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STRONG MIXING FOR THE PERIODIC LORENTZ GAS FLOW WITH INFINITE HORIZON

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Abstract. We establish strong mixing for the $\mathbb{Z}^d$-periodic, infinite horizon, Lorentz gas flow for continuous observables with compact support. The essential feature of this natural class of observables is that their support may contain points with infinite free flights. Dealing with such a class of functions is a serious challenge and there is no analogue of it in the finite horizon case. The mixing result for the aforementioned class of functions is obtained via new results: (1) mixing for continuous observables with compact support consisting of configurations at a bounded time from the closest collision; (2) a tightness-type result that allows us to control the configurations with long free flights. To prove (1), we establish a mixing local limit theorem for the Sinai billiard flow with infinite horizon, previously an open question. As far as we know, our approach to the tightness result has no analogue in the literature.

1. Introduction and main result

We are interested in mixing for the continuous time dynamics of the $\mathbb{Z}^d$-periodic Lorentz gas ($d \in \{1,2\}$). This model has been introduced by Lorentz in [20] to model the diffusion of electrons in a low conductive metal. It describes the behaviour of a point particle moving at unit speed in the plane $\mathcal{D}_2 := \mathbb{R}^2$ (when $d=2$) or on the tube $\mathcal{D}_1 := \mathbb{R}\times\mathbb{T}$ (when $d=1$, writing as usual $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ for the one-dimensional torus) between a $\mathbb{Z}^d$-periodic locally finite configuration of convex obstacles with disjoint closures and $\mathcal{C}^3$ boundary (with non-null curvature), with elastic collisions on them (pre-collisional and post-collisional angles being equal). We write Ω_d for the set of possible positions, that is the set of positions in $\mathcal{D}_d$ that are not inside an obstacle.

The set of configurations is the set $\widetilde{\mathcal{M}}$ of couples of position and unit velocity $(q,\vec{v}) \in \Omega_d\times\mathbb{S}^1$, identifying pre-collisional and post-collisional vectors at a collision time (rigorously, $\widetilde{\mathcal{M}}$ is the quotient of $\Omega_d\times\mathbb{S}^1$ by the equivalence relation identifying pre- and post-collisional vectors). The Lorentz gas flow $(\Phi_t)_t$ maps a configuration $(q,\vec{v})$ (corresponding to a couple position and velocity at time 0) to the configuration $\Phi_t(q,\vec{v}) = (q_t,\vec{v}_t)$ corresponding to the couple position and velocity at time t of a particle that was at time 0 at position q with velocity $\vec{v}$. This flow $(\Phi_t)_t$ preserves the infinite Lebesgue measure $\widetilde{\nu}$ on $\Omega_d\times\mathbb{S}^1$, normalized so that $\widetilde{\nu}\left((\Omega_2\cap[0,1[^2)\times\mathbb{S}^1\right) = 1$ if $d=2$ and so that $\widetilde{\nu}\left((\Omega_1\cap([0,1[\times\mathbb{T}))\times\mathbb{S}^1\right) = 1$ if $d=1$. It is natural to consider also the dynamics at collision times. The space $\widetilde{M}$ for this dynamics is the set of configurations $(q,\vec{v})\in\widetilde{\mathcal{M}}$ with $q\in\partial\Omega_d$. The collision map $\widetilde{T}:\widetilde{M}\to\widetilde{M}$, that

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maps a configuration at a collision time to the configuration at the next collision time, is referred to as the Lorentz gas map and preserves an infinite measure $\widetilde{\mu}$ absolutely continuous with respect to the Lebesgue measure. Let us write $\widetilde{W}_t : \widetilde{\mathcal{M}} \to \mathcal{D}_d$ for the map corresponding to the displacement up to time t:

$$\forall (q,\vec{v}) \in \widetilde{\mathcal{M}}, \quad \Phi_t(q,\vec{v}) = (q_t,\vec{v}_t) \quad \Rightarrow \quad \widetilde{W}_t(q,\vec{v}) = q_t - q.$$

If $d=1$ we set $\widetilde{W}'_t : \widetilde{\mathcal{M}} \to \mathbb{R}$ for the first coordinate of $\widetilde{W}_t$. If $d=2$, $\widetilde{W}_t$ takes its values in $\mathbb{R}^2$, we then just set $\widetilde{W}'_t = \widetilde{W}_t$. In both cases, $\widetilde{W}'_t$ is the natural projection of $\widetilde{W}_t$ on $\mathbb{R}^d$.

When every trajectory touches eventually at least one obstacle, we speak of *finite horizon* Lorentz gas. In the finite horizon case, it follows from [6, 7] that $\widetilde{W}_t$ satisfies a standard central Limit Theorem meaning that $(\widetilde{W}'_t/\sqrt{t})_t$ converges strongly in distribution,[1] as $t \to +\infty$, to a centered Gaussian random variable with non-degenerate variance matrix given by an infinite sum.

When there exists at least a trajectory that never touches an obstacle, we speak of *infinite horizon* Lorentz gas. In this article, we focus on the "fully dimensional" infinite horizon case, meaning that there exist at least d non-parallel unbounded trajectories touching no obstacle. In this case it follows from [34] by Szász and Varjú (see Subsection 2.3, in particular Proposition 2.5 for details) that[2]

$$\frac{\widetilde{W}'_t}{\sqrt{t\log t}} \Longrightarrow \mathcal{N}(0,\Sigma), \quad \text{as } t \to +\infty, \tag{1.1}$$

where Σ is a d-dimensional definite positive symmetric matrix which, furthermore, is given by an explicit formula in terms of the configuration of obstacles (recalled at the beginning of Section 6).

We are interested here in the question of strong mixing. We recall that an infinite measure preserving system $(\hat{X},\hat{T},\hat{\mu})$ is said to be strongly mixing if there exist a sequence $\widehat{a}_n \to \infty$ and a class of integrable functions f, g so that

$$\widehat{a}_n \int_{\hat{M}} f.g\circ \hat{T}^n \, d\hat{\mu} \to \int_{\hat{M}} f\, d\hat{\mu} \int_{\hat{M}} g\, d\hat{\mu}, \tag{1.2}$$

as $n \to +\infty$. The sequence $\widehat{a}_n$ gives the speed of convergence to 0 of $\int_{\hat{M}} f.g\circ\hat{T}^n\, d\hat{\mu}$. The first such rate was obtained in [35] for a very restrictive class of intermittent maps preserving an infinite measure. This was later generalized to larger classes of such maps in [23] and [17]. For other notions of mixing in the infinite measure set-up (such as local-global and global-global) were introduced in [19] (see also [13, 15] and references therein).

In the set-up of the *discrete* time Lorentz gas $(\widetilde{M},\widetilde{T},\widetilde{\mu})$, mixing in the sense of (1.2) is well understood in both finite and infinite horizon cases and it is a direct consequence of a mixing local limit theorem (MLLT) for the cell change (see e.g. [27, Section 3]). For the finite horizon case, we refer to [33] for the key LLT (which can be generalized in MLLT and thus provides mixing) and to [28] for expansions of any order. In the much more difficult set-up of infinite horizon case, we refer

[1] In this article, the strong convergence in distribution means the convergence in distribution with respect to any probability measure absolutely continuous with respect to the Lebesgue measure.

[2] The notation $\Longrightarrow \mathcal{N}(0,C)$ means the strong convergence in distribution to a Gaussian random variable of distribution $\mathcal{N}(0,C)$, that is centered with variance matrix C.

to [34] for LLT (which, again, leads to MLLT and mixing) and to [29] for error terms. There is a plethora of limit theorems known in the discrete time set-up with finite horizon case. Some results are also known for the discrete time Lorentz gas with infinite horizon case (in particular, [9, 34] and more recently, [29]).

Mixing for *continuous* time Lorentz gas $(\widetilde{\mathcal{M}}, (\Phi_t)_t, \widetilde{\nu})$ is seriously more challenging. Even in the set-up of the *finite* horizon Lorentz gas flow, mixing in the sense of (1.2) was open until the work of [12] and very recently, expansions of any order have been obtained in [15]. Strictly speaking, the work [12] focused on a mixing local limit theorem (MLLT) for the Sinai billiard flow with *finite* horizon, but as in [15], mixing in the sense of (1.2) and MLLT are equivalent. For related, but weaker, results on MLLT for group extensions of suspension flows with bounded roof function, not applicable as such to Sinai billiards, we refer to [2].

Nothing is known about the mixing for the Lorentz gas flow with *infinite* horizon. In this paper we address this open question and establish

Theorem 1.1. *For any continuous compactly supported functions $f, g : \Omega_d \times \mathbb{S}^1 \to \mathbb{R}$,*

$$\int_{\widetilde{\mathcal{M}}} f.g \circ \Phi_t \, d\widetilde{\nu} \sim \frac{\int_{\widetilde{\mathcal{M}}} f \, d\widetilde{\nu} \int_{\widetilde{\mathcal{M}}} g \, d\widetilde{\nu}}{(2\pi t \log t \det(\Sigma))^{\frac{d}{2}}}, \quad \text{as } t \to +\infty, \tag{1.3}$$

where Σ is the variance matrix appearing in (1.1).

Theorem 1.1 gives mixing for observables with support that may contain configurations with infinite free flights. In the set-up of the Lorentz gas flow with infinite horizon, this class of observables is the natural one. Theorem 1.1 can be rephrased in terms of vague convergence (see comments after Corollary 2.3). The main ingredients, which are new and important results on their own, used in the proof of Theorem 1.1 are

(1) Strong mixing for observables f, g with supports uniformly 'close' to a collision time (Corollary 2.3 of Proposition 2.2), i.e. the supports of f, g are at a bounded time of the closest collision time (either in the past or in the future). This mixing result is an easy consequence of a MLLT for the Sinai billiard flow with infinite horizon. The present MLLT, Proposition 2.2 and its variant Theorem 2.6, are first main results and are established via two joint local limit results for the Sinai billiard map on the cell change function and flight time together: (a) Joint MLLT, Lemma 3.4; (b) Joint Local Large Deviation, Lemma 3.5.
(2) A tightness type result, Theorem 5.1, that allows f, g to have any compact support in $\widetilde{\mathcal{M}}$. In particular, the supports of f, g can contain configurations of particles that will never hit an obstacle. The proof of Theorem 5.1 provided in Section 5 exploits a very delicate decomposition of the type of possible free flights along with Joint MLLT *with good error terms* for the Sinai billiard map (as in Lemma 3.2 and Corollary 3.3), combined with a series of subtle new estimates (including a large deviation estimate). We emphasize that the Joint MLLT with *error terms* (not only the JMLLT mentioned in the above item, and even with sharper error terms than the one obtained in [29] in the non-joint MLLT) together with several other new technical estimates are required ingredients for the proof of Theorem 5.1.

We conclude the introductory section with a very brief summary of the various results along with an outline of the paper.

In Section 2, we introduce most of the required notations, and state MLLTs for the Sinai billiard flow as in Proposition 2.2 for the cell change function, and as in Theorem 2.6 for the flight function. In Section 2 we also record a consequence of Proposition 2.2, namely Corollary 2.3 that proves mixing for continuous observables with compact support consisting of configurations at a bounded time from the closest collision; in short, this gives mixing for continuous observables supported on a region on which the free flights (either in the past or in the future) are uniformly bounded.

In Section 3, we state the joint limit results (Joint CLT, Joint MLLT with *error terms*, Joint LLD) for the Sinai billiard map, for the couple formed by the cell change function with the flight time. Using the statement of these key technical ingredients, in Sections 4 and 5 we prove Theorem 2.6, and Theorem 1.1.

The proofs of the technical key results stated in Section 3 are included in Sections 6, 7 and in Appendix A. While the joint LLD, Lemma 3.5, follows by slightly modifying the proof of LLD for the cell change function obtained in [22], all the other technical results obtained in this paper, namely, the Joint CLT and the Joint MLLT with *error terms* stated in Section 3 are new and require serious new ideas and work.

In Section 5 we state and prove the tightness result Theorem 5.1. At the beginning of Section 5, we use the statement of Theorem 5.1 to complete the proof of Theorem 1.1. The role and the novelty of Theorem 5.1 has been already summarized in item (2) above. Finally, we mention that, in Section 5, as a byproduct of certain technical lemmas we obtain a large deviation result, namely Proposition 5.2, which is of independent interest.

Notation. We use "big O" and $\ll$ notation interchangeably, writing $b_n = \mathcal{O}(c_n)$ or $b_n \ll c_n$ if there are constants $C > 0$, $n_0 \geq 1$ such that $b_n \leq Cc_n$ for all $n \geq n_0$. As usual, $b_n = o(c_n)$ means that there exists ε_n such that, for all n large enough, $b_n = c_n\varepsilon_n$ and $\lim_{n\to+\infty} \varepsilon_n = 0$ and $b_n \sim c_n$ means that $b_n = c_n + o(c_n)$. Unless otherwise specified, given $x \in \mathbb{R}^d$, we let $|x|$ be the usual Euclidean norm of x. Throughout this article, when $d = 1$, we identify $\mathbb{Z}^1$ (resp. $\mathbb{R}^1$) with $\mathbb{Z} \times \{0\}$ (resp. $\mathbb{R} \times \{0\}$). In particular, for any $(q, z) \in \Omega_1 \times \mathbb{R}$, the notation $q + z$ means $q + (z, 0)$.

2. MLLT for the Sinai billiard flow and mixing for the $\mathbb{Z}^d$-extension flow

2.1. **Notations and previous results.**

Let $d \in \{1, 2\}$. The domain Ω_d of the $\mathbb{Z}^d$-periodic Lorentz gas is given by $\Omega_d := \mathcal{D}_d \setminus \bigcup_{i=1}^I \bigcup_{\ell\in\mathbb{Z}^d} (\mathcal{O}_i + \ell)$ where $\mathcal{O}_1, \dots, \mathcal{O}_I$ is a non-empty finite family of convex open sets with $\mathcal{C}^3$ boundary of non-null curvature such that the obstacles $\mathcal{O}_i + \ell$ have pairwise disjoint closures. We recall that we are interested in the fully dimensional infinite horizon and so assume throughout that the interior of the billiard domain Ω_d contains at least d unbounded corridors (made of unbounded parallel lines) the directions of which are not parallel to each other.

Sinai billiard. Quotienting the system $(\widetilde{\mathcal{M}}, (\Phi_t)_t)$ by $\mathbb{Z}^d$ (for the position), we obtain the Sinai billiard flow $(\mathcal{M}, (\phi_t)_t)$ (see [32]) which describes the evolution of point particles moving at unit speed in $\Omega := \Omega_d/\mathbb{Z}^d = \mathbb{T}^2 \setminus \bigcup_{i=1}^I \overline{\mathcal{O}_i}$ with elastic reflection off $\partial\Omega$ (where $\overline{\mathcal{O}_i}$ is the image of $\mathcal{O}_i$ by the canonical projection $p_d : \mathcal{D}_d \to \mathbb{T}^2$). The flow $(\phi_t)_t$ preserves the probability measure ν on

$\mathcal{M} = (\Omega \times \mathbb{S}^1)/\equiv$ that is proportional to the Lebesgue measure, where $\equiv$ is the equivalence relation identifying pre- and post-collisional vectors. The Poincaré map of ϕ_t with Poincaré section $\partial\Omega \times \mathbb{S}^1$ is the Sinai billiard map (M, T, μ), where the two-dimensional phase space $M = \{(q, \vec{v}) \in \mathcal{M} : q \in \partial\Omega\}$ (position in $\partial\Omega$ and unit post-collisional velocity vector) is identified with $\partial\Omega \times (-\pi/2, \pi/2)$ (we parametrise here the post-collisional velocity vector by its angle with the normal to $\partial\Omega$). This map T sends a post-collisional vector to the post-collisional vector corresponding to the next collision. This map preserves the probability measure μ with density $\cos\varphi/(2|\partial\Omega|)$ at the point $(q, \varphi) \in \partial\Omega \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. The flight time between consecutive collisions is the return time of $(\phi_t)_t$ to M and we denote it by $\tau : M \to \mathbb{R}_+$. In this notation, we have the following identification

$$T(x) = (\phi_\tau)(x) = \phi_{\tau(x)}(x).$$

We set $\tau_n := \sum_{k=0}^{n-1} \tau \circ T^k$, with the usual convention $\tau_0 := 0$. For any $x \in \mathcal{M}$, we set $N_t(x) \in \mathbb{N}_0$ for the *collisions number* in the time interval $(0, t]$ starting from the configuration x. We observe that, for $x \in M$, this quantity satisfies

$$\tau_{N_t(x)}(x) \le t < \tau_{N_t(x)+1}(x). \tag{2.1}$$

Furthermore, for all $x \in M$ and all $u \in [0, \tau(x))$ and any $t \in [0, +\infty)$, $N_t(\phi_u(x)) = N_{t+u}(x)$. With these notations, the Sinai billiard flow $(\mathcal{M}, (\phi_t)_t, \nu)$ is isomorphic to the suspension flow $(\widehat{\mathcal{M}}, (\widehat{\phi}_t)_t, \widehat{\nu})$, given by

$$\widehat{\mathcal{M}} = \{(x, u) \in M \times [0, +\infty) : 0 \le u < \tau(x)\},$$
$$\widehat{\phi}_s(x, u) = (T^{N_{s+u}(x)}(x), s + u - \tau_{N_{s+u}(x)}(x)),$$
$$\widehat{\nu} = (\mu \times \mathrm{Leb})/\mu(\tau), \quad \text{where } \mu(\tau) := \int_M \tau \, d\mu,$$

via the isomorphism $(x, u) \in \widehat{\mathcal{M}} \mapsto \phi_u(x) \in \mathcal{M}$ (this map is injective, its image is the set of configurations in $\mathcal{M}$ that do not belong to an infinite free flight).

$\mathbb{Z}^d$-extension and cell change function. We recall that the $\mathbb{Z}^d$-periodic Lorentz gas map $(\widetilde{M}, \widetilde{T}, \widetilde{\mu})$ can be represented by the $\mathbb{Z}^d$-extension of the Sinai billiard map (M, T, μ) by the *cell change* function κ that can be defined as follows. For any $\ell \in \mathbb{Z}^d$, we call ℓ-cell the set $\mathcal{C}_\ell$ of configurations $(q, v) \in \widetilde{M}$ such that $q \in \bigcup_{i=1}^I (\partial O_i + \ell)$. Because of the $\mathbb{Z}^d$-periodicity of the model, there exists $\kappa : M \to \mathbb{Z}^d$, called the *cell change function*, such that

$$\widetilde{x} = (q, \vec{v}) \in \mathcal{C}_\ell \quad \Rightarrow \quad \widetilde{T}(x) \in \mathcal{C}_{\ell + \kappa(p_d(q), \vec{v})}. \tag{2.2}$$

Note that, for any $\widetilde{x} \in \widetilde{M}$, there exists a unique $x = ((q, \vec{v}), \ell) \in M \times \mathbb{Z}^d$ such that[3] $\widetilde{x} = (p_{d,0}^{-1}(q) + \ell, \vec{v})$, where $p_{d,0}$ denotes the restriction of p_d to $\bigcup_{i=1}^I \partial\mathcal{O}_i$ ($\vec{v}$ is the velocity of $\widetilde{x}$, setting $\widetilde{q}$ for the position of $\widetilde{x}$, (q, ℓ) is such that $q = p_d(\widetilde{q})$ and $\widetilde{x} \in \mathcal{C}_\ell$). Formula (2.2) can be rewritten under the form

$$\forall ((q, \vec{v}), \ell) \in M \times \mathbb{Z}^d, \ T(q, \vec{v}) = (q', \vec{v}') \ \Rightarrow \ \widetilde{T}(p_{d,0}^{-1}(q) + \ell, \vec{v}) = \left(p_{d,0}^{-1}(q') + \ell + \kappa(q, \vec{v}), \vec{v}'\right). \tag{2.3}$$

[3] Recall that, if $d = 1$, we identify $\mathbb{Z}^1$ with $\mathbb{Z} \times \{0\}$, meaning that for any $q' \in \mathcal{D}_1$ and any $\ell \in \mathbb{Z}^1$, the notation $q' + \ell$ means $q' + (\ell, 0)$.

This gives the identification of $(\widetilde{M},\widetilde{T},\widetilde{\mu})$ by the $\mathbb{Z}^d$-extension of (M,T,μ) by $\kappa: M\to\mathbb{Z}^d$. A direct and classical induction ensures that, for any $((q,\vec{v}),\ell)\in M\times\mathbb{Z}^d$ and any $n\in\mathbb{N}$,

$$(2.4)\quad T^n(q,\vec{v})=(q'_n,\vec{v}'_n)\ \Rightarrow\ \widetilde{T}^n(p_{d,0}^{-1}(q)+\ell,\vec{v})=\left(p_{d,0}^{-1}(q'_n)+\ell+\kappa_n(q,\vec{v}),\vec{v}'_n\right),$$

where we set $\kappa_n:=\sum_{j=0}^{n-1}\kappa\circ T^j$.

2.2. **Mixing for the Lorentz gas seen as a suspension flow.** We will use crucially the fact established in the previous section that $(\widetilde{\mathcal{M}},(\Phi_t)_t,\widetilde{\nu})$ can be represented as a suspension flow by $(x,\ell)\mapsto\tau(x)$ over $(\widetilde{M},\widetilde{T},\widetilde{\mu})$ which itself can be represented as a $\mathbb{Z}^d$ extension of (M,T,μ) by κ. Thus, we can represent $\widetilde{\mathcal{M}}$ by $\widehat{\mathcal{M}}\times\mathbb{Z}^d$. In this part, we state a mixing local limit theorem for κ_n and see how we can use it to easily derive Theorem 1.1 in the case of functions f,g with support at a bounded time from a collision, i.e. for functions that are compactly supported in $\widehat{\mathcal{M}}\times\mathbb{Z}^d$. As detailed in Section 5, these functions form a much more restrictive class than the ones of Theorem 1.1. To state these results, we shall introduce two classes of sets $\mathcal{F}$ (resp. $\widetilde{\mathcal{F}}$) that will correspond to the set of measurable sets of configurations in $\mathcal{M}$ (resp. $\widetilde{\mathcal{M}}$) with previous collision in some fixed subset of M (resp. some fixed cell of $\widetilde{M}$), at some time in a fixed bounded time interval.

Definition 2.1. Let $\mathcal{F}$ be the class of measurable subsets A of $\mathcal{M}$ of the form $A=\phi_I(A_0)=\{\phi_u(x),\ x\in A_0,\ u\in I\}$ that are represented in $\widehat{\mathcal{M}}$ by $A_0\times I\subset\widehat{\mathcal{M}}$ (implying that $I\subset[0,\inf_{A_0}\tau)$), with $A_0\subset M$ a measurable set satisfying $\mu(\partial A_0)=0$ and with I a bounded interval.

Let $\widetilde{\mathcal{F}}$ be the set of subsets of $\widetilde{\mathcal{M}}$ corresponding to $A_0\times I\times\{\ell\}\subset\widehat{\mathcal{M}}\times\mathbb{Z}^d$, with $\phi_I(A_0)\in\mathcal{F}$ and $\ell\in\mathbb{Z}^d$, that is sets of the form

$$\left\{\Phi_u(p_{d,0}^{-1}(q)+\ell,\vec{v}):(q,\vec{v})\in A_0,u\in I\right\}\ \text{with}\ \phi_I(A_0)\in\mathcal{F},\ \ell\in\mathbb{Z}^d.$$

We state now a MLLT for κ_{N_t} defined on $\mathcal{M}$ by

$$\forall(x,u)\in\widehat{\mathcal{M}},\ \forall t\in[0,+\infty),\quad \kappa_{N_t}(\phi_u(x)):=\kappa_{N_t(\phi_u(x))}(x)=\kappa_{N_{t+u}(x)}(x).$$

This observable κ_{N_t} will be understood as the cell change during the time interval $(0,t]$.

Proposition 2.2. *Let $A,B\in\mathcal{F}$ and let K be a bounded subset of $\mathcal{D}_d$ with $Leb(\partial K)=0$. Then*

$$(2.5)\qquad \forall\ell\in\mathbb{Z}^d,\quad (t\log t)^{\frac{d}{2}}\nu\left(A\cap\{\phi_t\in B,\ \kappa_{N_t}=\ell\}\right)\sim\widetilde{g}_d(0)\,\nu(A)\nu(B),$$

as $t\to\infty$, where $\widetilde{g}_d$ is the density of the d-dimensional Gaussian distribution $\mathcal{N}(0,\Sigma)$ appearing in (1.1).

This result is contained in a more general MLLT stated in Proposition 4.1 (applied with $w_t=\ell$, $w=0$, $K=\{0\}$). An immediate consequence of Proposition 2.2 is the following light version of Theorem 1.1 for compactly supported observables in the 'extended suspension' $\widehat{\mathcal{M}}\times\mathbb{Z}^d$; in particular, the supports of these functions only contain configurations that have hit or will hit an obstacle in a bounded time.

Corollary 2.3. *Let $n\in\mathbb{N}$, setting*

$$E_{\pm n}=\left\{\Phi_{\pm u}(q+\ell,\vec{v})\in\widetilde{\mathcal{M}}:q\in\bigcup_{i=1}^{I}\partial O_i,\ u\in[0,n],\ \ell\in\mathbb{Z}^d,\ |\ell|\le|n|\right\},$$

then, for any $f,g:\widetilde{\mathcal M}\to\mathbb R$ that are μ-a.e. continuous functions and supported respectively in E_{-n} and in E_n,

$$\int_{\widetilde{\mathcal M}} f.g\circ\Phi_t\,d\widetilde\nu \sim \frac{\int_{\widetilde{\mathcal M}} f\,d\widetilde\nu \int_{\widetilde{\mathcal M}} g\,d\widetilde\nu}{(2\pi t\log t\det(\Sigma))^{\frac d2}}, \tag{2.6}$$

as $t\to+\infty$.

Proof. Let A,B be two sets belonging to $\widetilde{\mathcal F}$ corresponding to respectively $A_0\times I\times\{\ell_0\}$ and $B_0\times J\times\{\ell_0'\}$ in $\widehat{\mathcal M}\times\mathbb Z^d$. We observe that

$$\widetilde\nu\,(A\cap\Phi_{-t}(B))=\nu(\phi_I(A_0)\cap\{\phi_t\in\phi_J(B_0),\,\kappa_{N_t}=\ell_0'-\ell_0\}).$$

Thus, it follows from (2.5) that

$$(t\log t)^{\frac d2}\widetilde\nu\,(A\cap\Phi_{-t}(B))\sim\widetilde g_d(0)\nu(\phi_I(A_0))\nu(\phi_J(A_0))=\widetilde g_d(0)\widetilde\nu(A)\widetilde\nu(B).$$

This result extends directly to any finite union $A,B\subset\widetilde{\mathcal M}$ of sets belonging to $\widetilde{\mathcal F}$, implying Krickeberg mixing as defined in [18] for the family of sets $(E_n)_{n\ge1}$. It follows from [18, Section 2] (see also, [24, Section 9] for the Krickeberg argument written for suspension flows) that (2.6) holds true for any f,g supported in some E_n and μ-almost everywhere continuous. To end the proof of Corollary 2.3, we notice that, Φ being invertible, if f is supported in E_{-n}, then $f\circ\Phi_{-n}$ is supported on E_n and we finally conclude with the use the following formula

$$\int_{\widehat{\mathcal M}} f.g\circ\Phi_t\,d\widetilde\nu=\int_{\widehat{\mathcal M}} f\circ\Phi_{-n}.g\circ\Phi_{t-n}\,d\widetilde\nu,$$

since $(t-n)\log(t-n)\sim t\log t$. □

The mixing result in Corollary 2.3 can be rephrased in terms of the vague convergence of the family of μ_t to $\mu\otimes\mu$ where μ_t is the measure on $(\widehat{\mathcal M})^2$ defined by $\mu_t(A'\times B')=\mu(A'\cap\Phi_{-t}B')$ for $A',B'\in\widetilde{\mathcal F}$ (this is a consequence of the Portmanteau theorem as in, for instance, [30]), and the same applies for Theorem 1.1.

Remark 2.4. We remark that mixing of the type of Corollary 2.3 has been previously obtained in [36] for $\mathbb Z$-extensions of Gibbs Markov semiflows with roof and displacement functions in the domain of a non-standard CLT. The method of proof in [36] is very different; in particular, it does not go via a MLLT for the base map.

2.3. **MLLT for the infinite horizon Sinai flow.** In this section we state the MLLT for a natural cocycle of the Sinai billiard flow, which corresponds to the displacement.

Free flight. Due to the $\mathbb Z^d$-periodicity, the *free flight* $\widetilde V:\widetilde M\to\mathcal D_d$ which is defined by

$$\forall(q,\vec v)\in\widetilde M,\quad \widetilde T(q,\vec v)=(\widetilde q,\vec v_1)\quad\Rightarrow\quad\widetilde V(q,\vec v)=\widetilde q-q \tag{2.7}$$

goes to the quotient by $\mathbb Z^d$, i.e. there exists $V:M\to\mathcal D_d$ such that

$$\widetilde V(q,\vec v)=V\,(p_d(q),\vec v)\,. \tag{2.8}$$

When $d=2$, this quantity is related to the flight time τ via the following identity

$$\text{if } d=2,\quad \tau=|V|. \tag{2.9}$$

Let us show that the free flight V is cohomologous to the cell change κ. It follows from (2.3), (2.7) and (2.8) that, for all $x = (q, \vec{v}) \in M$, if $T(q, \vec{v}) = (q', \vec{v}')$, then

$$\begin{aligned} V(x) &= \widetilde{V}\left(p_{d,0}^{-1}(q), \vec{v}\right) \\ &= p_{d,0}^{-1}(q') + \kappa(q, \vec{v}) - p_{d,0}^{-1}(q) = \kappa(x) + H_0(T(x)) - H_0(x), \end{aligned} \tag{2.10}$$

with $H_0(q, \vec{v}) = p_{d,0}^{-1}(q)$. Proceeding as for $\widetilde{W}_t$ in Section 1, if $d = 1$ we set $V' : \mathcal{M} \to \mathbb{R}$ for the first coordinate of V, and if $d = 2$, V takes its values in $\mathbb{R}^2$, we then just set $V' = V$. The following non-standard CLT was proved in [34] for V':

$$a_n^{-1} \sum_{j=0}^{n-1} V' \circ T^j \implies \mathcal{N}(0, \Sigma_0), \tag{2.11}$$

where $a_n = \sqrt{n \log n}$ and where $\Sigma_0 \in \mathbb{R}^{d\times d}$ is a positive-definite symmetric d-dimensional matrix (see (6.3) and (6.4)) for precise formulas). An important ingredient of [34] is that V lies in the domain of a non-standard CLT; that is, there exists $c > 0$ such that

$$\mu(|V| > t) \sim ct^{-2}. \tag{2.12}$$

Displacement function W_t. We have already defined in Section 1 the displacement function $\widetilde{W}_t : \widetilde{\mathcal{M}} \to \mathcal{D}_d$ and $\widetilde{W}'_t : \widetilde{\mathcal{M}} \to \mathbb{R}^d$ its projection on $\mathbb{R}^d$. Due to the $\mathbb{Z}^d$-periodicity of our model, both displacement functions go the quotient by $\mathbb{Z}^d$, i.e. there exists $W_t : \mathcal{M} \to \mathcal{D}_d$ and $W'_t : \mathcal{M} \to \mathbb{R}^d$ such that

$$\forall (q, \vec{v}) \in \widetilde{\mathcal{M}}, \quad \widetilde{W}_t(q, \vec{v}) = W_t(p_d(q), \vec{v}) \quad \text{and} \quad \widetilde{W}'_t(q, \vec{v}) = W'_t(p_d(q), \vec{v}).$$

Observe that W_t is a cocycle:

$$\forall x \in \mathcal{M},\ \forall t, s \geq 0, \quad W_{t+s}(x) = W_s(x) + W_t(\phi_s(x)) \tag{2.13}$$

and that

$$\forall x = (q, \vec{v}) \in M, \quad V(x) = W_\tau(x) := W_{\tau(x)}(x). \tag{2.14}$$

Thus the non-standard CLT for V' stated in (2.11) implies a non-standard CLT for W'_t via the relation (2.14) together with the classical scheme of lifting limit theorems from the induced map to the original system (map or flow) [16, 21]. This leads to the following result where we use the notation $a_t := \sqrt{t \log t}$.

Proposition 2.5 (CLT [34]). *As $t \to +\infty$, $a_t^{-1} W'_t \implies \mathcal{N}(0, \Sigma)$ where $\Sigma \in \mathbb{R}^{d\times d}$, $\Sigma = \Sigma_0/\mu(\tau)^{1/2}$ with Σ_0 as in (2.11).*

Let $v_0 : \mathcal{M} \to \mathbb{S}^1$ be the velocity map which is given by $v_0(q, \vec{v}) = \vec{v}$. Note that

$$\text{if } d = 2, \quad W_t := \int_0^t v_0 \circ \phi_s\, ds. \tag{2.15}$$

If $d = 1$, then W_t is the equivalent class in $\mathcal{D}_1$ (that is, W_t is the canonical projection) of the ergodic integral $\int_0^t v_0 \circ \phi_s\, ds$.

