^* follows from the fact that it is the *superior envelope* of the family $\{u_t : t \in \mathbb{R}\}$ of extended continuous functions $\alpha \in \mathbb{R} \xrightarrow{u_t} \alpha t - \phi(t) \in \overline{\mathbb{R}}$. Furthermore, since $\phi_X(0) = 0$, one has $\phi_X^* \ge 0$. Thus ϕ_X^* is a rate function in the sense of Definition 6.3.4. By Theorem 6.3.8, one has the weak large deviations upper bound for the sequence $\Big\{\dfrac{1}{n}X_n\Big\}$ with the rate function ϕ_X^*, i.e., for all compact $F \Subset \mathbb{R}$, one has the following

$$\limsup_{n\to\infty} \frac{1}{n}\log \nu\Big\{\frac{1}{n}X_n \in F\Big\} \le -\inf_F(\phi_X^*) =: -\phi_X^*(F). \tag{6.20}$$

Thus one can immediately obtain the statement of the theorem from Theorem 6.A.

□

6.6 Proof of Theorem 6.C: Lower bound

The goal of this section is to prove Theorem 6.C. For any real number q and a natural number n denote

$$\mathscr{Z}_n(q) := \int_\Omega e^{qX_n}\, d\nu.$$

We will employ the Large Deviations technique. Namely, for every q, by using Cramer's exponential tilting method, we will construct a measure μ_q (Lemma 6.6.3) such that for every $\omega \in \Omega$, $n \in \mathbb{N}$

$$\mu_q(B_n(\omega,\epsilon)) \le \frac{e^{qX_n(\omega)}}{\mathscr{Z}_n(q)}\nu(B_n(\omega,\epsilon)) \times K(n,q,\epsilon), \tag{6.21}$$

where the uniform 'correction' factor $K(n,q,\epsilon)>0$ is subexponential in n.

Once (6.21) is established, we immediately obtain that for all $\omega\in\Omega$, one has

$$-\frac{1}{n}\log\mu_q(B_n(\omega,\epsilon))\geq -\frac{qX_n(\omega)}{n}+\frac{1}{n}\log\mathscr{Z}_n(q)-\frac{1}{n}\log\nu(B_n(\omega,\epsilon))+o(1).$$

Therefore, for $\omega\in K_\alpha$ we then conclude

$$\underline{h}_{\mu_q}(T,\omega)=\lim_{\epsilon\to 0}\liminf_{n\to\infty}-\frac{1}{n}\log\mu_q(B_n(\omega,\epsilon))\geq h_{\mathsf{top}}(T,\Omega)-q\alpha+\phi_X(q). \tag{6.22}$$

Furthermore, for a given α, we will identify q such that $\mu_q(K_\alpha)=1$ (Lemma 6.6.7). As usual, $q=q_\alpha$ is such that $\alpha=\phi'(q_\alpha)$. Therefore, by the entropy distribution principle (Theorem 6.2.3), we can conclude that

$$h_{\mathsf{top}}(T,K_\alpha)\geq -q_\alpha\alpha+\phi_X(q_\alpha)+h_{\mathsf{top}}(T,\Omega)=h_{\mathsf{top}}(T,\Omega)-\phi_X^*(\alpha)=h_{\mathsf{top}}(T,\Omega)-I_X(\alpha). \tag{6.23}$$

After combining this with the upper bound, we obtain the claim of Theorem 6.C. Now, let us turn to the rigorous proof.

6.6.1 A fundamental inequality

In this subsection, we shall establish a compound inequality for the dynamical systems which is introduced in Subsection 6.2.4. Recall that for any $\epsilon>0$ there exists $M_1(\epsilon)\in\mathbb{N}$ such that for all $x\in\Omega$ and every $n\in\mathbb{N}$, $T^{n+M_1}(B_n(x,\epsilon))=\Omega$ since T is strongly topologically exact transformation, and there exists $M_2(\epsilon)\in\mathbb{N}$ such that for all $x,x'\in\Omega$ and every $n_1,n_2\in\mathbb{N}$, $B_{n_1}(x,\epsilon)\cap T^{-(n_1+M_2)}\big(B_{n_2}(x',\epsilon)\big)\neq\emptyset$ since T has the weak specification property (c.f. equations (6.5) and (6.6)). Let $M_1(\epsilon)$ and $M_2(\epsilon)$ be the smallest integers satisfying these properties, respectively. For $\epsilon>0$, let

$$M(\epsilon):=\max\{M_1(\epsilon),M_2(\epsilon/2)\}. \tag{6.24}$$

If no confusion arises, we just use M for $M(\epsilon)$ without stressing its dependence on ϵ. One has the following lemma for the expansive dynamical systems with certain mixing properties.

Lemma 6.6.1. *For any sufficiently small ϵ, there exists a positive constant $C_1(\epsilon)$ such that for any $n\in\mathbb{N}$ all $x\in\Omega$ and for any measurable function $Y:\Omega\to[0,+\infty)$ one has*

$$\frac{C_1(\epsilon)^{-1}}{\nu(B_n(x,\varepsilon))}\int_{B_n(x,\varepsilon)}Y\circ T^{n+M}\,d\nu(y)\leq\int_\Omega Y\,d\nu\leq\frac{C_1(\epsilon)}{\nu(B_n(x,\varepsilon))}\int_{B_n(x,\varepsilon)}Y\circ T^{n+M}\,d\nu.$$

Proof. We assume that T is expansive. Let ρ_{exp} denote the expansivity constant of T. Consider $\epsilon\in(0,\rho_{exp}/2)$, $n\in\mathbb{N}$, $x\in\Omega$ and a measurable function $Y:\Omega\to[0,+\infty)$. Let $\nu_{x,n}$ be the conditional probability measure on $B_n(x,\epsilon)$, i.e.,

$\nu_{x,n}(A) := \nu(A|B_n(x,\epsilon)) = \dfrac{\nu(A)}{\nu(B_n(x,\varepsilon))}$ for a Borel set $A \subset B_n(x,\epsilon)$. Set $\tau_{x,n} := \nu_{x,n} \circ T^{-(n+M)}$, then $\tau_{x,n}$ is always a probability measure on Ω since $T^{n+M}(B_n(x,\epsilon)) = \Omega$. Note that $\nu_{x,n}$ and $\tau_{x,n}$ also depend on ϵ, but we omit this to avoid cumbersome notations in our subsequent expressions. It is easy to check that

$$\int_\Omega Y(z)d\tau_{x,n}(z) = \frac{1}{\nu(B_n(x,\varepsilon))}\int_{B_n(x,\varepsilon)} Y(T^{n+M}z)d\nu(z). \tag{6.25}$$

Therefore, the claim of the lemma is equivalent to showing the existence of a constant $C_1(\epsilon) > 0$ independent of n and x such that

$$C_1(\epsilon)^{-1}\int_\Omega Y(y)d\tau_{x,n}(y) \le \int_\Omega Y(y)d\nu(y) \le C_1(\epsilon)\int_\Omega Y(y)d\tau_{x,n}(y). \tag{6.26}$$

We now present the proof of this statement in three steps.

Step 1: We start by showing that there exists a constant $C_2(\epsilon) > 0$, dependent only on ϵ, such that for every $k \in \mathbb{N}$ and $y \in \Omega$,

$$C_2(\epsilon)^{-1} \le \frac{\tau_{x,n}(B_k(y,\epsilon))}{\nu(B_k(y,\epsilon))} \le C_2(\epsilon). \tag{6.27}$$

Consider arbitrary $k \in \mathbb{N}$, $y \in \Omega$. By the choice of $M := M(\epsilon)$, we have

$$T^{-(n+M)}B_k(y,\varepsilon/2) \cap B_n(x,\varepsilon/2) \neq \emptyset.$$

Then for any $z \in T^{-(n+M)}B_k(y,\varepsilon/2) \cap B_n(x,\varepsilon/2)$, one has

$$B_{n+M+k}(z,\varepsilon/2) \subset T^{-(n+M)}B_k(y,\varepsilon) \cap B_n(x,\varepsilon).$$

Since ν is an Ahlfors-Bowen measure (c.f. (6.4)), we can get

$$D(\epsilon/2)^{-1}e^{-(n+M+k)h} \le \nu(B_{n+M+k}(z,\varepsilon/2)) \le \nu(T^{-(n+M)}B_k(y,\varepsilon) \cap B_n(x,\varepsilon)). \tag{6.28}$$

Hence

$$\begin{aligned}\tau_{x,n}(B_k(y,\epsilon)) &= \frac{\nu(T^{-(n+M)}B_k(y,\varepsilon) \cap B_n(x,\varepsilon))}{\nu(B_n(x,\varepsilon))}\\ &\ge \frac{D(\epsilon/2)^{-1}e^{-(n+M+k)h}}{D(\epsilon)e^{-nh}}\\ &\ge D(\varepsilon/2)^{-1}D(\varepsilon)^{-2}e^{-Mh}\nu(B_k(y,\varepsilon)).\end{aligned} \tag{6.29}$$

