Logarithmic Hochschild homology and cohomology

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Introduction

In this thesis, we study Hochschild homology and cohomology, which have nice expressions and properties for smooth and proper schemes and separated Deligne-Mumford stacks. The goal of this work is to adapt it to such spaces when they are obtained as compactifications of non-proper ones recording the boundaries, in the language of logarithmic geometry.

For simplicity, this introduction covers the main results present in this work stated for schemes. However, most of them apply to Deligne-Mumford stacks as well. They will be expressed in this more general version in the body of this thesis.

Logarithmic schemes

A classical way to encode the boundaries of non-proper algebraic schemes is via logarithmic algebraic geometry, which is the formalism that we work with. In this framework, the classical notion of smoothness is replaced by the one of logarithmic smoothness.

Logarithmic schemes are schemes with a logarithmic structure, i.e. a sheaf of monoids which is a submonoid of the structure sheaf (regarded with the multiplication) containing the units. Examples of logarithmically smooth logarithmic schemes include (proper) toric varieties, equipped by additional structure determined by the dense torus, and pairs made of a smooth (and proper) variety and a s.n.c. divisor (defining the logarithmic structure). Moreover, logarithmically smooth spaces typically include those with “mild” singularities (nodal) when equipped with an appropriate logarithmic structure. In fact, for such singularities, one may picture logarithmic smoothness as the smoothness of the strata of the logarithmic structure.

Logarithmic schemes naturally arise from compactifications to the sense of [Nag56], which associates smooth varieties with pairs of a proper variety and a (possibly empty) s.n.c. divisor, such that the original variety is embedded in the proper one with the divisor as complement. Such compactifications are, a priori, not unique. Think, for instance, of the affine plane $\mathbb{A}^2$ admitting both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ as compactifications. The pairs $(\mathbb{P}^2, \mathbb{P}^1)$ and $(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \cup \mathbb{P}^1)$ give two different logarithmic schemes, both associated with $\mathbb{A}^2$ in the above sense.

Hochschild homology and cohomology

Hochschild homology and cohomology are tools of crucial importance in commutative and noncommutative algebra and algebraic geometry. The geometric approach to the subject is to define them through derived complexes (the Hochschild homology and cohomology complexes), whose global sections give the Hochschild homology and cohomology vector spaces.

Definition 1. Let X be a smooth algebraic variety over $S = \operatorname{Spec} k$. Denote by $\Delta : X \to X \times_S X$ the diagonal morphism. The Hochschild homology (resp. cohomology) derived complex is

$$\mathbf{L}\Delta^*\mathbf{R}\Delta_*\mathcal{O}_X \quad (\text{resp. } \Delta^!\mathbf{R}\Delta_*\mathcal{O}_X).$$

The Hochschild homology (resp. cohomology) of X is the graded k-vector space $\mathbf{R}\Gamma(X, \mathbf{L}\Delta^*\mathbf{R}\Delta_*\mathcal{O}_X)$ (resp. $\mathbf{R}\Gamma(X, \Delta^!\mathbf{R}\Delta_*\mathcal{O}_X)$).

They have a nice "sheafy" expression for smooth varieties in terms of the cotangent and tangent bundles, and when the varieties are moreover proper the Hochschild homology and cohomology spaces have good properties. This is the content of the HKR theorem ([HKR62], [AC12]). A direct consequence is that the dimensions of the Hochschild homology spaces for smooth and proper varieties give information on its Hodge diamond. The low-degree Hochschild cohomology spaces play a role in deformation theory, encoding their first-order deformations and their obstructions ([Ger64], [Tod09]). The Hochschild homology and cohomology spaces are preserved under derived equivalence ([Cal03]) and may be defined for abstract derived categories and abelian categories ([LVdB05], [Kel21]). To conclude, Hochschild homology and cohomology may be defined for noncommutative schemes. In the noncommutative framework, the homology spaces play the role of Hodge cohomology in the commutative setup. A closely related notion is the one of cyclic homology, which may be interpreted as the noncommutative counterpart of the de Rham cohomology ([Lod98]).

Logarithmic Hochschild homology and cohomology

This thesis may be resumed as an extended affirmative answer to the following question.