MLLT for the displacement function. Let us see that, due to (2.14), the coboundary equation (2.10) for $V-\kappa$ leads to a similar equation involving W. We consider the function $H_1:\mathcal{M}\to\mathcal{D}_d$ mapping $\mathbf{x}\in\mathcal{M}$ to the position of its representant in $\mathcal{D}_d$ with previous collision in $\mathcal{C}_0$, that is

$$\forall((q,\vec{v}),u)\in\widehat{\mathcal{M}},\quad H_1(\phi_u(q,\vec{v}))=\mathfrak{p}\left(\Phi_u\left(p_{d,0}^{-1}(q),\vec{v}\right)\right)=p_{d,0}^{-1}(q)+W_u(q,\vec{v}), \tag{2.16}$$

where $\mathfrak{p}:\widetilde{\mathcal{M}}\to\Omega_d$ is the natural projection. In other words, if $d=2$, then

$$H_1(\phi_u(q,\vec{v}))=H_0(q,\vec{v})+W_u(q,\vec{v})=p_{d,0}^{-1}(q)+u\vec{v}; \tag{2.17}$$

if $d=1$, $H_1(\phi_u(q,\vec{v}))$ is the class of $p_{d,0}^{-1}(q)+u\vec{v}$ in $\mathcal{D}_1$. Recall that we set $N_t:\mathcal{M}\to\mathbb{N}_0$ for the *collisions number* in the time interval $(0,t]$ (see in particular (2.1)). The above defined function H_1 satisfies the following important property:

$$\forall(x=(q,\vec{v}),u)\in\widehat{\mathcal{M}},\quad W_t(\phi_u(x))=\kappa_{N_{t+u}(x)}(x)+H_1(\phi_t(\phi_u(x)))-H_1(\phi_u(x)). \tag{2.18}$$

Indeed, setting $N:=N_{t+u}(x)$, we notice that

$$\phi_t(\phi_u(x))=\phi_{u'}(q',\vec{w}),\quad \text{with } u':=u+t-\tau_N(x),\ (q',\vec{w}):=T^N(x)=\phi_{\tau_N}(x), \tag{2.19}$$

and $(T^N(x),u')$ is in $\widehat{\mathcal{M}}$. Therefore, it follows from (2.13) and (2.14) that

$$\begin{aligned}W_t(\phi_u(x))&=W_{t+u}(x)-W_u(x)=W_{u'}(\phi_{\tau_N}(x))+W_{\tau_N}(x)-W_u(x)\\&=W_{u'}(T^N(x))+W_{\tau_N}(x)-W_u(x)\\&=W_{u'}(T^N(x))+V_N(x)-W_u(x).\end{aligned}$$

Finally, using (2.10) and (2.17), we obtain that

$$\begin{aligned}W_t(\phi_u(x))&=W_{u'}(T^N(x))+\kappa_N(x)+H_0(T^N(x))-H_0(x)-W_u(x)\\&=H_1(\phi_t(\phi_u(x)))+\kappa_N(x)-H_1(\phi_u(x)),\end{aligned}$$

as announced.

Recall that $a_t=\sqrt{t\log t}$.

Theorem 2.6 (MLLT for W_t)**.** *Let $A,B\in\mathcal{F}$ and let K be a bounded subset of $\mathcal{D}_d$ with $Leb(\partial K)=0$. Let $w\in\mathbb{R}^d$ and let $w_t\in\mathbb{R}^d$ such that $\lim_{t\to+\infty}w_t/a_t=w$. Then*[4]

$$\begin{aligned}&a_t^d\nu\left(A\cap\{\phi_t\in B,\,W_t\in w_t+K\}\right)\\&\sim\widetilde{g}_d\left(w\right)\int_{A\times B}\#((K+w_t+H_1(\mathbf{x})-H_1(\mathbf{y}))\cap\mathbb{Z}^d)\,d\nu(\mathbf{x})\,d\nu(\mathbf{y}),\end{aligned} \tag{2.20}$$

as $t\to\infty$, where $\widetilde{g}_d$ is the density of the d-dimensional Gaussian distribution $\mathcal{N}(0,\Sigma)$ appearing in Proposition 2.5 *and where H_1 is the function that has been defined in* (2.16)*.*

The proof of Theorem 2.6 is provided in Section 4, and will appear as a consequence of an analogous result (Proposition 4.1) stated for κ_{N_t} instead of W_t.

[4]Again, in this formula, if $d=1$, the notation w_t+K means $(w_t,0)+K$ and $\mathbb{Z}^d$ means $\mathbb{Z}\times\{0\}$.

3. Statements of the Joint LLT with error term and the joint LLD for the billiard map

Let $d \in \{0,1,2\}$. In this section we state the main technical results that will be used in the proofs of Theorem 2.6 (MLLT for the Sinai flow) and Theorem 1.1 (mixing for the Lorentz gas), including those used in the proof of the key tightness-type result Theorem 5.1 (stated in Section 5). We are interested in joint MLLT and LLD for the pair

$$\widehat{\Psi} = \widehat{\Psi}^{(d)} := (\kappa, \widetilde{\tau}) : M \to \mathcal{D}_d \times \mathbb{R}, \quad \text{with } \widetilde{\tau} := \tau - \mu(\tau), \quad \text{if } d \in \{1,2\}$$

or for

$$\widehat{\Psi} = \widehat{\Psi}^{(0)} := \widetilde{\tau}, \quad \text{if } d = 0.$$

Note that $\int_M \widehat{\Psi}\, d\mu = 0$. When $d \in \{1,2\}$, our joint limit results are related to the fact that the sum of $(\kappa\circ T^k, \widetilde{\tau}\circ T^k)_k$ satisfies a CLT with non-standard normalization a_n^{-1}. In particular, as clarified in the proof of Sublemma 6.1, the vector $\widehat{\Psi}$ is so that $\mu(|\widehat{\Psi}| > t) \sim ct^{-2}$. As usual, we write $\widehat{\Psi}_n = \sum_{j=0}^{n-1} \widehat{\Psi} \circ T^j$ and similarly for V_n, $\tau_n, \widetilde{\tau}_n$. We start with a non-degenerate CLT with non-standard scaling for $\widehat{\Psi}_n$.

Lemma 3.1 (Joint CLT for the billiard map)**.** $a_n^{-1}\widehat{\Psi}_n \implies \mathcal{N}(0, \Sigma_{d+1})$ *as* $n \to \infty$, *where* $\Sigma_{d+1} \in \mathbb{R}^{(d+1)\times(d+1)}$ *is positive-definite (see* (6.5) *for an explicit formula).*

This result is proved in Section 6 by adapting the proof of the CLT for V_n established in [34] via [3], writing $\widehat{\Psi}_n$ as a function of the two dimensional cell change plus a Lipschitz function.

Let us state a MLLT for $\widehat{\Psi}_n$ with a uniform error term. We write Λ_{d+1} for the Haar measure on $\mathbb{Z}^d \times \mathbb{R}$ given by the product of the counting measure on $\mathbb{Z}^d$ and of the Lebesgue measure on $\mathbb{R}$.

Lemma 3.2 (Joint MLLT for the billiard map)**.** *Let* $p > 2$ *and* $R > 0$. *We take* $\mathfrak{a}_n$ *such that* $\mathfrak{a}_n^2 = 2n\log(\mathfrak{a}_n) \sim n\log n = a_n^2$. *Assume* $G, H : M \to \mathbb{R}$ *are two bounded dynamically Hölder continuous functions and that* $h : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ *is integrable with compactly supported Lipschitz Fourier transform* $\widehat{h} : \mathbb{T}^d \times \mathbb{R} \to \mathbb{C}$. *There exists* $a_0 > 0$ *(depending only on* p *and on the Hölder exponent of* G *and* H*) such that, for all* $k_n < n/4$,

$$\begin{aligned}
&\mathbb{E}_\mu\left[G.h(\widehat{\Psi}_n - L).H\circ T^n\right] \\
&\quad = \mathfrak{a}_n^{-d-1}\mathbb{E}_\mu[H]\,\mathbb{E}_\mu[G]\left(g_{d+1}\left(\frac{L}{\mathfrak{a}_n}\right)\int_{\mathbb{Z}^d\times\mathbb{R}} h\, d\Lambda_{d+1} + \mathcal{O}\left((\log n)^{-1} + \frac{k_n}{n}\right)\right) \\
&\qquad + \mathcal{O}\Big(e^{-a_0 k_n}\|G\|_{Holder}\|H\|_{Holder} \\
&\qquad + a_n^{-d-2}(k_n\|G\|_{L^1}\|H\|_{L^p} + \|H\|_{L^1}\|\widehat{\Psi}_{2k_n}.G\|_{L^1(\mu)})\Big),
\end{aligned}$$

uniformly in $L \in \mathbb{Z}^d \times \mathbb{R}$, *in* (n, k_n) *as above, and in* h *such that* $\mathrm{supp}(\widehat{h}) \subset B(0,R)$ *and* $\|\widehat{h}\|_{Lipschitz} \le R$, *where* g_{d+1} *is the density for the* $(d+1)$*-dimensional Gaussian in Lemma* 3.1.

This result is proved in Section 7. The scheme of its proof follows the one of the MLLT established in [29, Theorem 2.2] but there are at least two main differences. First, here we need to obtain a Joint MLLT, which is different from [29, Theorem 2.2] which was a MLLT with error terms for the cell change function. The new ingredients needed to deal with the Joint MLLT (with error) are summarized in Section 6. Second, the error term obtained in [29] is not sharp enough for the present purposes. To establish Lemma 3.2, we need to be much more careful with the error terms all throughout the proof and this requires entirely new estimates, all obtained in Section 7.

A consequence of Lemma 3.2 is

Corollary 3.3. *Under the assumptions of Lemma 3.2, then*

$$\begin{aligned}\mathbb{E}_\mu &\left[G.h(\widehat{\Psi}_n - L).H\circ T^n\right] - \mathbb{E}_\mu[G]\mathbb{E}_\mu[H]\mathbb{E}_\mu[h(\widehat{\Psi}_n - L)] \\ &= \mathcal{O}\left(e^{-a_0 k_n}\|G\|_{Holder}\|H\|_{Holder}\right. \\ &\quad \left.+a_n^{-d-2}(k_n\|G\|_{L^1}\|H\|_{L^p} + \|H\|_{L^1}\|\widehat{\Psi}_{2k_n}.G\|_{L^1(\mu)})\right).\end{aligned}$$

Proof. We observe that

$$\begin{aligned}\mathbb{E}_\mu &\left[G.h(\widehat{\Psi}_n - L).H\circ T^n\right] - \mathbb{E}_\mu[G]\mathbb{E}_\mu[H]\mathbb{E}_\mu\left[h(\widehat{\Psi}_n - L)\right] \\ &= \mathbb{E}_\mu[(G-\mathbb{E}_\mu[G]).h(\widehat{\Psi}_n - L).H\circ T^n])] \\ &\quad + \mathbb{E}_\mu[G]\,\mathbb{E}_\mu[h(\widehat{\Psi}_n - L).(H-\mathbb{E}_\mu[H])\circ T^n])],\end{aligned}$$

and we apply Lemma 3.2 to the two terms of the right hand side of the above equality, since $\|\widehat{\Psi}_{2k_n}\|_{L^1(\mu)} = \mathcal{O}(k_n)$, the function $\widehat{\Psi}$ being integrable. □

The following MLLT for $\widehat{\Psi}_n$ will be shown (in Section 7) from Lemma 3.2.

Lemma 3.4. *Let $A_0, B_0 \subset M$ be measurable sets such that $\mu(\partial A_0) = \mu(\partial B_0) = 0$. Let $K \subset \mathbb{Z}^d\times\mathbb{R}$ be a bounded set with $\Lambda_{d+1}(\partial K) = 0$ (boundary in $\mathbb{Z}^d\times\mathbb{R}$). Then, for any $L > 0$,*

(3.1)

$$a_n^{d+1}\mu\left(A_0\cap T^{-n}(B_0)\cap\{\widehat{\Psi}_n(x)\in z+K\}\right) - g_{d+1}\left(\frac{z}{a_n}\right)\mu(A_0)\mu(B_0)\Lambda_{d+1}(K)$$

converges to 0 uniformly in $z\in\mathbb{Z}^d\times\mathbb{R} : |z|\le La_n$, as $n\to+\infty$.

The joint LLD we shall need is

Lemma 3.5 (Joint LLD for the billiard map)**.** *Let $U\subset\mathbb{R}^{d+1}$ be an open ball. Then*

$$\mu(\widehat{\Psi}_n\in z+U) \ll \frac{n}{a_n^{d+1}}\frac{\log(2+|z|)}{1+|z|^2} \tag{3.2}$$

uniformly in $n\ge 1$ and $z\in\mathbb{R}^{d+1}$.

The proof of this result, given in Section A.4, is a more or less obvious adaptation of the proof of [22] with the additional complication that $\widehat{\Psi}_n, \tau_n$ are non-lattice valued. As already mentioned in Section 1, this is the only result of the current paper that does not require any novelty, but just a straightforward adaptation with some care.

4. Proof of MLLT for the Sinai flow (Proposition 2.2 and Theorem 2.6)

In this section we assume $d = 1$ or $d = 2$. In this section, we complete the proof of Theorem 2.6 by stating and proving the following result (assuming, for the moment, the statement of the results stated in Section 3).

Proposition 4.1. *Let $A, B \in \mathcal{F}$. Let K be a bounded subset of $\mathbb{R}^d$, let $w \in \mathbb{R}^d$ and $w_t \in \mathbb{R}^d$ be such that $\lim_{t\to+\infty} w_t/a_t = w$. Then*

$$a_t^d\ \nu\left(A \cap \{\phi_t \in B,\ \kappa_{N_t} \in w_t + K\}\right) \sim \widetilde{g}_d\,(w)\,\nu(A)\nu(B)\#((K + w_t) \cap \mathbb{Z}^d), \tag{4.1}$$

where $\widetilde{g}_d$ is the density of the Gaussian limit of Proposition 2.5 and with the following natural convention

$$\forall (x, u) \in \widehat{\mathcal{M}}, \quad (\kappa_{N_t})(\phi_u(x)) := \kappa_{N_{t+u(x)}}(x).$$

Using Proposition 4.1 we complete

Proof of Theorem 2.6. We first recall that we need to show that

$$a_t^d \nu\left(A \cap \{\phi_t \in B,\ W_t \in w_t + K\}\right) \sim \widetilde{g}_d\,(w)\,\mathcal{I}(A \times B, w_t), \tag{4.2}$$

where K is so that $Leb(\partial K) = 0$, where H_1 is as in (2.16) and where we set

$$\mathcal{I}(A \times B, w_t) := \int_{A\times B} \#((K + w_t + H_1(\mathbf{x}) - H_1(\mathbf{y})) \cap \mathbb{Z}^d)\, d\nu(\mathbf{x})\, d\nu(\mathbf{y}).$$

For any positive integer m, we partition A (resp. B) in a finite number of atoms $A_{k,m} \in \mathcal{F}$ and $B_{k,m} \in \mathcal{F}$ of diameter at most $1/m$, and consider the sets

$$K^-_{i,j,m} := \left\{z \in \mathbb{R}^d : \forall (x, y) \in A_{i,m} \times B_{j,m},\ z + H_1(y) - H_1(x) \in K\right\}$$

and

$$K^+_{i,j,m} := \left\{z \in \mathbb{R}^d : \exists (x, y) \in A_{i,m} \times B_{j,m},\ z + H_1(y) - H_1(x) \in K\right\}.$$

Note that H_1 is Lipschitz continuous in (x, u) and bounded on A and B. It follows from (2.18) that

$$\begin{aligned}
&\nu\left(A \cap \{\phi_t \in B,\ W_t \in w_t + K\}\right)\\
&= \nu\left(A \cap \{\phi_t \in B,\ \kappa_{N_t} + H_1 \circ \phi_t - H_1 \in w_t + K\right)\\
&\le \sum_{i,j} \nu\left(A_{i,m} \cap \{\phi_t \in B_{j,m},\ \kappa_{N_t} \in w_t + K^+_{i,j,m}\}\right)
\end{aligned} \tag{4.3}$$

and, analogously,

$$\nu\left(A \cap \{\phi_t \in B,\ W_t \in w_t + K\}\right) \ge \sum_{i,j} \nu\left(A_{i,m} \cap \{\phi_t \in B_{j,m},\ \kappa_{N_t} \in w_t + K^-_{i,j,m}\}\right). \tag{4.4}$$

By Proposition 4.1,

$$\begin{aligned}
&\nu\left(A_{i,m} \cap \{\phi_t \in B_{j,m},\ \kappa_{N_t} \in w_t + K^\pm_{i,j,m}\}\right)\\
&\sim a_t^{-d}\widetilde{g}_d(w)\#((w_t + K^\pm_{i,j,m}) \cap \mathbb{Z}^d)\nu(A_{i,m})\nu(B_{j,m}).
\end{aligned} \tag{4.5}$$

Furthermore,

$$\#((w_t + K^-_{i,j,m}) \cap \mathbb{Z}^d)\nu(A_{i,m})\nu(B_{j,m}) \le \mathcal{I}(A_{i,m} \times B_{j,m}, w_t) \tag{4.6}$$

$$\mathcal{I}(A_{i,m} \times B_{j,m}, w_t) \le \#((w_t + K^+_{i,j,m}) \cap \mathbb{Z}^d)\nu(A_{i,m})\nu(B_{j,m}). \tag{4.7}$$

Let $(x, y, z) \in \bigcup_{i,j}\left(A_{i,m} \times B_{j,m} \times \left(\mathbb{Z}^d \cap (w_t + (K^+_{i,j,m} \setminus K^-_{i,j,m}))\right)\right)$. Then there exist $\mathbf{x}, \mathbf{x}' \in A_{i,m}$ and $\mathbf{y}, \mathbf{y}' \in B_{j,m}$ such that $z - w_t + H_1(\mathbf{y}) - H_1(\mathbf{x}) \in K$ but $z - w_t + H_1(\mathbf{y}') - H_1(\mathbf{x}') \notin K$. Recall that H_1 is Lipschitz. Thus the above conditions means that $z \in \mathbb{Z}^d$ and that (x, y) is at distance smaller than $1/m$ of $\mathcal{E}_{w_t - z} := \{(\mathbf{x}, \mathbf{y}) : H_1(\mathbf{y}) - H_1(\mathbf{x}) \in w_t - z + \partial K\}$. This z should be one of the elements of $\mathbb{Z}^d$ contained in the ball of radius $\sup_A |H_1| + \sup_B |H_1| + \sup_{s\in K} |s|$ around w_t. But, for each such z, the measure of this neighbourhood of $\mathcal{E}_{w_t-z}$ satisfies

$$\nu^{\otimes 2}\left((\mathcal{E}_{w_t-z})^{[1/m]}\right) \leq \sup_{|u|\leq 3\sup_{A\cup B}|H_1|+\sup_{s\in K}|s|} \nu\left(H_1^{-1}\left((u+\partial K)\right)^{[1/m]}\right)$$
$$\leq \sup_{|u|\leq 3\sup_{A\cup B}|H_1|+\sup_{s\in K}|s|} Leb\left(((u+\partial K))^{[1/m]}\right),$$

which converges to 0 as $m \to +\infty$ since $Leb(\partial K) = 0$. Since the number of possible z is uniformly bounded, we have proved that

$$\lim_{m\to+\infty} \sup_t \sum_{i,j} \#((w_t + (K^+_{i,j,m} \setminus K^-_{i,j,m})) \cap \mathbb{Z}^d)\nu(A_{i,m})\nu(B_{j,m}) = 0. \tag{4.8}$$

The desired conclusion (4.2) follows from (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8). □

4.1. **Proof of Proposition 4.1.** Recall that $A = \phi_I(A_0)$ and $B = \phi_J(B_0)$ with $A_0, B_0 \subset M$ such that $\mu(\partial A_0) = \mu(\partial B_0) = 0$ and $I, J \subset \mathbb{R}$ two bounded intervals. We start by proving the lemma for $w_t \in \mathbb{Z}^d$ and $K = \{0\}$. We follow a decomposition somewhat similar to [1, Proof of Lemma 4.3], see also [12, Proof of Theorem 3.1] and [2, Proof of Theorem 1], with the obvious difference that one needs to figure out how to exploit the Joint LLT 3.4 and the Joint LLD 3.5.

Writing $\mathbf{x} = \phi_u(x)$ with $(x,u) \in \widehat{\mathcal{M}}$, we use the product structure of the measure $\widehat{\nu}$ and partition the set considering the different values taken by N_t:

$$\nu\left(A \cap \{\phi_t \in B,\, \kappa_{N_t} = w_t\}\right) = \frac{1}{\mu(\tau)} \sum_{n\geq 0} \int_I Q_n(t,u)\, du, \tag{4.9}$$

where

$$Q_n(t,u) := \mu\left(A_0 \cap T^{-n}B_0 \cap \{\kappa_n = w_t,\, \widetilde{\tau}_n \in u + t - n\mu(\tau) - J\}\right)$$
$$= \mu\left(A_0 \cap T^{-n}(B_0) \cap \left\{\widehat{\Psi}_n \in (w_t, t - n\mu(\tau)) + \{0\} \times J_u\right\}\right),$$

with $J_u = u - J$, recalling that $\widehat{\Psi}_n = (\kappa_n, \widetilde{\tau}_n)$. For L large, we split the sum as

$$\nu\left(A \cap \{\phi_t \in B,\, \kappa_{N_t} = w_t\}\right) = S_1(t,L) + S_2(t,L),$$

where

$$S_1(t,L) := \frac{1}{\mu(\tau)} \sum_{n:|n-t/\mu(\tau)|\leq La_t} \int_I Q_n(t,u)\, du,$$
$$S_2(t,L) := \frac{1}{\mu(\tau)} \sum_{n:|n-t/\mu(\tau)|> La_t} \int_I Q_n(t,u)\, du.$$

The main ingredient needed for the Proof of Proposition 4.1 is

Lemma 4.2.

(a) $\lim_{L\to\infty} \lim_{t\to\infty} a_t^d S_1(t,L) = \widetilde{g}_d(w)\nu(A)\nu(B)$,
(b) $\lim_{L\to\infty} \limsup_{t\to\infty} a_t^d S_2(t,L) = 0$.

The proof of Lemma 4.2 is provided in the paragraph 4.2 below. Equipped with the statement of Lemma 4.2 we can complete

Proof of Proposition 4.1. Note that (4.1) for $w_t \in \mathbb{Z}^d$ and $K = \{0\}$ follows directly from Lemma 4.2 due to (4.9). It remains to go from this special case to the general case. Let $w_t \in \mathbb{R}^d$ and consider a bounded subset K of $\mathbb{R}^d$. Then $(w_t + K) \cap \mathbb{Z}^d$ contains at most $(\mathrm{diam}(K)+1)^2$ integers, we can label them $w_{t,i}$ for $i = 1, \dots, (\mathrm{diam}(K)+1)^2$ (ordering them e.g. by their first coordinate, and then by their second, and completing if necessary by the successors of the last one for this order). Then

$$a_t^d\ \nu\left(A \cap \{\phi_t \in B,\, \kappa_{N_t} \in w_t + K\}\right) = a_t^d \sum_{i=1}^{\#((w_t+K)\cap\mathbb{Z}^d)} \nu\left(A \cap \{\phi_t \in B,\, \kappa_{N_t} = w_{t,i}\}\right)$$
$$\sim \#((w_t + K) \cap \mathbb{Z}^d)\widetilde{g}_d(w)\nu(A)\nu(B),$$

applying (4.1) with $K = \{0\}$ for each sequence $(w_{t,i})_t$. Indeed, since K is a bounded set, $\lim_{t\to+\infty} w_t/a_t = w$ and $\lim_{t\to+\infty} a_t = +\infty$, we obtain that $\lim_{t\to+\infty} w_{t,i}/a_t = w$ for all i. □

4.2. **Proof of Lemma 4.2.** We will use Lemmas 3.4 and 3.5.

Proof of Lemma 4.2(a). This will follow from Lemma 3.4. We consider the range $|n - t/\mu(\tau)| \le La_t$. Since $\frac{w_t}{a_n} = \frac{w_t}{a_t}\frac{a_t}{a_n} \sim \sqrt{\mu(\tau)}w$ and since $\frac{t-n\mu(\tau)}{a_n}$ is bounded, it follows from Lemma 3.4 that

$$a_n^{d+1}\frac{Q_n(t,u)}{\mu(\tau)} \sim \frac{1}{\mu(\tau)} g_{d+1}\left(\sqrt{\mu(\tau)}w, \frac{t-n\mu(\tau)}{a_n}\right)\mu(A_0)\mu(B_0)\Lambda_{d+1}(\{0\}\times J)$$
$$\sim \frac{\mu(\tau)}{|I|} g_{d+1}\left(\sqrt{\mu(\tau)}w, \frac{t-n\mu(\tau)}{a_n}\right)\nu(A)\nu(B),$$

uniformly in n such that $|n - t/\mu(\tau)| \le La_t$. Hence

$$a_t^d S_1(t,L) \sim \sum_{n:|n-t/\mu(\tau)|\le La_t} \frac{(\mu(\tau))^{1+\frac{d}{2}}}{a_n} g_{d+1}\left(w\sqrt{\mu(\tau)}, \frac{t-n\mu(\tau)}{a_n}\right)\nu(A)\nu(B).$$

Approximating Riemann sums by Riemann integrals, the right hand side converges, as $t \to +\infty$, to

$$(\mu(\tau))^{\frac{d}{2}} \int_{-L\mu(\tau)^{3/2}}^{L\mu(\tau)^{3/2}} g_{d+1}\left(w\sqrt{\mu(\tau)}, z\right)\,dz\,\nu(A)\nu(B)$$

which itself converges to $\widetilde{g}_d(w)\nu(A)\nu(B)$ as $L \to +\infty$, as announced, since $\widetilde{g}_d(w) = (\mu(\tau))^{\frac{d}{2}} \int_{\mathbb{R}} g_{d+1}(w\sqrt{\mu(\tau)}, z)\,dz$. □

For the proof of Lemma 4.2(b), note that $Q_n(t,u) \le \widetilde{Q}_n(t)$ where

$$\widetilde{Q}_n(t) = \sup_{u\in I} \mu\left(\widehat{\Psi}_n \in (w_t, t - n\mu(\tau)) + \{0\} \times J_u\right).$$

Lemma 4.2(b) is an immediate consequence of Sublemmas 4.3 and 4.4.

Sublemma 4.3. *For any* $c \in (0, 1/\mu(\tau))$,

$$\lim_{L\to\infty} \limsup_{t\to\infty} a_t^d \sum_{n>ct:La_t<|n-t/\mu(\tau)|} \widetilde{Q}_n(t) = 0.$$

Proof. In this range, $n \gg t$, so $\frac{n}{a_n^{d+1}} = n^{-\frac{d-1}{2}}(\log n)^{-\frac{d+1}{2}} \ll \frac{t}{a_t^{d+1}}$. Thus, for any L large enough, using Lemma 3.5 with $|z|_\infty = |t - n\mu(\tau)|$, we obtain that

$$\begin{aligned}(4.10)\qquad &\sum_{n>ct: La_t<|n-t/\mu(\tau)|} \widetilde{Q}_n(t) \\ &\ll \frac{t}{a_t^{d+1}} \sum_{n>ct: La_t<|n-t/\mu(\tau)|} \frac{|\log|t-n\mu(\tau)||}{1+|t-n\mu(\tau)|^2} \ll \frac{t}{a_t^{d+1}} \frac{\log(a_t)}{La_t} \ll \frac{1}{La_t^d},\end{aligned}$$

since $\log u/u^2$ has primitive $-(1+\log u)/u$ and since $a_t^2 = t\log t \sim 2t \log a_t$. $\square$

Sublemma 4.4. *For any $c \in (0, 1/\mu(\tau))$, $\lim_{L\to\infty} \limsup_{t\to\infty} a_t^d \sum_{n<ct} \widetilde{Q}_n(t) = 0$.*

Proof. In this range, $t - n\mu(\tau) \approx t \gg n$. Thus it follows from Lemma 3.5 that

$$\begin{aligned}\sum_{n<ct} \widetilde{Q}_n(t) &\ll \sum_{n<ct} \frac{n}{a_n^{d+1}} \frac{1+\log t}{t^2} \\ &\ll \frac{1+\log t}{t^2} \sum_{n<ct} \frac{1}{n^{\frac{d-1}{2}}(\log n)^{\frac{d+1}{2}}} \\ &\ll \frac{1+\log t}{t^2} \frac{t^{\frac{3-d}{2}}}{(\log t)^{\frac{d+1}{2}}} \ll t^{-\frac{d+1}{2}}(\log t)^{-\frac{d-1}{2}} = o(a_t^{-d}).\end{aligned}$$

$\square$

5. Proof of mixing for the Lorentz gas (Theorem 1.1)

In this section we assume $d = 1$ or $d = 2$. Corollary 2.3 states mixing for functions with compact support in the suspension $\widehat{\mathcal{M}} \times \mathbb{Z}^d$. To deal with the natural class of functions (with compact support in the manifold) in Theorem 1.1 we crucially rely on the following tightness-type result, which is the most delicate part of this work.

Theorem 5.1. *Let $K_0 > 0$ be fixed and let B_0 be the set of $\mathbf{x} \in \widetilde{\mathcal{M}}$ with position at a distance at most K_0 of the origin. For any positive integer R_0, let B_{R_0} be the set of configurations $\mathbf{x} \in \widetilde{\mathcal{M}}$ belonging to B_0 and with no collision during the time interval $[0, R_0)$. Then*

$$\lim_{R_0\to+\infty} \limsup_{t\to+\infty} a_t^d \widetilde{\nu}\left(B_{R_0} \cap \Phi_{-t}(B_0)\right) = 0.$$

Before proceeding to the proof of Theorem 5.1, let us see how Theorem 1.1 follows from Corollary 2.3 and Theorem 5.1.

Proof of Theorem 1.1. Assume that f and g are non-negative with support in B_0. We will use Corollary 2.3 and the sets $E_{\pm n}$ defined therein. Observe that $B_0 \backslash E_{-n} \subset B_n$. Let $(f_n)_n$ (resp. $(g_n)_n$) be an increasing sequence of continuous functions supported in E_{-2n} (resp. E_{2n}) coinciding with f (resp. g) on E_{-n} (resp. E_n) and converging pointwise to f (resp. g). Thus it will follow from Theorem 5.1 and time-reversibility of Φ that

$$\begin{aligned}&\lim_{n\to+\infty} \limsup_{t\to+\infty} \left| a_t^d \int_{\widetilde{\mathcal{M}}} [f.g\circ\Phi_t - f_n.g_n \circ \Phi_t]\, d\widetilde{\nu}\right| \\ (5.1)\qquad &\le \lim_{n\to+\infty} \limsup_{t\to+\infty} 2\|f\|_\infty \|g\|_\infty a_t^d \widetilde{\nu}\left(B_n \cap \Phi_{-t}(B_0)\right) = 0.\end{aligned}$$

Thus, for every n,

$$\begin{aligned}
&\limsup_{t\to+\infty}\left|a_t^d\int_{\widetilde{\mathcal{M}}}f.g\circ\Phi_t-\widetilde{g}_d(0)\int_{\widetilde{\mathcal{M}}}f\,d\widetilde{\nu}\int_{\widetilde{\mathcal{M}}}g\,d\widetilde{\nu}\right|\\
&\quad\le\limsup_{t\to+\infty}\left|a_t^d\int_{\widetilde{\mathcal{M}}}[f.g\circ\Phi_t-f_n.g_n\circ\Phi_t]\,d\widetilde{\nu}\right|\\
&\quad+\limsup_{t\to+\infty}\left|a_t^d\int_{\widetilde{\mathcal{M}}}f_n.g_n\circ\Phi_t\,d\widetilde{\nu}-\widetilde{g}_d(0)\int_{\widetilde{\mathcal{M}}}f_n\,d\widetilde{\nu}\int_{\widetilde{\mathcal{M}}}g_n\,d\widetilde{\nu}\right|\\
&\quad+\widetilde{g}_d(0)\left|\int_{\widetilde{\mathcal{M}}}f_n\,d\widetilde{\nu}\int_{\widetilde{\mathcal{M}}}g_n\,d\widetilde{\nu}-\int_{\widetilde{\mathcal{M}}}f\,d\widetilde{\nu}\int_{\widetilde{\mathcal{M}}}g\,d\widetilde{\nu}\right|\\
&\quad\le\limsup_{t\to+\infty}\left|a_t^d\int_{\widetilde{\mathcal{M}}}[f.g\circ\Phi_t-f_n.g_n\circ\Phi_t]\,d\widetilde{\nu}\right|\\
&\quad+\widetilde{g}_d(0)\left|\int_{\widetilde{\mathcal{M}}}f_n\,d\widetilde{\nu}\int_{\widetilde{\mathcal{M}}}g_n\,d\widetilde{\nu}-\int_{\widetilde{\mathcal{M}}}f\,d\widetilde{\nu}\int_{\widetilde{\mathcal{M}}}g\,d\widetilde{\nu}\right|,
\end{aligned}$$

where we used Corollary 2.3 applied to f_n, g_n in the last inequality. Since this holds true for any n, we conclude by taking the limit as $n\to+\infty$ thanks to (5.1) and to the dominated convergence theorem. □

The rest of this section is devoted to the proof of Theorem 5.1. Recall that K_0 is fixed and that we have to estimate $\widetilde{\nu}(B_{R_0}\cap\Phi_{-t}(B_0))$. The strategy of our proof is divided in two steps. In a first step (corresponding to Subsection 5.2), we explain how we can neglect "bad" configurations with first or last long free flights (Lemma 5.4) or having a small number of collisions within the time interval $[0,t]$ (Lemmas 5.3 and 5.6). In a second step (corresponding to Subsection 5.3), we will estimate the probability of the set of "good" configurations belonging to $B_{R_0}\cap\Phi_{-t}(B_0)$ by writing (as in the proof of Proposition 4.1 but with additional sums and complications) this set as a union of sets, the measure of which corresponds to the measure of a set that can be expressed in terms of the Sinai billiard map T.

In the process of proving Theorem 5.1, we obtain the following large deviation result (proved in Section 5.2), which is interesting in its own right.

Proposition 5.2. *For c_1 small enough,*

$$\mu(\tau_{\lfloor c_1 t\rfloor}>t)\le\nu(N_t\le c_1 t)/\min\tau=\mathcal{O}(\log t/t).$$

The bound $\mathcal{O}(\log t/t)$ is optimal because τ is in the domain of non-standard CLT with normalization $\sqrt{t\log t}$. This bound is in accord with the optimal result in *the i.i.d. scenario* with $\sqrt{t\log t}$ normalization, see [5, 31].