In order to get an upper bound in (6.27), set $B' := T^{-(n+M)}B_k(y,\varepsilon) \cap B_n(x,\varepsilon)$. Then since $\{B_{n+M+k}(z,2\epsilon) : z \in B'\}$ is a cover of $\overline{B'}$, by the the compactness of Ω and the Vitali covering lemma (Lemma 6.5.1), there exists a finite disjoint subcollection $\mathcal{G} := \{B_{n+M+k}(z_i, 2\epsilon)\}_{z_i \in B'}$ such that $\{B_{n+M+k}(z_i, 6\epsilon)\}_{z_i \in B'}$ covers B'. Denote $\mathcal{S} := \{T^n z_i : B_{n+M+k}(z_i, 2\epsilon) \in \mathcal{G}\}$. Because of the disjointness of $\mathcal{G}$ and by the definition of B', it is easy to check that $\sharp\mathcal{S} = \sharp\mathcal{G}$ and $\mathcal{S}$ is a $(M, 2\epsilon)$-separated set. Therefore, if we denote the cardinality of a maximal $(M, 2\epsilon)$-separated set in Ω by $\mathfrak{s}(M, 2\epsilon)$, then it is clear that $\sharp\mathcal{G} = \sharp\mathcal{S} \le \mathfrak{s}(M, 2\epsilon)$. From these and (6.4), it follows that

$$\nu(B') \le \sum_i \nu(B_{n+M+k}(z_i, 6\epsilon)) \le D(6\epsilon)\mathfrak{s}(M, 2\epsilon)e^{-(n+M+k)h}. \tag{6.30}$$

This bound then implies

$$\begin{aligned}\tau_{x,n}(B_k(y,\epsilon)) = \frac{\nu(B')}{\nu(B_n(x,\epsilon))} &\le \frac{D(6\epsilon)\mathfrak{s}(M,2\epsilon)e^{-(n+M+k)h}}{D(\epsilon)^{-1}e^{-nh}} \\ &\le D(6\epsilon)D(\epsilon)^2\mathfrak{s}(M,2\epsilon)\,\nu(B_k(y,\epsilon)).\end{aligned} \tag{6.31}$$

Combining (6.29), (6.31), the constant

$$C_2(\epsilon) := \max\{D(\varepsilon/2)D(\varepsilon)^2 e^{Mh}, D(6\epsilon)D(\epsilon)^2\mathfrak{s}(M, 2\epsilon)\}$$

satisfies the inequality (6.27).

Step 2: In this step, we shall prove (6.26) for continuous non-negative functions: namely, if the constant $C_1(\epsilon) := C_2(\epsilon)D(2\epsilon)D(\epsilon)$ and $\psi \in C(\Omega, \mathbb{R}_+)$, then

$$C_1(\epsilon)^{-1}\int_\Omega \psi\, d\tau_{x,n} \le \int_\Omega \psi\, d\nu \le C_1(\epsilon)\int_\Omega \psi\, d\tau_{x,n}. \tag{6.32}$$

Consider an arbitrary $N \in \mathbb{N}$ and let E_N be a maximal $(N, 2\epsilon)$-separated set in Ω. Thus $\{B_N(x, 2\epsilon) : x \in E_N\}$ is a cover of Ω and $\{B_N(x, \epsilon) : x \in E_N\}$ are pairwise disjoint. By successively applying (6.4) and (6.27) to these collections of the balls, one can obtain

$$\int_\Omega \psi\, d\tau_{x,n} \ge C_2(\epsilon)^{-1}D(2\epsilon)^{-1}D(\epsilon)^{-1}\int_\Omega \psi\, d\nu - C_2(\epsilon)^{-1}(v_{N,2\epsilon}(\psi) + D(2\epsilon)D(\epsilon)v_{N,\epsilon}(\psi)). \tag{6.33}$$

Since $v_{N,2\epsilon}(\psi) \to 0$, $v_{N,\epsilon}(\psi) \to 0$ as $N \to \infty$ (c.f. Remark 6.2.11), one obtains $\int_\Omega \psi\, d\tau_{x,n} \ge C_2(\epsilon)^{-1}D(2\epsilon)^{-1}D(\epsilon)^{-1}\int_\Omega \psi\, d\nu$. The opposite inequality is proved in exactly the same way.

Step 3: Now, let $\tilde{Y}$ be any bounded non-negative measurable function on Ω. By Lusin's theorem and Tietz's extension theorem ([17]), one can approximate $\tilde{Y}$ by non-negative continuous functions $\psi_k : \Omega \to \mathbb{R}$, $k \in \mathbb{N}$ such that $\int_\Omega \psi_k\, d\nu \to$

$\int_\Omega \tilde{Y}\, d\nu$ and $\int_\Omega \psi_k\, d\tau_{x,n} \to \int_\Omega \tilde{Y}\, d\tau_{x,n}$ as $k \to \infty$. Hence it can be obtained from (6.32) that

$$C_1(\epsilon)^{-1} \int_\Omega \tilde{Y}\, d\tau_{x,n} \le \int_\Omega \tilde{Y}\, d\nu \le C_1(\epsilon) \int_\Omega \tilde{Y}\, d\tau_{x,n}. \tag{6.34}$$

Now we conclude (6.26) from (6.34). Consider a non-negative measurable function Y on Ω, and let $Y_N := Y \cdot \mathbb{1}_{Y \le N}$ for $N \in \mathbb{N}$. Then Y_N is a bounded non-negative function, and $\{Y_N\}$ converges monotonically to Y as $N \to \infty$. Therefore, by the monotone convergence theorem and from (6.34), one gets (6.26). □

6.6.2 A "tilted" measure and its relationship with the reference measure

For any real number $q \in \mathbb{R}$ and a natural number $n \in \mathbb{N}$, define a probability measure $\mu_{n,q}$ by Cramer's tilting method as

$$d\mu_{n,q} := \frac{e^{qX_n}}{\mathbb{E}_\nu e^{qX_n}} d\nu = \frac{e^{qX_n}}{\mathscr{Z}_n(q)} d\nu.$$

For any $q \in \mathbb{R}$, consider the sequence $\{\mu_{n,q}\}_{n \ge 1}$ of probability measures. Since $\mathscr{M}(\Omega)$ is a compact set in weak* topology, there is a convergent subsequence $\{\mu_{N_j,q}\}_j$, and we denote the weak*—limit of this subsequence by μ_q. In this subsection, we shall establish an important relation (c.f. Lemma 6.6.3) between μ_q and ν under the conditions of Theorem 6.C. First, we prove an important lemma.

Lemma 6.6.2. *For all $n \in \mathbb{N}$, $q \in \mathbb{R}$, there exists a positive constant $C_4(n,q)$ with $\lim_{n\to\infty} \frac{\log C_4(n,q)}{n} = 0$ such that for any $m \in \mathbb{N}$,*

$$C_4(n,q)^{-1} \le \frac{\mathscr{Z}_{n+M+m}(q)}{\mathscr{Z}_n(q)\mathscr{Z}_m(q)} \le C_4(n,q), \tag{6.35}$$

where $M \in \mathbb{N}$ is as in (6.24).

Proof. Fix a sufficiently small ϵ. Consider $n, m \in \mathbb{N}$, $q \in \mathbb{R}$. Note that by the weak Bowen condition, each X_k is a bounded function. In what follows, $||X_k||$ denotes the supremum norm of a function X_k. Since the sequence X is almost additive, one has

$$\mathscr{Z}_{n+M+m}(q) = \int_\Omega e^{qX_{n+M+m}(y)} d\nu(y) \le e^{|q|(A_n + A_M + ||X_M||)} \int_\Omega e^{qX_n(y)} e^{qX_m(T^{n+M}y)} d\nu(y). \tag{6.36}$$

Let E be a maximal (n,ε)-separated set in Ω. Then one has that $\bigcup_{x\in E} B_n(x,\varepsilon)=\Omega$ and $\{B_n(x,\epsilon/2)\}_{x\in E}$ is a disjoint collection of Bowen balls. Therefore,

$$\begin{aligned}\int_\Omega e^{qX_n(y)}e^{qX_m(T^{n+M}y)}d\nu(y) &\le \sum_{x\in E}\int_{B_n(x,\varepsilon)} e^{qX_n(y)}e^{qX_m(T^{n+M}y)}d\nu(y)\\ &\le e^{|q|v_{n,\epsilon}(X_n)}\sum_{x\in E} e^{qX_n(x)}\int_{B_n(x,\varepsilon)} e^{qX_m(T^{n+M}y)}d\nu(y)\\ &\le e^{|q|v_{n,\epsilon}(X_n)}C_1(\varepsilon)\mathscr{Z}_m(q)\sum_{x\in E} e^{qX_n(x)}\nu(B_n(x,\varepsilon)),\end{aligned} \tag{6.37}$$

where the second inequality holds since $\{X_n\}$ satisfies weak Bowen's condition and the last inequality holds by Lemma 6.6.1 applied to $Y=X_m$. Now we estimate the sum $\sum_{x\in E} e^{qX_n(x)}\nu(B_n(x,\varepsilon))$ from above. One has

$$\begin{aligned}\sum_{x\in E} e^{qX_n(x)}\nu(B_n(x,\varepsilon)) &\le \sum_{x\in E} e^{qX_n(x)}D(\varepsilon/2)D(\varepsilon)\nu(B_n(x,\varepsilon/2))\\ &\le D(\varepsilon/2)D(\varepsilon)e^{|q|v_{n,\epsilon/2}(X_n)}\sum_{x\in E}\int_{B_n(x,\varepsilon/2)} e^{qX_n(y)}d\nu\\ &\le D(\varepsilon/2)D(\varepsilon)e^{|q|v_{n,\epsilon/2}(X_n)}\mathscr{Z}_n(q).\end{aligned} \tag{6.38}$$

After combining (6.36), (6.37) and (6.38)

$$\frac{\mathscr{Z}_{n+M+m}(q)}{\mathscr{Z}_n(q)\mathscr{Z}_m(q)} \le C_5(n,q), \tag{6.39}$$

where $C_5(n,q):=e^{|q|(A_n+A_M+\|X_M\|)}e^{|q|(v_{n,\epsilon/2}(X_n)+v_{n,\epsilon}(X_n))}C_1(\varepsilon)D(\varepsilon/2)D(\varepsilon)$.