Question 2. Are there Hochschild-like invariants for logarithmic schemes, coinciding with the usual ones for schemes with the trivial logarithmic structure and with an HKR-like decomposition in terms of the logarithmic cotangent and tangent bundles?

The main difficulty that one encounters trying to define similar endofunctors which are logarithmically compatible is that there are many attempts to define logarithmic analogous of the derived category of coherent sheaves ([MS80, Yok95], [TV18], [Vai17]) which appear to be convenient for some purposes but less appropriate for others. The author has no reason to believe that any of these definitions should prevail over the others. Therefore, our techniques avoid the use of such a derived category. In particular, the derived pullback and pushforward for morphisms of logarithmic schemes are just those for the underlying scheme and do not extend to functors recording any logarithmic information. To make similar endofunctors to those in Definition 1 encode the logarithmic data, we alter the diagonal morphism, replacing Δ by i defined by the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & B & \longrightarrow & X \times X \\ & \searrow & \downarrow & \ulcorner & \downarrow \\ & & \Theta_X & \longrightarrow & \Theta_{X \times X} \end{array} , \qquad (1)$$

where Θ_X and $\Theta_{X\times X}$ are the Artin fans of X and $X \times X$ respectively and B is the pullback in the category of logarithmic algebraic stacks. This alteration of the diagonal, which we call strict or logarithmic diagonal, does not modify the logarithmic structure.

Artin fans

In the above construction, the Artin fans of X and $X \times X$ appear. They are logarithmic algebraic stacks encoding the logarithmic structure of the logarithmic scheme they are associated with, i.e. such that the morphism $X \to \Theta_X$ is strict. For a toric variety V with dense torus $\mathbb{G}_m^k \simeq T \subset V$, the Artin fan is simply given by the quotient stack $[V/T]$. Such quotients are called Artin cones. Geometrically, it coincides with the fan.

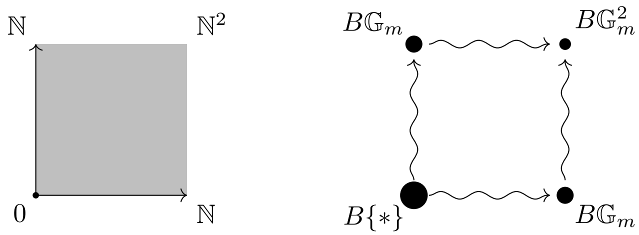


Figure 0.1: The fan of the affine plane $\mathbb{A}^2$ as toric variety, and its Artin fan.

Logarithmically smooth logarithmic schemes admit étale covers by toric varieties. the Artin fan is obtained by Artin cones glueing étale locally. A more extensive definition and treatment of the topic will be given in Section 2.3.

Working with Artin fans can be complicated, as the construction is not functorial. A famous example of the failure of this property is given by the Whitney umbrella [ACM$^+$15, Subsection 5.4.1]. To make sure that the bottom arrow in Diagram (1) exists and makes the diagram commute, we prove the following result.

Theorem A (Corollary 2.3.30)**.** Let X and Y be two logarithmically smooth quasicompact logarithmic schemes. Then the Artin fan of the product $X \times Y$ is isomorphic to the product of the Artin fans of the factors:

$$\Theta_{X\times Y} \xrightarrow{\sim} \Theta_X \times \Theta_Y .$$

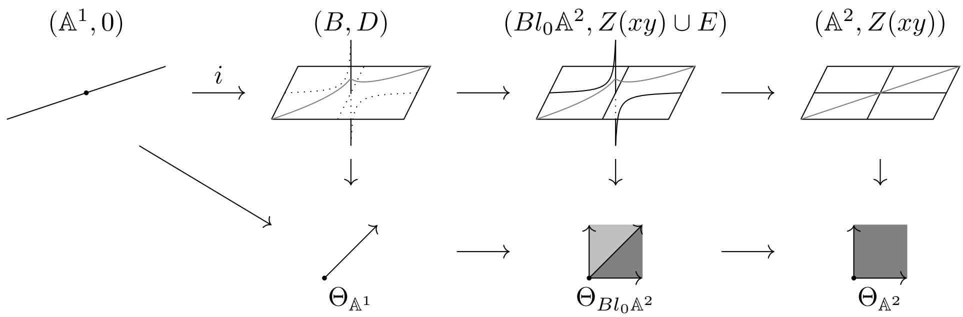


Figure 0.2: Construction of the logarithmic diagonal of the logarithmic scheme given by the pair $(\mathbb{A}^1, 0)$. For this example, it can be obtained by two subsequent pullbacks, first factoring the diagonal of the Artin fan through the subdivision of the fan. The target of the strict diagonal obtained like so is the open $B = Bl_0\mathbb{A}^2 \setminus Z(xy)$ with logarithmic structure induced by the exceptional divisor minus the points where it intersects the axis $D = E \setminus (E \cap Z(xy))$. Here we are representing the Artin fans via the fans as toric varieties.

This result arises as a corollary of the more general Theorem 2.3.29. Theorem A allows us to define the strict diagonal.

A formality result

After constructing the strict diagonal via Diagram (1), one can define the Hochschild homology and cohomology endofunctors as self-intersections along the strict diagonal.

Definition 3 (definition 3.1.7)**.** Let X be a logarithmically smooth, weakly logarithmically separated (definition 2.3.24), quasicompact logarithmic scheme. The logarithmic Hochschild homology (resp. cohomology) endofunctor of the derived cateogry of unbounded coherent sheaves of X is

$$\mathbf{L}i^*\mathbf{R}i_* : \mathscr{D}(X) \to \mathscr{D}(X) \quad (\text{resp. } i^!\mathbf{R}i_* : \mathscr{D}(X) \to \mathscr{D}(X)).$$

The logarithmic Hochschild homology (resp. cohomology) of X is

$$\mathrm{HH}^{\ell}_{\bullet}(X) = \mathbf{R}\Gamma(X, \mathbf{L}i^*\mathbf{R}i_*\mathscr{O}_X) \quad (\text{resp. } \mathrm{HH}^{\bullet}_{\ell}(X) = \mathbf{R}\Gamma(X, i^!\mathbf{R}i_*\mathscr{O}_X)).$$

The reason why $B := X \times X \times_{\Theta_{X \times X}} \Theta_X \simeq X \times_{\Theta_X} X$ is a suitable choice for the space where to regard the diagonal morphism is given by the following list of properties:

- the morphism $i : X \to B$ is strict: this means that it induces an isomorphism between the logarithmic structures;
- the morphism $B \to X \times X$ is logarithmically étale: this implies that the two spaces have the isomorphic logarithmic cotangent complexes; for easy

examples, one may think of the morphism to be a logarithmic blow-up (see Figure 0.2);

- the morphism $B \simeq X \times_{\Theta_X} X \to X \times_{\mathcal{L}} X$ is a open immersion, where $\mathcal{L}$ is the logarithmic algebraic stack $\mathcal{T}or$ from [Ols03], classifying (fs) logarithmic structures;
- the morphism i is a l.c.i. closed embedding: this allows us to use the techniques of [AC12].

With the aid of the properties listed above, we are able to prove the following statement. For brevity, we formulate it for the logarithmic Hochschild homology, but we point out that it has a cohomological version, that the reader can find in this thesis.

Theorem B (Corollary 3.2.6, Theorem 3.2.7)**.** Let X be a quasicompact, weakly logarithmically separated, logarithmically smooth logarithmic scheme.

- The endofunctor $\mathbf{L}i^*\mathbf{R}i_*$ is formal: there exists an isomorphism of dg endofunctors $\mathcal{D}(X) \to \mathcal{D}(X)$
$$\mathbf{L}i^*\mathbf{R}i_*(-) \xrightarrow{\sim} (-) \otimes \mathrm{Sym}(\Omega_X^{1,\log}[1]) \simeq (-) \otimes \bigwedge^{\bullet} \Omega_X^{1,\log}[\bullet].$$
Here, Sym is the derived symmetric algebra.
- The logarithmic Hochschild homology of $\mathcal{O}_X$ can be computed in terms of the logarithmic cotangent bundle:
$$\mathrm{HH}_{\bullet}^{\ell}(X) = \bigoplus_{q-p=\bullet} H^p(X, \Omega_X^{q,\log}).$$

Theorem B has some nice consequences. First, there is a natural transformation

$$\mathbf{L}\Delta^*\mathbf{R}\Delta_*(-) \to \mathbf{L}i^*\mathbf{R}i_*(-)$$

induced by the unit of the adjunction along the morphism $B \to X \times X$, giving rise to a long exact sequence imitating an Eilenberg–Steenrod axiom for smooth pairs (X, D), relating the Hochschild homology spaces of X and D, and the logarithmic Hochschild homology of the logarithmic scheme associated with the pair.