5.1. **Notations and recalls for the proof of Theorem 5.1.** Before entering deeper in the proof, let us introduce some needed notations. Recall that κ stands for the cell change (with values in $\mathbb{Z}^d$). It will be useful to consider $\widetilde{\kappa}:M\to\mathbb{Z}^2$ for the cell-change for the $\mathbb{Z}^2$-periodic Lorentz gas; so that $\kappa=\pi_d(\widetilde{\kappa})$ where $\pi_2=Id$ and $\pi_1:\mathbb{R}^2\to\mathbb{R}$ is the canonical projection on the first coordinate. Note that, when $d=2$, $\widetilde{\kappa}=\kappa$. We extend the definition of $\widetilde{\kappa}$ to $\widetilde{\mathcal{M}}$ by setting $\widetilde{\kappa}(\Phi_u(q,\vec{v}))=\widetilde{\kappa}(p_d(q),\vec{v})$ for every $x=(q,\vec{v})\in\widetilde{M}$ and every $u\in[0,\tau(x))$. Let us write $\widetilde{N}_t(\mathbf{x})$ for the number of collisions in the time interval $(0,t]$ for a trajectory starting from $\mathbf{x}\in\widetilde{\mathcal{M}}$. Recall that H_0 is the bounded coboundary defined in (2.10). Throughout the rest of this section we fix c_1 so that

$$c_1 \in (0, 1/(1000\mu(\tau))) \text{ and } 2c_1\|H_0\|_\infty < 1/100. \tag{5.2}$$

We consider the constant a_0 appearing in Lemma 3.2. Up to decreasing if necessary its value, it follows from e.g. [10, Theorem 7.37, Remark 7.38] that there exists $C_0' > 0$ such that

(5.3)
$$\forall n' \in \mathbb{N}_0, \quad Cov\left(f((\widetilde{\kappa}\circ T^m)_{m\geq n'}), g((\widetilde{\kappa}\circ T^{m'})_{m'\leq 0})\right) \leq C_0'\|f\|_\infty\|g\|_\infty e^{-a_0 n'},$$

for any bounded measurable functions f, g. by noticing that $f((\widetilde{\kappa}\circ T^m)_{m\geq 0})$ is constant on stable curves and that $g((\widetilde{\kappa}\circ T^{m'})_{m'\leq -1})$ is constant on unstable curves and as such, these functions are bounded by their infinite norms in the respective spaces $\mathcal{H}^-$ and $\mathcal{H}^+$ considered in [10].

We fix $K > 0$ so that

$$C_0' e^{-a_0\lfloor K\log t\rfloor} \leq t^{-100}. \tag{5.4}$$

We recall that it is proved in [34] that

$$\mu(\widetilde{\kappa} = z) = \mathcal{O}(|z|^{-3}), \tag{5.5}$$

and that the set $\mathfrak{C}$ of unit vectors of $\mathbb{R}^2$ corresponding to the corridor directions in $\mathcal{D}_2$ (i.e. the direction of a line in $\mathbb{R}^2$ touching no obstacle) is finite. Finally, recall that by [34, Propositions 11–12, Lemma 16],

(5.6) $\forall V > 0,$
$$\mu\left(\widetilde{\kappa} = z, \exists |j| \leq V\log(|z|+2),\ j\neq 0,\ |\widetilde{\kappa}|\circ T^j > |z|^{4/5}\right) = \mathcal{O}\left(|z|^{-3-\frac{2}{45}}\right).$$

5.2. **Control of "bad" configurations.** Lemma 5.3 allows us to neglect trajectories with no collision before time t.

Lemma 5.3. *The following estimate holds true as $t \to +\infty$,*

$$\widetilde{\nu}\left(B_0 \cap \Phi_{-t}(B_0) \cap \{\widetilde{N}_t = 0\}\right) = o(a_t^{-d}).$$

Proof. When $d = 2$, the lemma is immediate: as soon as $t > 4K_0$, $\widetilde{N}_t \geq 1$, otherwise, at time t, the trajectory cannot be $2K_0$-close of its initial position, at time 0.

When $d = 1$, we can have $\widetilde{N}_t = 0$ because of possible long free-flights in the vertical direction that remain at a bounded distance. However, using the representation of $(\widetilde{\mathcal{M}}, (\Phi_t)_t, \widetilde{\nu})$ as a suspension flow over a $\mathbb{Z}$-extension,

$$\begin{aligned}\widetilde{\nu}\left(B_0 \cap \{\widetilde{N}_t = 0\}\right) &\leq (2K_0+1)\nu(\widetilde{N}_t = 0)\\ &\leq \frac{2K_0+1}{\mu(\tau)}\int_0^{+\infty}\mu(\tau > s+t)\,ds\\ &\ll \int_0^{+\infty}(1+s+t)^{-2}\,ds \ll t^{-1} = o(a_t^{-1}),\end{aligned}$$

where we used (2.12) and (2.9). $\square$

Lemma 5.4 ensures that we can neglect trajectories with long first or long last free flight.

Lemma 5.4. *There exists a constant $C' > 0$ such that, for all R_0 and t large enough,*

$$\widetilde{\nu}\left(B_0 \cap \{|\widetilde{\kappa}| > a_t^d \log R_0\}\right) \le C' a_t^{-d} / \log R_0.$$

Remark 5.5. Note that, since Φ_t preserves the measure $\widetilde{\nu}$, we also have

$$\widetilde{\nu}\left(\Phi_{-t}\left(B_0 \cap \{|\widetilde{\kappa}| > a_t^d \log R_0\}\right)\right) \le C' a_t^{-d} / \log R_0.$$

Proof of Lemma 5.4. Using again the representation of $(\widetilde{\mathcal{M}}, (\Phi_t)_t, \widetilde{\nu})$ as a suspension flow over a $\mathbb{Z}^d$-extension, we note that

$$\begin{aligned}
&\widetilde{\nu}\left(B_0 \cap \{|\widetilde{\kappa}| > a_t^d \log R_0\}\right) \le (2K_0+1)^2 \nu\left(|\widetilde{\kappa}| > a_t^d \log R_0\right) \\
&\le \frac{(2K_0+1)^2}{\mu(\tau)} \mathbb{E}_\mu\left[\tau 1_{\{|\widetilde{\kappa}| > a_t^d \log R_0\}}\right] \\
&\ll \int_{a_t^d \log R_0}^{+\infty} \mu(\tau > s)\, ds + a_t^d \log R_0 \mu(|\widetilde{\kappa}| > a_t^d \log R_0)) \\
&\ll \int_{a_t^d \log R_0}^{+\infty} s^{-2}\, ds + (a_t^d \log R_0)^{-1} \ll a_t^{-d} / \log R_0,
\end{aligned}$$

using (2.12). □

Lemma 5.6 deals with the remaining range, namely $n = N_t \le c_1 t$ with c_1 as in (5.2).

Lemma 5.6. *For all $R_0 > 0$,*

$$\widetilde{\nu}\left(B_0 \cap \Phi_{-t}(B_0) \cap \{\widetilde{N}_t \le c_1 t\}\right) = o(a_t^{-d}),$$

as $t \to +\infty$.

To prove Lemma 5.6, we will deal separately with the cases $d = 1$ and $d = 2$. The main ingredients of the proofs below in these two cases come down to a very delicate decomposition of the involved sum along with fine estimates via the use of (5.6) and of Lemma 3.5 (joint LLD). The proof of Lemma 5.6 for $d = 1$ will use the following intermediate results.

Sublemma 5.7. *Recall that K satisfies* (5.4). *Then*

$$\widetilde{\nu}\left(B_0 \cap \{\widetilde{N}_{t/100} \le K \log t\}\right) \le (2K_0+1)^2 \nu(N_{t/100} \le K \log t) \ll (\log t)/t.$$

Proof of the sublemma. Recall that B_0 has diameter $2K_0$. The first inequality comes from the fact that B_0 contains at most $(2K_0+1)^2$ copies of $\mathcal{M}$. Let us prove the second inequality. Observe that

$$\nu(N_{t/100} \le K \log t) = \frac{1}{\mu(\tau)} \mathbb{E}_\mu\left[\tau.\mathbf{1}_{\{\tau_{\lfloor K \log t \rfloor + 1} \ge t/100\}}\right]$$

$$\ll \mathbb{E}_\mu\left[\tau.\mathbf{1}_{\bigcup_{k=0}^{\lfloor K \log t \rfloor} \{\tau \circ T^k > t/(100(1+K\log t)\}}\right] \tag{5.7}$$

$$\begin{aligned}
&\ll \mathbb{E}[\tau.\mathbf{1}_{\{\tau \ge t/(100(1+K \log t))\}}] \\
&\quad + \mathbb{E}\left[\tau.\mathbf{1}_{\bigcup_{k=1}^{\lfloor K \log t \rfloor} \{\tau < t/(100(1+K\log t)),\ \tau \circ T^k \ge t/(100(1+K \log t))\}}\right].
\end{aligned} \tag{5.8}$$

Now, proceeding as in the proof of Lemma 5.4, it follows from (2.12) that for all $t>2$,[5]

$$\begin{aligned}\mathbb{E}_\mu\left[\tau.\mathbf{1}_{\{\tau>\frac{t}{100(K\log t+1)}\}}\right]&=\int_0^{+\infty}\mu\left(\tau.\mathbf{1}_{\{\tau>\frac{t}{100(K\log t+1)}\}}>z\right)dz\\&=\int_0^{\frac{t}{100(K\log t+1)}}\mu\left(\tau>\frac{t}{100(K\log t+1)}\right)dz+\int_{\frac{t}{100(K\log t+1)}}^{+\infty}\mu(\tau>z)\,dz\\&\ll(t/\log t)^{-1},\end{aligned}\tag{5.9}$$

providing a control of the first term of the right hand side of (5.8). For the second term of the right hand side of (5.8), we distinguish the case of small (resp. big) values of τ. Set $m:=(1+\frac{1}{45})^{-1}$. On the one hand,

$$\mathbb{E}_\mu\left[\tau 1_{\{\tau\le t^m\}}.\mathbf{1}_{\bigcup_{k=1}^{\lfloor K\log t\rfloor}\{\tau\circ T^k\ge t/(100(1+K\log t))\}}\right]\ll t^m\log t(t/\log t)^{-2}=t^{m-2}(\log t)^3,\tag{5.10}$$

where we used $\tau\le t^m$ and $\mu\left(\bigcup_{k=1}^{\lfloor K\log t\rfloor}\{\tau\circ T^k\ge z\}\right)\le K\log t\,\mu(\tau>z)\ll z^{-2}\log t$. On the other hand, since $\tau-|\widetilde{\kappa}|$ is uniformly bounded by some constant L_0,

$$\begin{aligned}&\mathbb{E}_\mu\left[\tau 1_{\{t^m\le\tau\le t/(100(1+K\log t))\}}.\mathbf{1}_{\bigcup_{k=1}^{\lfloor K\log t\rfloor}\{\tau\circ T^k\ge t/(100(1+K\log t))\}}\right]\\&\le\sum_{z:t^m-L_0\le|z|\le t/(100(1+K\log t))+L_0}(|z|+L_0)\mu\Big(\widetilde{\kappa}=z,\\&\qquad\exists k=1,\dots,K\log t,\ |\widetilde{\kappa}|\circ T^k>\frac{t}{100K\log t}-L_0\Big)\\&\ll\sum_{z\in supp(\widetilde{\kappa}):|z|\ge t^m-L_0}(|z|+L_0)|z|^{-3-\frac{2}{45}}\ll t^{-m(1+\frac{2}{45})}=t^{m-2},\end{aligned}\tag{5.11}$$

where we apply (5.6) with $V=\frac{2K}{m}$ (indeed, for t large enough, $K\log t\le V\log(t^m-L_0+2)$). Thus, the last bound of the sublemma follows from (5.8), (5.9), (5.10) and (5.11) since $m-2=-\frac{47}{46}<-1$. □

Lemma 5.6 in the case $d=1$ follows from the following result.

Sublemma 5.8.

$$\widetilde{\nu}\left(B_0\cap\{\widetilde{N}_t\le c_1t\}\right)\le(1+2K_0)^2\nu(N_t\le c_1t)\ll(\log t)/t.$$

Proof of the sublemma. Again the first inequality follows from the fact that B_0 contains at most $(2K_0+1)^2$ copies of $\mathcal{M}$. The main issue is to establish the last upper bound. Since the flow ϕ preserves ν, $\nu(N_t\le c_1t)=\nu(N_t\circ\phi_{-t/2}\le c_1t)$. Furthermore, the fact that $N_t\circ\phi_{-t/2}\le c_1t$ means that there are at most c_1t collisions in the time interval $[-t/2;t/2]$, so at most c_1t collisions in both time intervals $[-t/2;0]$ and $[0;t/2]$, which implies that both $\tau_{\lfloor c_1t\rfloor}$ and $\tau-\tau_{-\lfloor c_1t\rfloor}$ are

[5] We use here again the classical formula $\mathbb{E}_\mu[X]=\int_0^{+\infty}\mu(X>z)\,dz$ valid for any positive measurable $X:M\to[0,+\infty)$.

larger than $t/2$, writing as usual $\tau_{-k} := -\sum_{m=-k}^{-1} \tau \circ T^m$ and $\tau_m(\phi_u(x)) := \tau_m(x)$ for all $(x,u) \in \widehat{\mathcal{M}}$. Therefore

$$\nu(N_t \le c_1 t) = \nu(N_t \circ \phi_{-t/2} \le c_1 t) \le \nu(\tau_{\lfloor c_1 t\rfloor} > t/2, \tau - \tau_{-\lfloor c_1 t\rfloor} > t/2). \tag{5.12}$$

In what follows we show that this quantity is $o(a_t^{-1})$. By Lemma 5.4, $\nu(\tau > a_t^2) \ll a_t^{-2}$ and by Sublemma 5.7, $\nu(\tau_{\lfloor K\log t\rfloor} > t/100) \ll (\log t)/t$ and $\nu(\tau - \tau_{-\lfloor K\log t\rfloor} > t/100) \ll (\log t)/t$ (up to time reversibility of Φ). This combined with (5.12) ensures that

$$\nu(N_t \le c_1 t) \le p_t + \mathcal{O}((\log t)/t), \tag{5.13}$$

with

$$\begin{aligned} p_t &:= \nu\left(\tau < a_t^2, \tau_{\lfloor c_1 t\rfloor} > t/2, \tau - \tau_{-\lfloor c_1 t\rfloor} > t/2, \tau_{\lfloor K\log t\rfloor}\right. \\ &\quad \left. < t/100, \tau - \tau_{-\lfloor K\log t\rfloor} < t/100\right) \\ &= \sum_{a'=0}^{a_t^2} \mu\left(\tau > a', \tau_{\lfloor c_1 t\rfloor} > t/2, \tau - \tau_{-\lfloor c_1 t\rfloor} > t/2, \tau_{\lfloor K\log t\rfloor}\right. \\ &\quad \left. < t/100, \tau - \tau_{-\lfloor K\log t\rfloor} < t/100\right) \\ &\le \sum_{a'=0}^{a_t^2} \mu\Big(|\widetilde{\kappa}| > a' - 2\|H_0\|_\infty, \tau_{\lfloor c_1 t\rfloor} > t/2, \\ &\qquad , \tau + \tau_{-\lfloor c_1 t\rfloor} > t/2, \tau_{\lfloor K\log t\rfloor} < t/100, \tau - \tau_{-\lfloor K\log t\rfloor} < t/100\Big), \end{aligned}$$

with the function H_0 appearing in (2.10) (since $|V| = \tau$ when $d = 2$). Observe that

$$\tau_{\lfloor c_1 t\rfloor} \circ T^{\lfloor K\log t\rfloor} > \tau_{\lfloor c_1 t\rfloor - \lfloor K\log t\rfloor} \circ T^{\lfloor K\log t\rfloor} = \tau_{\lfloor c_1 t\rfloor} - \tau_{\lfloor K\log t\rfloor}$$

and that

$$-\tau_{-\lfloor c_1 t\rfloor} \circ T^{-\lfloor K\log t\rfloor} > -\tau_{-\lfloor c_1 t\rfloor + \lfloor K\log t\rfloor} \circ T^{-\lfloor K\log t\rfloor} = -\tau_{-\lfloor c_1 t\rfloor} + \tau - \tau + \tau_{-\lfloor K\log t\rfloor}.$$

It follows that

$$\begin{aligned} p_t \le \sum_{a'=0}^{a_t^2 \log R_0} \mu\Big(&|\widetilde{\kappa}| > a' - 2\|H_0\|_\infty, \tau_{\lfloor c_1 t\rfloor} \circ T^{\lfloor K\log t\rfloor} \\ &> 49t/100, |\tau_{-\lfloor c_1 t\rfloor}| \circ T^{-\lfloor K\log t\rfloor} > 49t/100\Big). \end{aligned}$$

Recall that, for all $m \in \mathbb{Z}$, $|\widetilde{\kappa}_m| \ge \tau_m - 2m\|H_0\|_\infty$ and that, due to (5.2), $2c_1\|H_0\|_\infty < 1/100$. Thus,

$$\begin{aligned} p_t \le \sum_{a'=0}^{a_t^2 \log R_0} \mu\Big(&|\widetilde{\kappa}| > a' - 2\|H_0\|_\infty, |\widetilde{\kappa}|_{\lfloor c_1 t\rfloor} \circ T^{\lfloor K\log t\rfloor} \\ &> 48t/100, -|\widetilde{\kappa}|_{-\lfloor c_1 t\rfloor} \circ T^{-\lfloor K\log t\rfloor} > 48t/100\Big). \end{aligned}$$

Thus, using (5.3) (twice) combined with (5.4),

$$p_t \le \sum_{a'=0}^{a_t^2 \log R} \left(\mu\left(|\widetilde{\kappa}| > a' - 2\|H_0\|_\infty\right)\left(\mu\left(|\widetilde{\kappa}|_{\lfloor c_1 t\rfloor} > 48t/100\right)\right)^2 + \mathcal{O}(t^{-100})\right). \tag{5.14}$$

By Lemma 3.5 (with $d = 0$) and since τ is μ-integrable,

$$\begin{aligned}\mu\left(|\widetilde{\kappa}|_{\lfloor c_1 t\rfloor} > 48t/100\right) &\le \mu\left(\tau_{\lfloor c_1 t\rfloor} > 47t/100\right)\\ &\le \mu(\tau_{\lfloor c_1 t\rfloor} > t^2) + \mu\left(47t/100 < \tau_{\lfloor c_1 t\rfloor} < t^2\right)\\ &\ll \frac{\mathbb{E}_\mu[\tau_{\lfloor c_1 t\rfloor}]}{t^2} + \sum_{k=2t/5}^{t^2} \frac{t}{\sqrt{t\log t}} \frac{\log(k - \lfloor c_1 t\rfloor \mu(\tau))}{1 + (k - \lfloor c_1 t\rfloor\mu(\tau))^2}\\ &\ll t^{-1} + \sum_{k\ge t(2/5 - c_1\mu(\tau))} \frac{t}{\sqrt{t\log t}} \frac{\log t}{1+k^2}\\ &\ll \frac{t}{\sqrt{t\log t}} \frac{\log t}{t} = \sqrt{\frac{\log t}{t}}.\end{aligned}$$

Combining this with (5.14) and (5.13), we infer

$$\nu(N_t < c_1 t) \le \sum_{a'=0}^{a_t^2 \log R_0} \left((a'+1)^{-2}\frac{\log t}{t} + \mathcal{O}(t^{-100})\right) + \mathcal{O}((\log t)/t) = \mathcal{O}(\log t/t).$$

□

We take the line below to quickly complete

Proof of Proposition 5.2. We observe that

$$\begin{aligned}\mu(\tau_{\lfloor c_1 t\rfloor} > t) &\le \frac{\widehat{\nu}(\{(x,u) \in \widehat{\mathcal{M}} : \tau_{\lfloor c_1 t\rfloor}(x) > t,\ u \le \min\tau\})}{\min\tau}\\ &\le \frac{\widehat{\nu}(\{(x,u) \in \widehat{\mathcal{M}} : N_t(x) < \lfloor c_1 t\rfloor,\ u \le \min\tau\})}{\min\tau}\\ &\le \frac{\nu(N_t \le \lfloor c_1 t\rfloor)}{\min\tau} \le \frac{\nu(N_t \le c_1 t)}{\min\tau},\end{aligned}$$

and conclude due to Sublemma 5.8. □

We continue with

Proof of Lemma 5.6. When $d = 1$, the result follows from Sublemma 5.8 since

$$\widetilde{\nu}\left(B_0 \cap \Phi_{-t}(B_0) \cap \{\widetilde{N}_t \le c_1 t\}\right) \le \widetilde{\nu}\left(B_0 \cap \{\widetilde{N}_t \le c_1 t\}\right) \ll (\log t)/t = o(a_t^{-1}).$$

Unfortunately this estimate is not enough when $d = 2$. We assume from now on throughout this proof that $d = 2$. Recall that we have to prove that

$$\widetilde{\nu}\left(B_0 \cap \Phi_{-t}(B_0) \cap \{\widetilde{N}_t \le c_1 t\}\right) = o(a_t^{-2}) = o((t\log t)^{-1}) \quad \text{(assuming } d = 2).$$

We start with some preliminary calculation that will allow us to argue that we can neglect the configurations with more than one long free flight (of length larger than $t/(100(K\log t)^2)$) among the $K\log t$ future and past collision times. Let us write

$$D_t := \{\exists k, \ell : k \neq \ell; |k|, |\ell| \le K\log t, \min(\tau\circ T^k, \tau\circ T^\ell) > t/(100(K\log t)^2)\} \subset \mathcal{M}$$

and $\widetilde{D}_t$ for the corresponding event in $\widetilde{\mathcal{M}}$.

Sublemma 5.9. *For all* $\varepsilon \in (0, \frac{1}{45})$,

$$\widetilde{\nu}\left(B_0 \cap \Phi_{-t/2}(\widetilde{D}_t)\right) \le (2K_0+1)^2\nu(D_t) = o(t^{-1-\frac{1}{45}+\varepsilon}) = o(a_t^{-2}), \quad \textit{as } t \to +\infty.$$

Proof of the sublemma. Again, as in the proofs of Sublemmas 5.7 and 5.8, the first inequality follows from the fact that $\widetilde{\mathcal{M}}$ is made of at most $(2K_0+1)^2$ copies of $\mathcal{M}$ and that ϕ preserves the measure ν. It remains to prove the last estimate. Using the suspension flow representation and the Hölder inequality applied for any $p<2<q$ such that $\frac{1}{p}+\frac{1}{q}=1$ and close enough to 2, we observe that

$$\nu(D_t) \le 2 \sum_{-K\log t\le k<\ell\le K\log t} \mathbb{E}_\mu[\tau.\mathbf{1}_{\tau\circ T^k>t/(100(K\log t)^2),\,\tau\circ T^\ell>t/(100(K\log t)^2)}]$$

$$\le 2 \sum_{-K\log t\le k<\ell\le K\log t} \|\tau\|_{L^p}\left(\mu(\tau\circ T^k>t/(100(K\log t)^2), \tau\circ T^\ell>t/(100(K\log t)^2))\right)^{\frac{1}{q}}$$

$$\le 2 \sum_{-K\log t\le k,\ell\le K\log t:k\ne\ell} \|\tau\|_{L^p}\left(\sum_{i\in Supp(\widetilde{\kappa}):|i|>t/(100(K\log t)^2)} \mu(\widetilde{\kappa}\circ T^k=i, |\widetilde{\kappa}|\circ T^\ell>|i|^{\frac{4}{5}}\right)^{\frac{1}{q}}.$$

Finally applying (5.6) with $V=4K$ and t large enough so that $2K\log t< V\log(2+\frac{t}{100(K\log t)^2})$, we conclude that, for t large enough,

$$\nu(D_t)\ll(\log t)^2\left(\sum_{i>t/(100(K\log t)^2)} i^{-3-\frac{2}{45}}\right)^{\frac{1}{q}},$$

and this bound holds true for an arbitrary real number $q>2$. □

We are back to the proof of Lemma 5.6 assuming that $d=2$. We will decompose the quantity we have to estimate in $p_{t,1}+p_{t,2}$, distinguishing the case where the free flight at time $t/2$ is larger or smaller than $t/4$.

Estimate when the free flight at time $t/2$ is larger than $t/4$. In this part, we study

$$p_{t,1} := \widetilde{\nu}\left(B_0\cap\Phi_{-t}(B_0)\cap\{\widetilde{N}_t\le c_1t, \tau\circ\Phi_{\frac{t}{2}}>t/4\}\right). \tag{5.15}$$

We will use the fact that we can neglect the trajectories such that $\tau\circ\Phi_{t/2}>a_t^3$ since, due to Lemma 5.4,

$$\widetilde{\nu}(B_0\cap\{\tau\circ\Phi_{t/2}>a_t^3\})\le(2K_0+1)^2\nu\left(\tau\circ\phi_{t/2}>a_t^3\right)\ll a_t^{-3}. \tag{5.16}$$

It follows from (5.16) and from Sublemma 5.9 that

$$p_{t,1}=\widetilde{\nu}\left(B_0\cap\Phi_{-t}(B_0)\cap\left\{\widetilde{N}_t\le c_1t\right\}\cap\Phi_{-\frac{t}{2}}\left(\left\{\tau\in\left[\frac{t}{4},a_t^3\right]\right\}\setminus\widetilde{D}_t\right)\right)+o(a_t^{-2}). \tag{5.17}$$

Let us study the event appearing in the above formula.

The fact that $\tau\circ\Phi_{\frac{t}{2}}>t/4$ and that $\Phi_{t/2}\notin D_t$ implies that the $K\log t$ free flights just before and just after the one occurring at time $t/2$ have all length smaller than $t/(100(K\log t)^2)$ and thus that

$$\tau_{\lfloor K\log t\rfloor}\circ\widetilde{T}\circ\Phi_{t/2}<t/(100(\lfloor K\log t\rfloor))\quad\text{and}\quad|\tau_{-\lfloor K\log t\rfloor}|\circ\Phi_{t/2}<t/(100(\lfloor K\log t\rfloor)). \tag{5.18}$$

Recall that the configuration is in $B_0\cap\Phi_{-t}(B_0)$ and satisfies $\tau\circ\Phi_{t/2}>t/4$. Since $\widetilde{N}_t\le c_1t$ and since we are in dimension 2, the free flight of length $t/4$ made at time $t/2$ has to be canceled by the sum of the other (at most $(\lfloor c_1t\rfloor-1)$) free flights made during the time interval $[0;t]$. So,

$$|\tau_{-\lfloor c_1t\rfloor}|\circ\Phi_{t/2}>t/8\quad\text{or}\quad(\tau_{\lfloor c_1t\rfloor}-\tau)\circ\Phi_{t/2}>t/8. \tag{5.19}$$

The combination of Conditions (5.18) and (5.19) implies that at least one of the two next conditions should hold true

$$|\tau_{-\lfloor c_1 t\rfloor}|\circ \widetilde{T}^{-\lfloor K\log t\rfloor}\circ\Phi_{t/2} > |\tau_{-\lfloor c_1 t\rfloor}|\circ\Phi_{t/2} - |\tau_{-\lfloor K\log t\rfloor}|\circ\Phi_{t/2} > 11t/100$$

or

$$\tau_{\lfloor c_1 t\rfloor}\circ \widetilde{T}^{\lfloor K\log t\rfloor}\circ\Phi_{t/2} > (\tau_{\lfloor c_1 t\rfloor}-\tau)\circ\Phi_{t/2} - \tau_{\lfloor K\log t\rfloor}\circ\widetilde{T}\circ\Phi_{t/2} > 11t/100.$$

Note that the second condition above corresponds to the first one above one up to composing by Φ_t and up to using time reversal. Therefore,

$$\begin{aligned} p_{t,1} &\le 2\widetilde{\nu}\left(B_0\cap\left\{\tau\circ\Phi_{\frac{t}{2}}\in\left[\frac{t}{4};a_t^3\right],\ \tau_{\lfloor c_1 t\rfloor}\circ\widetilde{T}^{\lfloor K\log t\rfloor}\circ\Phi_{t/2}>\frac{11t}{100}\right\}\right)+o(a_t^{-2})\\ &\le 2(2K_0+1)^2\nu\left(\tau\circ\phi_{\frac{t}{2}}\in\left[\frac{t}{4};a_t^3\right],\ \tau_{\lfloor c_1 t\rfloor}\circ T^{\lfloor K\log t\rfloor}\circ\phi_{t/2}>\frac{11t}{100}\right)+o(a_t^{-2})\end{aligned}$$

using again the fact that B_0 is made of at most $(2K_0+1)^2$ copies of $\mathcal{M}$. Now using the $\phi_{t/2}$-invariance of ν, we obtain

(5.20)

$$\begin{aligned} p_{t,1} &\le 2(2K_0+1)^2\nu\left(\tau\in\left[\frac{t}{4};a_t^3\right],\ \tau_{\lfloor c_1 t\rfloor}\circ T^{\lfloor K\log t\rfloor}>\frac{11t}{100}\right)+o(a_t^{-2})\\ &\ll o(a_t^{-2})+\sum_{a'=t/4}^{a_t^3}\mu\left(\tau>a',\tau_{\lfloor c_1 t\rfloor}\circ T^{\lfloor K\log t\rfloor}>11t/100\right)\\ &\ll o(a_t^{-2})+\sum_{a'=t/4}^{a_t^3}\mu\left(|\kappa|>a'-2\|H_0\|_\infty,|\kappa|_{\lfloor c_1 t\rfloor}\circ T^{\lfloor K\log t\rfloor}>10t/100\right)\\ &\ll o(a_t^{-2})+\sum_{a'=t/4}^{a_t^3}\left(\mu\left(|\kappa|>a'-2\|H_0\|_\infty\right)\mu\left(|\kappa|_{\lfloor c_1 t\rfloor}>10t/100\right)+\mathcal{O}(t^{-100})\right)\end{aligned}$$

(5.21)

$$\ll o(a_t^{-2})+\sum_{a'=t/4}^{a_t^3}(a')^{-2}\mu\left(|\kappa|_{\lfloor c_1 t\rfloor}>t/10\right).$$

But, using Lemma 3.5 (with $d=0$) and the μ-integrability of τ,

$$\begin{aligned}\mu\left(|\widetilde{\kappa}|_{\lfloor c_1 t\rfloor}>t/10\right)&\le\mu\left(\tau_{\lfloor c_1 t\rfloor}>9t/100\right)\\ &\le\mu\left(\tau_{\lfloor c_1 t\rfloor}>t^3\right)+\mu\left(9t/100<\tau_{\lfloor c_1 t\rfloor}\le t^3\right)\\ &\ll\frac{\mathbb{E}_\mu[\tau_{c_1 t}]}{t^3}+\sum_{k=9t/100}^{t^3}\frac{t}{\sqrt{t\log t}}\frac{\log(k-\lfloor c_1 t\rfloor\mu(\tau))}{1+(k-\lfloor c_1 t\rfloor\mu(\tau))^2}\\ &\ll t^{-2}+\sum_{k\ge t(9/100-c_1\mu(\tau))}\frac{t}{\sqrt{t\log t}}\frac{\log t}{1+k^2}\\ &\ll t^{-2}+\frac{t}{\sqrt{t\log t}}\frac{\log t}{t}=\sqrt{\frac{\log t}{t}}.\end{aligned}\tag{5.22}$$

This together with (5.21) implies that

$$p_{t,1} = o(a_t^{-2}). \tag{5.23}$$

Estimate when the free flight at time $t/2$ is smaller than $t/4$. Let

$$p_{t,2} := \widetilde{\nu}\left(B_0 \cap \Phi_{-t}(B_0) \cap \{\widetilde{N}_t \le c_1 t, \tau\circ\Phi_{\frac{t}{2}} \le t/4\}\right).$$

Using again the fact that B_0 is made of at most $(2K_0+1)^2$ copies of $\mathcal{M}$ and that $\phi_{t/2}$ preserves ν,

$$\begin{aligned} p_{t,2} &\le (2K_0+1)^2 \nu(N_t \le c_1 t, \tau\circ\phi_{t/2} \le t/4) \\ &\le (2K_0+1)^2 \nu\left(N_{t/2} \le c_1 t, N_{-t/2} \le c_1 t, \tau \le t/4\right). \end{aligned} \tag{5.24}$$

This together with Sublemma 5.9 and time reversibility gives that

$$\begin{aligned} &\nu(N_t < c_1 t, \tau\circ\phi_{t/2} \le t/4) \\ &\le o(a_t^{-2}) + \nu\left(N_{t/2} \le c_1 t, N_{-t/2} \le c_1 t, \tau\right. \\ &\left.\le t/4, \min(\tau_{\lfloor K\log t\rfloor} - \tau, |\tau_{-\lfloor K\log t\rfloor})| \le t/100\right) \\ &\le o(a_t^{-2}) + 2\nu\left(N_{t/2} \le c_1 t,\ N_{-t/2} \le c_1 t,\ \tau \le t/4,\ |\tau_{-\lfloor K\log t\rfloor}| < t/100\right). \end{aligned}$$

Since we also know that

$$\begin{aligned} &|\tau_{-\lfloor c_1 t\rfloor}| \circ T^{-\lfloor K\log t\rfloor} \\ &= |\tau_{-\lfloor K\log t\rfloor - \lfloor c_1 t\rfloor} - \tau_{-\lfloor K\log t\rfloor}| > |\tau_{-\lfloor c_1 t\rfloor} - \tau| - |\tau_{-\lfloor K\log t\rfloor} - \tau| > \frac{t}{2} - \frac{t}{4} - \frac{t}{100}, \end{aligned}$$

we obtain

$$\begin{aligned} &\nu(N_t \le c_1 t, \tau\circ\phi_{t/2} \le t/4) \\ &\le o(a_t^{-2}) + 2\nu(N_{t/2} \le c_1 t,\ |\tau_{-\lfloor c_1 t\rfloor}| \circ T^{-\lfloor K\log t\rfloor} > 24t/100, \tau \le t/4). \end{aligned}$$

Let

$$A_{a',t'} = \{\tau \ge a' - 4\|H_0\|_\infty, \tau_{\lfloor c_1 t\rfloor} - a' > 48t/100\}.$$

Using again the representation by a suspension flow and the correlation estimate (5.3) combined with (5.4),

$$\begin{aligned} &\nu(N_t \le c_1 t, \tau\circ\phi_{t/2} \le t/4) + o(a_t^{-2}) \\ &\le \sum_{a'=0}^{\lfloor t/4\rfloor} \mu\left(\tau \ge a', \tau_{\lfloor c_1 t\rfloor} - a' \ge t/2, |\tau_{-\lfloor c_1 t\rfloor}| \circ T^{-\lfloor K\log t\rfloor} > 24t/100\right) \\ &\le \sum_{a'=0}^{\lfloor t/4\rfloor} \mu\left(|\kappa| \ge a' - 2\|H_0\|_\infty, |\kappa|_{\lfloor c_1 t\rfloor} - a'\right. \\ &\left.\ge 49t/100, \left||\kappa|_{-\lfloor c_1 t\rfloor}\right| \circ T^{-\lfloor K\log t\rfloor} > 23t/100\right) \\ &\le \sum_{a'=0}^{\lfloor t/4\rfloor} \left(\mu\left(A_{a',t}\right)\mu\left(|\tau_{-\lfloor c_1 t\rfloor}| > 22t/100\right) + \mathcal{O}(t^{-100})\right) \\ &\le \sum_{a'=0}^{\lfloor t/4\rfloor} \mu\left(A_{a',t}\right)\sqrt{\frac{\log t}{t}}, \end{aligned}$$

where in the last line we have used (5.22).