With similar reasoning, one gets the following inequalities

$$\mathscr{Z}_{n+M+m}(q)\ge e^{-|q|(A_n+A_M+\|X_M\|)}\int_\Omega e^{qX_n(y)}e^{qX_m(T^{n+M}y)}d\nu(y), \tag{6.40}$$

$$\begin{aligned}&\int_\Omega e^{qX_n(y)}e^{qX_m(T^{n+M}y)}d\nu(y)\nu(y)\ge\\ &\ge e^{-|q|v_{n,\epsilon/2}(X_n)}C_1(\varepsilon/2)^{-1}\mathscr{Z}_m(q)\sum_{x\in E} e^{qX_n(x)}\nu(B_n(x,\varepsilon/2)),\end{aligned} \tag{6.41}$$

and

$$\sum_{x\in E} e^{qX_n(x)}\nu(B_n(x,\varepsilon/2))\ge e^{-|q|v_{n,\epsilon}(X_n)}D(\varepsilon/2)^{-1}D(\varepsilon)^{-1}\mathscr{Z}_n(q). \tag{6.42}$$

After combining these inequalities, one has

$$\frac{\mathscr{Z}_{n+M+m}(q)}{\mathscr{Z}_n(q)\mathscr{Z}_m(q)} \geq C_6(n,q)^{-1}, \tag{6.43}$$

where $C_6(n,q) := e^{|q|(A_n+A_M+\|X_M\|)}e^{|q|(v_{n,\epsilon/2}(X_n)+v_{n,\epsilon}(X_n))}C_1(\varepsilon/2)D(\varepsilon/2)D(\varepsilon)$.

From (6.39) and (6.43), it is clear that $C_4(n,q) := \max\{C_5(n,q), C_6(n,q)\}$ satisfies the conditions of the lemma. □

Lemma 6.6.3. *For any sufficiently small $\epsilon > 0$ and for any $n \in \mathbb{N}$, $q \in \mathbb{R}$ there exists a constant $C_7(n,q,\epsilon) > 0$ such that $\lim_{n\to\infty} \frac{\log C_7(n,q,\epsilon)}{n} = 0$ and for all $x \in \Omega$,*

$$\mu_q(B_n(x,\varepsilon)) \leq C_7(n,q,\epsilon)\frac{e^{qX_n(x)}}{\mathscr{Z}_n(q)}\,\nu(B_n(x,\varepsilon)). \tag{6.44}$$

Proof. Consider again $\epsilon \in (0, \rho_{exp}/2)$, where ρ_{exp} is the expansivity constant of the transformation T, and $x \in \Omega$, $n \in \mathbb{N}$, $q \in \mathbb{R}$. By the Portmanteau theorem, we have

$$\mu_q(B_n(x,\varepsilon)) \leq \liminf_{j\to\infty} \int_{B_n(x,\varepsilon)} \frac{e^{qX_{N_j}(y)}}{\mathscr{Z}_{N_j}(q)}\,d\,\nu(y). \tag{6.45}$$

Now we estimate the numerator $\int_{B_n(x,\varepsilon)} e^{qX_{N_j}(y)}d\,\nu(y)$ and denominator $\mathscr{Z}_{N_j}(q)$ separately. By Lemma 6.6.2, one has

$$\frac{1}{\mathscr{Z}_{N_j}(q)} = \frac{1}{\mathscr{Z}_{n+M+N_j-n-M}(q)} \leq C_4(n,q)\frac{1}{\mathscr{Z}_n(q)\mathscr{Z}_{N_j-n-M}(q)}. \tag{6.46}$$

For the integral in the numerator, the following estimations are valid

$$\begin{aligned}\int\limits_{B_n(x,\varepsilon)} e^{qX_{n+M+N_j-n-M}(y)}d\,\nu(y) &\leq e^{|q|\tilde{A}_{n,M}} \int\limits_{B_n(x,\varepsilon)} e^{q(X_n(y)+X_M(T^n y)+X_{N_j-n-M}(T^{n+M}y))}d\,\nu(y)\\ &\leq e^{|q|(\tilde{A}_{n,M}+\|X_M\|)} \int\limits_{B_n(x,\varepsilon)} e^{q(X_n(y)+X_{N_j-n-M}(T^{n+M}y))}d\,\nu(y)\\ &\leq C_8(n,q,\epsilon)e^{qX_n(x)}\int_{B_n(x,\varepsilon)} e^{qX_{N_j-n-M}(T^{n+M}y)}d\,\nu(y),\end{aligned} \tag{6.47}$$

where $\tilde{A}_{n,M} = A_n + A_M$ and $C_8(n,q,\epsilon) := e^{|q|(\tilde{A}_{n,M}+\|X_M\|+v_{n,\epsilon}(X_n))}$. Note that

$$\int_{B_n(x,\varepsilon)} e^{qX_{N_j-n-M}(T^{n+M}y)}d\,\nu(y) \leq C_1(\varepsilon)\mathscr{Z}_{N_j-n-M}(q)\,\nu(B_n(x,\varepsilon)). \tag{6.48}$$

Thus it follows from (6.47) that

$$\int_{B_n(x,\varepsilon)} e^{qX_{N_j}(y)} d\nu(y) \le C_1(\epsilon)C_8(n,q,\epsilon)e^{qX_n(x)}\nu(B_n(x,\varepsilon))\mathscr{Z}_{N_j-n-M}(q). \quad (6.49)$$

Hence it is easy to see from (6.45)-(6.46) that the constant

$$C_7(n,q,\epsilon) := C_1(\epsilon)C_4(n,q)C_8(n,q,\epsilon)$$

satisfies (6.44), and by Lemma 6.6.2 and the weak Bowen condition, one gets $\lim_{n\to\infty} \frac{\log C_7(n,q,\epsilon)}{n} = 0.$ □

6.6.3 Differentiability of the log-moment generating function

Lemma 6.6.4. *For all $q \in \mathbb{R}$, the limit $\lim_{n\to\infty} \frac{1}{n}\log\mathscr{Z}_n(q)$ exists and finite. Thus, $\phi_X = \lim_{n\to\infty} \frac{1}{n}\log\mathscr{Z}_n$ is a finite convex function. Furthermore, $\phi_X = I_X^*$ and $\phi_X^* = I_X$.*

First, we state a slight generalization of the classical Fekete's lemma which is used to verify the existence and finiteness of the limit $\lim_{n\to\infty} \frac{1}{n}\log\mathscr{Z}_n(q)$.

Lemma 6.6.5 ([18])**.** *Let $\{a_n\}_{n\ge 0}$ be a sequence of real numbers. Assume that there exists a natural number $N \in \mathbb{N}$ and real numbers $D, \Delta_n \ge 0$, $n \in \mathbb{N}$ with $\lim_{n\to\infty} \frac{\Delta_n}{n} = 0$ such that for all $m \in \mathbb{Z}_+$ and $n \ge N$*

$$a_{n+m} \le a_n + a_m + D + \Delta_n. \quad (6.50)$$

Then the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists and belongs to $\mathbb{R}\bigcup\{-\infty\}$.

Proof of Lemma 6.6.4. By using Lemma 6.6.2, it can be shown for $q \in \mathbb{R}$, $n \ge M+1$ and any $m \in \mathbb{Z}_+$ that

$$\log\mathscr{Z}_{n+m}(q) \le \log\mathscr{Z}_n(q) + \log\mathscr{Z}_m(q) + 2\log C_4(n-M,q). \quad (6.51)$$

Thus by Lemma 6.6.5, the limit $\lim_{n\to\infty} \frac{1}{n}\log\mathscr{Z}_n(q)$ exists and $\lim_{n\to\infty} \frac{1}{n}\log\mathscr{Z}_n(q) < +\infty$. By applying a similar argument to the sequence $\{-\log\mathscr{Z}_n(q)\}$, one shows that $\lim_{n\to\infty} \frac{1}{n}\log\mathscr{Z}_n(q) > -\infty$.