Another implication of Theorem B is the invariance of the endofunctor under pullback along logarithmically étale morphisms. In particular, the pullback along logarithmic blow-ups induces an isomorphism of the logarithmic Hochschild homology vector spaces. This fact suggests that there should be a relation between the logarithmic derived category of a logarithmic scheme and the one of its logarithmic blow-up. In the non-logarithmic case, we know by [BO95] that, for the blow-up $Bl_{(0,0)}\mathbb{P}^2 \to \mathbb{P}^2$ with exceptional divisor $j : E \to Bl_{(0,0)}\mathbb{P}^2$, we have

$$\mathcal{D}^b(Bl_{(0,0)}\mathbb{P}^2) = \langle j_*\mathcal{O}_E(-1), \mathcal{D}^b(\mathbb{P}^2)\rangle.$$

Supposing that the logarithmic derived category of the blow-up is subject to a similar decomposition, the component associated with the exceptional divisor should have trivial logarithmic Hochschild homology.

Finally, there is a logarithmic Künneth formula, stating that the logarithmic Hochschild homology graded vector space of a finite product of logarithmic schemes is isomorphic to the tensor products of the logarithmic Hochschild homology graded vector spaces of the factors

$$\mathrm{HH}^{\ell}_{\bullet}(X \times Y) \simeq \mathrm{HH}^{\ell}_{\bullet}(X) \otimes \mathrm{HH}^{\ell}_{\bullet}(Y).$$

Moreover, the approach to logarithmic Hochschild homology and cohomology via the strict diagonal morphism $i : X \to B$ allows us to revisit the construction using the perspective of the loop space [BZN12, TV07, BM93], naturally giving as logarithmic loop space of logarithmically flat logarithmic schemes the derived algebraic stack

$$L^{\log}_X := X \times^{\mathbf{R}}_B X.$$

The logarithmic loop space allows us to mimic the classical theory defining cyclic, negative and periodic homology theories in a compatible way with the logarithmic Hochschild homology.

We point out that an alternative definition of logarithmic Hochschild homology has been formulated in [BLPØ23], via a topological approach. The two definitions may be seen to coincide in the logarithmically smooth case using the HKR theorem, and are conjecturally equivalent in broader generality. However, the geometric approach followed in this thesis allows us to prove some properties typically appearing in the field, like some of those mentioned above.

An invariance result

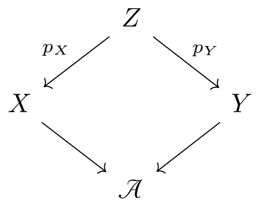
One of the properties of ordinary Hochschild homology and cohomology that is not easy to state in its logarithmic version is the derived invariance, proved for instance in [Cal03]. The reason behind this is once again the lack of a broadly accepted logarithmic version of the derived category. For the simple reason that the logarithmic Hochschild homology and cohomology spaces differ from the ordinary ones, we cannot expect the invariance to hold whenever the underlying schemes are derived equivalent.

We describe a condition, that we call strict logarithmic derived equivalence, under which invariance of logarithmic Hochschild homology and cohomology hold.

Definition 4 (definition 4.3.1)**.** Let X and Y be logarithmic schemes, whose underlying schemes are smooth and proper. We say that X and Y are strictly logarithmically derived equivalent if there are

- an Artin fan $\mathscr{A}$ and strict morphisms $\Theta_X \to \mathscr{A}$ and $\Theta_Y \to \mathscr{A}$;

- a smooth and proper logarithmic scheme Z and a commutative diagram



  where p_X and p_Y are strict and proper;

- an equivalence of type

$$\mathbf{R}p_{Y,*}(\mathbf{L}p_X^*(-) \otimes \mathscr{P}) : \mathscr{D}^b(X) \to \mathscr{D}^b(Y)$$

  for some $\mathscr{P} \in \mathscr{D}^b(Z)$.