The previous displayed estimate gives that

$$\nu(N_t \le c_1 t, \tau\circ\phi_{t/2} \le t/4) \le o(a_t^{-2}) + \nu(N_{48t/100+4\|H_0\|_\infty} \le c_1 t)\sqrt{\frac{\log t}{t}} = o(a_t^{-2}),$$

which together with (5.24) ensures that $p_{t,2} = o(a_t^{-2})$, which combined with (5.23) ends the proof of Lemma 5.6 in the case $d = 2$. □

5.3. **Control on "good" configurations.** Due to Lemmas 5.6 and 5.4 and to Remark 5.5, it remains to study the $\widetilde{\nu}$-measure of the set of configurations $\mathbf{x} \in B_{R_0} \cap \Phi_{-t}(B_0)$, that have at least $c_1 t$ collisions in the time interval $[0, t)$ and with first and last free flights both smaller than $a_t^d \log R$. Lemma 5.10 provides the domination of this measure by a sum. Let us write $\mathfrak{C}$ for the set of *unit direction of corridors* in $\mathcal{D}_2$. We recall that this set is finite (see e.g. [34]).

Lemma 5.10. *There exist a positive integer L_0, a positive real number C_0 and a compact set $K' \subset \mathbb{Z}^d \times \mathbb{R}$ such that, for all integers $R_0 > L_0$ and all $t > 0$,*

$$\widetilde{\mu}\left(B_{R_0} \cap \Phi_{-t}(B_0) \cap \{\widetilde{N}_t > c_1 t,\ |\widetilde{\kappa}| \le a_t^d \log R,\ |\widetilde{\kappa}\circ\Phi_t| \le a_t^d \log R\}\right) \le C_0 \sum_{\vec{w}_1,\vec{w}_2\in\mathfrak{C}} \sum_{n=\lfloor c_1 t\rfloor}^{\lfloor t/\min\tau\rfloor} \sum_{a=R_0-L_0}^{\lfloor a_t^d \log R_0\rfloor} \sum_{b=0}^{\lfloor a_t^d \log R_0\rfloor} \mu(A'_{a,b,n,t}(\vec{w}_1,\vec{w}_2,K')),$$

where we set

$$A'_{a,b,n,t}(\vec{w}_1,\vec{w}_2,K') = \Big\{\widehat{\Psi}_n \in (-\pi_d(a\vec{w}_1 + b\vec{w}_2), t - n\mu(\tau) - b - a) + K', \tag{5.25}$$
$$|\widetilde{\kappa}\circ T^{-1}| \ge a, |\widetilde{\kappa}\circ T^n| \ge b\Big\}.$$

Proof. Let $\mathbf{x} \in B_{R_0} \cap \Phi_{-t}(B_0) \cap \{\widetilde{N}_t > c_1 t,\ |\widetilde{\kappa}| \le a_t^d \log R,\ |\widetilde{\kappa}\circ\Phi_t| \le a_t^d \log R\}$. We will parametrise $\mathbf{x}$ by $(x, -u, \ell) \in M \times (-\infty, -R_0] \times \mathbb{Z}^d$, with $u \in [0, \tau(T^{-1}(x)))$. We write $\mathbf{x}$ under the form $\Phi_{-u}(\widetilde{x})$ with $\widetilde{x} \in \widetilde{M}$ corresponding to the configuration of the particle at the next (future) collision time. This configuration $\widetilde{x}$ belongs to some cell $\mathcal{C}_\ell$ with $\ell \in \mathbb{Z}^d$ and thus $\widetilde{x}$ can be rewritten under the form $(p_{d,0}^{-1}(q) + \ell, \vec{v})$ for some $x = (q, \vec{v}) \in M$ (as explained in Section 2.1). By construction $u \in [R_0, \tau(T^{-1}(x)))$. We parametrise $\Phi_t(\mathbf{x})$ by $((T^n(x), s), \ell') \in \widehat{\mathcal{M}} \times \mathbb{Z}^d$, as follows. Recalling that N_t is the lap number introduced above (2.1), we write $\Phi_t(\mathbf{x})$ under the form $\Phi_s(\widetilde{T}^n(\widetilde{x}))$ with $n = N_t(\phi_{-u}(x)) - 1$ and $s \in [0, \tau(T^n(x)))$. Due to our assumptions on $\mathbf{x}$, we know that $N_t(\phi_{-u}(x)) \ge \lfloor c_1 t\rfloor + 1$ so that $n \ge \lfloor c_1 t\rfloor$. Moreover, $N_t \le 1 + t/\min\tau$. Thus

$$\lfloor c_1 t\rfloor \le n \le t/\min\tau. \tag{5.26}$$

It follows from (2.9) and (2.10) that $\tau - |\widetilde{\kappa}|$ is uniformly bounded. Recall that $u \le \tau(T^{-1}(x))$ and $s \le \tau(T^n(x))$. We discretise u, s by setting

$$a := \max(0, \lfloor u\rfloor - \|\tau - |\widetilde{\kappa}|\|_\infty) \le |\widetilde{\kappa}(T^{-1}(x))| \le a_t^d \log R_0, \tag{5.27}$$

$$b := \max(0, \lfloor s\rfloor - \|\tau - |\widetilde{\kappa}|\|_\infty) \le |\widetilde{\kappa}(T^n(x))| \le a_t^d \log R_0. \tag{5.28}$$

With the previous notations,

$$\widetilde{\tau}_n(x) = t - n\mu(\tau) - u - s = t - n\mu(\tau) - a - b + \mathcal{O}(1), \tag{5.29}$$

where $\mathcal{O}(1)$ is uniformly bounded (independently of $\mathbf{x}$). Furthermore, there exist two unit directions of corridors $\vec{w}_1, \vec{w}_2 \in \mathfrak{C}$ that are "close" to be collinear to, respectively, the first and last free flight, meaning that

$$\widetilde{\kappa}(T^{-1}(x)) = |\widetilde{\kappa}(T^{-1}(x))|\vec{w}_1 + \mathcal{O}(1) \quad \text{and} \quad \widetilde{\kappa}(T^n(x)) = |\widetilde{\kappa}(T^n(x))|\vec{w}_2 + \mathcal{O}(1),$$

with $\mathcal{O}(1)$ uniformly bounded (independently of $\mathbf{x}$). This implies that $\ell = a\pi_d(\vec{w}_1) + \mathcal{O}(1)$ and $\ell' = -b\pi_d(\vec{w}_2) + \mathcal{O}(1)$, where again $\mathcal{O}(1)$ is uniformly bounded (independently of $\mathbf{x}$). Thus, for a given $(a, b, \vec{w}_1, \vec{w}_2)$, only a uniformly bounded number of values of (ℓ, ℓ') are possible. Second, this implies also that

$$\kappa_n + \pi_d(a\vec{w}_1 + b\vec{w}_2) = \ell' - \ell + a\pi_d(\vec{w}_1) + b\pi_d(\vec{w}_2) = \mathcal{O}(1). \tag{5.30}$$

Recalling that $\widehat{\Psi}_n = (\kappa_n, \widetilde{\tau}_n)$, it follows from (5.27), (5.28), (5.29) and (5.30) that we can find a compact set K' independent of $\mathbf{x}$ such that, with previous notations,

$$x \in A'_{a,b,n,t}(\vec{w}_1, \vec{w}_2, K').$$

The bounds on n, a, b come from respectively (5.26), (5.27) (and $u \geq R_0$) and (5.28). The multiplicative constant C_0 comes from the bounded number of possible values of $(\ell, \ell', \lfloor u \rfloor, \lfloor s \rfloor)$ once a and b are fixed. This ends the proof of the lemma. $\square$

Since $\mathfrak{C}$ is finite, it is enough to fix $\vec{w}_1, \vec{w}_2$ and to prove that

$$\lim_{R_0 \to +\infty} \limsup_{t \to +\infty} a_t^d \sum_{n=\lfloor c_1 t \rfloor}^{\lfloor t/\min \tau \rfloor} \sum_{a=R_0}^{\lfloor a_t^d \log R_0 \rfloor} \sum_{b=0}^{\lfloor a_t^d \log R_0 \rfloor} \mu(A'_{a,b,n,t}(\vec{w}_1, \vec{w}_2, K')) = 0. \tag{5.31}$$

The aimed result (5.31) will be proved via the next technical lemma, the proof of which uses a splitting of the summation over n, a, b in smaller ranges.

Lemma 5.11. *There exists $C' > 0$ such that for all R_0*

$$\sum_{n=\lceil c_1 t \rceil}^{\lfloor t/\min \tau \rfloor} \sum_{a,b=0,\dots,\lfloor a_t^d \log(R_0) \rfloor : \max(a,b) \geq R_0} \mu(A'_{a,b,n,t}(\vec{w}_1, \vec{w}_2, K')) \leq C' a_t^{-d} R_0^{-\frac{2}{45}} + o(a_t^{-d}),$$

as $t \to +\infty$.

Proof. Note that, by measure preserving and time reversal, $(\widetilde{\kappa} \circ T^{-1}, \widehat{\Psi}_n, \widetilde{\kappa} \circ T^n)$ has the same distribution as $(-\kappa \circ T^n, (-\widetilde{\kappa}_n, \widetilde{\tau}_n), -\widetilde{\kappa} \circ T^{-1})$ and so

$$\mu(A'_{a,b,n,t}(\vec{w}_1, \vec{w}_2, K')) = \mu(A'_{b,a,n,t}(-\vec{w}_2, -\vec{w}_1, K''))$$

with $K'' = \{(\ell, r) \in \mathbb{Z}^d \times \mathbb{R} : (-\ell, r) \in K'\}$, with the notation (5.25). Thus, up to replacing $(a, b, \vec{w}_1, \vec{w}_2, K')$ by $(b, a, -\vec{w}_2, -\vec{w}_1, K'')$, it is enough to prove that there exists C_0'' such that, for all R_0,

$$\sum_{n=\lceil c_1 t \rceil}^{\lfloor t/\min \tau \rfloor} \sum_{a=R_0}^{\lfloor a_t^d \log(R_0) \rfloor} \sum_{b=0}^{a} \mu(A'_{a,b,n,t}(\vec{w}_1, \vec{w}_2, K')) \leq C_0'' a_t^{-d} R_0^{-\frac{2}{45}} + o(a_t^{-d}), \quad \text{as } t \to +\infty. \tag{5.32}$$

The main ingredients of the proof of this are Lemma 3.2 together with its Corollary 3.3 and an argument similar to the one used in the proof of Lemma 4.2. To exploit Lemma 3.2, recall (5.25) and note that

$$\begin{aligned}(5.33)\quad &\mu(A'_{a,b,n,t}(\vec{w}_1,\vec{w}_2,K'))\\ &\le \mathbb{E}_\mu\left[\mathbf{1}_{|\widetilde{\kappa}|\circ T^n\ge b}\mathbf{1}_{|\widetilde{\kappa}|\circ T^{-1}\ge a}.h\left(\widehat{\Psi}_n-(-\pi_d(a\vec{w}_1+b\vec{w}_2),t-b-a-n\mu(\tau))\right)\right],\end{aligned}$$

where $h:\mathbb{Z}^d\times\mathbb{R}\to(0,\infty)$ is the integrable function with compactly supported Fourier transform given by $h(y_1,\dots,y_{d+1}):=K''\mathbf{1}_{|y_1|,|y_d|\le K''}\frac{(1-\cos(y_{d+1}/K''))}{y_{d+1}^2}$ for some suitable $K''>0$. Recall (5.3) and (5.4) and assume that $n>4\lfloor K\log t\rfloor$. Let $k_n=k_t=\lfloor K\log t\rfloor$. It follows from (5.33) and Corollary 3.3, that, for any $\varepsilon_0\in\left(0,\frac12-\frac1{45}\right)$ and any $n=\lfloor c_1t\rfloor,\dots,t/\min\tau$,

$$(5.34)\quad \mu(A'_{a,b,n,t}(\vec{w}_1,\vec{w}_2,K'))\ll\mu(|\widetilde{\kappa}|\ge a)\mu(|\widetilde{\kappa}|\ge b)\widetilde{Q}^{(0)}_{n,a,b}(t)$$

$$(5.35)\quad +\mathcal{O}\Big(t^{-100}+\log t\,a_t^{-d-2}(\mu(|\widetilde{\kappa}|\ge a)\mu(|\widetilde{\kappa}|\ge b))^{\frac12-\varepsilon_0}$$

$$(5.36)\quad +a_t^{-d-2}\mu(|\widetilde{\kappa}|\ge b)\left\|\widehat{\Psi}_{2k_n}1_{\{|\widetilde{\kappa}|\circ T^{-1}\ge a\}}\right\|_{L^1}\Big),$$

$$(5.37)\quad \text{with}\quad \widetilde{Q}^{(0)}_{n,a,b}(t):=\mathbb{E}_\mu\left[h\left(\widehat{\Psi}_n-(-\pi_d(a\vec{w}_1+b\vec{w}_2),t-b-a-n\mu(\tau))\right)\right].$$

Estimating the term in the right hand side of (5.34)**.**

We claim that, adapting carefully the argument of Lemma 4.2,

$$(5.38)\qquad \sup_{a,b}\sum_{n=\lceil c_1t\rceil}^{\lfloor t/\min\tau\rfloor}\widetilde{Q}^{(0)}_{n,a,b}(t)=\mathcal{O}(a_t^{-d}),$$

which implies that

$$(5.39)\qquad \sum_{n=\lceil c_1t\rceil}^{\lfloor t/\min\tau\rfloor}\sum_{b\ge0,a\ge R_0}\mu(|\widetilde{\kappa}|\ge a)\mu(|\widetilde{\kappa}|\ge b)\widetilde{Q}^{(0)}_{n,a,b}(t)=\mathcal{O}(a_t^{-d}R_0^{-1}).$$

We prove the claim (5.38). Fix $L>0$.

First, it follows from Lemma 3.2 that

$$\widetilde{Q}^{(0)}_{n,a,b}(t)\ll a_t^{-d-1}\left(g_{d+1}\left(\frac{-\pi_d(a\vec{w}_1+b\vec{w}_2),t-b-a-n\mu(\tau)}{a_t}\right)+(\log t)^{-1}\right)$$

and so that

$$\begin{aligned}\sum_{n:|t-b-a-n\mu(\tau)|<La_t\mu(\tau)}\widetilde{Q}^{(0)}_{n,a,b}(t)&\ll a_t^{-d}\int_{\mathbb{R}}g_{d+1}\left(\frac{-\pi_d(a\vec{w}_1+b\vec{w}_2)}{a_t},y\right)dy+o(a_t^{-d})\\ (5.40)\qquad &\ll a_t^{-d}\|g_d\|_\infty+o(a_t^{-d})\ll a_t^{-d},\end{aligned}$$

with $g_d:=\int_{\mathbb{R}}g_{d+1}(\cdot,y)\,dy$.

Second, if $n>c_1t$ and $La_t<|n-(t-b-a)/\mu(\tau)|$, then, as soon as t large enough, it follows from Lemma 3.5 that, setting $v_{a,b,n}:=(-\pi_d(a\vec{w}_1+b\vec{w}_2),t-b-a-n\mu(\tau))$,

$$\begin{aligned}\widetilde{Q}^{(0)}_{n,a,b}(t)&\ll\sum_{k_3\in\mathbb{Z}}\frac{1}{1+k_3^2}\mu\left(\widehat{\Psi}_n-v_{a,b,n}=(0,0,k_3)+\mathcal{O}(1)\right)\\ (5.41)\qquad &\ll\frac{t}{a_t^{d+1}}\sum_{k_3\in\mathbb{Z}}\frac{1}{1+k_3^2}\frac{\log(2+|v_{a,b,n}+k_3|)}{(2+|v_{a,b,n}+k_3|)^2}.\end{aligned}$$

But, on the one hand,

$$\sum_{k_3:|v_{a,b,n}+k_3|\ge|v_{a,b,n}|/2}\frac{1}{1+k_3^2}\frac{\log(2+|v_{a,b,n}+k_3|)}{(2+|v_{a,b,n}+k_3|)^2}\ll\frac{\log(4+|v_{a,b,n}|)}{(1+|v_{a,b,n}|)^2}\sum_{k_3}\frac{1}{1+k_3^2}$$
$$\ll\frac{\log(4+|v_{a,b,n}|)}{(1+|v_{a,b,n}|)^2},\tag{5.42}$$

and, on the other hand,

$$\sum_{k_3:|v_{a,b,n}+k_3|<|v_{a,b,n}|/2}\frac{1}{1+k_3^2}\frac{\log(2+|v_{a,b,n}+k_3|)}{(2+|v_{a,b,n}+k_3|)^2}$$
$$\ll\sum_{u:|u|<|v_{a,b,n}|/2}\frac{1}{1+|u-v_{a,b,n}|^2}\frac{\log(2+|u|)}{(2+|u|)^2}\tag{5.43}$$
$$\ll\frac{1}{|v_{a,b,n}|^2}\sum_{u:|u|<|v_{a,b,n}|/2}\frac{\log(2+|u|)}{(2+|u|)^2}\ll\frac{1}{|v_{a,b,n}|^2}.$$

It follows from (5.41), (5.42) and (5.43) that

$$\widetilde{Q}^{(0)}_{n,a,b}(t)\ll\frac{\log(2+|v_{a,b,n}|)}{(1+|v_{a,b,n}|)^2},$$

as $\widetilde{Q}_n(t)$ in Sublemma 4.3. Therefore

$$\sum_{n>c_1t:La_t<|n-(t-b-a)/\mu(\tau)|}\widetilde{Q}^{(0)}_{n,a,b}(t)$$
$$\ll\frac{t}{a_t^{d+1}}\sum_{n:La_t<|n-(t-a-b)/\mu(\tau)|}\frac{|\log|n\mu(\tau)-(t-a-b)||}{1+|n\mu(\tau)-(t-a-b)|^2}$$
$$\ll\frac{t}{a_t^{d+1}}\frac{\log(a_t)}{La_t}\ll\frac{1}{La_t^d}\ll a_t^{-d},\tag{5.44}$$

since $\log u/u^2$ has primitive $-(1+\log u)/u$ and since $a_t^2=t\log t\sim 2t\log a_t$. The claim (5.38) follows from (5.40) and (5.44).

Estimating the terms in (5.35). The first term leads to

$$\sum_{n=\lceil c_1t\rceil}^{\lfloor t/\min\tau\rfloor}\sum_{a=R_0}^{\lfloor a_t^d\log R_0\rfloor}\sum_{b=0}^{a}t^{-100}\ll t^{-99}a_t^{2d}(\log R_0)^2=o(a_t^{-d}).\tag{5.45}$$

It remains to estimate the contribution of the second part of (5.35). Note that

$$\sum_{a\ge R_0}\mu(|\widetilde{\kappa}|\ge a)\left(\sum_{b=0}^{a}\mu(|\widetilde{\kappa}|\ge b)^{\frac12-\varepsilon_0}\right)\ll\sum_{a\ge R_0}a^{-2}\left(\sum_{b=0}^{a}(b+1)^{-1+2\varepsilon_0}\right)$$
$$\ll\sum_{a\ge R_0}a^{-2+2\varepsilon_0}=R_0^{-1+2\varepsilon_0},$$

since $\varepsilon_0<\frac12$. Since we also know that $t\log t=a_t^2$, we obtain that

$$\sum_{n=\lceil c_1t\rceil}^{\lfloor t/\min\tau\rfloor}\sum_{a\ge R_0}\sum_{b=0}^{a}\log t\,a_t^{-d-2}\mu(\widetilde{\kappa}\ge a)\mu(\widetilde{\kappa}\ge b)^{\frac12-\varepsilon_0}\ll a_t^{-d}R_0^{-(1-2\varepsilon_0)},\tag{5.46}$$

with $1-2\varepsilon_0>0$.

Estimating the term in (5.36)**.** We claim that

$$\sum_{a\geq R_0}\left\|\widehat{\Psi}_{2k_n}1_{\{|\widetilde{\kappa}|\circ T^{-1}\geq a\}}\right\|_{L^1}\ll R_0^{-\frac{2}{45}}\log t. \tag{5.47}$$

Since we also know that $t\log t=a_t^2$ and that $\sum_{b\geq 0}\mu(|\widetilde{\kappa}|\geq b)=\mathbb{E}[|\widetilde{\kappa}|]<\infty$, we obtain that

$$\sum_{n=\lceil c_1 t\rceil}^{\lfloor t/\min\tau\rfloor}\sum_{a\geq R_0}\sum_{b\geq 0}a_t^{-d-2}\mu(|\widetilde{\kappa}|\geq b)\left\|\widehat{\Psi}_{2k_n}1_{\{|\widetilde{\kappa}|\circ T^{-1}\geq a\}}\right\|_{L^1}=\mathcal{O}\left(a_t^{-d}R_0^{-\frac{2}{45}}\right). \tag{5.48}$$

We now prove the claim (5.47). First, compute that

$$\begin{aligned}\sum_{a\geq R_0}\left\|\widehat{\Psi}_{2k_n}1_{\{|\widetilde{\kappa}|\circ T^{-1}\geq a\}}\right\|_{L^1}&\leq\sum_{\ell=0}^{2k_n-1}\sum_{a\geq R_0}\sum_{a'\geq a}\sum_{b'\geq 1}b'\mu(|\widetilde{\kappa}|\circ T^{-1}=a',|\widetilde{\kappa}|\circ T^{\ell}=b')\\&\leq\sum_{\ell=1}^{2k_n}\sum_{a'\geq R_0}\sum_{a=R_0}^{a'}\sum_{b'\geq 1}b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^{\ell}=b')\\&\leq\sum_{\ell=1}^{2k_n}\sum_{a'\geq R_0}\sum_{b'\geq 1}a'b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^{\ell}=b'),\end{aligned}$$

where the sum over a',b' is taken over the positive real numbers (non-necessarily integer) such that the summand is non-null.

We claim that, uniformly in n and in $\ell=1,\dots,2k_n$,

$$\sum_{a'\geq R_0}\sum_{b'\geq 1}a'b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^{\ell}=b')\ll R_0^{-\frac{2}{45}}. \tag{5.49}$$

The previous two displayed equations give the claim (5.47).

It remains to prove the claim (5.49). We proceed via considering all relevant cases of a',b'.

Case 1 (Contribution of the a',b' such that $b'^{\frac{4}{5}}\leq a'\leq b'$).

$$\begin{aligned}&\sum_{b'\geq 1}\sum_{a'\in[\max(R_0,(b')^{4/5});b']}a'b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^{\ell}=b')\\&\leq\sum_{b'\geq R_0}b'\mathbb{E}_\mu\left[\sum_{a'\in[(b')^{4/5};b']}a'\mathbf{1}_{|\widetilde{\kappa}|=a'}\mathbf{1}_{|\widetilde{\kappa}|\circ T^{\ell}=b'}\right]\\&\leq\sum_{b'\geq R_0}|b'|^2\mu(|\widetilde{\kappa}|\geq(b')^{\frac{4}{5}},|\widetilde{\kappa}|\circ T^{\ell}=b'),\end{aligned}$$

where in the last equation we used that

$$\sum_{a'\in[a_-;a_+]}a'1_{\{|\widetilde{\kappa}|=a'\}}\leq a_+1_{\{|\widetilde{\kappa}|\geq a_-\}}. \tag{5.50}$$

Thus applying (5.6) with $V:=100/a_0$, we obtain

$$\mu(|\widetilde{\kappa}|\geq(b')^{\frac{4}{5}},|\widetilde{\kappa}|\circ T^{\ell}=b')\ll|b'|^{-3-\frac{2}{45}},$$

uniformly in $\ell\leq\frac{100\log(2+|b'|)}{a_0}$, and it follows from (5.3) combined with (5.5) that

$$\mu(|\widetilde{\kappa}|\geq(b')^{\frac{4}{5}},|\widetilde{\kappa}|\circ T^{\ell}=b')\leq\mu(|\widetilde{\kappa}|\geq(b')^{\frac{4}{5}})\mu(|\widetilde{\kappa}|\circ T^{\ell}=b')+C'e^{-a_0\ell}\ll|b'|^{-3-\frac{2}{45}}$$

uniformly in $\ell \ge \frac{100\log(2+|b'|)}{a_0}$. Therefore

$$\sum_{a'\in[\max(R_0,(b')^{4/5});b']} a'b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^\ell=b') \ll \sum_{b'\ge R_0} |b'|^2\,|b'|^{-3-\frac{2}{45}} \tag{5.51}$$

$$\ll \sum_{b'\ge R_0} |b'|^{-1-\frac{2}{45}} \ll R_0^{-\frac{2}{45}}. \tag{5.52}$$

Case 2 (Contribution of the a',b' such that $(a')^{\frac45} < b' < a'$).

$$\begin{aligned}
&\sum_{b'\ge 1}\ \sum_{a'\in[\max(R_0,b');(b')^{\frac54}]} a'b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^\ell=b')\\
&\le \sum_{a'\ge R_0}\ \sum_{b'\in[(a')^{\frac45};a']} a'b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^\ell=b')\\
&\le \sum_{a'\ge R_0} |a'|^2\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^\ell>(a')^{\frac45})
\end{aligned}$$

$$\ll \sum_{a'\ge R_0} |a'|^2|a'|^{-3-\frac{2}{45}} \ll \sum_{a'\ge R_0} |a'|^{-1-\frac{2}{45}} \ll R_0^{-\frac{2}{45}}, \tag{5.53}$$

using again (5.50) and (5.6) again with $V=100/a_0$ when $\ell\le V\log(2+|a')$ and (5.3) otherwise.

Case 3 (Contribution of the a',b' such that $a'<(b')^\gamma<b'$ for some $\gamma\in(0,1)$ (e.g. $\gamma=\frac45$)).

$$\begin{aligned}
&\sum_{b'\ge 1}\ \sum_{a'\in[R_0;(b')^\gamma]} a'b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^\ell=b')\\
&\le \sum_{b'\ge R_0^{\frac1\gamma}}\ \sum_{a'\in[R_0;(b')^\gamma]} a'b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^\ell=b')\\
&\le \sum_{b'\ge R_0^{\frac1\gamma}} |b'|^\gamma b'\mu(|\widetilde{\kappa}|\ge R_0,|\widetilde{\kappa}|\circ T^\ell=b')
\end{aligned}$$

$$\ll \sum_{b'\in Supp(|\widetilde{\kappa}|):b'\ge R_0^{\frac1\gamma}} (b')^\gamma|b'|\,|b'|^{-3} \ll R_0^{\frac1\gamma(-1+\gamma)}, \tag{5.54}$$

using (5.5).

Case 4 (Contribution of the a',b' such that $b'<(a')^\gamma<a'$ with $\gamma\in(0,1)$ (e.g. $\gamma=\frac45$)).

$$\begin{aligned}
&\sum_{a'\ge R_0}\ \sum_{b'\in[1;(a')^\gamma]} a'b'\mu(|\widetilde{\kappa}|=a',|\widetilde{\kappa}|\circ T^\ell=b') \le \sum_{a'\ge R_0}(a')^\gamma|a'|\mu(|\widetilde{\kappa}|=a')\\
&\ll \sum_{a'\in Supp(|\widetilde{\kappa}|):a'\ge R_0} (a')^\gamma|a'||a'|^{-3} \ll R_0^{-1+\gamma}.
\end{aligned} \tag{5.55}$$

The claim (5.49) follows from (5.52), (5.53), (5.54) and (5.55), ending the proof of (5.47) and so of (5.48). Estimate (5.32) and so the lemma then follows from (5.34), (5.39), (5.45), (5.46) and (5.48). □

5.3.1. *Concluding the proof of Theorem* 5.1. The conclusion follows from Lemmas 5.4 (and the comment thereafter), 5.6, 5.10 and 5.11.

6. Proof of joint CLT (Lemma 3.1)

Let $d \in \{0,1,2\}$. In this section we show that arguments established in [3] and [29] can be adapted to the study of $\widehat{\Psi}$ instead of κ. The main idea comes down to a basic observation, namely that $\widehat{\Psi}$ can be written as the sum of a vector in $\mathbb{Z}^{d+1}$ that 'behaves like' κ and of a bounded function. The mentioned vector in $\mathbb{Z}^{d+1}$ is precisely $(\kappa, |\widetilde{\kappa}| - \mathbb{E}_\mu[|\widetilde{\kappa}|])$ which, as κ, is constant on good sets and has a similar tail probability. In particular, the distribution of $\widehat{\Psi}$ is in the domain of a non-standard CLT with normalization $\sqrt{n \log n}$. The details are provided around equation (6.12).

We will prove the convergence in distribution of $(\widehat{\Psi}_n/\sqrt{n\log n})_n$ by establishing the pointwise convergence of its characteristic function, with the use of Fourier perturbed operator on the quotient tower constructed by Young in [37] (see [8]) as Szász and Varjú did in [34] to establish the CLT and LLT for κ. It follows from (2.10) that $\tau = |V| = |\widetilde{\kappa}| + \mathcal{O}(1)$, where $\widetilde{\kappa} : M \to \mathbb{Z}^2$ is the cell change in the $\mathbb{Z}^2$-periodic Lorentz gas (see Subsection 5.1).

We have already recalled several properties of $\widetilde{\kappa}$. Let us recall, in particular, the precise tail of $\widetilde{\kappa}$ (this is partially recalled in (5.5)). By [34] completed by [29], there exist $L_0 > 0$ and a finite set $\mathcal{E}$ made of $(L,w) \in (\mathbb{Z}^d)^2$ with w prime such that

$$|\widetilde{\kappa}| > L_0 \quad \Rightarrow \quad \exists (L,w) \in \mathcal{E},\ \exists N \in \mathbb{N}^*\ \widetilde{\kappa} = L + Nw \tag{6.1}$$

and

$$\mu(\widetilde{\kappa} = L + Nw) = c_{L,w} N^{-3} + \mathcal{O}(N^{-4}), \quad \text{as } N \to +\infty, \tag{6.2}$$

with $c_{L,w} > 0$. This set $\mathcal{E}$ parametrises the set of corridors mentioned in Section 5 (the set $\mathfrak{C}$ therein corresponds to the set of unit vectors proportional to some w such that there exists $L \in \mathbb{Z}^2$ such that $(L,w) \in \mathcal{E}$). Then, when $d = 2$, the variance matrix Σ_0 (for the Sinai billiard map) appearing in the Central Limit Theorem for the displacement given by (2.11) corresponds to the following quadratic form

$$\text{If } d = 2, \quad \forall t \in \mathbb{R}^2, \quad \langle \Sigma_0 t, t\rangle := \frac{1}{2} \sum_{(L,w)\in\mathcal{E}} c_{L,w} \langle t, w\rangle^2. \tag{6.3}$$

It is not degenerate since, when $d = 2$, we assume the existence of at least two non-parallel corridors, and so of two non-parallel w, w' such that there exists $L, L' \in \mathbb{Z}^2$ such that $(L,w), (L',w') \in \mathcal{E}$.

When $d = 1$, setting $\pi_1(w_1,w_2) = w_1$, Σ_0 is given by the formula

$$\text{If } d = 1, \quad \Sigma_0 := \frac{1}{2} \sum_{(L,w)\in\mathcal{E}} c_{L,w} (\pi_1(w))^2 \tag{6.4}$$

which is non-null since we assumed the existence of at least an unbounded line touching no obstacle.

We recall that the variance matrix Σ for the flow appearing in (1.1) is given by $\Sigma = \Sigma_0/\sqrt{\mu(\tau)}$.

The variance matrix Σ_{d+1} of the limit of $a_n^{-1}\widehat{\Psi}_n$ will appear to be given by the following pretty similar formula:

$$\forall t \in \mathbb{R}^{d+1}, \quad \langle \Sigma_{d+1} t, t\rangle := \frac{1}{2} \sum_{(L,w)\in\mathcal{E}} c_{L,w} \langle t, (\pi_d(w), |w|)\rangle^2, \tag{6.5}$$

with, as in Section 5.1,

$$\forall w \in \mathbb{Z}^2, \quad \pi_2(w) = w \quad \text{and} \quad \pi_1(w_1, w_2) = w_1,$$

and with the convention

$$\forall (w,z) \in \mathbb{Z}^2 \times \mathbb{R}, \quad (\pi_0(w), |w|) = w \quad \text{and more generally} \quad (\pi_0(w), z) = z.$$

Throughout this section, we fix some (arbitrary) $q \in [1,2)$, and some $b_q > 2$ so that

$$\frac{1}{b_q} + \frac{1}{q} < 1. \tag{6.6}$$

This choice will determine the choice of the Banach space on the Young tower.

6.1. **Expression of the characteristic functions via Fourier perturbed operator.** We observe that

$$|\widehat{\Psi}(x) - \widehat{\Psi}(y)| \le d(x,y) + d(T(x), T(y)),$$

for any x, y in the same connected component of $M \setminus (\mathcal{S}_0 \cup T^{-1}(\mathcal{S}_0))$, where $\mathcal{S}_0$ is the set of post-collisional vectors tangent to $\partial\Omega$. We recall that the diameter of the connected components of $M \setminus \bigcup_{k=-n}^{n} T^{-k}(\mathcal{S}_0)$ is $\mathcal{O}(\beta_1^n)$ for some $\beta_1 \in (0,1)$.