Clearly, each $\log\mathscr{Z}_n$ is a convex function, thus ϕ_X is also a convex function as the pointwise limit of convex functions.

Since for all $q \in \mathbb{R}$, $\phi_X(q) < +\infty$, and $\left\{\frac{1}{n}X_n\right\}_{n\in\mathbb{N}}$ satisfies LDP with the good rate function I_X, we immediately obtain from Theorem 6.3.6 that $I_X^* = \phi_X$. Note that $I_X = \phi_X^*$ also holds by the Fenchel duality lemma ([14]), since I_X is convex and lower semi-continuous. □

Now we show that ϕ_X is a differentiable function. In fact, the following holds true.

Lemma 6.6.6. *The function ϕ_X is continuously differentiable on $\mathbb{R}$, i.e., $\phi_X \in C^1(\mathbb{R})$.*

Proof. By Lemma 6.6.4, we have that $I_X^* = \phi_X$. Thus since I_X is an essentially strictly convex rate function, ϕ_X is an essentially smooth function by Theorem 6.3.3. Hence since $\mathrm{dom}(\phi_X) = \mathbb{R}$ by Lemma 6.6.4, we have that ϕ_X is a differentiable function on $\mathbb{R}$. Note that ϕ_X is a convex function, therefore, $\phi_X \in C^1(\mathbb{R})$ by Darboux's intermediate value theorem. □

By the monotonicity of ϕ_X', we define $\underline{\alpha} := \lim_{t\to-\infty} \phi_X'(t)$ and $\overline{\alpha} := \lim_{t\to+\infty} \phi_X'(t)$. Then by convexity of ϕ_X, it is easy to verify the following limits

$$\underline{\alpha} = \lim_{q\to-\infty} \frac{\phi_X(q)}{q}, \text{ and } \overline{\alpha} = \lim_{q\to+\infty} \frac{\phi_X(q)}{q}. \tag{6.52}$$

6.6.4 A measure supported on the level set and a proof of Theorem 6.C

Lemma 6.6.7. *Under the conditions of Theorem 6.C, for every $\alpha \in (\underline{\alpha}, \overline{\alpha})$, there exists $q \in \mathbb{R}$ such that $\mu_q(K_\alpha) = 1$. In particular, $(\underline{\alpha}, \overline{\alpha}) \subset \mathscr{L}_X$.*

The proof of the lemma relies on the following well-known result in the theory of convex functions.

Proposition 6.6.8 ([16, 45])**.** *For any convex function Λ defined on $\mathbb{R}$, we have*
(1) $\beta q \le \Lambda^*(\beta) + \Lambda(q)$ *for any* $q, \beta \in \mathbb{R}$,
(2) *if Λ is differentiable at q then $\beta q = \Lambda^*(\beta) + \Lambda(q) \Longleftrightarrow \beta = \Lambda'(q)$.*

Proof of Lemma 6.6.7. Note that the for every $q \in \mathbb{R}$, the limit $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_\nu e^{qX_n}$ exists by Lemma 6.6.4. One ([19]) can prove that the following limit

$$C_q(\beta) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mu_q} e^{\beta X_n},$$

also exists, and $C_q(\beta) = \phi_X(q+\beta) - \phi_X(q)$, $\forall \beta, q \in \mathbb{R}$. Then we have $C_q'(\beta) = \phi_X'(q+\beta)$ and $C_q(0) = 0$, and thus $C_q^*(\gamma) \ge 0$ for any $\gamma \in \mathbb{R}$. Moreover, $C_q^*(\alpha) = 0$ if and only if $\alpha = C_q'(0) = \phi_X'(q)$. The following two observations show that μ_q is supported on the level set K_α:

- For any nonempty closed set $F \subset \mathbb{R}$ such that $\alpha \notin F$, $\eta_F := \inf_{\gamma\in F} C_q^*(\gamma) > 0$.

- Since the limit $\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}_{\mu_q}e^{\beta X_n}$ exists for any $\beta\in\mathbb{R}$, by the first part of Theorem 6.1.3, for any closed set F, we have,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_q(\{\omega\in\Omega:\frac{1}{n}X_n(\omega)\in F\})\le-\eta_F<0.$$

In fact, for $m\in\mathbb{N}$, consider $F_m:=\mathbb{R}\setminus(\alpha-\frac{1}{m},\alpha+\frac{1}{m})$. From the above observations and the Borel-Cantelli lemma, one has for every $m\in\mathbb{N}$ that

$$\mu_q\Big(\bigcap_{j\in\mathbb{N}}\bigcup_{n\ge j}\Big\{\frac{1}{n}X_n\in F_m\Big\}\Big)=0. \tag{6.53}$$

Therefore,

$$\mu_q(K_\alpha)=\lim_{m\to\infty}\mu_q\Big(\bigcup_{j\in\mathbb{N}}\bigcap_{n\ge j}\Big\{\frac{1}{n}X_n\in F_m^c\Big\}\Big)=1. \tag{6.54}$$

□

Proof of Theorem 6.C. The first claim: $(\underline{\alpha},\overline{\alpha})\subset\mathscr{L}_X$ is proven in Lemma 6.6.7. Now we show that $\mathscr{L}_X\subset[\underline{\alpha},\overline{\alpha}]$. Take any $\alpha\in\mathscr{L}_X$, then by Theorem 5.B, $\phi_X^*(\alpha)\le h=h_{\mathsf{top}}(T,\Omega)$. Thus by Young's inequality (the first statement of Proposition 6.6.8), one has for all $q\in\mathbb{R}$,

$$\alpha q\le\phi_X^*(\alpha)+\phi_X(q)\le h+\phi_X(q). \tag{6.55}$$

Hence for $q<0$,

$$\alpha\ge\frac{h}{q}+\frac{\phi_X(q)}{q},\ \text{and for } q>0,\ \ \alpha\le\frac{h}{q}+\frac{\phi_X(q)}{q}. \tag{6.56}$$

Furthermore, from (6.52) and (6.56), for $q<0$, we conclude that $\alpha\ge\lim_{q\to-\infty}\big(\frac{h}{q}+\frac{\phi_X(q)}{q}\big)=\underline{\alpha}$, and for $q>0$, we conclude that $\alpha\le\lim_{q\to+\infty}\big(\frac{h}{q}+\frac{\phi_X(q)}{q}\big)=\overline{\alpha}$.

The second claim: By combining Lemma 6.6.3, Lemma 6.6.7, Proposition 6.6.8 and Theorem 6.2.3, one concludes the proof of the second claim (c.f. (6.21)-(6.23)).

□

6.7 Examples and Discussion

In this section, we shall discuss some of the known results of multifractal formalism, and show how they can be treated by the results of this chapter.

6.7.1 Lyapunov exponents for products of matrices.

We shall adopt the setup in ([23],[24], [27]). Consider a map $M : \Omega \to M_d(\mathbb{R})$, where $\Omega := \{1, ..., q\}^{\mathbb{Z}_+}$ is a full shift space with the shift transformation T, and $M_d(\mathbb{R})$ is the algebra of $d \times d$ real matrices. For $n \in \mathbb{N}$, $x \in \Omega$, set $M_n(x) := M(T^{n-1}x)\, M(T^{n-2}x) \cdots M(x)$. We impose the following two conditions on the map M:

(C1): $M_n(x) \neq O$ holds for all $n \in \mathbb{N}$ and $x \in \Omega$, where O is the zero matrix;

(C2): M depends only on finitely many coordinates, i.e., M is locally constant.

Note that condition (C1) is automatically satisfied if, for example, the map M takes values in the set $GL_d(\mathbb{R})$ of irreducible matrices, or the set $M_d^+(\mathbb{R})$ of $d \times d$ positive matrices. In [24], the authors imposed an irreducibility condition which also implies the condition (C1). Under the condition (C1), one can define the (upper) Lyapunov exponent of the cocycle (T, M) as: for $x \in \Omega$,

$$\lambda_M^+(x) := \lim_{n\to\infty} \frac{1}{n} \log \|M_n(x)\|,$$

provided the limit exists, where $\|\cdot\|$ is a matrix norm. Let $K_\alpha := \{x \in \Omega : \lambda_M^+(x) = \alpha\}$, and $\mathscr{L}_M := \{\alpha \in \mathbb{R} : K_\alpha \neq \emptyset\}$. We denote $X_n(x) := \log\|M_n(x)\|$ for all $n \in \mathbb{N}$, $x \in \Omega$. Then it is clear that X_n is a locally constant function since M is only dependent on finitely many coordinates, therefore, the sequence $\{X_n\}$ satisfies the Bowen condition. In [27], the authors introduced the pressure function P_M of the cocycle (T, M) by

$$P_M(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in\Omega_n} \sup_{x\in[\omega]} \|M(T^{n-1}x)\, M(T^{n-1}x) \cdots M(x)\|^t, \ \ \forall\, t \in \mathbb{R},$$

where $\Omega_n := \{1, ..., q\}^n$, $n \in \mathbb{N}$. Note that the above limit exists and

$$P_M(t) = \log q + \phi_X(t),\ t \in \mathbb{R}, \tag{6.57}$$

where ϕ_X is the log moment generating function of the sequence $\{X_n\}$, i.e., $\phi_X(t) = \lim_{n\to\infty} \frac{1}{n} \log \int_\Omega \exp(tX_n)d\nu$, $\ t \in \mathbb{R}$, and ν is the uniform Bernoulli measure on Ω. Then by applying Theorem 5.B, one can obtain that

$$h_{\mathrm{top}}(T, K_\alpha) \leq \log q - \phi_X^*(\alpha) = \inf_{q\in\mathbb{R}} \{-\alpha q + P_M(q)\} \tag{6.58}$$

for all $\alpha \in \mathscr{L}_M$. Note that the last inequality generalizes the upper bound of Theorem 1.1 in [24]. In the same theorem, the authors also proved the lower bound under much stronger conditions than the conditions we have imposed above.