Notice that every strict logarithmic derived equivalence is also a derived equivalence. This suggests the insufficiency of this condition for a comprehensive notion of logarithmic derived equivalence. However, we expect that any good definition of logarithmic derived equivalence should be satisfied by strict logarithmic derived equivalences.

We prove the invariance result under this condition.

Theorem C (theorem 4.3.6)**.** If X and Y are strictly logarithmically derived equivalent, separated, weakly logarithmically separated and logarithmically flat logarithmic schemes, then their logarithmic Hochschild homology and cohomology spaces are isomorphic.

Imitating the strategy of [Cal03], we define a logarithmic version of the logarithmic Serre functor relating logarithmic Hochschild homology and cohomology and we show that strict logarithmic derived equivalences conjugate them.

A formality result for orbifolds

The HKR theorem may be refined for orbifolds. An orbifold is a quotient stack obtained by the action of a finite group G on a scheme X. In particular, it is a Deligne-Mumford stack. We stress that, unlike in other references, we require them to be global quotients, instead of just local ones.

It is proved in [ACH19] that the Hochschild homology endofunctor for an orbifold $[X/G]$ may be decomposed as

$$\bigoplus_{g \in G} \mathbf{L}\Delta_g^* \mathbf{R}\Delta_* \simeq \bigoplus_{g \in G} \mathbf{R}q_* \left(\mathbf{L}p^*(-) \otimes \mathrm{Sym}((\Omega_X^1)_g[1]) \right)$$

where $\Delta_g : X \to X \times X$ is the twisted diagonal and q and p are the pullbacks of Δ and Δ_g respectively. After taking the global sections, the above expression provides the HKR decomposition for orbifolds.

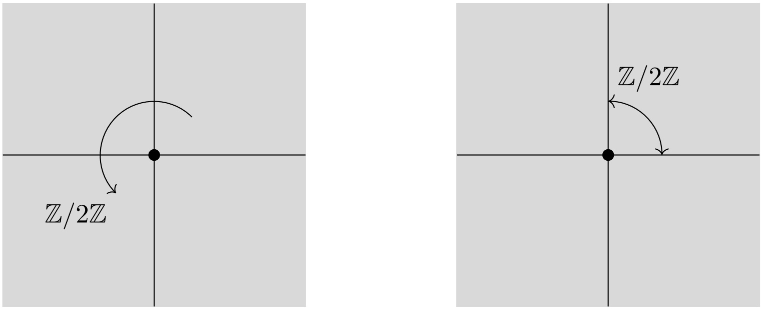


Figure 0.3: On the left, the firm action of $\mathbb{Z}/2\mathbb{Z}$ on the logarithmic scheme associated with $(\mathbb{A}^2, Z(xy))$, rotating the plane of 180°; this action maps each axis onto itself by reflecting it. On the right, the non-firm action of $\mathbb{Z}/2\mathbb{Z}$ on the same logarithmic scheme given by the reflection across the line $x = y$, swapping the two axes.

In order to prove a similar statement for logarithmic Hochschild homology, we need to understand the intersection functors $\mathbf{L}i_g^*\mathbf{R}i_*$. This problem is hard to tackle with our techniques in the very general case. The reason behind this difficulty is the fact that the action of G on X induces an action of $G \times G$ on $X \times X$, but this does not always lift to the strict diagonal $B = X \times_{\Theta_X} X$. For this reason, to produce a result along the lines of [ACH19], we restrict to firm actions, i.e. actions that induce the trivial action on the Artin fan.

For such actions, we prove the following theorem.

Theorem D (Corollary 5.1.16, Corollary 5.1.17)**.** Let X be a quasicompact, weakly logarithmically separated, logarithmically smooth logarithmic scheme and G a finite group acting firmly on X (definition 5.1.3).