As in [34], we consider the towers constructed by Young in [37] (see also [8]). We recall some facts on Young towers and introduce some notations that we shall use in the remainder of this paper. We let $(\Delta, f_\Delta, \mu_\Delta)$ be the hyperbolic tower, which is an extension of (M, T, μ) by $\pi : \Delta \mapsto M$ (with $\pi(x,\ell) = T^\ell(x)$) and write $\left(\overline{\Delta}, f_{\overline{\Delta}}, \mu_{\overline{\Delta}}\right)$ for the quotient tower (obtained from Δ by quotienting out the stable manifolds). The quotient tower is identified with $\overline{\Delta} := \{(x,\ell) \in \Delta : x \in \overline{Y}\}$, where $\overline{Y}$ is an unstable curve of a well chosen set $Y \subset M$, and write $\overline{\pi} : \Delta \to \overline{\Delta}$ for the projection corresponding to the holonomy along the stable curves of Δ. The dynamical system $(\Delta, f_\Delta, \mu_\Delta)$ is given by

- The space Δ is the set of couples $(x,\ell) \in Y \times \mathbb{N}_0$ such that $\ell < R(x)$, where R is a return time to Y.
- The map f_Δ is given by $f_\Delta(x,\ell) = (x, \ell+1)$ if $\ell < R(x)-1$ and $f_\Delta(x, R(x)-1) = (T^{R(x)}(x), 0)$.
- The probability measure μ_Δ is given by $\mu_\Delta(A \times \{\ell\}) = \mu_\Delta(A \cap \{R > \ell\})/\mathbb{E}_\mu[R.1_Y]$, for any measurable set $A \subset Y$.

We assume that the greatest common divisor (g.c.d.) of R is 1, which can be done because of total ergodicity[6] of T; this assumption is not essential, since one can also deal directly with $g.c.d.(R) \neq 1$, but it helps simplifying the proofs and notation throughout the remainder of this paper.

[6]The idea of using the total ergodicity of T for constructing a new tower with $g.c.d.(R) = 1$ was suggested in [37, Section 4] and used in [33] for ensuring aperiodicity of the version of κ on $\overline{\Delta}$. The details of such a tower construction are contained in [26, Appendix B].

The partition $\mathcal{P}$ on Δ consists of a union of partitions of the different levels which become finer and finer as one goes up in the tower. The partition $\mathcal{P}$ is used to define a separation time $s(\cdot,\cdot)$ on Δ:

$$s(x,y) := \inf\{n \geq -1 : \mathcal{P}(f_\Delta^{n+1}(x)) \neq \mathcal{P}(f_\Delta^{n+1}(y))\}.$$

The separation time $s(x,y)$ satisfies the following property: $\pi(x)$ and $\pi(y)$ are in the same connected component of $M \setminus (\mathcal{S}_0 \cup T^{-1}(\mathcal{S}_0))$ if $s(x,y) \geq 0$. In particular if $s(x,y) > 2n$, then $\pi(f_\Delta^n(x))$ and $\pi(f_\Delta^n(y))$ are in the same connected component of $M \setminus \bigcup_{k=-n}^n T^{-k}(\mathcal{S}_0)$. Since the atoms of the partition $\mathcal{P}$ are unions of stable curves, this separation time has a direct correspondent $\overline{s}(\cdot,\cdot)$ on the quotient tower $\overline{\Delta}$. Let P be the transfer operator of $\left(\overline{\Delta}, f_{\overline{\Delta}}, \mu_{\overline{\Delta}}\right)$, i.e. P is defined on $L^1(\mu_{\overline{\Delta}})$ by

$$\int_{\overline{\Delta}} H.P(G)\, d\mu_{\overline{\Delta}} = \int_{\overline{\Delta}} H \circ f_{\overline{\Delta}}.G\, d\mu_{\overline{\Delta}}.$$

Let $\beta \in (\beta_1^{\frac{1}{4}}, 1)$ and close enough to 1. It follows from [37] and [8] that there exists $\varepsilon' > 0$ such that, for all $\varepsilon \in]0, \varepsilon'[$, P is quasicompact on the Banach space $\mathcal{B} = \mathcal{B}_\varepsilon$ corresponding to the set of functions of the form $e^{\varepsilon\omega} H$, with $H \in \mathcal{B}_0$, where $\omega(x,\ell) = \ell$ and where $\mathcal{B}_0$ is the Banach space of bounded functions $H : \overline{\Delta} \to \mathbb{C}$ that are Lipschitz continuous with respect to the ultrametric $\beta^{\overline{s}(\cdot,\cdot)}$ (the space $\mathcal{B}_0$ corresponds to the space $\mathcal{B}_\varepsilon$ when $\varepsilon = 0$). The space $\mathcal{B}$ is then endowed with the norm $\|\cdot\|_{\mathcal{B}}$ given by

$$\|H\|_{\mathcal{B}} = \|e^{-\varepsilon\omega} H\|_{\mathcal{B}_0}. \tag{6.7}$$

Recall b_q satisfies (6.6). Choose ε small enough so that $e^{\varepsilon\omega} \in \mathbb{L}^{b_q}(\mu_{\overline{\Delta}})$ which implies that $\mathcal{B}$ is continuously embedded in $\mathbb{L}^{b_q}(\mu_{\overline{\Delta}})$ since

$$\|H\|_{L^{b_q}} \leq \|e^{\varepsilon\omega}\|_{L^{b_q}} \|e^{-\varepsilon\omega} H\|_\infty \leq \|e^{\varepsilon\omega}\|_{L^{b_q}} \|H\|_{\mathcal{B}}. \tag{6.8}$$

(This particular choice of b_q will be used in the proof of Sublemma 6.2.) Since we assume that $g.c.d.(R) = 1$, 1 is the only (dominating) eigenvalue of modulus 1 of P, and it is simple and isolated in the spectrum of P. In particular, there exists $\theta \in (0,1)$ (depending on (β, ε)) such that

$$\|P^n - \mathbb{E}_{\mu_{\overline{\Delta}}}[\cdot] 1_{\overline{\Delta}}\|_{\mathcal{L}(\mathcal{B})} = \mathcal{O}(\theta^n), \quad \text{as } n \to +\infty. \tag{6.9}$$

Let $t \in \mathbb{R}^{d+1}$. Recall that via [4, eq. (6.2) verified in Corollary 9.4],

$$\widehat{\Psi} \circ \pi = \overline{\Psi} \circ \overline{\pi} + \chi \circ f_\Delta - \chi, \tag{6.10}$$

with $\overline{\Psi} = (\overline{\kappa}, \overline{\tau}) : \overline{\Delta} \to \mathbb{Z}^d \times \mathbb{R}$, where $\overline{\tau}$ is the version of $\widetilde{\tau} = \tau - \mu(\tau)$ on $\overline{\Delta}$ given by

$$\overline{\tau} := \widetilde{\tau} \circ \pi + \sum_{n \geq 1} \left(\tau \circ \pi \circ f_\Delta^n - \tau \circ \pi \circ f_\Delta^{n-1} \circ f_{\overline{\Delta}}\right)$$

and where $\chi = (\mathbf{0}, \chi_0)$ with $\mathbf{0}$ the null element of $\mathbb{Z}^d$ and with

$$\chi_0 := \sum_{n \geq 0} \left(\tau \circ \pi \circ f_\Delta^n - \tau \circ \pi \circ f_\Delta^n \circ \overline{\pi}\right).$$

By [4, Proof of Lemma 8.3] (see also Appendix B) $\overline{\tau}$ is locally Lipschitz continuous (on each atom of Young's partition) with respect to the ultrametric $\beta^{s(\cdot,\cdot)}$, and $\chi : \Delta \to \{0\}^d \times \mathbb{R}$ is bounded and Lipschitz in the following sense:

$$\sup_{k \geq 1} \sup_{x,y : s(x,y) > 2k} \frac{|\chi(f_\Delta^k(x)) - \chi(f_\Delta^k(y))|}{\beta^k} < \infty. \tag{6.11}$$

It follows from the coboundary equation (6.10) that

$$\mathbb{E}_\mu[e^{i\langle t,\frac{\widehat{\Psi}_n}{a_n}\rangle}] = \mathbb{E}_{\mu_\Delta}\left[e^{-i\langle \frac{t}{a_n},\chi\rangle}e^{i\langle \frac{t}{a_n},\overline{\Psi}_n\rangle\circ\overline{\pi}}e^{i\langle \frac{t}{a_n},\chi\circ f_\Delta^n\rangle}\right].$$

Let $\overline{K}:\overline{\Delta}\to\mathbb{Z}^2$ be the version of $\widetilde{\kappa}$ on $\overline{\Delta}$, i.e. the function such that $\overline{K}\circ\overline{\pi}=\widetilde{\kappa}\circ\pi$. It follows from (2.10) and (6.10) that the function $\Theta:\overline{\Delta}\to\mathbb{R}^{d+1}$ defined by

$$\Theta := \overline{\Psi}-\Upsilon,\quad \text{with } \Upsilon := \left(\pi_d(\overline{K}), |\overline{K}|-\mathbb{E}_{\mu_{\overline{\Delta}}}[|\overline{K}|]\right), \tag{6.12}$$

is bounded and Lipschitz, and Υ is constant on partition elements (as $\overline{\kappa}=\pi_d(\overline{K})$ corresponding to the cell change). Since both $\overline{\Psi}$ and Υ have mean zero, $\mathbb{E}_{\mu_{\overline{\Delta}}}[\Theta]=0$.

We define the Fourier-perturbed operators $P_t,\widetilde{P}_t\in\mathcal{L}(\mathcal{B})$ by $P_tv=P(e^{i\langle t,\overline{\Psi}\rangle}v)$, $\widetilde{P}_tv = P(e^{i\langle t,\Upsilon\rangle}v)$ for $t\in\mathbb{R}^{d+1}$. By [34] (which exploits [8, 37]), up to enlarging the value of $\theta\in(0,1)$ appearing in (6.9), there exist $\beta_0\in(0,\pi]$, a continuous function $t\mapsto\lambda_t\in\mathbb{C}$ and two families of operators $(\Pi_t)_t$ and $(U_t)_t$ acting on $\mathcal{B}$ such that $t\mapsto\Pi_t\in\mathcal{L}(\mathcal{B},L^1(\overline{\Delta}))$ is continuous and such that, for every $t\in[-\beta_0,\beta_0]^{d+1}$ and every positive integer n,

$$P_t^n=\lambda_t^n\Pi_t+U_t^n,\quad \widetilde{P}_t^n=\widetilde{\lambda}_t^n\widetilde{\Pi}_t+\widetilde{U}_t^n, \tag{6.13}$$

$$\text{with}\quad \sup_{t\in[-\beta_0,\beta_0]^d}\left(\|U_t^n\|_{\mathcal{B}}+\|\widetilde{U}_t^n\|_{\mathcal{B}}\right)=\mathcal{O}(\theta^n),\quad \text{as } n\to+\infty. \tag{6.14}$$

Set $k=k_n:=(\log n)^2$ and $F_{k,u}(x):=e^{i\langle u,\overline{\Psi}_k(\overline{\pi}(x))+\chi\circ f_\Delta^k(x)\rangle}$. It follows from (6.10) that

$$\begin{aligned}\mathbb{E}_\mu[e^{i\langle t,\frac{\widehat{\Psi}_n}{a_n}\rangle}] &= \mathbb{E}_{\mu_\Delta}\left[e^{-i\langle \frac{t}{a_n},\chi\circ f_\Delta^k\rangle}e^{i\langle \frac{t}{a_n},\overline{\Psi}_n\rangle\circ\overline{\pi}\circ f_\Delta^k}e^{i\langle \frac{t}{a_n},\chi\circ f_\Delta^{n+k}\rangle}\right]\\ &= \mathbb{E}_{\mu_\Delta}\left[F_{k,-\frac{t}{a_n}}e^{i\langle \frac{t}{a_n},\overline{\Psi}_n\rangle\circ\overline{\pi}}F_{k,\frac{t}{a_n}}\circ f_\Delta^n\right].\end{aligned}$$

We approximate $F_{k,u}(x)$ by its conditional expectation $\widehat{F}_{k,u}(x)=\overline{F}_{k,u}(\overline{\pi}(x))$ on the set $\{y\in\Delta : s(x,y)>2k\}$, where s is the separation time on Δ as recalled earlier in this section. Since $e^{i\langle u,\overline{\Psi}\rangle}$ is bounded and Lipschitz on $\overline{\Delta}$ and since $e^{i\langle t,\chi\rangle}$ is bounded and Lipschitz on Δ (in the sense of (6.11)), it follows that

$$\begin{aligned}\mathbb{E}_\mu[e^{i\langle t,\frac{\widehat{\Psi}_n}{a_n}\rangle}] &= \mathbb{E}_{\mu_{\overline{\Delta}}}\left[\overline{F}_{k,-\frac{t}{a_n}}e^{i\langle \frac{t}{a_n},\overline{\Psi}_n\rangle}\overline{F}_{k,\frac{t}{a_n}}\circ f_{\overline{\Delta}}^n\right]+\mathcal{O}(\beta^k)\\ &= \mathbb{E}_{\mu_{\overline{\Delta}}}\left[\overline{F}_{k,\frac{t}{a_n}}P_{\frac{t}{a_n}}^n(\overline{F}_{k,-\frac{t}{a_n}})\right]+\mathcal{O}(\beta^k)\\ &= \lambda_{t/a_n}^{n-2k}\mathbb{E}_{\mu_{\overline{\Delta}}}\left[\overline{F}_{k,t}\Pi_{\frac{t}{a_n}}(P_{\frac{t}{a_n}}^{2k}(\overline{F}_{k,-\frac{t}{a_n}}))\right]+\mathcal{O}(\beta^k+\theta^{n-2k}),\end{aligned}$$

as $n\to+\infty$. Furthermore $\|\overline{F}_{k,u}\|_\infty\le 1$ and $P_{\frac{t}{a_n}}^{2k}(\overline{F}_{k,-\frac{t}{a_n}})$ are uniformly (in k,n) Lipschitz with respect to Young's ultrametric $\beta^{s(\cdot,\cdot)}$. Thus by continuity of $t\mapsto\Pi_t\in\mathcal{L}(\mathcal{B},L^1(\overline{\Delta}))$,

$$\mathbb{E}_\mu[e^{i\langle t,\frac{\widehat{\Psi}_n}{a_n}\rangle}] = \lambda_{\frac{t}{a_n}}^{n-2k}\left(\mathbb{E}_{\mu_{\overline{\Delta}}}\left[\overline{F}_{k,\frac{t}{a_n}}\right]\mathbb{E}_{\mu_{\overline{\Delta}}}\left[P_{\frac{t}{a_n}}^{2k}(\overline{F}_{k,-\frac{t}{a_n}}))\right]+o(1)\right),$$

as $n\to+\infty$. Observe that, for every $t\in\mathbb{R}^{d+1}$,

$$\mathbb{E}_{\mu_{\overline{\Delta}}}\left[\overline{F}_{k,\frac{t}{a_n}}\right]=\mathbb{E}_{\mu_\Delta}\left[e^{i\langle \frac{t}{a_n},\overline{\Psi}_k\circ\overline{\pi}+\chi\circ f_\Delta^k\rangle}\right]=1+o(1)\quad \text{as } n\to+\infty,$$

due to the dominated convergence theorem (using the fact that $\lim_{k\to+\infty}\overline{\Psi}_k/k=0$ $\mu_{\overline{\Delta}}$-almost-surely). Analogously

$$\forall t\in\mathbb{R}^{d+1},\quad \mathbb{E}_{\mu_{\overline{\Delta}}}\left[P^{2k}_{\frac{t}{a_n}}(\overline{F}_{k,-\frac{t}{a_n}}))\right]=\mathbb{E}_{\mu_\Delta}\left[e^{i\langle\frac{t}{a_n},\overline{\Psi}_k\circ\overline{\pi}-\chi\rangle\circ f^k_\Delta}\right]=1+o(1),\quad \text{as } n\to+\infty.$$

Thus

$$\mathbb{E}_\mu[e^{i\langle t,\frac{\widehat{\Psi}_n}{a_n}\rangle}]=\lambda^{n-2k}_{\frac{t}{a_n}}+o(1). \tag{6.15}$$

An important observation that will allow us to adapt the results of [29] to the present context is that

$$P_t-\widetilde{P}_t=P\left(\left(e^{i\langle t,\overline{\Psi}\rangle}-e^{i\langle t,\Upsilon\rangle}\right)\cdot\right)=\widetilde{P}_t\left(\left(e^{i\langle t,\Theta\rangle}-1\right)\cdot\right).$$

6.2. **Regularity of the dominating eigenvalues and its spectral projector.** In this part, we prove that for $q\in[1,2)$ chosen before (6.6),

$$\|\Pi_t-\Pi_0\|_{\mathcal{B}\to L^q(\mu_{\overline{\Delta}})}=\mathcal{O}(t)\quad\text{and}\quad\lambda_t=1-\log(1/|t|)\langle t,\Sigma_{d+1}t\rangle+\mathcal{O}(t^2),$$

as $t\to 0$. We do so, via the following several steps: we first establish in Sublemma 6.1 an equivalent of λ_t-1, and we use it to establish in Sublemma 6.2 the announced estimate of $\|\Pi_t-\Pi_0\|_{\mathcal{B}\to L^q(\mu_{\overline{\Delta}})}$ that we finally use to establish in Sublemma 6.3 the announced expansion of λ_t.

To obtain such estimates, we will control the error between λ_t and $\widetilde{\lambda}_t$, and between Π_t and $\widetilde{\Pi}_t$. Here we crucially exploit that Υ and κ satisfy similar properties. This allows us to adapt some results obtained for κ in [29, 34] with the use of [3].

We start by studying $P_t-\widetilde{P}_t$. Observe that $(e^{i\langle t,\overline{\Theta}\rangle}-1)\cdot\in\mathcal{L}(\mathcal{B})$ is dominated by $\left\|e^{i\langle t,\overline{\Theta}\rangle}-1\right\|_{\mathcal{B}_0}$. This implies that $\left\|\widetilde{P}_t-P_t\right\|_{\mathcal{L}(\mathcal{B})}=\mathcal{O}(|t|)$, and thus that

$$\left\|\Pi_t-\widetilde{\Pi}_t\right\|_{\mathcal{B}}=\mathcal{O}(t), \tag{6.16}$$

using the usual Cauchy integral expression for Π_t and $\widetilde{\Pi}_t$. In particular, the announced estimate on $\|\Pi_t-\Pi_0\|_{\mathcal{B}\to L^q(\mu_{\overline{\Delta}})}$ will follow from the same estimate for $\|\widetilde{\Pi}_t-\widetilde{\Pi}_0\|_{\mathcal{B}\to L^q(\mu_{\overline{\Delta}})}$.

Lemma 3.1 follows immediately from (6.15), combined with the continuity of $t\mapsto\Pi_t\in\mathcal{L}(\mathcal{B}\to L^1(\mu_{\overline{\Delta}}))$ and from Sublemma 6.1.

Sublemma 6.1. *As $t\to 0$, $1-\lambda_t\sim\log(1/|t|)\langle t,\Sigma_{d+1}t\rangle$, where Σ_{d+1} is given by* (6.5).

Proof of Sublemma 6.1. To study the expansion of $t\mapsto\lambda_t$, in both Sublemmas 6.1 and 6.3, we consider the normalized eigenvectors $v_t,\widetilde{v}_t$ of respectively $P_t,\widetilde{P}_t$ associated with $\lambda_t,\widetilde{\lambda}_t$ given by $v_t=\frac{\Pi_t(1_\Delta)}{\mathbb{E}_{\mu_\Delta}[\Pi_t(1_\Delta)]}$ and $\widetilde{v}_t=\frac{\widetilde{\Pi}_t(1_\Delta)}{\mathbb{E}_{\mu_\Delta}[\widetilde{\Pi}_t(1_\Delta)]}$, and we will use the following expressions

$$\lambda_t=\mathbb{E}_{\mu_\Delta}[P_t(v_t)]=\mathbb{E}_{\mu_\Delta}[e^{i\langle t,\overline{\Psi}\rangle}v_t]\quad\text{and}\quad\widetilde{\lambda}_t=\mathbb{E}_{\mu_\Delta}[\widetilde{P}_t(\widetilde{v}_t)]=\mathbb{E}_{\mu_\Delta}[e^{i\langle t,\overline{\Upsilon}\rangle}\widetilde{v}_t].$$

Therefore

$$\lambda_t-\widetilde{\lambda}_t=I_1(t)+I_2(t), \tag{6.17}$$

with

$$I_1(t) := \int_{\overline{\Delta}} e^{i\langle t,\overline{\Psi}\rangle}(v_t - \widetilde{v}_t)\, d\mu_{\overline{\Delta}} = \int_{\overline{\Delta}} (1 - e^{i\langle t,\overline{\Psi}\rangle})(\widetilde{v}_t - v_t)\, d\mu_{\overline{\Delta}},$$

and

$$I_2(t) := \int_{\overline{\Delta}} (e^{i\langle t,\overline{\Psi}\rangle} - e^{i\langle t,\Upsilon\rangle})\widetilde{v}_t\, d\mu_{\overline{\Delta}}.$$

As argued below, $I_1(t)$ and $I_2(t)$ are $\mathcal{O}(|t|^2)$. Regarding I_2, we first note that $(e^{i\langle t,\overline{\Psi}\rangle} - e^{i\langle t,\Upsilon\rangle})\cdot \in \mathcal{L}(\mathcal{B})$ is dominated by the Lipschitz norm of $(e^{i\langle t,\overline{\Psi}\rangle} - e^{i\langle t,\Upsilon\rangle}) = e^{i\langle t,\Upsilon\rangle}(e^{i\langle t,\Theta\rangle} - 1)$. It follows from the definitions of $v_t, \widetilde{v}_t$ and (6.16) that

$$\|v_t - \widetilde{v}_t\|_{\mathcal{L}(\mathcal{B})} = \mathcal{O}(t), \quad \text{as } t \to 0.$$

Recall that $\mathcal{B} \subset L^{b_q}$ with $b_q > 2$ fixed satisfying (6.6). Further, note that by (6.2), Υ and thus $\overline{\Psi}$ are in L^q for any $q < 2$. By the choice of b_q, $b_q > q/(q-1)$. Hence,

$$|I_1(t)| \ll |t| \int_{\overline{\Delta}} |\overline{\Psi}|\, |\widetilde{v}_t - v_t|\, d\mu_{\overline{\Delta}} \ll |t|\, \|\overline{\Psi}\|_{L^q} \|\widetilde{v}_t - v_t\|_{L^{\frac{q}{q-1}}} \ll |t|\, |\widetilde{v}_t - v_t\|_{\mathcal{B}} \ll t^2.$$

Next, recalling the definition of Θ in (6.12),

$$I_2(t) = \int_{\overline{\Delta}} e^{i\langle t,\Upsilon\rangle}(e^{i\langle t,\Theta\rangle} - 1)\widetilde{v}_t\, d\mu_{\overline{\Delta}} = I_2^1(t) + I_2^2(t)$$

with

$$I_2^1(t) := it \int_{\overline{\Delta}} e^{i\langle t,\Upsilon\rangle}\, \Theta\, \widetilde{v}_t\, d\mu_{\overline{\Delta}}, \quad \text{and} \quad I_2^2(t) := \int_{\overline{\Delta}} e^{i\langle t,\Upsilon\rangle}(e^{i\langle t,\Theta\rangle} - 1 - it\Theta)\widetilde{v}_t\, d\mu_{\overline{\Delta}}.$$

Now, since Θ is bounded,

$$|I_2^2(t)| \ll |t|^2 \int_{\overline{\Delta}} |\Theta|^2\, |\widetilde{v}_t|\, d\mu_{\overline{\Delta}} \ll (|t|\, \|\Theta\|_\infty)^2 \int_{\overline{\Delta}} |\widetilde{v}_t|\, d\mu_{\overline{\Delta}} \ll |t|^2.$$

For I_2^1, write

$$\begin{aligned} I_2^1(t) &= it \int_{\overline{\Delta}} e^{i\langle t,\Upsilon\rangle}\, \Theta\, \widetilde{v}_0\, d\mu_{\overline{\Delta}} + it \int_{\overline{\Delta}} e^{i\langle t,\Upsilon\rangle}\, \Theta\, (\widetilde{v}_t - \widetilde{v}_0)\, d\mu_{\overline{\Delta}} \\ &= it \int_{\overline{\Delta}} \Theta\, d\mu_{\overline{\Delta}} + it \int_{\overline{\Delta}} (e^{i\langle t,\Upsilon\rangle} - 1)\, \Theta\, \widetilde{v}_0\, d\mu_{\overline{\Delta}} + \mathcal{O}(|t|^2) \\ &= 0 + \mathcal{O}(|t|^2), \quad \text{as } t \to 0, \end{aligned}$$

where we have used that $\int_{\overline{\Delta}} \Theta\, d\mu_{\overline{\Delta}} = 0$. Putting the above together, $|I_2(t)| \ll |t|^2$.

The bounds for I_1 and I_2 together with (6.17) give that

$$\lambda_t = \widetilde{\lambda}_t + \mathcal{O}(|t|^2), \quad \text{as } t \to 0. \tag{6.18}$$

It remains to show that $1 - \widetilde{\lambda}_t$ has the desired asymptotic, using the form of Υ in (6.12). To this end we observe that

$$1 - \widetilde{\lambda}_t = \mathbb{E}_{\mu_{\overline{\Delta}}}[1 - \widetilde{P}_t(\widetilde{v}_t)] = \mathbb{E}_{\mu_{\overline{\Delta}}}[(1 - e^{i\langle t,\Upsilon\rangle})\widetilde{v}_t] = I_1'(t) + I_2'(t), \tag{6.19}$$

with

$$I_1'(t) := \int_{\overline{\Delta}} (1 - e^{i\langle t,\Upsilon\rangle})\widetilde{v}_0\, d\mu_{\overline{\Delta}} \quad \text{and} \quad I_2'(t) := \int_{\overline{\Delta}} (1 - e^{i\langle t,\Upsilon\rangle})(\widetilde{v}_t - \widetilde{v}_0)\, d\mu_{\overline{\Delta}}.$$

We estimate $I_1'(t)$. Recall equations (6.1), (6.2) and that $\int_{\overline{\Delta}} \Upsilon \, d\mu_{\overline{\Delta}} = 0$. For any L, w, N, write $W_N(L,w) := (\pi_d(L+Nw), |L+Nw| - \mathbb{E}_\mu[|\widetilde{\kappa}|])$ and compute that

$$\begin{aligned}
I_1'(t) &= \int_{\overline{\Delta}} (1 - e^{i\langle t, \Upsilon\rangle} - i\langle t, \Upsilon\rangle) \, d\mu_{\overline{\Delta}} \\
&= \sum_{(L,w)\in\mathcal{E}} \sum_{N\geq 1} \left(e^{i\langle t, W_N(L,w)\rangle} - 1 - i\langle t, W_N(L,w)\rangle\right) \mu(\{\kappa = L+Nw\}) + \mathcal{O}(|t|^2) \\
&= \sum_{(L,w)\in\mathcal{E}} \sum_{N=1}^{1/|t|} \left(e^{i\langle t, W_N(L,w)\rangle} - 1 - i\langle t, W_N(L,w)\rangle\right) \left(c_{L,w}N^{-3} + \mathcal{O}(N^{-4})\right) \\
&\quad + O\left(|t|^2 + |t| \sum_{(L,w)\in\mathcal{E}} |w| \sum_{N>1/|t|} N.N^{-3}\right) \\
&= \sum_{(L,w)\in\mathcal{E}} \frac{c_{L,w}}{2} \sum_{N=1}^{1/|t|} \langle t, W_N(L,w)\rangle^2 N^{-3} + \mathcal{O}(|t|^2).
\end{aligned}$$

Since $\langle t, W_N(L,w)\rangle^2 N^{-3} = N^{-1}\langle t, (\pi_d(w), |w|)\rangle^2 + \mathcal{O}(|t|^2 N^{-2})$,

$$\begin{aligned}
I_1'(t) &= \sum_{(L,w)\in\mathcal{E}} \frac{c_{L,w}}{2} \log(1/|t|)\langle t, (\pi_d(w), |w|)\rangle^2 + \mathcal{O}(|t|^2) \\
&= \log(1/|t|)\langle \Sigma_{d+1} t, t\rangle + \mathcal{O}(|t|^2). \qquad (6.20)
\end{aligned}$$

For $I_2'(t)$, we just need to explain that the argument of [34] via [3], which provides the asymptotic of the eigenvalue associated to perturbation by κ instead of Υ as here, goes through. The required ingredients in the argument of [3, 34] are: (1) tail of κ and (2) the 'double probability' estimate (5.6). Regarding (1), we already know the tail of Υ: see equations (6.1) and (6.2). Regarding (2), an analogue of (5.6), we recall that we already know that this holds for $\widetilde{\kappa}$. Because of the expression of Υ (via $\widetilde{\kappa}$), the arguments used in [34, Proofs of Propositions 11–12] ensure that

$$\mu_{\overline{\Delta}}(A_{N,V}) = \mathcal{O}(N^{-3-\varepsilon}), \quad \text{as } N \to +\infty, \qquad (6.21)$$

where

$$A_{N,V} := \left\{\Upsilon = W_N(L,w), \ \exists |j| \leq V\log(n+2), \ |\Upsilon \circ f^j| > |W_N(L,w)|^{4/5}\right\}.$$

Equations (6.2) and (6.21) together with [3, Proof of Theorem 3.4] ensure that[7]

$$I_2'(t) = o\left(|t|^2 \log(1/|t|)\right), \quad \text{as } t \to 0, \qquad (6.22)$$

The conclusion from this together with (6.18), (6.19) and (6.20). □

In the remainder of this section, we prove a stronger version of Sublemma 6.1, along with a strong continuity estimate on Π_t, that will be essential in the proofs of Lemmas 3.2 and 3.4 carried out in Section 7.

Recall that $q \in [1,2)$ has been fixed at the beginning of the present section.

Sublemma 6.2.

$$\left\|\widetilde{\Pi}_t - \widetilde{\Pi}_0\right\|_{\mathcal{B}\to L^q(\mu_{\overline{\Delta}})} + \|\Pi_t - \Pi_0\|_{\mathcal{B}\to L^q(\mu_{\overline{\Delta}})} = \mathcal{O}(t) \quad \textit{as } t \to 0.$$

[7]More precisely, see *Estimate of λ_t (there) in the proof of* [3, Proof of Theorem 3.4]. In particular, see [3, Lemma 3.16 and 3.19].

Proof. Due to (6.16), it is enough to control $\left\|\widetilde{\Pi}_t - \widetilde{\Pi}_0\right\|_{\mathcal{B}\to L^q(\mu_{\overline{\Delta}})}$. We claim that with the choice of b_q (see (6.6)) and ε in the text before (6.8), [29, Proposition 5.4] applies to ensure that

$$\|1_Y(\widetilde{\Pi}_t - \widetilde{\Pi}_0)\|_{\mathcal{B}\to\mathcal{B}_0} = \mathcal{O}(|t|), \quad \text{as } t\to 0. \tag{6.23}$$

Using (6.23), we modify the proof of [29, Proposition 5.3] to conclude the proof of the sublemma. Set $\pi_0(x,\ell) := x$, recall that $\omega(x,\ell) = \ell$ and, using [29, Formula (44)], we write $(\widetilde{\Pi}_t - \widetilde{\Pi}_0)(w)(x) = I_{1,t}(x) + I_{2,t}(x) + I_{3,t}(x)$ with

$$\begin{aligned} I_{1,t}(x) &:= \left[(e^{i\langle t,\Upsilon_{\omega(x)}(\pi_0(x))\rangle} - 1)\widetilde{\Pi}_0(w)\right], \\ I_{2,t}(x) &:= \left[e^{i\langle t,\Upsilon_{\omega(x)}(\pi_0(x))\rangle}(\widetilde{\Pi}_t - \widetilde{\Pi}_0)(w)(\pi_0(x))\right], \\ I_{3,t}(x) &:= (\widetilde{\lambda}_t^{-\omega(x)} - 1)[e^{i\langle t,\Upsilon_{\omega(x)}(\pi_0(x))\rangle}\widetilde{\Pi}_t(w)(\pi_0(x))]. \end{aligned}$$

First,

$$\begin{aligned} \|I_{1,t}\|^q_{L^q(\mu_{\overline{\Delta}})} &= \mathbb{E}_{\mu_{\overline{\Delta}}}\left[\left((e^{i\langle t,\Upsilon_{\omega(\cdot)}(\pi_0(\cdot))\rangle} - 1)\right)^q\right]\|w\|^q_{L^1(\mu_{\overline{\Delta}})} \\ &\le |t|^q \mathbb{E}_{\mu_{\overline{\Delta}}}\left[\left|\Upsilon_{\omega(\cdot)}(\pi_0(\cdot))\right|^q\right]\|w\|^q_{L^1(\mu_{\overline{\Delta}})} \\ &\ll |t|^q \sum_{n\ge 0}\mathbb{E}_\mu\left[1_{Y\cap\{R>n\}}|\Upsilon_n|^q\right]\|w\|^q_{L^1(\mu_{\overline{\Delta}})} \\ &\ll |t|^q\sum_{n\ge 0}\mu_Y(R>n)^{1/q'}\||\Upsilon_n|^q\|_{L^{p'}(\mu_{\overline{Y}})}\|w\|^q_{L^1(\mu_{\overline{\Delta}})} \ll |t|^q\|w\|^q_{\mathcal{B}}, \end{aligned} \tag{6.24}$$

taking $p' > 1$ and $q' > 1$ such that $\frac{1}{p'} + \frac{1}{q'} = 1$ and $qp' < 2$, and using the fact that $\mu_{\overline{Y}}(R\ge n)$ decreases exponentially fast. Second, it follows from (6.23) that

$$\begin{aligned} \|I_{2,t}\|^q_{L^q(\mu_{\overline{\Delta}})} &\le \left\|(\widetilde{\Pi}_t - \widetilde{\Pi}_0)(w)(\pi_0(\cdot))\right\|^q_{L^q(\mu_{\overline{\Delta}})} \\ &\ll \|1_Y(\widetilde{\Pi}_t - \widetilde{\Pi}_0)(w)\|^q_\infty \ll |t|^q\|w\|^q_{\mathcal{B}}. \end{aligned} \tag{6.25}$$

Third, by Sublemma 6.1, there exists some $a' > 0$ such that, for t small enough, $1 > |\widetilde{\lambda}_t| > e^{-a'\frac{|t|^2\log(1/|t|)}{2}}$, and so

$$\begin{aligned} \|I_{3,t}\|^q_{L^q(\mu_{\overline{\Delta}})} &\le \left\|1_Y\widetilde{\Pi}_t(w)\right\|_\infty |\lambda_t - 1|^q\,\mathbb{E}_{\mu_{\overline{\Delta}}}\left[\left|\omega(\cdot)e^{a'\frac{|t|^2\log(1/|t|)}{2}(\omega(\cdot)-1)}\right|^q\right] \\ &\ll \left(|t|^2\log(1/|t|)\left\|\widetilde{\Pi}_t(w)\right\|_{\mathcal{B}}\right)^q\sum_{n\ge 1}\mu_Y(R>n)n^q e^{a'q\frac{n|t|^2\log(1/|t|)}{2}} \\ &\ll \left(|t|^2\log(1/|t|)\|w\|_{\mathcal{B}}\right)^q, \end{aligned} \tag{6.26}$$

provided $|t|$ is small enough, using again that $\mu_Y(R>n)$ decays exponentially fast in n. The conclusion follows from (6.24), (6.25) and (6.26).