In this setup, we can not treat the lower bound with Theorem 6.C, since the sequence $\{X_n\}$ is not almost additive in general. However, if the map M takes values in the set $M_d^+(\mathbb{R})$ of $d \times d$ positive matrices, then it was proven in [27], [23] that the sequence $\{X_n\}$ is in fact almost additive. Furthermore, in this case, D.J. Feng has proven in [23] that the pressure function P_M is differentiable, thus ϕ_X is also differentiable by (6.57). Then, in this particular setup, one can apply Theorem 6.C and obtain the equality in (6.58).

6.7.2 Ratios of the Birkhoff sums

In large deviation theory, the contraction principle describes how the large deviation rate function evolves if the corresponding sequence of random elements (probability measures, random variables) is transformed by a continuous map. More specifically, if $\mathscr{X}$ and $\mathscr{Y}$ are Hausdorff topological spaces and a family $\{\mu_n\}$ of probability measures on $\mathscr{X}$ satisfies the LDP with the rate function I, $F : \mathscr{X} \to \mathscr{Y}$ is a continuous map, then $\tau_n := \mu_n \circ F^{-1}$ satisfies LDP with the rate function $J(y) := \inf\limits_{x:\, F(x)=y} I(x)$. For example, suppose $X_n^{(1)} = S_n\phi$, $X_n^{(2)} = S_n\psi$, where ϕ, ψ are continuous functions on a compact metric space Ω and ψ is strictly positive. Suppose ν is a probability measure such that the (vector) sequence $X = (X_n^{(1)}, X_n^{(2)})$ satisfies the LDP with the rate function $I_X(\alpha, \beta)$. Then a new sequence $Z = \{Z_n\}_{n\geq 1}$ of random variables

$$Z_n := \frac{X_n^{(1)}}{X_n^{(2)}} = \frac{S_n\phi}{S_n\psi}, \;\; n \geq 1,$$

satisfies LDP with the rate function $J_Z(\gamma) = \inf\limits_{\alpha/\beta=\gamma} I_X(\alpha, \beta)$. Thus we can easily obtain many non-trivial observables satisfying LDP. In case, $X_n^{(1)} = S_n\phi$, $X_n^{(2)} = S_n\psi$, $\psi > 0$, one easily shows that $\{nZ_n\}$ satisfies the weak Bowen condition, therefore, Theorem 6.A applies, and hence $h_{\mathsf{top}}(T, K_\gamma^Z) \leq h_{\mathsf{top}}(T, \Omega) - \inf\limits_{\alpha/\beta=\gamma} I_X(\alpha, \beta) = h_{\mathsf{top}}(T, \Omega) - J_Z(\gamma)$. In this case, one can also easily obtain a lower bound as well. Indeed, for any α, β such that

$$K_{\alpha,\beta}^X := \{\omega \in \Omega : \frac{1}{n}(S_n\phi)(\omega) \to \alpha,\; \frac{1}{n}(S_n\psi)(\omega) \to \beta\},$$

we have that $K_{\alpha,\beta}^X \subseteq K_{\alpha/\beta}^Z$. Therefore, $h_{\mathsf{top}}(T, K_\gamma^Z) \geq h_{\mathsf{top}}(T, K_{\alpha,\beta}^X)$, where $\gamma = \alpha/\beta$. Note that Theorem 6.C has been established for random variables, but the same argument works for random vectors as well. (There are also other results for the level sets of the form $K_{\alpha,\beta}^X$, c.f., [40, 48].) Hence $h_{\mathsf{top}}(T, K_\gamma^Z) \geq h_{\mathsf{top}}(T, \Omega) - I_X(\alpha, \beta)$, and thus

$$\begin{aligned} h_{\mathrm{top}}(T, K_\gamma^Z) &\geq \sup_{\alpha/\beta=\gamma} [h_{\mathrm{top}}(T,\Omega) - I_X(\alpha,\beta)] \\ &= h_{\mathrm{top}}(T,\Omega) - \inf_{\alpha/\beta=\gamma} I_X(\alpha,\beta) \\ &= h_{\mathrm{top}}(T,\Omega) - J_Z(\gamma). \end{aligned}$$

Therefore, combining two inequalities conclude that $h_{\mathrm{top}}(T, K_\gamma^Z) = h_{\mathrm{top}}(T,\Omega) - J_Z(\gamma)$.

6.7.3 Almost additive sequences over a subshift of finite type.

In [6], the authors considered almost additive sequences of continuous potentials and they obtained an expression for the multifractal spectra of the level sets of the sequence in terms of a rate function. In this subsection, we shall discuss that the setup considered in [6] is covered by Theorem 6.C. Let us now recall the result of [6] using the notation of the present chapter.

Theorem 6.7.1 ([6]). *Let (Ω, T) be a topologically mixing subshift of finite type. Assume that $X := \{X_n\}_{n\in\mathbb{N}}$ is an almost additive sequence satisfying Bowen's condition, and X is not cohomologous to a constant, i.e., $\frac{X_n}{n}$ does not uniformly converge to a constant as $n \to \infty$. Furthermore, assume that $\lim_{n\to\infty} \frac{1}{n}\int_\Omega X_n \, d\nu = \inf_{n\in\mathbb{N}} \frac{1}{n}\int_\Omega X_n \, d\nu = 0$, where ν is the unique measure of maximal entropy for T. Then*

$$h_{top}(T, K_\alpha) = h_{top}(T,\Omega) - \phi_X^*(\alpha).$$

It is clear that the Bowen condition implies the weak Bowen condition, and a mixing subshift of finite type satisfies the condition (A3) in Theorem 6.C. In Theorem 6.C, it is also assumed that the sequence $\{\frac{1}{n}X_n\}_{n\in\mathbb{N}}$ satisfies LDP with an essentially strictly convex good rate function. Note that if the moment generating function ϕ_X of the sequence X is finite and differentiable on $\mathbb{R}$, then by Theorem 6.1.3 $\{\frac{1}{n}X_n\}_{n\in\mathbb{N}}$ satisfies LDP with a good rate function ϕ_X^*. Furthermore, by Theorem 6.3.3, ϕ_X^* is also an essentially strictly convex function. Below, we show that the conditions of Theorem 6.7.1 imply differentiability of the moment-generating function ϕ_X. In [6], it is observed that

$$\phi_X(t) = P_{\mathrm{top}}(T, tX) - h_{\mathrm{top}}(T,\Omega), \text{ for all } t \in \mathbb{R}, \tag{6.59}$$

where $P_{\text{top}}(T, tX)$ is *the topological pressure* (for the concept, see [6], [1], [25], [52]) of T of the almost additive sequence $X = \{X_n\}$, and

$$\phi_X(t) = \lim_{n\to\infty} \frac{1}{n} \log \int_{\Omega} e^{tX_n(x)} \nu(dx)$$

as usual. In [2], by generalising the well-known result in [46] to almost additive setting, the authors proved that $t \mapsto P_{\text{top}}(T, tX)$ is differentiable at 0 if the almost additive sequence X has unique *equilibrium state* (for the concept, see [6], [1], [25], [52]), and showed

$$\frac{d}{dt} P_{\text{top}}(T, tX)\Big|_{t=t_0} = \lim_{n\to\infty} \int_{\Omega} \frac{X_n}{n} d\mu_{t_0 X},$$

where μ_{t_0X} is the unique equilibrium state of $t_0X = \{t_0X_n\}$. In 2006, Barreira proved that if an almost additive sequence X satisfies Bowen's condition, then it has a unique equilibrium state [1], therefore, from the above one can conclude that the pressure function $t \in \mathbb{R} \mapsto P_{\text{top}}(T, tX)$ is differentiable everywhere on $\mathbb{R}$. Then from (6.59), the log-moment generating function ϕ_X is also differentiable on $\mathbb{R}$, and hence, the conditions of Theorem 6.7.1 imply the conditions of Theorem 6.C.