- Denoting the strict diagonal for the orbifold $[X/G]$ by

$$\iota : [X/G] \to [X/G] \times_{\Theta_{[X/G]}} [X/G],$$

  we have the decomposition of the Hochschild homology endofunctor

$$\mathbf{L}\iota^*\mathbf{R}\iota_*(-) \simeq \bigoplus_{g \in G} \mathbf{L}i_g^*\mathbf{R}i_*(-)$$

  where $i_g : X \to X \times_{\Theta_X} X$ denotes the strict twisted diagonal induced by $x \mapsto (x, g.x)$;

- There are natural isomorphisms of endofunctors

$$\mathbf{L}i_g^*\mathbf{R}i_*(-) \simeq \mathbf{R}q_* \left(\mathbf{L}p^*(-) \otimes \mathrm{Sym}\left(\left(\Omega_X^{1,\log} \right)_g [1] \right) \right);$$

- There are isomorphisms of vector spaces

$$\mathrm{HH}^{\ell}_{\bullet}\left([X/G]\right) = \left(\bigoplus_{g\in G} \mathrm{HH}^{\ell}_{n}\left(X^{g}_{\log}\right)\right)^{G} = \left(\bigoplus_{g\in G}\bigoplus_{q-p=n} H^{p}\left(X^{g}_{\log}, \Omega^{q,\log}_{X^{g}_{\log}}\right)\right)^{G}.$$

Structure of the thesis

This thesis is articulated in five chapters. In Chapter 1, we provide an overview of classical facts and methods about Hochschild homology and cohomology, including their definitions for different structures, the HKR theorem, some invariance results, their role in Hodge theory and the on of the cohomology in deformation theory, with a focus on the approach used in [AC12, ACH19] to reprove the HKR theorem. In Chapter 2, we establish the background on logarithmic geometry, starting with the classical approach of [Kat89, Ogu18], then revisiting it is the more modern lexicon of [Ols03, Ols05] and eventually introducing Artin fans and their role in logarithmic geometry and proving Theorem A. In Chapter 3, we give the definition of logarithmic Hochschild homology and cohomology and prove Theorem B, we prove some properties relying on the HKR theorem, we explain what these results mean for smooth pairs, we exhibit some examples of computation of logarithmic Hochschild homology, we define the logarithmic loop space and describe how it fits in the picture and conjecture over the Duflo isomorphism for the logarithmic Hochschild cohomology. In Chapter 4, we introduce the logarithmic Serre functor and use Fourier–Mukai theory to relate logarithmic Hochschild homology and cohomology and we prove Theorem C. In Chapter 5, we investigate intersection theory along l.c.i. morphisms between logarithmic schemes which are not smooth in general (but typically the l.c.i. morphism in question is strict and between schemes which are logarithmically smooth), we use it to prove Theorem D and we conduct an explicit computation for the orbifold obtained by the shuffle action of the symmetric group, displaying a HKR-like decomposition for this example too.

Conventions and notations

Unless differently stated, all schemes and algebraic stacks are meant over $S = \operatorname{Spec} k$ where k is an algebraically closed field of characteristic 0. Similarly, the symbols $\times, \otimes, \mathrm{Hom}$ stand for $\times_S, \otimes_k, \mathrm{Hom}_k$ respectively. By variety, we mean a separated scheme of finite type over S. The properties for schemes or algebraic stacks like flatness, smoothness, separateness are meant over S.

The fiber product of logarithmic schemes is by default meant in the category of fs logarithmic schemes, unless stated otherwise or clear from the context. We denote by $\times, \ulcorner$ the fiber product in the category of logarithmic schemes, by $\times^{\ell}, \ulcorner\ell$ the fiber product in the category of fs logarithmic schemes and by $\times^{\not\ell}, \ulcorner\not\ell$ the fiber product when the two coincide. Throughout the thesis, the stack classifying logarithmic structures will be denoted by $\mathcal{L}og$ and the stack following the classical literature, whilst the stack denoted by $\mathcal{L}$ is the one classifying fs logarithmic structures denoted by $\mathcal{T}or$ in [Ols03]. From chapter 3, all logarithmic schemes are assumed to be fs logarithmic schemes.

Unless differently stated or clear from the context, all functors are derived. The symbols Sym, $\times$, $\otimes$, Hom, f^*, $f_*, \ldots$ denote $\mathbf{L}$Sym, $\times^{\mathbf{L}}$, $\otimes^{\mathbf{L}}$, $\mathbf{R}$Hom, $\mathbf{L}f^*$, $\mathbf{R}f_*, \ldots$ respectively.