It remains to complete

Proof of the claim (6.23). Recall b_q satisfies (6.6) and that ε has been fixed in the text before (6.8); in particular, $\frac{1}{b_q} + \frac{1}{q} < 1$. There exists $p\in(2,b_q)$ so that $\frac{1}{p} + \frac{1}{q} < 1$. In particular, $1 < q\frac{p-1}{p}$ and so $2\frac{p-1}{p} > \frac{2}{q} > 1$. Let $\gamma\in\left(1,\frac{p-1}{p}\right)$. Let $h\in\mathcal{B}$. Note that $h = vw$, with $w := e^{-\varepsilon\omega}h\in\mathcal{B}_0$ and with $v := e^{\varepsilon\omega}\in L^{b_q}$ constant on partition elements.

With these choices, $h, v, p, b = b_q, \gamma$ satisfy the assumptions of [29, Proposition 5.4, Lemma C.2] which still holds true with the same proof in when replacing Π_t therein by the present $\widetilde{\Pi}_t$ (this is equivalent to replacing $\overline{\kappa}$ in [29] by the present $\overline{\Psi}$). Again, this adaptation is possible because $\overline{\Psi}$ is constant on partition elements and $\overline{\Psi}$ has a similar tail to that of $\overline{\kappa}$ (again, see equations (6.1) and (6.2)). (The similar tail ensures, in particular, that $\overline{\Psi}$. is as integrable as $\overline{\kappa}$.) As a consequence, [29, Proof of Lemma C.2] goes through with $\overline{\kappa}$ replaced by $\overline{\Psi}$. Moreover, because of the same properties of $\overline{\Psi}$, the proof of [29, Proposition 5.4] can be easily modified to prove the claim (6.23).

The adaptation of [29, Proposition 5.4] implies that

$$\|1_Y(\widetilde{\Pi}_t - \widetilde{\Pi}_0)(h)\|_{\mathcal{B}_0} \ll |t|\|1_Y\widetilde{\Pi}_0'(h)\|_{\mathcal{B}_0} + |t|^\gamma \left(\|h\|_{\mathcal{B}} + \|e^{-\varepsilon\omega}h\|_{\mathcal{B}_0}\|e^{\varepsilon\omega}\|_{L^{b_q}}\right).$$

By [29, Lemma C.2] with $\overline{\kappa}$ replaced by $\overline{\Psi}$,

$$\left(\|1_Y\widetilde{\Pi}_0'(h)\|_{\mathcal{B}_0} \le \|h\|_{\mathcal{B}} + \|e^{-\varepsilon\omega}h\|_{\mathcal{B}_0}\|e^{\varepsilon\omega}\|_{L^{b_q}}\right).$$

We conclude that

$$\|1_Y(\widetilde{\Pi}_t - \widetilde{\Pi}_0)(h)\|_{\mathcal{B}_0} \ll |t| \left(\|h\|_{\mathcal{B}} + \|e^{-\varepsilon\omega}h\|_{\mathcal{B}_0}\|e^{\varepsilon\omega}\|_{L^{b_q}}\right) \ll |t|\|h\|_{\mathcal{B}},$$

since $\|e^{-\varepsilon\omega}h\|_{B_0} = \|h\|_{\mathcal{B}}$. □

Sublemma 6.3. *As $t \to 0$, $1 - \lambda_t = \log(1/|t|)\langle t, \Sigma_{d+1}t\rangle + O(|t|^2)$.*

Proof. We keep the notations of the proof of Sublemma 6.1. It follows from (6.18), (6.19) and (6.20) that $1 - \lambda_t = \log(1/|t|)\langle t, \Sigma_{d+1}t\rangle + I_2'(t) + O(|t|^2)$. The proof that $|I_2'(t)| = O(|t|^2)$ follows exactly as in the proof of [29, Lemma 6.1] (replacing everywhere $\overline{\kappa}, P_t, \Pi_t, v_t, \lambda_t$ therein by $\Upsilon, \widetilde{P}_t, \widetilde{\Pi}_t, \widetilde{v}_t, \widetilde{\lambda}_t$ and following the proof line by line). This is due to the fact that Υ satisfies the following properties (that are also satisfied by $\overline{\kappa}$): Υ is constant on partition elements, equations (6.2), (6.22) and (5.6) hold, and the estimate on $\widetilde{\Pi}_t$ stated in Sublemma 6.2 holds. □

7. Proofs of joint MLLT (Lemmas 3.2 and 3.4)

Let $d \in \{0, 1, 2\}$. Compared to CLT, a specific property required to prove the MLLT is the non-arithmeticity (or minimality), which is treated in Lemma 7.1.

Lemma 7.1. *For every proper closed subgroup Γ of $\mathbb{Z}^d \times \mathbb{R}$ and for every $a \in \mathbb{Z}^d \times \mathbb{R}$,*

$$\mu\left(\widehat{\Psi} + g - g \circ T \notin a + \Gamma\right) > 0.$$

Proof. We adapt the proof of [11, Lemma A.3] to $d+1$-dimensional observable $\widehat{\Psi}$, with a slightly different presentation. Assume there exists a proper subgroup Γ of $\mathbb{Z}^d \times \mathbb{R}$, a measurable function $g = (g_1, \ldots, g_{d+1}) : M \to \mathbb{Z}^d \times \mathbb{R}$ and $a = (a_1, \ldots, a_{d+1}) \in \mathbb{Z}^d \times \mathbb{R}$ such that $\widehat{\Psi} + g - g \circ T \in a + \Gamma$ μ-almost surely.

- Let us prove that we can find a family $(v_1, \ldots, v_{d+1})$ of generators of Γ of the form

$$v_i := (\mathbf{e_i}, \alpha_i) \text{ for } i = 1, \ldots, d, \quad v_{d+1} := (\mathbf{0}, \alpha_{d+1}),$$

  where $\mathbf{e_i}$ is the i-th vector of the canonical basis of $\mathbb{R}^d$ and where $\mathbf{0}$ is the null element of $\mathbb{Z}^d$. Let $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$ be such that $v_i := (\mathbf{e_i}, \alpha_i) \in \Gamma$ for $i = 1, \ldots, d$. Such numbers exist since the projection of Γ on the first d

coordinates generates $\mathbb{Z}^d$ (since it has been proved in [34] that $\kappa : M \to \mathbb{Z}^d$ is non-arithmetic).

Observe that $\Gamma \cap (\{\mathbf{0}\} \times \mathbb{R})$ is a discrete subgroup of $\{\mathbf{0}\} \times \mathbb{R}$. Indeed it is a closed subgroup, and it cannot be $\{\mathbf{0}\} \times \mathbb{R}$ otherwise Γ would be $\mathbb{Z}^d \times \mathbb{R}$, since then any element $(a'_1, \ldots, a'_{d+1})$ of $\mathbb{Z}^d \times \mathbb{R}$ could be rewritten $\left(\sum_{i=1}^d a'_i.v_i\right) + \left(a'_{d+1} - \sum_{i=1}^d a'_i \alpha_i\right)(\mathbf{0}, 1)$. Hence, $\Gamma \cap (\{\mathbf{0}\} \times \mathbb{R})$ is discrete and has the form $\{\mathbf{0}\} \times (\alpha_{d+1}\mathbb{Z})$ for a non-negative real number α_{d+1}. Set $v_{d+1} := (\mathbf{0}, \alpha_{d+1}) \in \Gamma$.

Let us prove that $(v_1, \ldots, v_{d+1})$ generates the group Γ. Let

$$a' = (a'_1, \ldots, a'_{d+1}) \in \Gamma \subset \mathbb{Z}^d \times \mathbb{R}.$$

Set $w = a' - \sum_{i=1}^d a'_i v_i = (\mathbf{0}, \beta)$. By definition of α_{d+1}, there exists $m \in \mathbb{Z}$ such that $\beta = m\alpha_{d+1}$. Thus $a' \in \sum_{i=1}^{d+1} \mathbb{Z} v_i$.

- Let us prove that there exist $r, \alpha \in \mathbb{R}$ and two measurable functions $c' : M \to \mathbb{Z}$ and $G : M \to \mathbb{R}$ such that $\mu(0 < G < \min \tau) > 0$ and

$$\tau - \mu(\tau) = r + \alpha.c' + G - G \circ T.$$

It follows from the previous item that there exists a measurable function $c = (c_1, \ldots, c_{d+1}) : M \to \mathbb{Z}^{d+1}$ such that

$$\widehat{\Psi} + g - g \circ T = a + \sum_{i=1}^{d+1} c_i.v_i \quad \mu - a.s.,$$

by taking $c_i = \kappa_i + g_i - g_i \circ T - a_i$ for $i \in \{1, \ldots, d\}$ and

$$c_{d+1} = \left(\tau - \mu(\tau) + g_{d+1} - g_{d+1} \circ T - a_{d+1} - \sum_{i=1}^d \alpha_i.c_i\right) / \alpha_{d+1} \mathbf{1}_{\alpha_{d+1} \neq 0}. \tag{7.1}$$

Thus

$$\tau - \mu(\tau) + g_{d+1} - g_{d+1} \circ T = a_{d+1} + \sum_{i=1}^{d+1} \alpha_i.c_i.$$

Now we observe that $\kappa = -\kappa \circ \xi \circ T$ and $\tau = \tau \circ \xi \circ T$ with ξ the involution mapping $(q, \varphi) \in \partial\Omega \times [-\frac{\pi}{2}; \frac{\pi}{2}]$ to $(q, -\varphi)$, so the previous identity composed with $\xi \circ T$ becomes

$$\tau - \mu(\tau) + g_{d+1} \circ \xi \circ T - g_{d+1} \circ T \circ \xi \circ T = a_{d+1} + \sum_{i=1}^{d+1} \alpha_i.c_i \circ \xi \circ T.$$

Therefore, by taking the average of the two previous identities, we obtain

$$\tau - \mu(\tau) + \frac{g_{d+1} \circ \xi \circ T - g_{d+1} \circ T \circ \xi \circ T}{2} + \frac{g_{d+1} - g_{d+1} \circ T}{2} = a_{d+1} + \sum_{i=1}^{d+1} \alpha_i \frac{c_i + c_i \circ \xi \circ T}{2}.$$

But for $i = 1, \ldots, d$,

$$c_i + c_i \circ \xi \circ T = g_i - g_i \circ T - g_i \circ \xi \circ T + g_i \circ T \circ \xi \circ T$$

is a coboundary, and so, due to (7.1), $\alpha_{d+1}\frac{c_{d+1}+c_{d+1}\circ\xi\circ T}{2}-(\tau-\mu(\tau)-a_{d+1})$ is also a coboundary. Thus we have proved the existence of two measurable functions $c':M\to\mathbb{Z}$ and $G:M\to\mathbb{R}$ such that

$$\tau-\mu(\tau)=a_{d+1}+\frac{\alpha_{d+1}}{2}.c'+G-G\circ T, i.e.\ \tau=r+\frac{\alpha_{d+1}}{2}.c'+G-G\circ T,$$

with $r:=\mu(\tau)+a_{d+1}$. The condition $\mu(0<G<\min\tau)>0$ is obtained up to adding a constant to G. The above identity would contradict the non-arithmeticity of τ.

- We follow exactly the second part of the proof of [11, Lemma A.3].

  Set $\alpha:=\frac{\alpha_{d+1}}{2}$. For $\delta>0$, we consider the set $\mathcal{C}''_\delta$ of points $y=\phi_t(x)$ with $x\in M$, $0<t<\tau(x)$ and $|t-G(x)|<\delta$.

  It follows from the previous item that the first return time ζ to $\mathcal{C}''_0$ takes its values in $r\mathbb{N}+\alpha\mathbb{Z}$. Indeed if $y=\phi_t(x)\in\mathcal{C}''_0$ with $t=G(x)\in(0;\tau(x))$ and $\phi_s(y)=\phi_u(T^n(x))\in\mathcal{C}''_0$, with $u=t+s-\tau_n(x)=G(T^n(x))\in(0,\tau(T^n(x)))$, then $s=G(T^n(x))-t+\tau_n(x)=G(T^n(x))-G(x)+\tau_n(x)=nr+\alpha c'_n$.

  We choose $\varepsilon>0$ so that $r+\varepsilon\in\alpha\mathbb{Q}$. Let b be the smallest positive element of $\alpha\mathbb{Z}+(r+\varepsilon)\mathbb{Z}$, so that $\alpha\mathbb{Z}+(r+\varepsilon)\mathbb{Z}=b\mathbb{Z}$. Indeed, if $r+\varepsilon=\alpha\frac{p}{q}$ with $p,q\in\mathbb{Z}$, $q\neq 0$, then $\alpha\mathbb{Z}+(r+\varepsilon)\mathbb{Z}=\frac{\alpha}{q}(q\mathbb{Z}+p\mathbb{Z})=b\mathbb{Z}$, with $b:=\frac{|\alpha|\,gcd(p,q)}{|q|}$.

  The first return time to $\mathcal{C}'_0:=\{(x,G(x)),x\in M,G(x)\le\tau(x)+\varepsilon\}$ for the suspension flow $\phi^{(\varepsilon)}$ over (M,T) with roof function $\tau+\varepsilon$ is in $\alpha\mathbb{Z}+(r+\varepsilon)\mathbb{N}\subset b\mathbb{Z}$. Indeed, if $y=\phi_t(x)\in\mathcal{C}'_0$ with $t=G(x)\in(0;\tau(x)+\varepsilon)$ and $\phi^{(\varepsilon)}_s(y)=\phi^{(\varepsilon)}_u(T^n(x))\in\mathcal{C}'_0$, with $u=t+s-\tau_n(x)-n\varepsilon=G(T^n(x))\in(0,\tau(T^n(x))+\varepsilon)$, then $s=G(T^n(x))-t+\tau_n(x)+n\varepsilon=G(T^n(x))-G(x)+\tau_n(x)+n\varepsilon=nr+\alpha c'_n+n\varepsilon$.

  Thus, if $\delta\in(0,b/2)$, the return time of $\phi^{(\varepsilon)}$ to $\mathcal{C}'_\delta$ (defined as $\mathcal{C}''_\delta$ but for $\phi^{(\varepsilon)}$) occurs only at time t at distance at most 2δ of $b\mathbb{Z}$, which contradicts the mixing of $\phi^{(\varepsilon)}$. As explained in [11, Lemma A.2], the proof of the mixing of the suspension flow $\phi^{(\varepsilon)}$ follows the same line as the mixing of the billiard flow established in [10, Sections 6.10-6.11] thanks to the temporal distance (which remains unchanged if we replace τ by $\tau+\varepsilon$).

□

Proof of Lemma 3.2. Let $p>2$, G,H,h as in the assumptions of Lemma 3.2. To prove the joint MLLT, we will use estimates established in Section 6. We keep the notations of this section with the couple (q,p_q) being chosen so that $q\in[1,2)$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $b_q>p$ (this will imply that $\frac{1}{b_q}+\frac{1}{q}<1$).

On Δ and $\overline{\Delta}$, we keep the convention $u_n:=\sum_{k=0}^{n-1}u\circ f_\Delta^k$ and $\overline{u}_n:=\sum_{k=0}^{n-1}\overline{u}\circ f_{\overline{\Delta}}^k$ for any $u:\Delta\to\mathbb{R}^{d'}$ and $\overline{u}:\overline{\Delta}\to\mathbb{R}^{d'}$.

For simplicity, we keep the notation $G,H,\widehat{\Psi}$ for the functions defined on Δ (instead of M) corresponding to $G\circ\pi,H\circ\pi,\widehat{\Psi}\circ\pi$ respectively. Since the functions G and H are bounded and dynamically Hölder continuous on M, the functions G and H are also bounded and dynamically Hölder on Δ in the following sense: up to increasing the value of $\beta\in(0,1)$ in the Young Banach space $\mathcal{B}$ introduced in Section 6, G and H satisfy the following property

$$s(x,y)>2k\quad\Rightarrow\quad|G(f_\Delta^k(x))-G(f_\Delta^k(y))|<L_G\beta^k,\ |H(f_\Delta^k(x))-H(f_\Delta^k(y))|<L_H\beta^k. \tag{7.2}$$

Recall that ε in the definition of Young's Banach space $\mathcal{B}$ (see text before (6.8)) is so that $\mathcal{B}$ is continuously embedded in $\mathbb{L}^{b_q}(\mu_{\overline{\Delta}})$. Also, by Sublemma 6.2,

$$\|\Pi_t - \Pi_0\|_{\mathcal{B}\to L^q(\mu_{\overline{\Delta}})} = \mathcal{O}(t).$$

With the above notations, we are led to the study of the following quantity:

$$\mathbb{E}_{\mu_\Delta}[G.h(\widehat{\Psi}_n - L)H \circ f_\Delta^n].$$

It follows from the Fourier inversion theorem that

$$\mathbb{E}_{\mu_\Delta}[G.h(\widehat{\Psi}_n - L)H \circ f_\Delta^n] = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{T}^d\times\mathbb{R}} e^{-i\langle t,L\rangle}\widehat{h}(t)\mathbb{E}_{\mu_\Delta}\left[Ge^{i\langle t,\widehat{\Psi}_n\rangle}H\circ f_\Delta^n\right]dt, \tag{7.3}$$

with $\widehat{h}(t) := \sum_{\ell\in\mathbb{Z}^d}\int_{\mathbb{R}} h(\ell,x)e^{i\langle t,(\ell,x)\rangle}\,dx$.

- Step 1: Transition to the quotient
  - Approximation.
    Recall, from (6.10) that $\widehat{\Psi}\circ\pi = \overline{\Psi}\circ\overline{\pi}+\chi\circ f_\Delta-\chi$, with $\overline{\Psi}=(\overline{\kappa},\overline{\tau})$ with values in $\mathbb{Z}^d\times\mathbb{R}$ uniformly locally Hölder on each partition element, with χ bounded and dynamically Hölder in the sense of (6.11). Thus

$$\begin{aligned}
&\mathbb{E}_{\mu_\Delta}[Ge^{i\langle t,\widehat{\Psi}_n\rangle}H\circ f_\Delta^n] = \mathbb{E}_{\mu_\Delta}[G\circ f_\Delta^{k_n}e^{i\langle t,\overline{\Psi}_n\circ f_{\overline{\Delta}}^{k_n}\circ\overline{\pi}+\chi\circ f_\Delta^{k_n+n}-\chi\circ f_\Delta^{k_n}\rangle}H\circ f_\Delta^{k_n}\circ f_\Delta^n]\\
&=\mathbb{E}_{\mu_\Delta}\Big(e^{-i\langle t,\overline{\Psi}_{k_n}\circ\overline{\pi}\rangle}(Ge^{-i\langle t,\chi\rangle})\\
&\quad\circ f_\Delta^{k_n}.e^{i\langle t,\overline{\Psi}_n\circ\overline{\pi}\rangle}.\left((He^{i\langle t,\chi\rangle})\circ f_\Delta^{k_n}e^{i\langle t,\overline{\Psi}_{k_n}\circ\overline{\pi}\rangle}\right)\circ f_\Delta^n\Big).
\end{aligned}$$

    We approximate $\widehat{G}_{(k_n)}(t) := e^{-i\langle t,\overline{\Psi}_{k_n}\circ\overline{\pi}\rangle}(Ge^{-i\langle t,\chi\rangle})\circ f_\Delta^{k_n}$ by $G_{(k_n)}(t)\circ\overline{\pi}$ with

$$G_{(k_n)}(t)\circ\overline{\pi} := \mathbb{E}_{\mu_\Delta}[e^{-i\langle t,\overline{\Psi}_{k_n}\circ\overline{\pi}\rangle}(Ge^{-i\langle t,\chi\rangle})\circ f_\Delta^{k_n}|s(.,.)>2k_n]$$

    and analogously $e^{i\langle t,\overline{\Psi}_{k_n}\circ\overline{\pi}\rangle}(He^{i\langle t,\chi\rangle})\circ f_\Delta^{k_n}$ by $H_{(k_n)}(-t)\circ\overline{\pi}$.
  - Control of the error in this approximation.
    Since G and H and $e^{i\langle t,\chi\rangle}$ are uniformly bounded and dynamically Hölder in the sense of (7.2) and (6.11), it follows that

$$\left\|\widehat{G}_{(k_n)}(t)-G_{(k_n)}(t)\circ\overline{\pi}\right\|_\infty \le C\left(\|G\|_\infty|t|+L_G\right)\beta^{k_n}. \tag{7.4}$$

    Therefore

$$\begin{aligned}
&\left|\mathbb{E}_{\mu_\Delta}[G.e^{i\langle t,\widehat{\Psi}_n\rangle}.H\circ f_\Delta^n]-\mathbb{E}_{\mu_{\overline{\Delta}}}\left[G_{(k_n)}(t).e^{i\langle t,\overline{\Psi}_n\rangle}.H_{(k_n)}(-t)\circ f_{\overline{\Delta}}^n\right]\right|\\
&\qquad\le C'(1+|t|)\left(\|G\|_\infty\|H\|_\infty+L_H\|G\|_\infty+L_G\|H\|_\infty\right)\beta^{k_n}.
\end{aligned}$$

    Therefore

$$\mathbb{E}_{\mu_\Delta}[G.h(\widehat{\Psi}_n-L)H\circ f_\Delta^n] \tag{7.5}$$

$$=\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{T}^d\times\mathbb{R}} e^{-i\langle t,L\rangle}\widehat{h}(t)\mathbb{E}_{\mu_{\overline{\Delta}}}\left[G_{(k_n)}(t).e^{i\langle t,\overline{\Psi}_n\rangle}.H_{(k_n)}(-t)\circ f_{\overline{\Delta}}^n\right]dt$$

$$+\mathcal{O}\left(\beta^{k_n}\left(\|G\|_\infty\|H\|_\infty+L_H\|G\|_\infty+L_G\|H\|_\infty\right)\right), \tag{7.6}$$

    since $\int_{\mathbb{R}^d}(1+|t|).|\widehat{h}(t)|\,dt<\infty$.

- Step 2: Use of the transfer operator of $f_{\overline{\Delta}}$.

Due to (7.5), we are led to the study of the integral in $t \in \mathbb{R}^d$ of $\widehat{h}(t)$ multiplied by:

$$\mathbb{E}_{\mu_{\overline{\Delta}}}\left[G_{(k_n)}(t).e^{i\langle t,\overline{\Psi}_n\rangle}.H_{(k_n)}(-t)\circ f_{\overline{\Delta}}^n\right] = \mathbb{E}_{\mu_{\overline{\Delta}}}\left[H_{(k_n)}(-t).P_t^n(G_{(k_n)}(t))\right]$$
$$= \mathbb{E}_{\mu_{\overline{\Delta}}}\left[H_{(k_n)}(-t).P_t^{n-3k_n}(P_t^{3k_n}(G_{(k_n)}(t)))\right]. \tag{7.7}$$

We already know that $\|H_{(k_n)}(-t)\|_{L^p} \le \|H\|_{L^p}$. Let $m \ge 2$. Let us prove that there exists $C_0 > 0$ such that, for every $n \ge 3$, $t \in \mathbb{R}^{d+1}$ and $\bar{x}, \bar{y} \in \overline{\Delta}$ such that $\overline{s}(\bar{x}, \bar{y}) > 0$, the following inequality holds true

$$\left|P_t^{mk_n}(G_{(k_n)}(t))(\bar{x}) - P_t^{mk_n}(G_{(k_n)}(t))(\bar{y})\right| \le C_0(1+|t|)\left|P^{mk_n}(|G_{(k_n)}(t)|)(x)\right|\beta^{\overline{s}(\bar{x},\bar{y})}. \tag{7.8}$$

Indeed, we observe that

$$\begin{aligned}
&\left|P_t^{mk_n}(G_{(k_n)}(t))(\bar{x}) - P_t^{mk_n}(G_{(k_n)}(t))(\bar{y})\right| \\
&\le \sum_{\phi_{(mk_n)}} \left| e^{g_{mk_n}(\phi_{(mk_n)}(\bar{x}))} e^{i\langle t,\overline{\Psi}_{mk_n}(\phi_{(mk_n)}(\bar{x}))\rangle}(G_{(k_n)}(t))(\phi_{(mk_n)}(\bar{x})) \right. \\
&\left. - e^{g_{mk_n}(\phi_{mk_n}(\bar{y}))} e^{i\langle t,\overline{\Psi}_{mk_n}(\phi_{(mk_n)}(\bar{y}))\rangle}(G_{(k_n)}(t))(\phi_{(mk_n)}(\bar{y}))\right|,
\end{aligned}$$

where the sum is taken over the inverse branches $\phi_{(mk_n)}$ of $f_{\overline{\Delta}}^{mk_n}$ and where g satisfies

$$|e^{g(\overline{x})} - e^{g(\overline{y})}| \le L_g e^{g(x)}\beta^{\overline{s}(\overline{x},\overline{y})}, \text{if } \overline{s}(\overline{x},\overline{y}) > 1. \tag{7.9}$$

We conclude by noticing that

$$G_{(k_n)}(t)(\phi_{(mk_n)}(\bar{x})) = G_{(k_n)}(t)(\phi_{(mk_n)}(\bar{y})),$$

since $m \ge 2$ and that

$$\begin{aligned}
\left|e^{i\langle t,\overline{\Psi}_{mk_n}(\phi_{(mk_n)}(\bar{x}))\rangle} - e^{i\langle t,\overline{\Psi}_{mk_n}(\phi_{(mk_n)}(\bar{y}))\rangle}\right| &\le |t|L_{\overline{\Psi}}\sum_{j=0}^{mk_n-1}\beta^{\bar{s}(\bar{x},\bar{y})+mk_n-j} \\
&\le |t|L_{\overline{\Psi}}\beta^{\overline{s}(\bar{x},\bar{y})+1}/(1-\beta),
\end{aligned} \tag{7.10}$$

and

$$\begin{aligned}
&\left|e^{g_{mk_n}(\phi_{(mk_n)}(\bar{x}))} - e^{g_{mk_n}(\phi_{(mk_n)}(\bar{y}))}\right| \\
&\le e^{g_{mk_n}(\phi_{(mk_n)}(\bar{x}))}\left|e^{g_{mk_n}(\phi_{(mk_n)}(\bar{y}))-g_{mk_n}(\phi_{(mk_n)}(\bar{x}))} - 1\right| \\
&\le e^{g_{mk_n}(\phi_{(mk_n)}(\bar{x}))}L_g\sum_{j=0}^{mk_n-1}\beta^{\overline{s}(\overline{x},\overline{y})+mk_n-j+1} \\
&\le e^{g_{mk_n}(\phi_{(mk_n)}(\bar{x}))}L_g\frac{\beta^{\overline{s}(\overline{x},\overline{y})+1}}{1-\beta},
\end{aligned} \tag{7.11}$$

due to (7.9). This ends the proof of (7.8). Therefore

$$\|P^{2k_n}(|G_{(k_n)}|)\|_{\mathcal{B}} \le \|P^{2k_n}(|G_{(k_n)}|)\|_{\mathcal{B}_0} = \mathcal{O}\left(\|G\|_\infty\right),$$

and

$$
\begin{aligned}
\|P_t^{3k_n}(G_{(k_n)}(t))\|_{\mathcal{B}} &= \mathcal{O}\left((1+|t|)\|P^{3k_n}(|G_{(k_n)}|)\|_{\mathcal{B}}\right) \\
&= \mathcal{O}\left((1+|t|)\|P^{k_n}(P^{2k_n}(|G_{(k_n)}|))\|_{\mathcal{B}}\right) \\
&\le \mathcal{O}((1+|t|)(\mathbb{E}_\mu[|G|]+\mathcal{O}(\theta^{k_n}\|G\|_\infty))), \qquad (7.12)
\end{aligned}
$$

where we used (6.9) at the last line.

- Step 3: Restriction to a neighbourhood of 0.

Let $K>0$ be such that the support of $\widehat{g}$ is contained in $\mathbb{T}^d\times[-K,K]$. Using Sublemma 6.3, we consider $b_0\in(0;\min(1,\beta_0))$ (see (6.13), (6.14)) small enough so that there exists $a'>0$ such that, for all $t\in[-b_0,b_0]^{d+1}$, the following holds true

$$
P_t^n=\lambda_t^n\Pi_t+\mathcal{O}(\theta^n),\qquad \lambda_t=e^{-\Sigma_{d+1}t\cdot t\,\log(1/|t|)+\mathcal{O}(t^2)}, \tag{7.13}
$$

$$
\Pi_t-\mathbb{E}_{\mu_{\overline{\Delta}}}[\cdot]1_{\overline{\Delta}}=\mathcal{O}(|t|)\text{ in }\mathcal{L}((\mathcal{B},\|\cdot\|_{\mathcal{B}})\to L^q(\mu_{\overline{\Delta}})), \tag{7.14}
$$

$$
\theta\le e^{-2\Sigma_{d+1}t\cdot t\log(1/|t|)}\le|\lambda_t|\le e^{-a'|t|^2\log(1/|t|)}, \tag{7.15}
$$

and that $\forall y>x>b_0^{-1}$, $\frac{1}{2}(x/y)^\epsilon\le\log(x)/\log(y)\le 2(y/x)^\varepsilon$ (using for example Karamata's representation of slowly varying functions). This last condition will imply that, for every n large enough (so that $\mathfrak{a}_n>b_0^{-1}$) and for every $u\in[-b_0\mathfrak{a}_n,b_0\mathfrak{a}_n]^{d+1}$, the following inequalities hold true

$$
\frac{1}{2}\min(|u|^\varepsilon,|u|^{-\varepsilon})\le\log(\mathfrak{a}_n/|u|)/\log\mathfrak{a}_n\le 2\max(|u|^\varepsilon,|u|^{-\varepsilon}). \tag{7.16}
$$

Lemma 7.1 ensures that the spectral radius of P_t is smaller than 1 for every $t\neq 0$. This implies that there exists $\theta_0\in(0,1)$ such that $\sup_{b_0<|t|_\infty<K}\|P_t^n\|=\mathcal{O}(\theta_0^n)$ (by upper semicontinuity of the spectral radius). Thus

$$
\left|\int_{b_0<|t|_\infty<K}e^{-i\langle t,L\rangle}\widehat{h}(t)\mathbb{E}_{\mu_{\overline{\Delta}}}\left(H_{(k_n)}(-t).P_t^{n-3k_n}(P_t^{3k_n}(G_{(k_n)}(t)))\right)dt\right| \tag{7.17}
$$

$$
\le\|h\|_{L^1}K\mathcal{O}\left(\theta_0^{n-3k_n}\|H\|_{L^p}\sup_{|t|<b_0}\|P_t^{3k_n}(G_{(k_n)}(t))\|_{\mathcal{B}}\right), \tag{7.18}
$$

since L^p is continuously included in the dual of the Young space $\mathcal{B}$. Thus, we can focus on $[-b_0,b_0]^d$. It follows from (7.13) and (7.14) that, in $L^q(\mu_{\overline{\Delta}})$,

$$
\begin{aligned}
&P_t^{n-3k_n}(P_t^{3k_n}(G_{(k_n)}(t))) \\
&\quad=\lambda_t^{n-3k_n}\left(\mathbb{E}_{\mu_{\overline{\Delta}}}[P_t^{3k_n}(G_{(k_n)}(t))]+\mathcal{O}(|t|\,\|P_t^{3k_n}(G_{(k_n)}(t))\|_{\mathcal{B}})\right) \\
&\quad+\mathcal{O}\left(\theta^{n-3k_n}(1+|t|)\|P_t^{3k_n}(G_{(k_n)}(t))\|_{\mathcal{B}}\right),
\end{aligned}
$$

and so

$$
\int_{[-b_0,b_0]^{d+1}}e^{-i\langle t,L\rangle}\widehat{h}(t)\left(\mathbb{E}_{\mu_{\overline{\Delta}}}\left[H_{(k_n)}(-t).P_t^{n-3k_n}(P_t^{3k_n}(G_{(k_n)}(t)))\right]\right. \tag{7.19}
$$

$$
\left.-\mathbb{E}_{\mu_{\overline{\Delta}}}\left[H_{(k_n)}(-t)\right]\lambda_t^{n-3k_n}\mathbb{E}_{\mu_{\overline{\Delta}}}[P_t^{3k_n}(G_{(k_n)}(t))]\right)dt \tag{7.20}
$$

$$
=\mathcal{O}\left(\|H\|_{L^p}\mathfrak{a}_n^{-d-2}\sup_{|t|<b_0}\|P_t^{3k_n}(G_{(k_n)}(t))\|_{\mathcal{B}}\right),
$$

since

$$\int_{[-b_0,b_0]^{d+1}} \left(|t||\lambda_t|^{n-3k_n} + \theta^{n-3k_n}(1+|t|)\right)\, dt$$

$$\leq \int_{[-b_0,b_0]^{d+1}} |t|e^{-a'n|t|^2\log(1/|t|)}\, dt + \mathcal{O}\left(\theta^{\frac{n}{2}}\right)$$

$$\leq \mathfrak{a}_n^{-d-2}\int_{[-b_0\mathfrak{a}_n,b_0\mathfrak{a}_n]^{d+1}} |u|e^{-a'n|u/\mathfrak{a}_n|^2\log(|\mathfrak{a}_n|/|u|)}\, du + \mathcal{O}\left(\theta^{\frac{n}{2}}\right)$$

$$\leq \mathfrak{a}_n^{-d-2}\int_{[-b_0\mathfrak{a}_n,b_0\mathfrak{a}_n]^{d+1}} |u|e^{-\frac{a'|u|^2}{2\log(\mathfrak{a}_n)}\log(|\mathfrak{a}_n|/|u|)}\, du + \mathcal{O}\left(\theta^{\frac{n}{2}}\right)$$

$$\leq \mathfrak{a}_n^{-d-2}\int_{[-b_0\mathfrak{a}_n,b_0\mathfrak{a}_n]^{d+1}} |u|e^{-\frac{a'}{4}|u|^{2-\varepsilon}}\, du + \mathcal{O}\left(\theta^{\frac{n}{2}}\right)$$

$$= \mathcal{O}\left(\mathfrak{a}_n^{-d-2}\right), \tag{7.21}$$

where we used (7.16). Furthermore, it follows from the definition of $H_{(k_n)}$ that

$$\mathbb{E}_{\mu_{\overline{\Delta}}}\left[H_{(k_n)}(-t)\right] = \mathbb{E}_{\mu_\Delta}\left(e^{i\langle t,\overline{\Psi}_{k_n}\circ\overline{\pi}\rangle}(He^{i\langle t,\chi\rangle})\circ f_\Delta^{k_n}\right) \tag{7.22}$$

$$= \mathbb{E}_{\mu_\Delta}[H] + \mathcal{O}(k_n t\|H\|_{L^p}), \tag{7.23}$$

since χ is uniformly bounded and $\|\widehat{\Psi}_{k_n}\|_{L^q(\mu)} \ll k_n\|\widehat{\Psi}\|_{L^q(\mu)}$ since $q<2$. Moreover, due to (7.4), for all $t\in[-b_0;b_0]^{d+1}$,

$$\begin{aligned}\mathbb{E}_{\mu_{\overline{\Delta}}}[P_t^{3k_n}(G_{(k_n)}(t))] &= \mathbb{E}_{\mu_\Delta}[(e^{i\langle t,\overline{\Psi}_{2k_n}\circ\overline{\pi}-\chi\rangle}G)\circ f_\Delta^{k_n}] + \mathcal{O}(\beta^{k_n})\\ &= \mathbb{E}_\mu\left[G\right] + \mathcal{O}\left(t\|G\|_{L^1} + t\|\widehat{\Psi}_{2k_n}.G\|_{L^1(\mu)}\right).\end{aligned}$$