6.7.4 Multifractal Formalism in the absence of strict convexity of the rate function.

The Manneville-Pomeau map is one of the well-known examples of a non- uniformly expanding interval map. For a given $s \in (0,1)$, the Manneville–Pomeau map $T : [0,1] \to [0,1]$ is defined as

$$T(x) := x + x^{1+s} \mod 1.$$

It should be noted that T is topologically conjugate to a one-sided full shift, therefore, it has the specification property. It is also easy to check that T is strongly topologically exact and expansive. The *upper Lyapunov exponent* λ_T is defined as

$$\lambda_T(x) = \lim_{n\to\infty} \frac{1}{n} \log(T^n)'(x)$$

provided that the limit exists at a point $x \in [0,1]$. Set

$$K_\alpha := \{x \in \Omega : \lambda_T(x) = \alpha\}, \quad \mathscr{L}_T := \{\alpha \in \mathbb{R} : K_\alpha \neq \emptyset\}.$$

Define, $\psi(x) := \log T'(x) = \log(1 + (1+s)x^s)$, $x \in [0,1]$. Then it is easy to notice that the Lyapunov exponent λ_T exists at a point x if and only if the limit

$\lim_{n\to\infty}\frac{1}{n}S_n\psi(x)$ converges, and $\lim_{n\to\infty}\frac{1}{n}S_n\psi(x)=\lambda_T(x)$. It was shown in [44], [49] that the pressure function $P_\psi(q):=P(T,q\psi)$, $q\in\mathbb{R}$ exhibits a *first order phase transition* at the point $q=-1$, in a sense that P_ψ is not differentiable at $q=-1$, furthermore, it is strictly convex positive differentiable function in $(-1,\infty)$, and $P_\psi(q)=0$ for all $q<-1$. Since T is expansive and satisfies the specification, there is a unique measure of maximal entropy $\nu\in\mathscr{M}([0,1])$ for T, and it also becomes the Ahlfors-Bowen measure ([50]). Define $X_n:=S_n\psi$ for all $n\in\mathbb{N}$, and set $\phi_X(q):=\limsup_{n\to\infty}\frac{1}{n}\log\int_0^1 e^{qX_n}d\nu$, $q\in\mathbb{R}$. Then one can easily check that $\phi_X(q)=h_{\mathsf{top}}(T,[0,1])+P_\psi(q)$ for all $q\in\mathbb{R}$. Thus since $\{X_n\}$ satisfies the weak Bowen condition one obtains from Theorem 5.B that for all $\alpha\in\mathscr{L}_T$,

$$h_{\mathsf{top}}(T,K_\alpha)\le h_{\mathsf{top}}(T,[0,1])-\phi_X^*(\alpha)=-P_\psi^*(\alpha)=\inf_{q\in\mathbb{R}}\{P(T,q\psi)-\alpha q\}. \tag{6.60}$$

Now we discuss the lower bound. As we have already mentioned, ϕ_X is not differentiable everywhere; otherwise, the sequence $\{\frac{1}{n}X_n\}$ satisfies LDP with an essentially strictly convex good rate function. However, Lemma 6.6.3 and Lemma 6.6.7 are still applicable if $\alpha\in\{0\}\cup((P_\psi)'_+(-1),\ \overline{\alpha})$, where $(P_\psi)'_+(-1)>0$ is the right derivative of P_ψ at the point -1, and $\overline{\alpha}:=\lim_{q\to\infty}P'_\psi(q)$. Thus one can obtain from these lemmas that

$$h_{\mathsf{top}}(T,K_\alpha)=h_{\mathsf{top}}(T,[0,1])-\phi_X^*(\alpha)=-P_\psi^*(\alpha)=\inf_{q\in\mathbb{R}}\{P(T,q\psi)-\alpha q\}, \tag{6.61}$$

holds for all $\alpha\in\{0\}\cup((P_\psi)'_+(-1),\ \overline{\alpha})$.

We should note that in [48], the above equality was proven for all $\alpha\in(0,\overline{\alpha})$ by a different method.

6.7.5 Local or pointwise entropies.

Recall that the upper and the lower local entropies of an invariant measure μ at a point $x\in\Omega$ are defined as

$$\overline{h}_\mu(T,x):=\lim_{\epsilon\to0+}\limsup_{n\to\infty}-\frac{1}{n}\log\mu(B_n(x,\epsilon)),\ \ \underline{h}_\mu(T,x):=\lim_{\epsilon\to0+}\liminf_{n\to\infty}-\frac{1}{n}\log\mu(B_n(x,\epsilon)).$$

If $\overline{h}_\mu(T,x)=\underline{h}_\mu(T,x)$, we say that the local entropy of the measure μ at x exists and denote by $h_\mu(T,x)$ the common value. Finally, consider the α-level set $K_\alpha^{(\mu)}$ of the local entropies

$$K_\alpha^{(\mu)}:=\{x\in\Omega:h_\mu(T,x)=\alpha\},$$

and the entropy spectrum $\mathscr{E}_\mu(\alpha):=h_{\mathsf{top}}(T,K_\alpha^{(\mu)})$.

In [47], the authors introduced the notion of *weak entropy doubling condition*

$$C_n(\epsilon) = \sup_x \frac{\mu(B_n(x,\epsilon))}{\mu(B_n(x,\epsilon/2))} < \infty, \text{ and } \lim_n \frac{1}{n} \log C_n(\epsilon) = 0.$$

Note that the weak entropy doubling condition implies the weak Bowen condition for observables $X_n : \Omega \to \mathbb{R}$ given by

$$X_n(x) = -\log \mu(B_n(x,\epsilon)).$$

Indeed, it is easy to show that $v_{n,\epsilon}(X_n) \le \log C_n(2\epsilon)$.

In [47] it was shown that if μ is fully supported measures satisfying the (weak) entropy doubling condition, then for any α one has:

$$\mathscr{E}_\mu(\alpha) := h_{\mathsf{top}}(T, K_\alpha^{(\mu)}) \le H_\mu^*(\alpha),$$

where $H_\mu^*(\alpha)$ is the Legendre transform of the correlation entropy function

$$H_\mu(q) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log I_\mu(q,n,\epsilon),\ I_\mu(q,n,\epsilon) = \begin{cases} \displaystyle\int \mu(B_n(x,\epsilon))^{q-1} d\mu, & \text{if } q \ne 1, \\ \displaystyle\int \log \mu(B_n(x,\epsilon)) d\mu, & \text{if } q = 1. \end{cases}$$

Note the similarity between this result and the upper bound (Theorem 6.A) – under very similar mild regularity assumptions, one obtains an upper bound on the multifractal spectrum.

In order to obtain a lower bound on the multifractal spectrum of local entropies one needs additional assumptions on the measure μ. For example, μ is the so-called Bowen-Gibbs measure:

$$\frac{1}{C(\epsilon)} \le \frac{\mu(B_n(x,\epsilon))}{\exp(\sum_{k=0}^{n-1} \phi(T^k x))} \le C(\epsilon).$$

Then the multifractal analysis of level sets of local entropies is reduced to the analysis of level sets of Birkhoff's averages $K_\alpha = \{\omega : \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k\omega) \to \alpha\}$, see [47].

6.7.6 Weighted ergodic averages

Fan ([19]) recently studied multifractal spectra of a sequence $\{X_n\}$ in the form $X_n := \sum_{i=0}^{n-1} f_i \circ T^i$, where $f_i,\ i \in \mathbb{Z}_+$ are continuous functions on the shift space $\Omega := \mathscr{A}^{\mathbb{Z}_+}$, where $\mathscr{A}$ is a finite alphabet with $\#\mathscr{A} = q$. A motivation to study this kind of sequence has been the weighted ergodic averages $S_n^w f = \sum_{i=0}^{n-1} w_i f \circ T^i$, $n \in \mathbb{Z}_+$ of a potential $f \in C(\Omega)$, where $\{w_i : i \in \mathbb{Z}_+\} \subset \mathbb{R}$ are some fixed weights. This is indeed a special case of the former sequence if one sets $f_i = w_i f$ for all $i \in \mathbb{Z}_+$. To find the dimension spectra of the sequence $\{X_n\}$ in terms of the partition function, Fan made two assumptions:

(H1): The limit $\phi(t) = \lim_{n\to\infty} \frac{1}{n} \log \int_\Omega \exp(tX_n) d\mu$ exists for all $t \in \mathbb{R}$,

(H2): $\sup_{n\in\mathbb{Z}_+} \sup_{x_0^{n-1}=y_0^{n-1}} \sum_{i=0}^{n-1} |f_i(T^i(x)) - f_i(T^i(y))| < +\infty$.