Combining these last two estimates with (7.19) via (7.21), we infer that

$$\begin{aligned}&\int_{[-b_0,b_0]^{d+1}} e^{-i\langle t,L\rangle}\widehat{h}(t)\left(\mathbb{E}_{\mu_{\overline{\Delta}}}\left[H_{(k_n)}(-t).P_t^{n-3k_n}(P_t^{3k_n}(G_{(k_n)}(t)))\right]\right.\\ &\qquad\qquad \left.-\mathbb{E}_{\mu_\Delta}\left[H\right]\mathbb{E}_{\mu_\Delta}\left[G\right]\lambda_t^{n-3k_n}\right)\, dt\\ &= \mathcal{O}\left(\mathfrak{a}_n^{-d-2}\left[\|H\|_{L^p}\sup_{|t|<b_0}\|P_t^{3k_n}(G_{(k_n)}(t))\|_{\mathcal{B}} + k_n\|G\|_{L^1}\|H\|_{L^p}\right.\right.\\ &\qquad\qquad \left.\left.+\|H\|_{L^1}\|\widehat{\Psi}_{2k_n}.G\|_{L^1(\mu)}\right]\right).\end{aligned}$$

It follows from this last estimate combined with (7.5) and (7.7) that

$$\begin{aligned}&\mathbb{E}_{\mu_\Delta}[G.h(\widehat{\Psi}_n - L).H\circ f_\Delta^n]\\ &\quad - \mathfrak{a}_n^{-d-1}\mathbb{E}_{\mu_\Delta}\left[H\right]\mathbb{E}_{\mu_\Delta}[G]\int_{[-b_0\mathfrak{a}_n,b_0\mathfrak{a}_n]^{d+1}} e^{-i\mathfrak{a}_n^{-1}\langle t,L\rangle}\widehat{h}(t/\mathfrak{a}_n)\lambda_{t/\mathfrak{a}_n}^{n-3k_n}\, dt\end{aligned} \tag{7.24}$$

$$= \mathcal{O}\left(\beta^{k_n}\|G\|_{Holder}.\|H\|_{Holder} + \mathfrak{a}_n^{-d-2}\left[\|H\|_{L^p}\sup_{|t|<b_0}\|P_t^{3k_n}(G_{(k_n)}(t))\|_{\mathcal{B}}\right.\right. \tag{7.25}$$

$$\left.\left.+k_n\|G\|_{L^1}\|H\|_{L^p} + \|H\|_{L^1}\|\widehat{\Psi}_{2k_n}.G\|_{L^1(\mu)}\right]\right). \tag{7.26}$$

Thus, due to (7.12), the above formula (7.24) is bounded by

$$\mathcal{O}\left(\max(\beta,\theta)^{k_n}\|G\|_{Holder}.\|H\|_{Holder}\right.$$
$$\left.+\mathfrak{a}_n^{-d-2}\left[k_n\|G\|_{L^1}\|H\|_{L^p}+\|H\|_{L^1}\|\widehat{\Psi}_{2k_n}.G\|_{L^1(\mu)}\right]\right).$$

It remains to estimate

$$\int_{[-b_0\mathfrak{a}_n,b_0\mathfrak{a}_n]^{d+1}} e^{-i\mathfrak{a}_n^{-1}\langle t,L\rangle}\widehat{h}(t/\mathfrak{a}_n)\lambda_{t/\mathfrak{a}_n}^{n-3k_n}\,dt.$$

To this end, let us notice that, due to (7.16), for $t\in[-\mathfrak{a}_n b_0,\mathfrak{a}_n b_0]^d$,

(7.27)

$$\begin{aligned}\lambda_{t/\mathfrak{a}_n}^{n-3k_n} &= \lambda_{t/\mathfrak{a}_n}^{n}\lambda_{t/\mathfrak{a}_{n-3k_n}}^{-3k_n} = e^{-\frac{n}{\mathfrak{a}_n^2}\langle\Sigma_{d+1}t,t\rangle\,(\log(\mathfrak{a}_n/|t|))+\mathcal{O}(n|t|^2/\mathfrak{a}_n^2)}e^{\mathcal{O}(\frac{k_n}{n}\max(|t|^{2-\epsilon},|t|^{2+\epsilon}))}\\ &= e^{-\frac{1}{2\log(\mathfrak{a}_n)}\langle\Sigma_{d+1}t,t\rangle\,(\log\mathfrak{a}_n-\log(|t|))+\mathcal{O}(\max(|t|^{2-\epsilon},|t|^{2+\epsilon})\eta_n)}\\ &= e^{-\frac12\langle\Sigma_{d+1}t,t\rangle\,(1-\frac{\log|t|}{\log\mathfrak{a}_n})}+\mathcal{O}\left(e^{-\frac{a'\min(|t|^{2-\epsilon},|t|^{2+\epsilon})}{2}}\max(|t|^{2-\varepsilon},|t|^{2+\varepsilon})\eta_n\right),\end{aligned}$$

with $\eta_n:=\frac{1}{\log n}+\frac{k_n}{n}$. Therefore, since $\widehat{h}$ is Lipschitz continuous and writing $[\widehat{h}]_{Lip}$ for its Lipschitz constant, it follows that

$$\begin{aligned}&\int_{[-b_0\mathfrak{a}_n,b_0\mathfrak{a}_n]^{d+1}} e^{-i\mathfrak{a}_n^{-1}\langle t,L\rangle}\widehat{h}(t/\mathfrak{a}_n)\lambda_{t/\mathfrak{a}_n}^{n-3k_n}\,dt\\ &=\int_{\mathbb{R}^{d+1}} e^{-i\mathfrak{a}_n^{-1}\langle t,L\rangle}\left(\widehat{h}(0)+\mathcal{O}(t/\mathfrak{a}_n)\right)e^{-\frac12\langle\Sigma_{d+1}t,t\rangle\left(1-\mathbf{1}_{|t|<b_0\mathfrak{a}_n}\frac{\log(|t|)}{\log(\mathfrak{a}_n)}\right)}\,dt\\ &\quad+\mathcal{O}\left(\left(|\widehat{h}(0)|+\frac{[\widehat{h}]_{Lip}}{\mathfrak{a}_n}\right)\eta_n\right)\\ &=\widehat{h}(0)\int_{\mathbb{R}^{d+1}} e^{-i\mathfrak{a}_n^{-1}\langle t,L\rangle}e^{-\frac12\langle\Sigma_{d+1}t,t\rangle}\left(1+\frac{\mathbf{1}_{|t|<b_0\mathfrak{a}_n}}{2}\langle\Sigma_{d+1}t,t\rangle\frac{\log(|t|)}{\log(\mathfrak{a}_n)}\right.\\ &\quad\left.+O\left(\mathbf{1}_{|t|<b_0\mathfrak{a}_n}\max\left(1,e^{\frac12\langle\Sigma_{d+1}t,t\rangle\frac{\log(|t|)}{\log\mathfrak{a}_n}}\right)\frac{|t|^4(\log(|t|))^2}{(\log n)^2}\right)\right)dt\\ &\quad+O\left(|\widehat{h}(0)|\eta_n+\frac{[\widehat{h}]_{Lip}}{\mathfrak{a}_n}\right),\end{aligned}$$

where we used $e^x=1+x+O(\max(1,e^x)x^2)$. Thus

$$\begin{aligned}\text{(7.28)}\quad&\int_{[-b_0\mathfrak{a}_n,b_0\mathfrak{a}_n]^{d+1}} e^{-i\mathfrak{a}_n^{-1}\langle t,L\rangle}\widehat{h}(t/\mathfrak{a}_n)\lambda_{t/\mathfrak{a}_n}^{n-3k_n}\,dt\\ &=g_{d+1}\left(\frac{L}{\mathfrak{a}_n}\right)\int_{\mathbb{Z}^d\times\mathbb{R}} h\,d\lambda_{d+1}+\mathcal{O}\left(|\widehat{h}(0)|\eta_n+\frac{[\widehat{h}]_{Lip}}{\mathfrak{a}_n}\right),\end{aligned}$$

with $g_{d+1}(z):=\frac{e^{-\frac12\langle\Sigma_{d+1}z,z\rangle}}{\sqrt{(2\pi)^d\det\Sigma_{d+1}}}$.

This ends the proof of Lemma 3.2. □

Proof of Lemma 3.4. Let $A_0,B_0\subset M$ be measurable sets such that $\mu(\partial A_0)=\mu(\partial B_0)=0$. Let $K\subset\mathbb{Z}^d\times\mathbb{R}$ be a bounded set with $\Lambda_{d+1}(\partial K)=0$ (boundary in $\mathbb{Z}^d\times\mathbb{R}$) and let $z\in\mathbb{R}^{d+1}$ and $(z_n)_n$ be a sequence of $\mathbb{Z}^d\times\mathbb{R}$ such that

$\lim_{n\to+\infty} z_n/a_n = z$. Let us prove that

(7.29)
$$\lim_{n\to+\infty} a_n^{d+1}\mu\left(A_0\cap T^{-n}(B_0)\cap\{\widehat\Psi_n(x)\in z_n+K\}\right) = g_{d+1}(z)\mu(A_0)\mu(B_0)\lambda_{d+1}(K).$$

We will approximate A_0 and B_0 by $A_n^\pm$ and $B_n^\pm$ respectively, where A_n^- (resp. A_n^+) is the union of all connected components of $M\setminus\bigcup_{k=-m_n}^{m_n}T^{-k}(\mathcal{S}_0)$ contained in (resp. intersecting) A_0 with $m_n\to+\infty$, analogously with $B_n^\pm$ with respect to B_0. Since the diameter of these connected components is smaller than $C\vartheta^n$ for some $C>0$ and some $\vartheta\in(0,1)$ and since $\mu(\partial A_0)=\mu(\partial B_0)=0$, we conclude that $\mu(A_n^+\setminus A_n^-)$ and $\mu(B_n^+\setminus B_n^-)$ vanishes as $n\to+\infty$. Consider h as in Lemma 3.2 taking non-negative values. We set

$$\mathfrak{M}_n(A,B) := a_n^{d+1}\mathbb{E}_\mu[\mathbf{1}_A.h(\widehat\Psi_n-z_n).\mathbf{1}_{T^{-n}(B)}]$$

and

$$\mathfrak{M}(A,B) := \mu(A)\mu(B)g_{d+1}(z)\int_{\mathbb{Z}^d\times\mathbb{R}} h\,d\Lambda_{d+1}.$$

Since $h\ge 0$, these two quantities are increasing in A and in B. Thus

$$\begin{aligned}\mathfrak{M}_n(A_m^-,B_m^-)-\mathfrak{M}(A_m^+,B_m^+) &\le \mathfrak{M}_n(A_0,B_0)-\mathfrak{M}(A_0,B_0)\\ &\le \mathfrak{M}_n(A_m^+,B_m^+)-\mathfrak{M}(A_m^-,B_m^-).\end{aligned}$$

It then follows from Lemma 3.2 (with $k_n=\log n$) applied to $\mathbf{1}_{A_m^\pm},\mathbf{1}_{B_m^-}$ since these observables are dynamically Hölder and since $\mathfrak{a}_n\sim a_n$ that, for all $m\ge 1$,

$$\begin{aligned}&(\mu(A_m^-)\mu(B_m^-)-\mu(A_m^+)\mu(B_m^+))g_{d+1}(z)\int_{\mathbb{Z}^d\times\mathbb{R}} g\,d\lambda_{d+1}\\ &=\lim_{n\to+\infty}\mathfrak{M}_n(A_m^-,B_m^-)-\mathfrak{M}(A_m^+,B_m^+)\\ &\le\liminf_{n\to+\infty}\mathfrak{M}_n(A_0,B_0)-\mathfrak{M}(A_0,B_0),\end{aligned}$$

from which we conclude that $\liminf_{n\to+\infty}\mathfrak{M}_n(A_0,B_0)-\mathfrak{M}(A_0,B_0)\ge 0$, we proceed analogously with the lim sup exchanging exponents $+$ and $-$, and we conclude that

$$\lim_{n\to+\infty}\mathfrak{M}_n(A_0,B_0)=\mathfrak{M}(A_0,B_0)=0, \tag{7.30}$$

and extend this to the case of complex valued function g. Consider the function H_0 appearing in (2.10). Let $\delta=1/L>0$, where L is an integer such that $L>2\|H_0\|_\infty$ and $K\subset(-L+2\|H_0\|_\infty, L-2\|H_0\|_\infty)^d\times(-\frac{2\pi}{\delta},\frac{2\pi}{\delta})$. Let us consider the family of functions $(g_{\delta,\theta}:\mathbb{R}^{d+1}\to\mathbb{C})_\theta$ given by

$$g_{\delta,\theta}(x)=e^{i\langle\theta,x\rangle}h_\delta(x), \tag{7.31}$$

with $h_\delta:(x_1,\dots x_{d+1})\mapsto\frac{1-\cos(\delta x_{d+1})}{(2L-1)^2\pi\delta x_{d+1}^2}\mathbf{1}_{|x_1|,\dots,|x_d|<L}$ (using the density of Polya's distribution). The Fourier transform of $g_{\delta,\theta}$ is

$$t\mapsto\sum_{|k_1|,..,|k_d|<L}\frac{e^{i\sum_{i=1}^d k_it_i}}{(2L-1)}\max(0,1-|(t_{d+1}+\theta)/\delta|).$$

The above convergence result (7.30) with $h=g_{\delta,\theta}$ for all θ implies the convergence in distribution of $(\mathfrak{m}_n)_n$ to $\mathfrak{m}$ (since it ensures the convergence of characteristic functions), where $\mathfrak{m}_n$ has density $\frac{h_\delta}{a_n^{d+1}\mathbb{E}_\mu[\mathbf{1}_{A_0\cap T^{-n}(B_0)}h_\delta(\widehat\Psi_n-z_n)]}$ with respect to the image measure of $a_n^{d+1}\mathbf{1}_{A_0\cap T^{-n}B_0}\mu$ by $\widehat\Psi_n-z_n$, and where $\mathfrak{m}$ is the probability measure with density $h_\delta/g_{d+1}(z)$ with respect to $g_{d+1}(z)\Lambda_{d+1}$. Thus, since $K \subset (-L,L)^d \times (-\frac{2\pi}{\delta}, \frac{2\pi}{\delta})$, the previous distribution convergence implies that $\lim_{n\to+\infty} \int_K \frac{1}{h_\delta}\, d\mathfrak{m}_n = \int_K \frac{1}{h_\delta}\, d\mathfrak{m}$, i.e.

$$\lim_{n\to+\infty} \frac{a_n^{d+1}\mu\left(A_0, \widehat{\Psi}_n - z_n \in K, T^{-n}(B_0)\right)}{a_n^{d+1}\mathbb{E}_\mu[\mathbf{1}_{A_0\cap T^{-n}(B_0)} h_\delta(\widehat{\Psi}_n - z_n)]} = \Lambda_{d+1}(K),$$

and so, using again (7.30) for the denominator,

$$\lim_{n\to+\infty} \frac{a_n^{d+1}\mu\left(A_0, \widehat{\Psi}_n - z_n \in K, T^{-n}(B_0)\right)}{g_{d+1}(z)\mu(A_0)\mu(B_0)} = \Lambda_{d+1}(K).$$

This ends the proof of pointwise MLLT for $\widehat{\Psi}_n$ (7.29).

It remains to prove the uniformity in the convergence results. Assume that (3.1) does not converge to 0 uniformly in $z \in \mathbb{Z}^d \times \mathbb{R} : |z| \le La_n$, as $n \to +\infty$. Then, there would exist a sequence $(z_n)_n$ in $\mathbb{Z}^d \times \mathbb{R}$ such that $|z_n| < La_n$, a sequence of integers $m(n)$ and a real number $\eta > 0$ such that

$$\forall n \ge N_0, \quad \Big| a_n^{d+1}\mu\left(A_0 \cap T^{-n}(B_0) \cap \{\widehat{\Psi}_n \in z_n + K\}\right) \\ - g_{d+1}\left(\frac{z}{a_n}\right)\mu(A_0)\mu(B_0)\lambda_{d+1}(K)\Big| > \eta.$$

This ends the proof of Lemma 3.4. □

Appendix A. Proof of joint LLD (Lemma 3.5)

In this appendix, we prove Lemma 3.5. The proof is very similar to that of [22] except that the function $\overline{\Psi}$ is not constant on partition elements. For completeness, we explain in this appendix which adaptations have to be done to [22] to prove our joint LLD estimate stated in Lemma 3.5.

We recall that optimal LLD for the cell change κ (and so for the flight function V, due to (2.10)) have been obtained in [22]. More precisely, by [22, Theorem 1.1 and Remark 1.2], for any $h > 0$, there exists $C > 0$ so that $\mu(V_n \in B(x,h))) \le C\frac{n}{a_n^d}\frac{\log|x|}{1+|x|^2}$, for any $n \ge 1$ and $x \in \mathbb{R}^d$. Here $B(x,h)$ denotes an open ball in $\mathbb{R}^d$ of radius h centered at x. Similarly, $\mu(\kappa_n = N) \le C\frac{n}{a_n^d}\frac{\log|N|}{1+|N|^2}$ for all $N \in \mathbb{Z}^d$ and all $n \ge 1$.

The proof of LLD for κ in [22] relies strongly on the fact that κ goes to the quotient Young tower and that $\overline{\kappa}$ is constant on partition elements of the partition $\mathcal{P}$ for the Young tower Δ (as recalled in Section 6); the statement on V follows immediately since, up to a bounded coboundary, V is the same as κ. Due to (6.10), $\widehat{\Psi}\circ\pi$ can be written as $\overline{\Psi}\circ\overline{\pi} = (\overline{\kappa}, \overline{\tau})\circ\overline{\pi}$ plus a bounded coboundary. Thus, LLD for $\widehat{\Psi}$ will follow from LLD for $\overline{\Psi}$. The function $\overline{\tau}$, and thus $\overline{\Psi}$, is not constant on partition elements. However, as argued below, the argument in [22] goes through to provide LLD as in Lemma 3.5 for $\overline{\Psi}$ (and thus $\widehat{\Psi}$).

Throughout this section, let $d \in \{0,1,2\}$ and $U \subset \mathbb{R}^{d+1}$ be an open ball, as in the statement of Lemma 3.5. To avoid a clash of notation below, the z in the statement of Lemma 3.5 will be replaced by x. More precisely, here we shall prove that, for any bounded set $U \subset \mathbb{R}^{d+1}$,

$$\mu(\widehat{\Psi}_n \in x+U) \ll \frac{n}{a_n^{d+1}} \frac{(\log|x|)}{1+|x|^2} \quad \text{uniformly in } n \geq 1,\, x \in \mathbb{R}^{d+1}. \tag{A.1}$$

As recalled in [22, Remark 1.3], the LLD in the range $|x| \leq a_n$ follows from the involved LLT, while the range $|x| \geq a_n$ requires serious work.

A.1. **The range $|x| \leq a_n$.** It follows from the LLT estimate given in Lemma 3.2 that $\mu(\widehat{\Psi}_n \in B(x,h)) \ll a_n^{-d-1} \ll \frac{n}{a_n^{d+1}} \frac{|\log|x||}{1+|x|^2}$, where the first inequality holds true uniformly in $n \geq 1, x \in \mathbb{R}^{d+1}$ and where the second one holds true for $n \geq 1$ and $x \in \mathbb{R}^d$ such that $|x| \leq a_n$.

Indeed, $t \mapsto \frac{t^2}{|\log|t||}$ as limit $+\infty$ as $t \to +\infty$, has derivative $t \mapsto \frac{2\log t - 1}{t}$ and so it is increasing on $[e^{\frac{1}{2}}, +\infty[$. Thus, for n large enough, if $|x| \leq a_n$, then $\frac{|x|^2}{|\log|x||} \leq \frac{a_n^2}{|\log a_n|} = \frac{2n\log n}{\log n + \log\log n}$.

A.2. **The range $n \ll \log|x|$.** In this range we proceed similarly to [22, Lemma 3.1] obtain

Lemma A.1. *For any $\epsilon_1 > 0$ and any $q \geq 1$, there exists $C_q > 0$ so that, for every $x \in \mathbb{R}^{d+1}$ and every $n \geq 1$ such that $\epsilon_1 n \leq \log|x|$, $\mu(\widehat{\Psi}_n \in x+U) \leq \frac{C_q}{n^q|x|^2}$.*

Proof. There exists x_0 such that if $|x| > x_0$, then $|x| > 2(\operatorname{diam}(U) + \epsilon_1^{-1}\log|x|)$. There exists a constant C'_q such that $\mu(\widehat{\Psi}_n \in x+U) \leq \frac{C'_q}{n^q|x|^2}$ for all x, n such that $\epsilon_1 n \leq \log|x| \leq \log x_0$. It remains to treat the case $|x| \geq x_0$. One can observe that the proof of [22, Lemma 3.1] only uses the fact that $|\kappa|_\infty$ takes integer values, that $\mu(|\kappa|_\infty = p) \ll p^{-3}$ as $x \to +\infty$ and that $\mu(|\kappa|_\infty = p,\, |\kappa|_\infty \circ T^r \geq cp^{\frac{4}{5}}) \ll p^{-3-\frac{2}{45}}$. These properties are also satisfied by $(\kappa, \lfloor\widetilde{\tau}\rfloor)$. Thus, for every $q \geq 1$, there exists $C'_q > 0$ such that, for all $y \in \mathbb{R}^d$ and all $n \geq 1$ such that $\epsilon n \leq \log|y|$, $\mu((\kappa_n, (\lfloor\widetilde{\tau}\rfloor)_n) = y) \leq \frac{C''_q}{n^q|y|^2}$ and so

$$\mu(\Psi_n \in x+U) \leq \sum_{y \in (x+U+\{0\}^d\times[-n;0])\cap\mathbb{Z}^{d+1}} \frac{C''_q}{n^q|y|^2} \leq (n+1)(\operatorname{diam}(U)+1)^{d+1} \frac{4C''_q}{n^q|x|^2},$$

since y in the first sum above satisfies

$$|y| \geq |x| - \operatorname{diam}(U) - n \geq |x| - \operatorname{diam}(U) - \frac{\log|x|}{\epsilon_1} \geq |x|/2.$$

We conclude by taking e.g. $C_q := \max\left(C'_q, 8C''_{q+1}(\operatorname{diam}(U)+1)\right)$. $\square$

A.3. **The range $a_n \leq |x| \leq e^{\epsilon_1 n}$ for a particular ϵ_1.** This ϵ_1 is to be fixed so that it matches with the choice of ϵ_1 in [22, Proposition 6.2]. As we shall explain below, it does not play a role in the current argument, but see (A.4) for a particular choice.

Since $\widehat{\Psi} \circ \pi$ is equal to $\overline{\Psi} \circ \overline{\pi}$ plus a bounded coboundary, the desired LLD (A.1) for $\widehat{\Psi}$ will follow from LLD for $\overline{\Psi}$. So, in this range, we focus on LLD estimates for $\overline{\Psi}$. As clarified in Appendix B, $\overline{\Psi}$ is uniformly Lipschitz on Young's partition elements.

We work in the set-up of Section 6. As in Section 6, for simplicity, we assume that[8] the g.c.d. of R is 1.

Set $\delta = b_0/4$ with $b_0 \in (0, \min(1, \beta_0))$ (with β_0 introduced before (6.13)) and such that there is a constant $c > 0$ such that[9]

$$\text{(A.2)}\quad \forall t \in \mathbb{R}^d, \quad |t| < b_0 \;\Rightarrow\; |\lambda_t| \le e^{-c|t|^2 L_t}, \quad \text{with } L(t) := \log(|t|^{-1}) = |\log|t||.$$

Using (6.13) and (6.14), we consider a function $r : \mathbb{R}^{d+1} \to \mathbb{C}$ is C^2 with $\operatorname{supp} r \subset [-b_0, b_0]^{d+1}$ such that[10]

$$\text{(A.3)}\qquad \mu(\overline{\Psi}_n \in x + U) \le \int_{[-\delta,\delta]^{d+1}} e^{-i\langle t,x\rangle} r(t) P_t^n 1\, dt = A_{n,x} + \mathcal{O}(\theta^n),$$

with

$$A_{n,x} := \int_{[-\delta,\delta]^{d+1}} e^{-i\langle t,x\rangle} r(t) \lambda_t^n \Pi_t 1\, dt.$$

The desired LLD (A.1) in this range will follow from (A.3) together with the following estimates on λ_t and Π_t. The following result corresponds to the hardest estimate in the set-up of [22]. Let $\partial_j = \partial_{t_j}$ for $j = 1, \dots, d+1$. For $t, h \in \mathbb{R}^{d+1}$, $\mathbf{b} > 0$, set

$$M_{\mathbf{b}}(t,h) = |h| L_h \big\{ 1 + L_h\, |t|^2 L_t + |h|^{-\mathbf{b}|t|^2 L_t} L_h^2\, |t|^4 L_t^2 \big\}.$$

Lemma A.2 (Analogue of [22, Lemma 4.1])**.** *Let $j \in \{1, \dots, d\}$. The maps $t \mapsto \lambda_t$ and $t \mapsto \Pi_t : \mathcal{B}_0 \to L^1$ are C^1 on $[-b_0, b_0]^{d+1}$.*

Furthermore, there exist $C > 0$ and $\mathbf{b} > 0$ such that for all $t, h \in [-b_0, b_0]^{d+1}$,

$$|\partial_j \lambda_{t+h} - \partial_j \lambda_t| \le C M_{\mathbf{b}}(t,h), \qquad \|\partial_j \Pi_{t+h} - \partial_j \Pi_t\|_{\mathcal{B}_0 \mapsto L^1} \le C M_{\mathbf{b}}(t,h).$$

In [22], we obtained the same formula for $M_{\mathbf{b}}(t,h)$, while working with $\kappa \in \mathbb{Z}^d$ instead $\overline{\Psi} \in \mathbb{R}^{d+1}$). Given Lemma A.2 in the range $a_n \le |x| \le e^{\varepsilon_1 n}$ for

$$\text{(A.4)}\qquad \varepsilon_1 := c/\mathbf{b}$$

with $\mathbf{b} > 0$ as in Lemma A.2 and $c > 0$ as in (A.2), the desired LLD for $\overline{\Psi}$ (as in (A.1) with $\overline{\Psi}$ instead of $\widehat{\Psi}$) follows word for word as in [22, Section 6 via Corollary 4.3] (written for κ). Indeed, the proofs therein just use the statement of Lemma A.2, (A.2) and the fact that $a_n = \sqrt{n \log n}$.

A.4. **Proof of Lemma A.2.** We follow the proofs of [22]. We focus on the changes that are needed since $\overline{\Psi}$ is not constant on the atoms of the Young partition. We will indicate which part of proofs of [22] can be followed line by line. Throughout this section let $Q : L^1(\overline{Y}) \to L^1(\overline{Y})$ be the transfer operator corresponding to the Gibbs-Markov map $\overline{F} : \overline{Y} \to \overline{Y}$ (the base map of the quotient tower $(\overline{\Delta}, f_{\overline{\Delta}})$). We recall that R is the return time of $f_{\overline{\Delta}}$ to $\overline{Y}$, so that $F(\cdot) = f_{\overline{\Delta}}^{R(\cdot)}(\cdot)$. The proof

[8]This assumption is not essential. One could, as in [22] work without, but in that case (6.13) becomes slightly more complicated as there exists no longer a simple isolated eigenvalue at 1, but finitely many eigenvalues of finite multiplicity.

[9]The existence of such a couple (b_0, c) comes from Sublemma 6.1.

[10]The existence of such an r is guaranteed by a classical smoothing argument; see, for instance, [25, Footnote 1].

of [22, Lemma 4.1] of which Lemma A.2 is an analogue (in the set-up of Lemma 3.5) starts from the following renewal equation
(A.5)
$$\widehat{P}(z,t) = \sum_{n\geq 0} z^n P_t^n = \widehat{A}(z,t)\widehat{T}(z,t)\widehat{B}(z,t) + \widehat{E}(z,t), z \in \mathbb{C}, |z| \leq 1, t \in [-\delta_0, \delta_0]^{d+1},$$

where the operators $\widehat{A}, \widehat{T}, \widehat{B}, \widehat{E}$ and $\delta_0 > 0$ are to be defined/specified in the subsections to follow. In particular $\widehat{T}$ will be given by

$$\widehat{T}(z,t) = (I - \widehat{Q}(z,t))^{-1},$$

with

$$\widehat{Q}(z,t) := Q\left(z^{R(\cdot)} e^{i\langle t, \overline{\Psi}_R\rangle} \cdot\right) = \sum_{n\geq 1} z^n Q\left(\mathbf{1}_{\{R=n\}} e^{i\langle t, \overline{\Psi}_n\rangle} \cdot\right), \tag{A.6}$$

where we write $\overline{\Psi}_R$ for the function defined by

$$\forall y \in \overline{Y}, \quad \overline{\Psi}_{R(y)}(y) = \sum_{k=0}^{R(y)-1} \overline{\Psi}(y,k).$$

Following the approach in [22], the proof of Lemma A.2 consists in using (A.5) to clarify that

$$\lambda_t = (g_0(t))^{-1},$$

where $t \mapsto g_0(t)$ is continuous, satisfies $g_0(0) = 1$ and is such that 1 is the dominating eigenvalue of $\widehat{Q}(g_0(t), t)$ and that

$$\Pi_t = \lambda_t \widehat{A}(g_0(t), t)\widetilde{\pi}_0(t)\widehat{B}(g_0(t), t),$$

where $\widetilde{\pi}_0(t)$ is such that $\widehat{H}(z,t) := \widehat{T}(z,t) - (g_0(t) - z)^{-1}\widetilde{\pi}(t)$ is analytic in z. As in [22], the regularity (in terms of M_b) of the derivatives of λ (stated in Lemma A.2) will follow, via the use of the implicit function theorem, from the study of $\widehat{Q}(z,t)$ for z close to 1, and from the properties satisfied by $\overline{\Psi}$. The analogous property for Π will follow from the properties satisfied by $\lambda = 1/g_0$ and also from the study of the derivatives in t of $\widehat{A}, \widehat{T}, \widehat{B}, \widehat{E}$ for z close to 1.

A.4.1. *Renewal operators.* Throughout, we write α for the partition corresponding to the Gibbs Markov map $(\overline{F}, \overline{Y}, \alpha, \mu_{\overline{Y}})$. Also, let $s'(y,y')$ be the usual separation time of points $y, y' \in \overline{Y}$ (see, for instance, [22, Section 2] for definition) and for $\beta \in (0,1)$, let $d_\beta(y,y') := \beta^{s'(y,y')}$. Let $\mathcal{B}_1(\overline{Y})$ be the Banach space of bounded observables $v : \overline{Y} \to \mathbb{R}$ Lipschitz with respect to the metric d_β. It follows from the fact that R has exponential tail probability that $\mu_{\overline{Y}}(|\overline{\Psi}|_R > n) = \mathcal{O}(n^{-2})$ as $\mu(|\overline{\Psi}| > n)$ (see e.g. [29, Section 2] for this argument).

Proposition A.3. *There exists* $C_{\overline{\Psi}} > 0$ *such that*

$$\forall a \in \alpha, \quad \forall y, y' \in a, \quad \sum_{\ell=0}^{R(a)-1} |\overline{\Psi}(y,\ell) - \overline{\Psi}(y',\ell)| \leq C_{\overline{\Psi}} d_\beta(y,y').$$

Proof. Recall that $\overline{\Psi}$ is-Lipschitz with respect to Young's metric $\beta^{s(\cdot,\cdot)}$. Let us write $C'_{\overline{\Psi}}$ for its Lipschitz constant. Thus, for $a \in \alpha$ and y, $y' \in a$,

$$|\overline{\Psi}_R(y) - \overline{\Psi}_R(y')| \le \sum_{\ell=0}^{R(a)-1} |\overline{\Psi}(y,\ell) - \overline{\Psi}(y',\ell)| \\ \le C'_{\overline{\Psi}} \sum_{\ell=0}^{R(a)-1} \beta^{R(a)-\ell+\overline{s}(y,y')} \le \frac{C'_{\overline{\Psi}} d_\beta(y,y')}{1-\beta},$$

since $s'(y,y') \le \overline{s}(y,y') - R(a) + 1$. □

Recall that

$$Q(u)(y) = \sum_{a\in\alpha} \xi(y_a)u(y_a), \tag{A.7}$$

where y_a is the preimage of y under F that belongs to a, and with $\xi = e^{g_R}$ with $g_R(y) := \sum_{k=0}^{R(y)-1} g(y,k)$ where g satisfies (7.9). Thus

$$0 < \xi(y_a) = e^{g_R(y_a)} \le C\mu_{\overline{Y}}(a), \qquad |\xi(y_a) - \xi(y'_a)| \le C\mu_{\overline{Y}}(a) d_\beta(y,y'), \tag{A.8}$$

for all $y, y' \in \overline{Y}$, $a \in \alpha$, where we define g_R as we have defined $\overline{\Psi}_R$ considering the function g instead of $\overline{\Psi}$.

The next result extends [22, Proposition 5.1] which was stated, in the context therein, for a function u constant on elements of α (for which the local Lipschitz constant K_u is null).

Proposition A.4. *There exists $C > 0$ such that*

$$\|Q(u)\|_{\mathcal{B}_1(\overline{Y})} \le C\||u| + K_u\|_{L^1(\mu_{\overline{Y}})},$$

for every $u \in L^1(\overline{Y})$ such that $K_u \in L^1(\mu_{\overline{Y}})$ with

$$\forall a \in \pi,\ \forall x \in a, \quad K_u(x) := \sup_{y,y'\in a} \frac{|u(y) - u(y')|}{\beta^{s'(y,y')}}.$$

In particular, for all $v \in \mathcal{B}_1(\overline{Y})$ (set of Lipschitz functions on $\overline{Y}$),

$$\|Q(uv)\|_{\mathcal{B}_1(\overline{Y})} \le C\||uv| + K_{uv}\|_{L^1(\mu_{\overline{Y}})} \le C\||u| + K_u\|_{L^1(\mu_{\overline{Y}})}\|v\|_{\mathcal{B}_1(\overline{Y})}. \tag{A.9}$$

Proof of Proposition A.4. Let $y, y' \in \overline{Y}$. For $a \in \alpha$, we write $y_a, y'_a \in a$ for the respective preimages of y, y' under F. It follows from (A.7) and from the first part of (A.8) that

$$\|Q(u)\|_\infty \ll \sum_{a\in\alpha} \mu_{\overline{Y}}(a) \sup_a |u| \le \||u| + K_u\|_{L^1(\mu_{\overline{Y}})},$$

since, for all $y, x \in a \in \alpha$, $|u(y)| \le |u(x)| + K_u(x)$.

Next, let $a \in \alpha$ and $y, y' \in a$, using the second part of (A.8), we obtain

$$|Q(u)(y) - Q(u)(y')| \le \sum_{a\in\alpha} \mu_{\overline{Y}}(a)\ |u(y_a) - u(y'_a)| \\ \le \sum_{a\in\alpha} \mu_{\overline{Y}}(a) K_u(a) \beta^{s'(y_a,y'_a)} \\ \le \sum_{a\in\alpha} \mu_{\overline{Y}}(a) K_u(a) \beta^{s'(y,y')+1},$$

which ends the proof. □

For $z \in \mathbb{C}$ with $|z| \le 1$ and $t \in \mathbb{R}^d$, the operator $\widehat{Q}(z,t)$ formally defined in (A.6) defines an operator on $L^1(\mu_{\overline{Y}})$ and can be decomposed in $\widehat{Q}(z,t) = \sum_{n=1}^{\infty} z^n Q_{t,n}$, where we set

$$Q_{t,n} := Q\left(1_{\{R=n\}} e^{i\langle t, \overline{\Psi}_R\rangle} \cdot\right).$$

The next result replaces [22, Proposition 5.2]. The conclusion is the same, but, in our context, we have to deal with K_u, so we include entirely its proof.

Proposition A.5. *There exists $\delta_0 > 0$ such that when regarded as functions with values in the set of continuous linear operators on $\mathcal{B}_1(\overline{Y})$,*

(a) *$z \mapsto \widehat{Q}(z,t)$ is analytic on $B_{1+\delta_0}(0)$ for all $t \in \mathbb{R}^d$;*
(b) *$(z,t) \mapsto (\partial_z^k \widehat{Q})(z,t)$ is C^1 on $B_{1+\delta_0}(0) \times \mathbb{R}^d$ for $k = 0, 1, 2$;*
(c) *$z \mapsto (\partial_j \widehat{Q})(z,t)$ is C^1 on $B_{1+\delta_0}(0)$ uniformly in $t \in B_1(0)$ for $j = 1, \dots, d$.*

Proof. It suffices to show that there exist $a > 0$, $C > 0$ such that

$$\|Q_{t,n}\|_{\mathcal{B}_1(\overline{Y})} \le C(|t|+1)e^{-an}, \qquad \|\partial_j Q_{t,n}\|_{\mathcal{B}_1(\overline{Y})} \le C(|t|+1)e^{-an},$$

for all $t \in \mathbb{R}^d$, $j = 1, \dots, d$, $n \ge 1$. Since R is constant on partition elements, it follows from Proposition A.4 that

$$\|Q_{t,n}\|_{\mathcal{L}(\mathcal{B}_1(\overline{Y}))} \ll \left\|1_{\{R=n\}}(1 + K_{e^{i\langle t, \overline{\Psi}_R\rangle}})\right\|_{L^1(\mu_{\overline{Y}})},$$

and that

$$\begin{aligned}\|\partial_j Q_{t,n}\|_{\mathcal{L}(\mathcal{B}_1(\overline{Y}))} &= \left\| iQ\left(1_{\{R=n\}}(\overline{\Psi}_R)_j e^{i\langle t, \overline{\Psi}_R\rangle}\right)\right\|_{\mathcal{L}(\mathcal{B}_1(\overline{Y}))} \\ &\ll \left\|1_{\{R=n\}}\left(|\overline{\Psi}_R| + K_{(\overline{\Psi}_R)_j e^{i\langle t, \overline{\Psi}_R\rangle}}\right)\right\|_{L^1(\mu_{\overline{Y}})}.\end{aligned}$$

But it follows from Proposition A.3 that $K_{e^{i\langle t, \overline{\Psi}_R\rangle}} \le C_{\overline{\Psi}}|t|$ and that

$$K_{(\overline{\Psi}_R)_j e^{i\langle t, \overline{\Psi}_R\rangle}} \le C_{\overline{\Psi}}(1 + |t|\,|\overline{\Psi}_R|). \tag{A.10}$$

Therefore

$$\|Q_{t,n}\|_{\mathcal{L}(\mathcal{B}_1(\overline{Y}))} \ll (1+|t|)\mu_{\overline{Y}}(R=n),$$

and

$$\|\partial_j Q_{t,n}\|_{\mathcal{L}(\mathcal{B}_1(\overline{Y}))} \ll (1+|t|)\left\|1_{\{R=n\}}(1+|\overline{\Psi}_R|)\right\|_{L^1(\mu_{\overline{Y}})}.$$

We complete the proof by noticing that, since $\overline{\Psi} \in L^r(\overline{Y})$ for all $r < 2$ and R has exponential tails, there exists $a > 0$ such that $\|1_{\{R=n\}}(1+\overline{\Psi}_R)\|_1 \ll e^{-an}$. $\square$

For $z \in \mathbb{C}$ with $|z| \le 1$ and $t \in \mathbb{R}^d$, define

$$\widehat{A}(z,t) : L^1(\overline{Y}) \to L^1(\overline{\Delta}), \qquad \widehat{A}(z,t)(v) = \sum_{n=1}^{\infty} z^n A_{t,n}(v),$$

where $A_{t,n}(v)(y,\ell) = 1_{\{\ell=n\}} P_t^n(v)(y,\ell) = 1_{\{\ell=n\}} e^{i\langle t, \overline{\Psi}_n(y,0)\rangle} v(y)$.

Proposition A.6. *There exists $\delta_0 > 0$ such that when regarded as functions with values in the set of continuous linear operators from $L^\infty(\overline{Y})$ to $L^1(\overline{\Delta})$,*

(a) *$z \mapsto \widehat{A}(z,t)$ is analytic on $B_{1+\delta_0}$ $t \in \mathbb{R}^d$;*
(b) *$(z,t) \mapsto (\partial_z \widehat{A})(z,t)$ is C^1 on $B_{1+\delta_0}(0) \times \mathbb{R}^d$.*

Proof. The proof goes word for word as [22, Proof of Proposition 5.3] since it just uses the Hölder inequality combined with the fact that $\|1_{R>n}\overline{\Psi}_R\|_{L^1(\overline{Y})}$ decays exponentially fast in n as $n \to +\infty$. $\square$

For $z \in \mathbb{C}$ with $|z| \le 1$ and $t \in \mathbb{R}^d$, define

$$\widehat{B}(z,t) : L^1(\overline{\Delta}) \to L^1(\overline{Y}), \qquad \widehat{B}(z,t)(v) = \sum_{n=1}^{\infty} z^n B_{t,n}(v),$$

where

$$B_{t,n}(v) = 1_{\overline{Y}} P_t^n(1_{D_n} v), \qquad D_n = \{(y, R(y)-n) : y \in \overline{Y},\ R(y) > n\}.$$

Proposition A.7. *There exists $\delta_0 > 0$ such that when regarded as functions with values in the set of continuous linear operators from $\mathcal{B}_0$ to $\mathcal{B}_1(\overline{Y})$,*

(1) *$z \mapsto \widehat{B}(z,t)$ is analytic on $B_{1+\delta_0}(0)$ for all $t \in \mathbb{R}^d$;*
(2) *$(z,t) \mapsto (\partial_z \widehat{B})(z,t)$ is C^1 on $B_{1+\delta_0}(0) \times \mathbb{R}^d$.*

The proof is analogous to the one of [22, Proposition 5.4], but, again, we have to deal with the presence of K_u in Proposition A.4. So we detail this proof.

Proof of Proposition A.7. We observe that $B_{t,n}v = Q(1_{\{R>n\}} v_{t,n})$ where

$$v_{t,n}(y) := e^{i\langle t, \overline{\Psi}_n(y,R(y)-n)\rangle} v(y, R(y)-n).$$

Since R is constant on partition elements, it follows from Proposition A.4 that

$$\|B_{t,n}(v)\|_{\mathcal{B}_1(\overline{Y})} \ll \left\|1_{\{R>n\}}(|v_{t,n}| + K_{v_{t,n}})\right\|_{L^1(\mu_{\overline{Y}})} \le (|t|+1)\|1_{\{R>n\}}\|_{L^1(\mu_{\overline{Y}})} \tag{A.11}$$

and

$$\|\partial_j B_{t,n}(v)\|_{\mathcal{B}_1(\overline{Y})} \ll \left\|1_{\{R>n\}}(|\partial_j v_{t,n}| + K_{\partial_j v_{t,n}})\right\|_{L^1(\mu_{\overline{Y}})}. \tag{A.12}$$

But on $\{R > n\}$,

$$K_{v_{t,n}} \le (1 + |t| C_{\overline{\Psi}})\|v\|_{\mathcal{B}_0} \quad \text{and} \quad K_{\partial_j v_{t,n}} \le |\overline{\Psi}|_R K_{v_{t,n}} + C_{\overline{\Psi}}\|v\|_\infty, \tag{A.13}$$

where $C_{\overline{\Psi}}$ is the constant appearing in Proposition A.3. Indeed, for any $a \in \alpha$ and any $y, y' \in a$, writing $[v]_{\mathcal{B}_0}$ for the Lipschitz constant of v and using the fact that

$$\begin{aligned}\overline{s}((y, R(a)-n), (y', R(a)-n)) &= \overline{s}((F(y),0),(F(y'),0)) + n \ge s'(F(y),F(y')) + n \\ &= s'(y,y') + n - 1 \ge s'(y,y'),\end{aligned}$$

we observe that

$$\begin{aligned}&\left| e^{i\langle t, \overline{\Psi}_n(y,R(y)-n))\rangle} v(y,R(y)-n) - e^{i\langle t, \overline{\Psi}_n(y',R(y')-n))\rangle} v(y',R(y')-n) \right| \\ &\le \left(|t| C_{\overline{\Psi}}\|v\|_\infty + [v]_{\mathcal{B}_0}\right) \beta^{s'(y,y')},\end{aligned}$$

which ends the proof of the first part of (A.13). The second comes from the standard bound of the Lipschitz constant of a product. It follows from (A.11), (A.12) and (A.13) that

$$\|B_{t,n}(v)\|_{\mathcal{B}_1(\overline{Y})} \ll (1+|t|)\|1_{\{R>n\}}\|_{L^1(\mu_{\overline{Y}})}\|v\|_{\mathcal{B}_0}$$

and

$$\|\partial_j B_{t,n}(v)\|_{\mathcal{B}_1(\overline{Y})} \ll (1+|t|)\left\|1_{\{R>n\}}|\overline{\Psi}|_R\right\|_{L^1(\mu_{\overline{Y}})}\|v\|_{\mathcal{B}_0}.$$

The result follows from the fact that $\mu_{\overline{Y}}(R > n)$ decays exponentially fast in n as $n \to +\infty$ and from the fact that $|\overline{\Psi}|_R$ is $L^{2-\epsilon}$ for any $\epsilon > 0$. $\square$

For $z\in\mathbb{C}$ with $|z|\le 1$ and $t\in\mathbb{R}^d$, define

$$\widehat{E}(z,t):\mathcal{B}_0\to L^1(\overline{\Delta}),\qquad \widehat{E}(z,t)(v)=\sum_{n=1}^{\infty} z^n E_{t,n}(v),$$

where $E_{t,n}(v)(y,\ell)=1_{\{\ell>n\}}P_t^n(v)(y,\ell)$.

Proposition A.8. *There exists $\delta_0>0$ such that regarded as operators from $\mathcal{B}_0$ to $L^1(\overline{\Delta})$,*

(a) *$z\mapsto \widehat{E}(z,t)$ is analytic on $B_{1+\delta_0}(0)$ for all $t\in\mathbb{R}^d$;*
(b) *$(z,t)\mapsto \widehat{E}(z,t)$ is C^0 on $B_{1+\delta_0}(0)\times\mathbb{R}^d$;*

Proof. The proof goes word for word as [22, Proof of Proposition 5.5] since it just uses the Hölder inequality combined with the fact that $\|R1_{R>n}\|_{L^1(\overline{Y})}$ decays exponentially fast in n as $n\to+\infty$. $\square$

A.4.2. *Further estimates.* The results contained in this subsection are the analogue of [22, Proposition 5.6–5.9]. Since, we will have to deal with K_u coming from (A.9), some modifications are required in these proofs. We detail the parts corresponding to these modifications and indicate which parts of the proofs of [22] remain the same.

Proposition A.9. *There exist $C>0$, $\delta_0>0$ and $b>0$ such that*

$$\|\partial_j\partial_z\widehat{Q}(z,t+h)-\partial_j\partial_z\widehat{Q}(z,t)\|_{\mathcal{B}_1(\overline{Y})}\le C|h|L_h^2\{1+|h|^{-b\log|z|}L_h(|z|-1)\},$$

for all $t,h\in B_{\delta_0}(0)$, all $z\in\mathbb{C}$ with $1\le|z|\le 1+\delta$, and all $j=1,\dots,d$.

Proof. We observe that

$$\partial_j\partial_z Q(z,t+h)(v)-\partial_j\partial_z Q(z,t)(v)=iQ\left(w_{t,h}Rz^{R-1}v)\right),$$

with $w_{t,h}:=(\overline{\Psi}_R)_j e^{i\langle t,\overline{\Psi}_R\rangle}\left(e^{i\langle h,\overline{\Psi}_R\rangle}-1\right)$. It follows from Propositions A.4 and A.3 that

$$\begin{aligned}&\|\partial_j\partial_z Q(z,t)\|_{\mathcal{B}_1(\overline{Y})}\\ &\ll\sum_{n=1}^{\infty} n|z|^{n-1}\left\|1_{\{R=n\}}\left(|\overline{\Psi}_R(e^{i\langle h,\overline{\Psi}_R\rangle}-1)|+K_{w_{t,h}}\right)\right\|_{L^1(\mu_{\overline{Y}})},\\ &\ll\sum_{n=1}^{\infty} n|z|^{n-1}\left\|1_{\{R=n\}}\left(|\overline{\Psi}_R(e^{i\langle h,\overline{\Psi}_R\rangle}-1)|(1+(1+|t|)C_{\overline{\Psi}})+C_{\overline{\Psi}}|h|\left(|\overline{\Psi}_R|+C_{\overline{\Psi}}\right)\right)\right\|_{L^1(\mu_{\overline{Y}})}.\end{aligned}$$

Indeed, for all partition elements a and for all $y,y'\in a$,

$$\begin{aligned}&|w_{t,h}(y)-w_{t,h}(y')|\\ &\le\left|(\overline{\Psi}_R(y))_j e^{i\langle t,\overline{\Psi}_R(y)\rangle}-(\overline{\Psi}_R(y'))_j e^{i\langle t,\overline{\Psi}_R(y')\rangle}\right|\left|e^{i\langle h,\overline{\Psi}_R(y)\rangle}-1\right|\\ &\quad+\left|\overline{\Psi}_R(y')\right|\left|e^{i\langle h,\overline{\Psi}_R(y)\rangle}-e^{i\langle h,\overline{\Psi}_R(y')\rangle}\right|\\ &\le C_{\overline{\Psi}}(1+|t|)|\overline{\Psi}_R|\beta^{s'(y,y')}\left|e^{i\langle h,\overline{\Psi}_R(y)\rangle}-1\right|+|h|\left|\overline{\Psi}_R(y')\right|C_{\overline{\Psi}}\beta^{s'(y,y')},\end{aligned}$$

due to (A.10) and Proposition A.3; this gives the required domination of $K_{w_{t,h}}$. Thus, we have proved that

$$\|\partial_j\partial_z Q(z,t)\|_{\mathcal{L}(\mathcal{B}_1(\overline{Y}))} \ll \left\|\left(|\overline{\Psi}_R(e^{i\langle h,\overline{\Psi}_R\rangle}-1)|+|h|(1+|\overline{\Psi}_R|)\right) Rz^R\right\|_{L^1(\mu_{\overline{Y}})}$$

$$\ll \left(\left\||\overline{\Psi}|_R \min(|h|\,|\overline{\Psi}|_R,1)Rz^R\right\|_{L^1(\mu_{\overline{Y}})}+|h|\right), \tag{A.14}$$

provided δ is small enough since $\overline{\Psi}_R \in L^{2-\varepsilon}$ for all $\varepsilon \in (0,2)$ and since $\mu_{\overline{Y}}(R>n)$ decays exponentially fast in n as $n \to +\infty$. It remains to estimate the first term of the right hand side of (A.14). For any $x \in \overline{Y}$, let us write $\psi(x)$ for the supremum of the upper integer part of $|\overline{\Psi}|_R$ on the partition atom containing x. Then

$$\left\||\overline{\Psi}|_R \min(|h|\,|\overline{\Psi}|_R,1)Rz^R\right\|_{L^1(\mu_{\overline{Y}})} \le \left\|\psi \min(|h|\,\psi,1)Rz^R\right\|_{L^1(\mu_{\overline{Y}})} \tag{A.15}$$

provided δ is small enough. But

$$\left\|\psi \min(|h|\,\psi,1)Rz^R\right\|_{L^1(\mu_{\overline{Y}})} \ll \sum_{m,n=1}^{\infty} r_{m,n},$$

where

$$r_{m,n} = \mu_{\overline{Y}}(\psi=m, R=n)mn\min\{|h|m,1\}|z|^n. \tag{A.16}$$

The rest of the proof then follows the same lines as [22, Proposition 5.6]. □

Remark A.10. Similarly to [22, Remark 5.7], a simplified version of the argument used in the proof of Proposition A.9 (only for the derivative in j) gives

$$\|\partial_j\widehat{Q}(z,t+h)-\partial_j\widehat{Q}(z,t)\|_{\mathcal{B}_1(\overline{Y})} \le C|h|L_h\big\{1+|h|^{-b\log|z|}L_h(|z|-1)\big\}.$$

Proposition A.11. *There exist $C>0$, $\delta>0$ and $b>0$ such that*

$$\|\partial_j\widehat{A}(z,t+h)-\partial_j\widehat{A}(z,t)\|_{\mathcal{B}_1(\overline{Y})\mapsto L^1(\overline{\Delta})} \le C|h|L_h\big\{1+|h|^{-b\log|z|}L_h(|z|-1)\big\},$$

for all $t,h\in B_\delta(0)$, all $z\in\mathbb{C}$ with $1\le|z|\le 1+\delta$, and all $j=1,\dots,d$.

Proof. We have

$$(\widehat{A}(z,t)v)(y,\ell) = \sum_{n=1}^{\infty} z^n 1_{\{\ell=n\}} e^{i\langle t,\overline{\Psi}_n(y,0)\rangle}v(y) = z^\ell e^{i\langle t,\overline{\Psi}_\ell(y,0)\rangle}v(y),$$

for $\ell=0,\dots,R(y)-1$. Hence we can proceed as in the proof of Proposition A.9, except that there is one less factor of n (and so one less factor of L_h). □

Proposition A.12. *There exist $C>0$, $\delta>0$ and $b>0$ such that*

$$\|\partial_j\widehat{B}(z,t+h)-\partial_j\widehat{B}(z,t)\|_{\mathcal{B}_0\mapsto\mathcal{B}_1(\overline{Y})} \le C|h|L_h\big\{1+|h|^{-b\log|z|}L_h(|z|-1)\big\},$$

for all $t,h\in B_\delta(0)$, all $z\in\mathbb{C}$ with $1\le|z|\le 1+\delta$, and all $j=1,\dots,d$,

Proof. Let $v\in\mathcal{B}_0$. In the notation of Proposition A.7,

$$\partial_j\widehat{B}(z,t+h)(v)-\partial_j\widehat{B}(z,t)(v) = i\sum_{n\ge1} z^n Q(1_{\{R>n\}}(\partial_j v_{t+h,n}-\partial_j v_{t,n})) \tag{A.17}$$

with

$$v_{t,n}(y) := e^{i\langle t,\overline{\Psi}_n(y,R(y)-n)\rangle}v(y,R(y)-n).$$

Therefore

$$\text{(A.18)} \quad \partial_j v_{t+h,n} - \partial_j v_{t,n} = \left(\overline{\Psi}_n(\cdot, R(\cdot)-n)e^{i\langle t,\overline{\Psi}_n(\cdot,R(\cdot)-n)\rangle}\right)_j \left(e^{i\langle h,\overline{\Psi}_n(\cdot,R(\cdot)-n)\rangle}-1\right) v(\cdot, R(\cdot)-n).$$

It follows from Proposition A.4 that

$$\text{(A.19)} \quad \left\|Q(1_{\{R>n\}}\partial_j v_{t+h,n} - \partial_j v_{t,n})\right\|_{\mathcal{B}_1(\overline{Y})} \ll \left\|1_{\{R>n\}}\left(|\partial_j v_{t+h,n} - \partial_j v_{t,n})| + K_{\partial_j v_{t+h,n}-\partial_j v_{t,n}}\right)\right\|_{L^1(\mu_{\overline{Y}})}.$$

We proceed as in the proof of Proposition A.9. It follows from (A.18) that

$$\text{(A.20)} \quad \left\|1_{\{R>n\}}(\partial_j v_{t+h,n} - \partial_j v_{t,n})\right\|_{L^1(\mu_{\overline{Y}})} \le \left\|1_{\{R>n\}}\overline{\Psi}_n(\cdot, R(\cdot)-n)\left(e^{i\langle h,\overline{\Psi}_n(\cdot,R(\cdot)-n)\rangle}-1\right)\right\|_{L^1(\mu_{\overline{Y}})}\|v\|_\infty.$$

Furthermore, due to Proposition A.3, for all $a\in\pi$ and all $y,y'\in a$,

$$\begin{aligned}&|(\partial_j v_{t+h,n} - \partial_j v_{t,n})(y) - (\partial_j v_{t+h,n} - \partial_j v_{t,n})(y')|\\ &\ll \beta^{s(y,y')}\left\|1_{R>n}\left(1+|\overline{\Psi}_n(\cdot,R(\cdot)-n)|\right)\left(e^{i\langle h,\overline{\Psi}_n(\cdot,R(\cdot)-n)\rangle}-1\right)\right\|_{L^1(\mu)}(1+C_{\overline{\Psi}})\|v\|_{\mathcal{B}_0}\\ &+|h|\beta^{s(y,y')}C_{\overline{\Psi}}\left\|1_{R>n}(\overline{\Psi}_n(\cdot,R(\cdot)-n))_j\right\|_{L^1(\mu)}\|v\|_\infty.\end{aligned}$$

This combined with (A.19) and (A.20) ensures that

$$\begin{aligned}&\|Q(1_{R>n}\partial_j v_{t+h,n} - \partial_j v_{t,n})\|_{\mathcal{B}_1(\overline{Y})}\\ &\ll \left\|1_{\{R>n\}}(1+\overline{\Psi}_n(\cdot,R(\cdot)-n))\left(e^{i\langle h,\overline{\Psi}_n(\cdot,R(\cdot)-n)\rangle}-1\right)\right\|_{L^1(\mu_{\overline{Y}})}\|v\|_{\mathcal{B}_0}+|h|\|v\|_\infty\\ &\ll \left\|1_{\{R>n\}}\psi\min(|h|\,|\psi|)\right\|_{L^1(\mu_{\overline{Y}})}\|v\|_{\mathcal{B}_0}+|h|\|v\|_\infty.\end{aligned} \tag{A.21}$$

But

$$\sum_{n\ge1}\left\|z^n 1_{\{R>n\}}\psi\min(|h|\,|\psi|)\right\|_{L^1(\mu_{\overline{Y}})} = \sum_{m,n\ge1}\mu_{\overline{Y}}(\psi=m,R=n)m\min\{|h|m,1\}|z|^n,$$

which can be estimated as in the proof of Proposition A.9. We conclude by combining this estimate with (A.17) and (A.21). □

The rest of the proofs of [22] (corresponding to Section 5.2 therein that provide all the required spectral properties for $\widehat{Q}(z,t)$) go through unchanged.

Appendix B. Smoothness of $\overline{\tau}$ and χ

Recall that $\overline{\Delta}\subset\Delta$. Using the fact that $T\circ\pi=\pi\circ f_\Delta$ on Δ, and that $f_{\overline{\Delta}}=\overline{\pi}\circ f_\Delta$ on $\overline{\Delta}$, $\overline{\tau}$ and χ_0 defined in Section 6 can be rewritten as follows

$$\overline{\tau} := \widetilde{\tau}\circ\pi + \sum_{n\ge1}\left(\tau\circ T^n\circ\pi - \tau\circ T^{n-1}\circ\pi\circ\overline{\pi}\circ f_\Delta\right) \quad \text{on } \overline{\Delta}$$

and

$$\chi_0 := \sum_{n\ge0}\chi_{0,n}, \quad \text{with } \chi_{0,n} := (\tau\circ T^n\circ\pi - \tau\circ T^n\circ\pi\circ\overline{\pi}) \quad \text{on } \Delta.$$

First, observe that, for every $x \in \Delta$, $\pi(x)$ and $\pi(\overline{\pi}(x))$ are in the same stable manifold, thus $d(T^n(\pi(x)), T^n(\pi(\overline{\pi}(x)))) \le C_1\beta_1^n$, and so

$$\forall x \in \Delta, \quad |\chi_{0,n}| \le 2C_1\beta_1^n. \tag{B.1}$$

Analogously, for all $x \in \overline{\Delta}$,

$$\begin{aligned} &|\tau(T^n(\pi(x))) - \tau(T^{n-1}(\pi(\overline{\pi}(f_\Delta(x)))))| \qquad\qquad (B.2)\\ &\quad = |\tau(T^{n-1}(\pi(f_\Delta(x)))) - \tau(T^{n-1}(\pi(\overline{\pi}(f_\Delta(x)))))| \\ &\quad \le 2C_1\beta_1^{n-1}. \qquad\qquad (B.3)\end{aligned}$$

This ensures that χ_0 and $\overline{\tau}$ are well defined and we have proved the identity

$$\widetilde{\tau}\circ\pi - \chi_0 + \chi_0\circ f_\Delta = \widetilde{\tau}\circ\pi\circ\overline{\pi} + \sum_{n\ge 0}\left(\tau\circ T^{n+1}\circ\pi\circ\overline{\pi} - \tau\circ T^n\circ\pi\circ\overline{\pi}\circ f_\Delta\right) = \overline{\tau}\circ\overline{\pi}$$

since $\overline{\pi}\circ f_\Delta = f_{\overline{\Delta}}\circ\overline{\pi}$.

For any $x, y \in \overline{\Delta}$ such that $\overline{s}(x,y) = N \ge 2k$, then $T^N(\pi(x))$ and $T^N(\pi(y))$ are in a same unstable manifold and the same holds true for $T^{N-1}(\overline{\pi}(f_\Delta(x)))$ and $T^{N-1}(\overline{\pi}(f_\Delta(y)))$. This implies that, for every $n = 0, \dots, N$, $d(T^n(\pi(x)), T^n(\pi(y))) \le C_1\beta_1^{N-n}$ and $d(T^{n-1}(\pi(\overline{\pi}(f_\Delta(y)))), T^{n-1}(\pi(\overline{\pi}(f_\Delta(x)))) \le C_1\beta_1^{N-n-1}$. Therefore

$$\begin{aligned} |\overline{\tau}(x) - \overline{\tau}(y)| &\le |\widetilde{\tau}(\pi(x)) - \widetilde{\tau}(\pi(y))| \\ &\quad + 2\sum_{n\ge\lceil\overline{s}(x,y)/2\rceil+1} \|\tau\circ T^n\circ\pi - \tau\circ T^{n-1}\circ\pi\circ\overline{\pi}\circ f_\Delta\|_\infty \\ &\quad + \sum_{n=1}^{\lceil\overline{s}(x,y)/2\rceil} |\tau(T^n(\pi(x))) - \tau(T^n(\pi(y))) \\ &\quad - [\tau(T^{n-1}(\pi(\overline{\pi}(f_\Delta(x))))) - \tau(T^{n-1}(\pi(\overline{\pi}(f_\Delta(y)))))]| \\ &\le 2\left(\beta_1^{\overline{s}(x,y)} + 2\sum_{n\ge\lceil\overline{s}(x,y)/2\rceil+1}\beta_1^n + 2\sum_{n=1}^{\lceil\overline{s}(x,y)/2\rceil}\beta_1^{\overline{s}(x,y)-n-1}\right) \\ &\le 2\beta_1^{\frac{\overline{s}(x,y)}{2}}\left(1 + 4\frac{\beta_1^{-1}}{1-\beta_1}\right). \end{aligned}$$

Now, let us prove that χ_0 satisfies

$$\sup_{k\ge 1}\sup_{x,y: s(x,y)>2k} \frac{|\chi_0(f_\Delta^k(x)) - \chi_0(f_\Delta^k(y))|}{\beta^k} < \infty.$$

Observe that

$$\chi_0\circ f_\Delta^k = \sum_{n\ge 0}\chi_{0,n}\circ f_\Delta^k, \quad \text{and } \chi_{0,n}\circ f_\Delta^k = \tau\circ T^{n+k}\circ\pi - \tau\circ T^n\circ\pi\circ\overline{\pi}\circ f_\Delta^k. \tag{B.4}$$

Let $x, y \in \Delta$ be such that $s(x,y) = N > 2k$.

- for $n > k/2$, we observe that it follows from (B.1) that

$$|\chi_{0,n}(f_\Delta^k(x)) - \chi_{0,n}(f_\Delta^k(y))| \le 2\|\chi_{0,n}\|_\infty \le 4C_1\beta_1^n. \tag{B.5}$$

- for $n = 0, \dots, k/2$, we observe that $\pi(x), \pi(\overline{\pi}(x))$ are in a same stable manifold, $\pi(y), \pi(\overline{\pi}(y))$ are also in a same stable manifold, and that $T^N(\pi(\overline{\pi}(x)),$

$T^N(\pi(\overline{\pi}(y)))$ are in a same unstable manifold. Therefore

$$|\tau(T^{n+k}(\pi(x))) - \tau(T^{n+k}(\pi(\overline{\pi}(x))))| \le 2C_1\beta_1^{n+k},$$
$$|\tau(T^{n+k}(\pi(y))) - \tau(T^{n+k}(\pi(\overline{\pi}(y))))| \le 2C_1\beta_1^{n+k},$$
$$|\tau(T^{n+k}(\pi(\overline{\pi}(x)))) - \tau(T^{n+k}(\pi(\overline{\pi}(y))))| \le 2C_1\beta_1^{N-n-k} \le 2C_1\beta_1^{k-n}.$$

Thus

$$|\tau \circ T^{n+k}(\pi(x)) - \tau \circ T^{n+k}(\pi(y))| \le 6C_1\beta_1^{k-n}. \tag{B.6}$$

Furthermore $x'_k = \overline{\pi}(f_\Delta^k(x)), y'_k = \overline{\pi}(f_\Delta^k(y)) \in \overline{\Delta}$ and $\overline{s}(x'_k, y'_k) = N - k$. Thus $T^{N-k}(\pi(\overline{\pi}(f_\Delta^k(x))))$ and $T^{N-k}(\pi(\overline{\pi}(f_\Delta^k(y))))$ are in the same unstable manifold and so

$$|\tau(T^n(\pi(\overline{\pi}(f_\Delta^k(x))))) - \tau(T^n(\pi(\overline{\pi}(f_\Delta^k(x)))))| \le 2C_1\beta_1^{N-k-n} \le 2C_1\beta_1^{k-n}. \tag{B.7}$$

It follows from (B.4) combined with (B.5), (B.6) and (B.7) that

$$\left|\chi_0(f_\Delta^k(x)) - \chi_0(f_\Delta^k(y))\right| \le \sum_{n>k/2} 4C_1\beta_1^n + \sum_{n=0}^{k/2} 6C_1\beta_1^{k-n} \ll \beta_1^{\frac{k}{2}}.$$

We conclude since $\beta_1^{\frac{1}{2}} \le \beta$.

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