Recall that in the shift space, for all $Z \subset \Omega$, $h_{\mathrm{top}}(T, Z) = \log q \cdot \dim_H(Z)$. Under the assumptions (H1)-(H2), Fan obtained the following theorem reformulated in terms of topological entropies:

Theorem 6.7.2 ([19] **).** *If* $\lambda \geq 0$*,*

$$h_{top}(T,\Omega) - \phi^*(\max \partial\phi(\lambda)) \leq h_{top}(T, K_{\partial\phi(\lambda)}) \leq h_{top}(T,\Omega) - \phi^*(\min \partial\phi(\lambda)),$$

where $\partial\phi(\lambda)$ *is the set of all subgradients of* ϕ *at* λ*, and*

$$K_{[a;b]} = \left\{ \omega \in \Omega : a \leq \liminf_{n\to\infty} \frac{X_n(\omega)}{n} \leq \limsup_{n\to\infty} \frac{X_n(\omega)}{n} \leq b \right\} \text{ for } a, b \in \mathbb{R}.$$

If $\lambda < 0$*, we have similar estimates but we have to exchange the roles of* $\min \partial\phi(\lambda)$ *and* $\max \partial\phi(\lambda)$*. In particular, if* ϕ *is differentiable at* λ*, then for the level set* K_α*,* $\alpha = \phi'(\lambda)$*, one has*

$$h_{top}(T, K_\alpha) = h_{top}(T,\Omega) - \phi^*(\alpha). \tag{6.62}$$

Now we should note that condition (H2) implies the Bowen condition, and thus the weak Bowen condition as well. In fact, the condition (H2) is strictly stronger than the Bowen condition as the following example shows.

Example 6.7.3. *Let* $\mathscr{A} = \{0,1\}$*, and* $a_n = \frac{(-1)^n}{n^\gamma}$*,* $\gamma \in (1,2]$*,* $n \in \mathbb{N}$*. Define* $f(\omega) := \sum_{n\in\mathbb{N}} a_n \omega_n$ *for* $\omega \in \Omega$*, then* $f \in C(\Omega)$*. It is easy to notice that*

1) $\sup_{n\in\mathbb{N}} \sum_{j=1}^{\infty} (|a_j| + \ldots + |a_{j+n-1}|) = \sum_{j\in\mathbb{N}} j|a_j| = +\infty;$

2) $\sup_{n\in\mathbb{N}} \sum_{j=1}^{\infty} |a_j + \ldots + a_{j+n-1}| < +\infty.$

Thus

$$\sup_{n\in\mathbb{N}} \sup_{\omega_0^{n-1}=\bar{\omega}_0^{n-1}} \sum_{i=0}^{n-1} |(f\circ T^i)(\omega)-(f\circ T^i)(\bar{\omega})| = +\infty, \tag{6.63}$$

but $\sup_{n\in\mathbb{N}} v_n(f_n) < +\infty$, *where* $f_n := \sum_{i=0}^{n-1} f\circ T^i$.

Since the sequence $\{X_n\}$ satisfies the weak Bowen condition, Theorem 5.B generalizes the inequality $h_{\mathsf{top}}(T,K_\alpha) \le h_{\mathsf{top}}(T,\Omega) - \phi^*(\alpha)$ which is a part of (6.62) since it relaxes both assumptions (H1) and (H2).

It should be stressed that Theorem 6.C is not applicable in this setup since the sequence $\{X_n\}_{n\in\mathbb{N}}$ is not almost additive in general.

6.8 Final remarks

In this work, we began building a systematic framework to derive multifractal formalism directly from large deviation principles. Under standard multifractal conditions, we progressed this approach by establishing Theorems A and C. Nevertheless, we believe this program has the potential to extend well beyond the cases studied here; indeed, the conclusions of Theorems A and C should be applicable in more general settings. To encourage further development, we highlight below several promising setups that we consider especially suitable for advancing this program.

A) It would be interesting if one could obtain an analogue of the above results for the Hausdorff dimension instead of the topological entropy. However, the Hausdorff dimension might be more difficult to treat as it is not related to the dynamics of the underlying transformation.

B) In [38–40], the authors propose considering a more general form of the level sets than the standard $K_\alpha := \{\omega\in\Omega : \lim_{n\to\infty} \frac{X_n(\omega)}{n} = \alpha\}$. Specifically, for $C \subset \mathbb{R}$,

$$K_C := \{\omega\in\Omega : \text{ the set of limit points of the sequence } \{X_n(\omega)/n\}_{n\in\mathbb{N}} \text{ is } C\}. \tag{6.64}$$

Note that one restores K_α if $C = \{\alpha\}$.

It would be interesting to investigate the relationship between the *generalized multifractal spectra* of $\{X_n\}$,

$$C \subset \mathbb{R} \mapsto \mathscr{E}_X(C) := h_{\mathsf{top}}(T, K_C),$$

and the LDP rate function of $\{X_n\}$.

C) In [10], the author explored the entropy spectrum for the ratios of Birkhoff sums (see Subsection 6.7.2) for a family of pairs of potentials (ϕ_i, ψ_i), $i \in 1,\dots,d$.

In [10], instead of a specification condition, the author assumes the existence of a *dense family of continuous functions with unique equilibrium states.* This assumption slightly differs from the specification condition (see Definition 6.2.7 and Theorem 6.2.8) in the present chapter. Therefore, it will be worthwhile to explore whether our theorems in this chapter can be proven using the existence of a dense family of continuous potentials with unique equilibrium states, rather than relying on strong topological exactness.

D) The results of this chapter can be extended to the case that the reference measure is a Bowen–Gibbs measure, i.e., there exists potential $\zeta \in C(\Omega, \mathbb{R})$ such that for every $\epsilon > 0$ there exists a constant $C(\epsilon) \geq 1$ satisfying for all $n \in \mathbb{N}$ and $x \in \Omega$ that

$$C(\epsilon)^{-1} \exp(\zeta_n(x)) \leq \nu(B_n(x, \epsilon)) \leq C(\epsilon) \exp(\zeta_n(x)). \tag{6.65}$$

It is easy to see that the Ahlfors-Bowen measures satisfy the above property (6.65) with a constant potential $\zeta \equiv const$. To extend the results for such measures, instead of the topological entropy, one can consider another dimension characteristics of the sets related to the potential ζ. We note that the Pesin–Pitskel topological pressure [42] associated with the potential ζ is suitable for this purpose.

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Summary

This thesis explores the rigorous mathematical foundations of Gibbs measures, which are fundamental objects at the intersection of statistical mechanics and dynamical systems. Rooted in the pioneering works of Boltzmann and Gibbs, Gibbs measures provide a precise probabilistic description of the equilibrium states of systems with a vast number of interacting constituents. These ideas, formalised in the Dobrushin–Lanford–Ruelle (DLR) framework, lie at the heart of our understanding of phase transitions, long-term statistical behaviour, and the connections between microscopic interactions and macroscopic phenomena.

The thesis is structured into three interconnected parts. The first part develops the conceptual and technical foundations for several notions of Gibbsianity. It explores how Gibbs measures appear both as equilibrium distributions in statistical mechanics and as invariant measures in ergodic theory and dynamical systems. One of the key results is a constructive demonstration of a translation-invariant Gibbsian specification that can not be generated by any translation-invariant, uniformly absolutely summable interaction. This reveals an important structural mismatch and exposes conceptual limitations within the classical framework. To address this gap, the thesis establishes a new variational principle formulated entirely in terms of specifications, providing a robust alternative perspective that aligns more closely with the language of dynamical systems and does not rely on traditional interaction-based formulations. Moreover, it has been shown in the literature that certain long-range systems can admit equilibrium states that lack Gibbsian properties. To resolve this, the thesis identifies barely a minimal condition under which all equilibrium states of such long-range systems remain Gibbsian, thus clarifying the delicate boundary between general equilibrium measures and Gibbs states.

The second part turns to one-dimensional systems, which, despite their apparent simplicity, are a fertile ground for testing theoretical ideas about long-range interactions, phase transitions, and the spectral properties of transfer operators. Here, the Perron–Frobenius–Ruelle transfer operator plays a central role: it links the statistical structure of Gibbs measures with dynamical evolution and mixing properties. A significant contribution of this section is the development of new techniques for proving the existence and regularity of eigenfunctions of

transfer operators under minimal regularity assumptions – extending the classical theory to a class of long-range models for which previous results do not apply. These methods are applied in particular to the celebrated Dyson model, a prototypical example of a one-dimensional long-range Ising model that illustrates the rich interplay between statistical mechanics and dynamical systems.

The third part extends the study of Gibbs measures to multifractal analysis and large deviation principles in dynamical systems. Many phenomena of physical and mathematical interest depend on understanding rare events and the fine-scale statistical structure of systems over long time horizons. Using topological entropy as a dimension, the thesis develops a rigorous framework connecting the multifractal spectrum of general observables to large deviation properties. The results provide necessary and sufficient conditions for deducing the dimension and frequency of rare events directly from the large deviation behaviour of the system. This approach unifies ideas from ergodic theory, large deviations theory, and dimension theory, showing how the same core probabilistic structures underlie both typical and exceptional behaviour in complex systems.

Taken together, this thesis offers a cohesive and original development of modern Gibbs theory that bridges foundational concepts and contemporary challenges. By clarifying the subtle relation between specifications and interactions, establishing new results for transfer operators in long-range settings, and connecting multifractal properties to large deviation principles, it significantly advances our understanding of how local interactions, global symmetries, and rare events interplay in mathematical physics and dynamical systems.

Samenvatting

Deze scriptie onderzoekt de rigoureuze wiskundige fundamenten van Gibbs - maatregelen, die fundamentele objecten zijn op het snijvlak van de statistische mechanica en dynamische systemen. Geworteld in het baanbrekende werk van Boltzmann en Gibbs bieden Gibbs-maatregelen een nauwkeurige probabilistische beschrijving van de evenwichtstoestanden van systemen met een groot aantal onderling interagerende componenten. Deze ideeën, geformaliseerd in het Dobrushin – Lanford – Ruelle (DLR)-kader, vormen de kern van ons begrip van faseovergangen, langetermijn statistisch gedrag en de verbanden tussen microscopische interacties en macroscopische verschijnselen.

De scriptie is opgebouwd uit drie onderling verbonden delen. Het eerste deel ontwikkelt de conceptuele en technische basis voor verschillende noties van Gibbsianiteit. Het onderzoekt hoe Gibbs-maatregelen zowel verschijnen als evenwichtsverdelingen in de statistische mechanica als invariantiemaatregelen in de ergodentheorie en dynamische systemen. Een van de belangrijkste resultaten is een constructieve demonstratie van een translatie - invariante Gibbs-specificatie die niet kan worden voortgebracht door enige translatie-invariante, uniform absoluut sommeerbare interactie. Dit onthult een belangrijk structureel spanningsveld en legt conceptuele beperkingen bloot binnen het klassieke kader. Om dit hiaat aan te pakken, formuleert de scriptie een nieuw variatieprincipe dat volledig is opgesteld in termen van specificaties. Dit biedt een robuust alternatief perspectief dat nauwer aansluit bij de taal van dynamische systemen en niet afhankelijk is van traditionele, interactie-gebaseerde formuleringen. Bovendien is in de literatuur aangetoond dat bepaalde systemen met langeafstandsinteracties evenwichtstoestanden kunnen toelaten die geen Gibbs-eigenschappen bezitten. Om dit op te lossen, identificeert de scriptie een minimaal vereiste conditie waaronder alle evenwichtstoestanden van zulke langeafstandssystemen Gibbsiaans blijven, waarmee de delicate grens tussen algemene evenwichtsmaatregelen en Gibbs-toestanden wordt verduidelijkt.

Het tweede deel richt zich op eendimensionale systemen die, ondanks hun schijnbare eenvoud, een vruchtbare grond vormen om theoretische ideeën over langeafstandsinteracties, faseovergangen en de spectrale eigenschappen van transferoperatoren te toetsen. Hierin speelt de Perron–Frobenius–Ruelle transferop-

erator een centrale rol: deze verbindt de statistische structuur van Gibbs - maatregelen met de dynamische evolutie en mengingseigenschappen. Een belangrijke bijdrage van dit deel is de ontwikkeling van nieuwe technieken om het bestaan en de regulariteit van eigenfuncties van transferoperatoren aan te tonen onder minimale regulariteitsvoorwaarden — waarmee de klassieke theorie wordt uitgebreid naar een klasse van langeafstandsmodellen waarvoor eerdere resultaten niet van toepassing waren. Deze methoden worden in het bijzonder toegepast op het beroemde Dyson model, een prototypisch voorbeeld van een eendimensionaal langeafstands Isingmodel dat de rijke wisselwerking tussen statistische mechanica en dynamische systemen illustreert.

Het derde deel breidt de studie van Gibbs - maatregelen uit naar multifractale analyse en grote deviatieprincipes in dynamische systemen. Veel fenomenen van fysisch en wiskundig belang hangen samen met het begrijpen van zeldzame gebeurtenissen en de fijnmazige statistische structuur van systemen over lange tijdschalen. Met topologische entropie als dimensie ontwikkelt de scriptie een rigoureus kader dat het multifractale spectrum van algemene meetbare grootheden verbindt met eigenschappen van grote deviatie. De resultaten bieden noodzakelijke en voldoende voorwaarden om de dimensie en frequentie van zeldzame gebeurtenissen direct af te leiden uit het grote deviatiesgedrag van het systeem. Deze benadering verenigt ideeën uit de ergodentheorie, grootedeviatietheorie en dimensionale theorie, en laat zien hoe dezelfde probabilistische kernstructuren zowel typisch als uitzonderlijk gedrag in complexe systemen ondersteunen.

Gezamenlijk biedt deze scriptie een samenhangende en originele ontwikkeling van de moderne Gibbs-theorie die fundamentele concepten en hedendaagse uitdagingen met elkaar verbindt. Door de subtiele relatie tussen specificaties en interacties te verduidelijken, nieuwe resultaten vast te stellen voor transferoperatoren in langeafstandscontexten en multifractale eigenschappen te koppelen aan grote deviatieprincipes, draagt zij significant bij aan ons begrip van hoe lokale interacties, globale symmetrieën en zeldzame gebeurtenissen samenkomen in de mathematische fysica en dynamische systemen.

Acknowledgments

First, I would like to express my sincere gratitude to my supervisor, Prof. Evgeny Verbitskiy, for insightful discussions and regular meetings throughout my doctoral journey. I am also thankful to my other promoter, Prof. Frank den Hollander, for carefully reading my thesis and offering thoughtful advice and support whenever needed.

I am grateful to the members of the doctoral committee – Prof. Frank Redig, Prof. Christof Külske, and Dr. Loren Coquille – for taking the time to read and evaluate my thesis.

I would like to thank the secretariat of the MI, in particular Pim Overgaauw and Anja Verduijn, for their constant willingness to help and for their warm and friendly support.

A special thanks goes to Prof. Aernout van Enter for his patience in addressing both my professional and less-professional questions, for his early reviews of our manuscripts, and for providing valuable insights and suggestions to improve them.

I would also like to thank Prof. Roberto Fernández for his helpful comments and for taking the time to answer my questions.

I would also like to thank Prof. Stefano Luzzatto and Prof. Emanuel Carneiro from the ICTP for their consistent support during my job application process.

My sincere thanks go as well to Dr. Marks Ruziboev for our many enjoyable and inspiring conversations in mathematics and beyond. I also thank him for letting me know about the PhD position at Leiden University when it was announced.

Last but not least, I am grateful to my family for their unwavering encouragement and support throughout this journey.

Curriculum Vitae

Mirmukhsin Utkirbek ugli Makhmudov was born in Khorazm, Uzbekistan. He completed his primary education at a local village school in 2011 and then attended a local high school in Urganch, graduating in 2014.

After achieving a high score in the national university entrance exam, Mirmukhsin was awarded a full government scholarship to study in the Department of Mathematics at the National University of Uzbekistan (NUUz). During his undergraduate studies, he distinguished himself by winning several national and international mathematics competitions, including the 2nd Qori Niyoziy National Mathematics Competition and the 5th North Countries Universities Mathematical Competition. In 2018, he graduated with honours from the mathematics department. That same year, he earned the highest score in the department's master's entrance exam, securing a full scholarship for the master's program at NUUz.

Alongside his master's studies at NUUz, Mirmukhsin was offered a position at the Monetary Policy Department of the Central Bank of Uzbekistan as a first-category economist. His career at the CBU progressed in parallel with his academic work, and he was promoted twice – first to the lead economist and then to the chief economist role.

In 2020, Mirmukhsin completed his master's degree, writing a thesis under the supervision of Professor Rasul G'anixo'jaev on the topic "Small parametric dynamical systems", which resulted in a single-authored publication. That same year, he was awarded a Postgraduate Diploma Fellowship by the International Centre for Theoretical Physics (ICTP) in Trieste, Italy. He spent a year in the Mathematics section at ICTP, where he completed a diploma thesis under the supervision of Professor Stefano Luzzatto, focusing on the properties of topological pressure and its relation to invariant measures.

In September 2021, following his time at ICTP, Mirmukhsin was admitted to the PhD program at Leiden University. His research at Leiden focused on statistical mechanics, ergodic theory, and dynamical systems, with particular emphasis on thermodynamic formalism, the Dobrushin–Lanford–Ruelle (DLR) Gibbs formalism, and multifractal analysis. Throughout his PhD, Mirmukhsin presented his research at numerous national and international conferences, including the

Mark Kac Seminar, the 11th World Congress in Probability and Statistics, and the 9th European Congress of Mathematics. In addition to his research activities, he also assisted in teaching several university courses at Leiden.

Mirmukhsin's doctoral dissertation, "Gibbs States in Statistical Mechanics and Dynamical Systems" compiles results from five research projects, which also led to research papers listed below:

1. *The Eigenfunctions of the Transfer Operator for the Dyson model in a field*, J. Stat. Phys., 192(92), 2025, 1-20.

2. *Multifractal Formalism from Large Deviations*, 2024, joint with E. Verbitskiy and Q. Xiao

3. *On an extension of a theorem by Ruelle to long-range potentials*, 2024, joint with A. van Enter, R. Fernández and E. Verbitskiy

4. *Gibbs properties of equilibrium states*, 2025, joint with E. Verbitskiy

5. *Concentration inequalities and Transfer operators for supercritical Dyson models*, 2025.